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BLOW-UP OF SOLUTIONS TO PARABOLIC EQUATIONS WITH NONSTANDARD GROWTH CONDITIONS

S. ANTONTSEV AND S. SHMAREV

Abstract. We study the phenomenon of finite time blow-up in solutions of the homogeneous Dirichlet problem for the parabolic equation

$$u_t = \operatorname{div}\left(a(x,t)|\nabla u|^{p(x)-2}\nabla u\right) + b(x,t)|u|^{\sigma(x,t)-2}u$$

with variable exponents of nonlinearity $p(x)$, $\sigma(x,t) \in (1,\infty)$. Two different cases are studied. In the case of semilinear equation with $p(x) \equiv 2$, $a(x,t) \equiv 1$, $b(x,t) \geq b^- > 0$ we show that the finite time blow-up happens if the initial function is sufficiently large and either $\min_\Omega \sigma(x,t) = \sigma^-(t) > 2$ for all $t > 0$, or $\sigma^-(t) \geq 2$, $\sigma^-(t) \searrow 2$ as $t \to \infty$ and $\displaystyle\int_1^\infty \mathrm{e}^{s(2-\sigma^-(s))}\,ds < \infty$. In the case of the evolution $p(x)$-Laplace equation with the exponents $p(x)$, $\sigma(x)$ independent of t, we prove that every solution corresponding to a sufficiently large initial function exhibits a finite time blow-up if $a_t(x,t) \leq 0$, $b_t(x,t) \geq 0$, $\min \sigma(x) > 2$ and $\max p(x) \leq \min \sigma(x)$.

1. Introduction

This work addresses the blow-up phenomenon in solutions of nonlinear parabolic equations with variable nonlinearity. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz-continuous boundary Γ, $Q_T = \Omega \times (0,T]$ and $\Gamma_T = \Gamma \times (0,T]$. We consider the Dirichlet problem for the class of degenerate parabolic equations with variable exponents of nonlinearity

$$(1.1) \quad \begin{cases} u_t = \operatorname{div}\left(a(x,t)\,|\nabla u|^{p(x)-2}\,\nabla u\right) + b(x,t)\,|u|^{\sigma(x,t)-2}\,u & \text{for } (x,t) \in Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T. \end{cases}$$

The coefficients a, b and the exponents p, σ are given measurable functions of their arguments. It is assumed that these functions satisfy the following conditions:

$$(1.2) \qquad 0 < a^- \leq a(x,t) \leq a^+ \leq \infty,\ 0 \leq b^- \leq b(x,t) \leq b^+ \leq \infty.$$

$$(1.3) \qquad 1 < p^- \leq p(x) \leq p^+ < \infty,\ 1 < \sigma^- \leq \sigma(x,t) \leq \sigma^+ < \infty,$$

Equations of the type (1.1) appear in the mathematical modelling of various physical phenomena such as flows of electro-rheological or thermo-rheological fluids

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[2, 7, 26], processes of filtration through a porous medium. They are frequently used in the processing of digital images [1, 15, 21]. For a more detailed information on the possible applications of these models to the study of the real world processes we refer the reader to the papers [9, 26, 27] and the further references therein.

Equations of the type (1.1) are usually referred to as equations with nonstandard growth conditions. In the recent years, PDEs of this type have been intensively studied. The questions of existence, uniqueness and qualitative properties of solutions for elliptic and parabolic equations with variable nonlinearity were discussed by many authors and under different conditions on the data - see, for example, [3, 4, 5, 6, 8, 9, 10, 11, 12, 14].

It is known that parabolic equations with variable nonlinearity may possess, for certain ranges of the exponents, the localization (alias vanishing) properties which are intrinsic for the solutions of nonlinear equations with constant nonlinearity such as vanishing in a finite, finite speed of propagation on disturbances from the data or waiting time phenomena (see [10, 11, 12, 13]), but thus far only one work [23] has addressed the question of possible blow-up of solutions of the parabolic PDEs with nonstandard growth conditions. An excellent insight into the theory of blow-up behavior of solutions to parabolic equations with constant nonlinearity can be found in the monographs [19] and [28] (see also [16, 17, 18, 22, 24, 25, 29, 30]). Paper [23] deals with the solutions of the homogeneous Dirichlet problem for the semilinear parabolic equation

$$\begin{cases} u_t = \Delta u + f(x,u) \quad \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases}$$

where the source term is either a power,

$$f(x,u) = a(x)u^{p(x)} \quad \text{or is nonlocal: } f(x,u) = a(x)\int_\Omega u^{q(y)}(y,t)dy.$$

In the present paper, we consider a more general class of parabolic equations with nonstandard growth conditions. In Section 2 we give the definition of weak solution to problem (1.1) and remind the existence theorem.

In Section 3 we study problem (1.1) for the semilinear equations with $a(x,t) = 1$, $p(x,t) = 2$, and variable $\sigma(x,t)$, $b(x,t)$. The first result of this section extends the assertion of [23] to the case when the exponent of nonlinearity in the source term may depend on t. The second result is specific to case when the exponent σ depends on t. We show that the solutions of the semilinear problem (1.1) may blow-up even in the case when $\sigma(x,t) \searrow 2$ as $t \to \infty$ and the equation eventually becomes linear.

In Section 4 we present some examples and generalize the conclusions of Section 4 to the case when the Laplace operator is substituted by a linear elliptic operator of general form but with the coefficients independent of t. Another generalization concerns the form of the source term which can be nonlocal.

In Section 5 we establish sufficient condition of the blow-up for solutions of problem (1.1) assuming that the exponents of nonlinearity $p(x)$ and $\sigma(x)$ are independent of t, and satisfy the condition $p^+ < \sigma^-$. The coefficients $a(x,t)$, $b(x,t)$ are assumed differentiable in t and monotone: $a_t(x,t) \leq 0$, $b_t(x,t) \geq 0$.

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2. Preliminaries

Let $p(x,t)\in C^0(Q_T)$. We introduce the set of functions

$$L^{p(\cdot)}(Q_T)=\left\{u(x,t):\ u \text{ is measurable in } Q_T,\ A_{p(\cdot)}(u)\equiv\int_{Q_T}|u|^{p(x,t)}\,dxdt<\infty\right\}.$$

The set $L^{p(\cdot)}(Q_T)$ equipped with the norm (Luxemburg's norm)

$$\|u\|_{p(\cdot),Q_T}=\inf\left\{\lambda>0:\ \int_{Q_T}\left|\frac{u}{\lambda}\right|^{p(x,t)}\,dxdt<1\right\}$$

becomes a Banach space. For the elements of these spaces Hölder's inequality holds in the following form: for $f\in L^{p(\cdot)}(Q_T)$, $g\in L^{q(\cdot)}(Q_T)$ with $p(x)\in(1,\infty)$, $q(x)=\frac{p(x)}{p(x)-1}$

$$\left|\int_{Q_T} f\,g\,dx\right|\le\left(\frac{1}{p^-}+\frac{1}{p'^-}\right)\|f\|_{p(\cdot),Q_T}\|g\|_{q(\cdot),Q_T}.$$

The norms $\|\cdot\|_{p(\cdot),Q_T}$ can be estimated in terms of the integrals $A_p(u)$: for every $u\in L^{p(\cdot)}(Q_T)$

$$\min\left\{\|u\|^{p^+}_{p(\cdot),Q_T},\ \|u\|^{p^-}_{p(\cdot),Q_T}\right\}\le A_{p(\cdot)}(u)\le\max\left\{\|u\|^{p^+}_{p(\cdot),Q_T},\ \|u\|^{p^-}_{p(\cdot),Q_T}\right\}.$$

By $\mathbf{W}(Q_T)$ we denote the Banach space

$$\begin{cases}\mathbf{W}(Q_T)=\left\{u(x,t):\ u\in L^2(Q_T),\ |\nabla u|\in L^{p(x,t)}(Q_T),\ u=0 \text{ on } \Gamma\right\},\\ \|u\|_{\mathbf{W}(Q_T)}=\|u\|_{2,Q_T}+\|\nabla u\|_{p(\cdot),Q_T},\end{cases}$$

and by $\mathbf{W}'(Q_T)$ we denote the dual of $\mathbf{W}(Q_T)$ with respect to the scalar product in $L^2(Q_T)$. Let us consider the following problem:

$$\text{(2.1)}\qquad\begin{cases}u_t=\operatorname{div}\left(a(x,t,u)|\nabla u|^{p(x,t)-2}\nabla u\right)+d(x,t,u)\quad\text{in } Q_T,\\ u=0 \text{ on } \Gamma_T,\qquad u(x,0)=u_0(x) \text{ in } \Omega.\end{cases}$$

We assume that the exponents $p(x,t)$, $\sigma(x,t)$ and the coefficient $a(x,t,u)$ are subject to the following conditions:

$$\text{(2.2)}\qquad\begin{cases}a(x,t,u) \text{ is a Carathéodory function},\\ \text{there exist finite constants } p^{\pm} \text{ such that}\quad 1<p^-\le p(x,t)\le p^+,\\ p(\cdot) \text{ is continuous in } \overline{Q}_T \text{ with the logarithmic module of continuity:}\\ \forall\ z,\zeta\in\overline{Q}_T,\ |z-\zeta|<1,\\ \qquad |p(z)-p(\zeta)|\le\omega(|z-\zeta|),\quad \overline{\lim}_{\tau\to 0^+}\omega(\tau)\ln\frac{1}{\tau}<+\infty.\end{cases}$$

Definition 2.1. *A function $u(x,t)\in\mathbf{W}(Q_T)\cap L^\infty(0,T;\,L^2(\Omega))$ is called weak solution of problem* (2.1) *if for every test-function*

$$\zeta\in\{\eta(z):\ \eta\in\mathbf{W}(Q_T)\cap L^\infty\left(0,T;L^2(\Omega)\right),\ \eta_t\in\mathbf{W}'(Q_T)\},$$

and every $t_1,t_2\in[0,T]$, the following identity holds:

$$(2.3)\quad \int_{t_1}^{t_2}\int_\Omega \left(u\zeta_t - a(x,t,u)|\nabla u|^{p(x,t)-2}\nabla u\cdot\nabla\zeta - d(x,t,u)\zeta\right)dz = \int_\Omega u\zeta dx\Big|_{t_1}^{t_2}.$$

Theorem 2.1. [11, Theorem 4.3] *Let assumptions* (2.2) *be fulfilled and let the function* $d(x,t,u)$ *satisfy the growth condition*

$$|d(x,t,s)| \le d_0|s|^{\delta-1} + h(x,t) \quad \text{with some constants } d_0 > 0 \text{ and } \delta > 2.$$

Then for every $u_0 \in L^\infty(\Omega)$ *there exists* $\theta \in (0,T]$, *depending on* δ, d_0, $\|u_0\|_{L^\infty(\Omega)}$, $\|h\|_{L^1(0,\theta;L^\infty(\Omega))}$, *such that problem* (2.1) *has at least one weak solution* $u \in \mathbf{W}(Q_\theta)$ *such that* $u_t \in \mathbf{W}'(Q_\theta)$ *and* $\|u\|_{\infty,Q_\theta} < \infty$. *The solution can be continued to the interval* $[0,T^*]$ *where*

$$T^* = \sup\{\theta > 0 :\ \|u\|_{\infty,Q_\theta} < \infty\}.$$

3. Semilinear equation with nonlinear source

3.1. **Statement of problem and results.** Let us consider the semilinear problem

$$(3.1)\qquad \begin{cases} u_t = \Delta u + b(x,t)u^{\sigma(x,t)-1} & \text{in } Q_T,\\ u(x,0) = u_0(x) \ge 0 \text{ in } \Omega, & u = 0 \text{ on } \Gamma_T \end{cases}$$

with the coefficients $b(x,t)$, $\sigma(x,t)$ satisfying conditions (1.2), (1.3), (2.2). Under these conditions problem (3.1) has a local in time solution for every $u_0 \in L^\infty(Q)$. Moreover, $u \ge 0$ a.e. in Q_T, provided that $u_0 \ge 0$ in Ω - [11, Theorem 4.1]

To study the possibility of the blow-up we will apply the eigenvalue method of S. Kaplan - [20]. Let $\lambda > 0$, $\phi(x) \ge 0$ be the first eigenvalue and the corresponding eigenfunction of the Dirichlet problem for the Laplace operator in Ω:

$$(3.2)\qquad \Delta\phi = \lambda\phi \text{ in } \Omega, \qquad \phi = 0 \text{ on } \Gamma.$$

We normalize ϕ by the condition $\displaystyle\int_\Omega \phi(x)dx = 1$. Introduce the functions

$$(3.3)\qquad \begin{aligned} &\mu(t) = \int_\Omega u(x,t)\phi(x)dx,\\ &\alpha(t) = \left(\int_\Omega b^{\frac{1}{2-\sigma^-(t)}}(x,t)\phi(x)dx\right)^{2-\sigma^-(t)}, \qquad \sigma^-(t) = \min_{x\in\Omega}\sigma(x,t),\\ &\beta(t) = \int_\Omega b(x,t)\phi(x)dx \end{aligned}$$

and

$$(3.4)\qquad A(t) = \left(\alpha(t) - \frac{\lambda^{\sigma^-(t)-1}}{\sigma^-(t)-1}\right), \qquad B(t) = \beta(t) + \frac{\sigma^-(t)-2}{\sigma^-(t)-1}.$$

We will assume that

$$(3.5)\qquad A^- = \min_{t\ge 0}\ A(t) > 0, \qquad B^+ = \max_{t\ge 0}\ B(t) < \infty.$$

Definition 3.1. *We say that the solution $u(x,t)$ blows-up in a finite time if there exists an instant $t^* < \infty$ such that $\mu(t^*) = \infty$ and*

$$\mu(t) = \int_\Omega u\phi dx \leq \|u(\cdot,t)\|_{\infty,\Omega} \int_\Omega \phi dx = \|u(\cdot,t)\|_{\infty,\Omega} \to \infty \quad \text{as } t \to t^*.$$

Theorem 3.1. *Let the data of problem (3.1) satisfy the conditions*

$$\begin{cases} \forall\, t \geq 0 \quad 2 < \sigma^- = \min\limits_{x\in\Omega} \sigma(x,t) = const. \\ 0 < b^- \leq b(x,t) \leq b^+ < \infty, \end{cases} \tag{3.6}$$

$$\begin{cases} -\lambda\mu(0) + b^-\mu^{\sigma^- -1}(0) - b^+ > 0, \\ -\lambda + b^-(\sigma^- - 1)\mu^{\sigma^- -1}(0) > 0. \end{cases} \tag{3.7}$$

Then every weak solution blows-up at a moment $t^* \equiv t^*(\mu(0), \sigma^-, b^\pm) < \infty$.

Theorem 3.2. *Let condition (3.5) are fulfilled and, in addition,*

$$\begin{cases} \min\limits_{x\in\Omega} \sigma(x,t) = \sigma^-(t) \geq \sigma^- > 2, \\ g(t,\mu(0)) = A^-\mu^{\sigma^-(t)-1}(0) - B^+ > 0 \quad \text{for every } t \geq 0. \end{cases} \tag{3.8}$$

If either

$$\sigma^-(t) = \min_{x\in\Omega} \sigma(x,t) \geq \sigma^- = const > 2, \tag{3.9}$$

or

$$\begin{cases} \mu(0) > 1, \quad \sigma^-(t) \geq 2, \\ \sigma^-(t) \text{ is monotone decreasing and } \sigma^-(t) \to 2 \text{ as } t \to \infty, \\ \displaystyle\int_1^\infty e^{s(2-\sigma^-(s))} ds < \infty, \end{cases} \tag{3.10}$$

then every solution of problem (3.1) blows-up at a finite moment $t^* < \infty$.

3.2. **Proof of Theorems 3.1, 3.2.**

3.2.1. *The differential inequality for $\mu(t)$.* Let $u(x,t)$ be a weak solution of the semilinear problem (2.1) with $p(x,t) \equiv 2$. $a(x,t) \equiv 1$. By the definition, for every test-function $\phi(x) \in W_0^{1,2}(\Omega)$ and every $t, t+h < T^*$

$$\int_t^{t+h} \int_\Omega (u_t\phi + \nabla u \cdot \nabla\phi - d(x,t,u)\,\phi)\, dxdt = 0. \tag{3.11}$$

Let us choose the eigenfunction ϕ for the test-function in (3.11), divide the resulting equality by h, and let $h \to 0$. Applying the Lebesgue differentiation theorem we find that for a.e. $t < T^*$

$$\begin{aligned} \mu'(t) &= \int_\Omega u_t\phi dx = -\int_\Omega \nabla u \cdot \nabla\phi dx + \int_\Omega b(x,t) u^{\sigma(x,t)-1}\phi(x) dx \\ &= -\lambda\mu + \int_\Omega b u^{\sigma(x,t)-1}\phi dx. \end{aligned} \tag{3.12}$$

Using the representation

$$I = \int_\Omega b u^{\sigma(x,t)-1}\phi dx = \int_{\Omega\cap(u\geq 1)} b u^{\sigma(x,t)-1}\phi dx + \int_{\Omega\cap(u<1)} b u^{\sigma(x,t)-1}\phi dx, \tag{3.13}$$

we evaluate I in the following way: since $\sigma^- > 2$

$$\begin{aligned} I &\geq \int_{\Omega\cap(u\geq 1)} b u^{\sigma^- -1}\phi dx = \int_\Omega b u^{\sigma^- -1}\phi dx - \int_{\Omega\cap(u<1)} b u^{\sigma^- -1}\phi dx \\ &\geq \int_\Omega b u^{\sigma^- -1}\phi dx - \int_\Omega b\phi dx. \end{aligned}$$

Applying the inverse Hölder's inequality

$$\int_\Omega |u||v| dx \geq \left(\int_\Omega |u|^q dx\right)^{\frac{1}{q}} \left(\int_\Omega |v|^{\frac{q}{q-1}} dx\right)^{\frac{q-1}{q}} \tag{3.14}$$

with the exponent $q = \frac{1}{\sigma^- -1} \in (0,1)$, we may estimate

$$\begin{aligned} \int_\Omega b u^{\sigma^- -1}\phi dx &= \int_\Omega u^{\sigma^- -1}\phi^{\sigma^- -1} b\phi^{2-\sigma^-(t)} dx \\ &\geq \left(\int_\Omega u\phi dx\right)^{\sigma^- -1} \left(\int_\Omega b^{\frac{1}{2-\sigma^-}}(x,t)\phi dx\right)^{2-\sigma^-} \\ &= \alpha(t)\mu^{\sigma^- -1}(t). \end{aligned} \tag{3.15}$$

Gathering these formulas we arrive at the ordinary differential inequality for the function $\mu(t)$:

$$\mu'(t) \geq -\lambda\mu(t) + \alpha(t)\mu^{\sigma^- -1}(t) - \beta(t) \equiv f(\mu(t)), \tag{3.16}$$

with the functions $\alpha(t)$, $\beta(t)$ defined in (3.3). Notice that if condition $\min_{x\in\Omega}\sigma(x,t) = \sigma^- = const$ is substituted by the weaker condition $\min_{x\in\Omega}\sigma(x,t) = \sigma^-(t) > 2$ the same arguments lead to the inequality

$$\mu'(t) \geq -\lambda\mu(t) + \alpha(t)\mu^{\sigma^-(t)-1}(t) - \beta(t) \equiv F(t,\mu(t)). \tag{3.17}$$

3.2.2. *Analysis of the differential inequality - Theorem* 3.1. Under conditions (3.6)

$$\begin{aligned} &0 < b^- \leq \alpha(t) = \left(\int_\Omega b^{\frac{1}{2-\sigma^-}}(x,t)\phi(x) dx\right)^{2-\sigma^-}, \\ &0 \leq \beta(t) = \int_\Omega b(x,t)\phi(x) dx \leq b^+ < \infty, \end{aligned}$$

and (3.16) yields the inequality with constant coefficients and exponents:

$$\mu'(t) \geq -\lambda\mu(t) + b^-\mu^{\sigma^- -1}(t) - b^+ \equiv f(\mu(t)), \quad \sigma^- = const > 2. \tag{3.18}$$

The function $f(s)$ is concave and attains its minimum at the point

$$y_* = \left(\frac{\lambda}{b^-(\sigma^- - 1)}\right)^{\frac{1}{\sigma^- - 2}}.$$

Conditions (3.7) mean that $f(\mu(0)) > 0$, $f'_\mu(\mu(0)) > 0$ and inequality (3.18) guarantee that $\mu(t)$ is a strictly positive and increasing function of t, and $f(\mu(t))$ is strictly positive for all $t \geq 0$. Dividing the both parts of (3.18) by $f(\mu(t))$ and integrating, we have:

$$J(\mu(t)) = \int_{\mu(0)}^{\mu(t)} \frac{ds}{f(s)} \geq t.$$

Since the integral $J(s)$ is convergent at $s = \infty$, this inequality is possible only if there exists t^* such as $\mu(t) \to \infty$ as $t \to t^*$.

Remark 3.1. *Conditions* (3.7) *are surely fulfilled for all sufficiently large* $\mu(0)$.

3.2.3. *Analysis of the differential inequality - Theorem* 3.2. In this case $\mu(t)$ satisfies (3.17). Applying Young's inequality

$$a\,b \leq \frac{1}{p}\,(\epsilon a)^p + \frac{p-1}{p}\left(\frac{b}{\epsilon}\right)^{\frac{p}{p-1}}$$

with $a = \mu$, $b = \lambda$, $\epsilon = 1$, $p = \sigma^-(t) - 1$, we have:

$$\mu(t)\lambda \leq \frac{1}{\sigma^-(t)-1}\,(\lambda\mu(t))^{\sigma^-(t)-1} + \frac{\sigma^-(t)-2}{\sigma^-(t)-1}.$$

Plugging this inequality into (3.17), we obtain

$$\begin{aligned} \mu'(t) &\geq F(t,\mu(t)) \geq A(t)\mu^{\sigma^-(t)-1}(t) - B(t) \\ &\geq A^-\mu^{\sigma^-(t)-1}(t) - B^+ \equiv g(t,\mu(t)) \end{aligned} \tag{3.19}$$

with the coefficients $A(t)$, $B(t)$, A^-, B^+ defined in (3.4), (3.5). Since

$$\partial_\mu g(t,\mu) = A^-\left(\sigma^-(t) - 1\right)\mu^{\sigma^-(t)-1} > 0,$$

the function $g(t,\mu)$ is increasing as a function of μ, whence, by virtue of the first condition in (3.8) and the inequality $\mu' \geq g(t,\mu)$, we have $g(t,\mu(t)) > g(t,\mu(0)) > 0$ for every $t > 0$. It follows that $\mu'(t) > 0$ for all $t > 0$. If condition (3.9) is fulfilled, the conclusion about the finite time blow-up of the solution u follows exactly like in the proof of Theorem 3.1. Let us assume that condition (3.10) is fulfilled. Since $g(t,\mu)$ in increasing as a function of μ, μ is an increasing function of t, and $g(t,\mu) \to \infty$ as $\mu \to \infty$, there exists t' such that $g(t,\mu) \geq \frac{1}{2}A^-\mu^{\sigma^-(t)-1}(t)$ for all $t \geq t'$. Inequality (3.19) gives

$$\mu'(t) \geq \frac{1}{2}A^-\mu^{\sigma^-(t)-1}(t) \quad \text{for } t \geq t', \qquad \mu(t') \geq \mu(0) > 1.$$

Let us introduce the new independent variable $\theta = \frac{A^-}{2}(t - t')$ and denote $\gamma(\theta) = \sigma^-(t)$, $\nu(\theta) \equiv \mu(t)$. For the function $\nu(\theta)$ we have the conditions

$$\nu'(\theta) \geq \nu^{\gamma(\theta)-1}(\theta), \quad \nu(\theta) \geq 1 \quad \text{for } \theta > 0, \tag{3.20}$$

which yield the inequality $\nu'(\theta) \geq \nu(\theta)$. Integration of this inequality gives

$$\ln\,\nu(\theta) \geq \ln\left(\frac{\nu(\theta)}{\nu(0)}\right) \geq \theta,$$

and for the monotone decreasing function $\gamma(\theta)$ we have: $\gamma(\theta) \geq \gamma\,(\ln\,\nu(\theta))$. In the result we have the autonomous inequality for the $\nu(\theta)$:

$$\nu'(\theta) \geq \nu^{\gamma(\ln\,\nu(\theta))-1}, \quad \nu(\theta) \geq 1 \quad \text{for } \theta \geq 0.$$

Integrating and changing the variable of integration we finally obtain the inequality

$$I(\ln\,\nu(\theta)) \equiv \int_0^{\ln\,\nu(\theta)} \frac{d\tau}{\mathrm{e}^{\tau(\gamma(\tau)-2)}} \geq \int_{\nu(0)}^{\nu(\theta)} \frac{ds}{s^{\gamma(\ln\,s)}-1} \geq \theta.$$

If $I(\infty) < \infty$, this inequality leads to a contradiction unless there exists a finite θ^* such that $\nu(\theta) \to \infty$ as $\theta \to \theta^*$. The proof of Theorem 3.2 is completed.

4. Generalizations

4.1. **An example.** Let us illustrate the assertion of Theorem 3.2 by the following example: let u be a weak solution of problem (3.1) for the equation

$$u_t = \Delta u + u^{2+\epsilon(t)}, \tag{4.1}$$

where $\epsilon(t)$ is a monotone decreasing positive function such that $\epsilon(t) \to 0$ as $t \to \infty$. We assume that the initial function u_0 is as large as is required in Theorem 3.2. Moreover, increasing $\mu(0)$ we may guarantee that in (3.19) $g(t, \mu(t)) \geq \frac{1}{2}A^-\mu^{1+\epsilon(t)}(t)$ for all $t \geq 0$, so that the sufficient condition of the finite time blow-up of the solution u reduces to the following claim:

$$\int_1^{\infty+} \frac{d\tau}{\mathrm{e}^{\epsilon(\tau)\tau}} < \infty.$$

The simplest convergence test shows that this condition is fulfilled if, say, $\epsilon(\tau) = \alpha\frac{\ln\,\tau}{\tau}$ with any $\alpha > 1$.

4.2. **The ordinary differential inequality.** The proof of Theorems 3.1, 3.2 is based on the study of the properties of the functions satisfying the ordinary differential inequality (3.16). For the sake of presentation, we made several simplifying assumptions which made it possible to reduce (3.16) to an inequality with constant coefficients. It may happen, however, that the behavior of the variable coefficient $b(x, t)$ may influence the possibility of the blow-up of solutions to problem (3.1). Let us consider the simplified inequality (3.16),

$$\mu'(t) \geq \alpha(t)\mu^{\sigma^-(t)-1}, \qquad \mu(0) > 1, \tag{4.2}$$

assuming that the coefficient b is independent of x: $b(x, t) \equiv b(t)$. Introducing the new independent variable

$$\tau = \int_0^t \alpha(s)\,ds \equiv \int_0^t \left(\int_\Omega b^{\frac{1}{2-\sigma^-(s)}}(s)\phi(x)dx\right)^{2-\sigma^-(s)} ds = \int_0^t b(s)ds$$

and the functions $\nu(\tau) = \mu(t)$, $\gamma(\tau) = \sigma^-(t)$, we obtain the inequality

$$\nu'(\tau) \geq \nu^{\gamma(\tau)-1}, \quad \nu(\tau) \geq 1.$$

Arguing like in the proof of Theorem 3.2, we conclude that the function $\mu(t)$ blows-up at a finite instant t^* if, for example,

$$\infty > \int_0^\infty \frac{d\tau}{\mathrm{e}^{\tau(\gamma(\tau)-2)}} \geq \tau = \int_0^t b(s)\,ds \to \infty \quad \text{as } t \to \infty.$$

This means that the blow-up is possible even in the case that $b(t) \to 0$ as $t \to \infty$.

4.3. **Regional blow-up.** The conclusions about the blow-up of solutions of problem (3.1) remain true if instead of the whole domain Ω we restrict the study to a subdomain. Let us assume that there exists a subdomain $D \subset \Omega$, $\operatorname{meas} D > 0$, $\partial D \in C^1$, and let $\phi > 0$ in D and λ be the first eigenfunction and the corresponding eigenvalue of the problem

$$-\Delta\phi = \lambda\phi \text{ in } D, \qquad \phi = 0 \text{ on } \partial D. \tag{4.3}$$

Let us introduce the function

$$\mu(t) = \int_D u\,\phi\,dx.$$

Given a solution $u \in \mathbf{W}(Q_T)$, we may formally consider the semilinear equation (3.1) (at least for small times) as the heat equation with the bounded free term $f(x,t) \equiv b(x,t)\,u^{\sigma(x,t)-1}$. It follows then from the classical parabolic theory that $u \in W_2^{1,2}(\omega \times \theta)$ for every subdomain $\omega \subset \Omega$ with the sufficiently smooth boundary $\partial\omega$ and every $\theta < t^*$. This observation justifies the forthcoming arguments. Let us multiply the equation by the function ϕ and integrate over D:

$$\begin{aligned}\mu'(t) &= \int_D u_t\phi\,dx = \int_D u\Delta\phi\,dx - \int_{\partial D} u\,(\nabla\phi,\,\mathbf{n})\,dS + \int_D b\,u^{\sigma(x,t)}\,\phi\,dx \\ &= -\lambda\mu - \int_{\partial D} u\,(\nabla\phi,\,\mathbf{n})\,dS + \int_D b\,u^{\sigma(x,t)}\phi\,dx,\end{aligned}$$

where $\mathbf{n}$ denotes the outward normal to ∂D. Since $\phi \geq 0$ in D, then $(\nabla\phi,\,\mathbf{n}) \leq 0$ on ∂D, and for nonnegative solution u

$$-\int_{\partial D} u\,(\nabla\phi,\,\mathbf{n})\,dS \geq 0.$$

The differential inequality for $\mu(t)$ takes on the form

$$\mu'(t) \geq -\lambda\mu + \int_D b\,u^{\sigma(x,t)}\,\phi\,dx$$

and its analysis follows the proof of Theorem 3.1.

4.4. **Equations with nonlocal reaction terms.** Let us consider the problem

$$
\text{(4.4)} \qquad \begin{cases} u_t = \triangle u + f(x,t,u) \quad \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T, \end{cases}
$$

where

$$
f(x,t,u) = \sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1} + \sum_{i=N+1}^{Q} c_i(x,t) \int_\Omega d_i(s,t) u^{\sigma_i(s,t)-1} ds,
$$

with $b_k \geq 0$, $c_i \geq 0$, $d_i \geq 0$, $Q \leq n$. Multiplying (4.4) by u and integrating over Ω we arrive at the relation (cf. with (3.12))

$$
\text{(4.5)} \qquad \mu'(t) = -\lambda \mu + I_1 + I_2,
$$

where

$$
\begin{aligned}
I_1 &= \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1} \right) \phi \, dx, \\
I_2 &= \int_\Omega \left(\sum_{i=N+1}^{Q} c_i(x,t) \int_\Omega d_i(s,t) u^{\sigma_i(s,t)-1} ds \right) \phi \, dx.
\end{aligned}
$$

I_1, I_2 are estimated from below in the following way (cf. with (3.13), (3.14), (3.15)):

$$
\begin{aligned}
I_1 &\geq \int_{\Omega \cap (u \geq 1)} \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k(x,t)-1} \right) \phi dx \geq \int_{\Omega \cap (u \geq 1)} \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k^-(x,t)-1} \right) \phi dx \\
&= \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k^-(x,t)-1} \right) \phi dx - \int_{\Omega \cap (u<1)} \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k^-(x,t)-1} \right) \phi dx \\
&\geq \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t) u^{\sigma_k^-(x,t)-1} \right) \phi(x) dx - \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t) \right) \phi(x) dx \\
&\geq \sum_{k=1}^{N} \left(\int_\Omega b_k^{-\frac{1}{\sigma_k^-(x,t)-2}}(x,t) \phi(x) dx \right)^{-\frac{1}{\sigma_k^-(x,t)-2-}} \left(\int_\Omega u(x,t) \phi(x) dx \right)^{\sigma_k^-(x,t)-1} \\
&\quad - \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t) \right) \phi dx = \sum_{k=1}^{N} \alpha_k(t) \mu^{\sigma_k^-(t)-1} - \beta(t),
\end{aligned}
$$

$$
\begin{aligned}
I_2 \geq & \sum_{i=N+1}^{Q} \int_\Omega c_i(x,t)\phi(x,t) \left(\int_\Omega u(s,t)\phi(s)ds\right)^{\sigma_i^- -1} \\
& \times \left(\int_\Omega d_i^{\overline{\frac{1}{2-\sigma_i^-(t)}}}(s,t)\phi^{\frac{\sigma_i-1}{\sigma_i-2}}(s,t)ds\right)^{2-\sigma_i^-} dx \\
& - \sum_{i=N+1}^{Q} \int_\Omega \left(c_i(x,t)\int_\Omega d_i(s,t)ds\right)\phi dx \\
& = \sum_{i=N+1}^{Q} \mu^{\sigma_i^- -1}(t)\theta_i(t)\int_\Omega c_i(x,t)\phi(x)dx - \beta(t) \\
& = \sum_{i=N+1}^{Q} \alpha_i(t)\mu^{\sigma_i^- -1} - \beta(t),
\end{aligned}
$$

where for $k = 1, \dots, N$

$$
\alpha_k(t) = \left(\int_\Omega b_k^{\overline{\frac{1}{2-\sigma_k^-(t)}}}(x,t)\phi(x)dx\right)^{2-\sigma_k^-(t)}, \qquad \beta(t) = \int_\Omega \left(\sum_{k=1}^{N} b_k(x,t)\right)\phi dx,
$$

and for $k = N+1, \dots, Q$

$$
\alpha_k(t) = \int_\Omega c_i\phi\, dx \left(\int_\Omega d_i^{\overline{\frac{1}{2-\sigma_i^-}}} \phi^{\frac{\sigma_i-1}{\sigma_i-2}}\, dx\right)^{2-\sigma_i^-}, \quad \beta(t) = \sum_{i=N+1}^{Q} \int_\Omega \left(c_i\int_\Omega d_i ds\right)\phi\, dx.
$$

Gathering these formulas we arrive at the nonlinear ODI

$$
\mu'(t) \geq -\lambda\mu + \sum_{i=1}^{Q} \alpha_i(t)\mu^{\sigma_i^-(t)-1}(t)(t) - \beta(t),
$$

which can be studied like (3.17).

5. Evolution equations of $p(x)$-Laplace type

5.1. **Assumptions and result.** Let us consider the problem

$$
(5.1) \qquad \begin{cases} u_t = \operatorname{div}\left(a(x,t)|\nabla u|^{p(x)-2}\nabla u\right) + b(x,t)|u|^{\sigma(x)-2}u & \text{in } Q_T, \\ u(x,0) = u_0(x) \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma_T \end{cases}
$$

with the coefficients and exponents satisfying conditions (1.2), (1.3). Let us introduce the functions

$$
(5.2) \qquad f(t) = \frac{1}{2}\int_0^t \int_\Omega |u(x,\tau)|^2\, dx d\tau, \qquad E(t) = \int_\Omega \left(\frac{a}{p}|\nabla u|^p - \frac{b}{\sigma}|u|^\sigma\right) dx.
$$

Theorem 5.1. *Let the exponents $p(x)$, $\sigma(x)$ satisfy the conditions*

$$\sigma^- > 2 \quad \text{and} \quad p^+ = \max_\Omega p(x) \le \sigma^- = \min_\Omega \sigma(x). \tag{5.3}$$

Let us assume, in addition to (1.2)*, that the coefficients a, b are differentiable in t and monotone:*

$$a_t(x,t) \le 0, \quad b_t(x,t) \ge 0, \quad \int_0^T \left(\max_{x\in\Omega} |a_t(x,t)| + |b_t(x,t)| \right) dt < \infty. \tag{5.4}$$

Finally, let $|u_0|^{\sigma(x)} \in L^1(\Omega)$, $|\nabla u_0|^{p(x)} \in L^1(\Omega)$. If

$$E(0) = \int_\Omega \left(\frac{a(x,0)}{p(x)} |\nabla u_0|^{p(x)} - \frac{b(x,0)}{\sigma(x)} |u_0|^{\sigma(x)} \right) dx \le 0, \tag{5.5}$$

then every weak solution $u \in \mathbf{W}(Q_T)$ blows-up in a finite time: there exists $t^* \equiv t^*(\Omega, \|u_0\|_\infty) < \infty$ such that $\|u(\cdot,t)\|_{\infty,\Omega} \to \infty$ as $t \nearrow t^*$.

5.2. **The energy relations.** According to Theorem 2.1 the solution $u \in \mathbf{W}(Q_T)$ can be taken for the test-function in the integral identity (2.3), which gives the first energy relation:

$$\frac{1}{2}\int_\Omega |u|^2 \, dx + \int_0^t \int_\Omega \left(a\,|\nabla u|^p - b\,|u|^\sigma \right) dxdt = \frac{1}{2}\int_\Omega |u_0|^2 \, dx. \tag{5.6}$$

To derive the second energy estimate we rely on the following result:

Lemma 5.1. *Let the exponents and coefficients of problem* (5.1) *satisfy the conditions of Theorem* 5.1*. Then the weak solution of problem* (5.1) *satisfies the estimate*

$$\forall \, \text{a.e.}\, t > 0 \qquad E(t) + \int_0^t \int_\Omega |u_t|^2 \, dxdt \le E(0). \tag{5.7}$$

Proof. The assertion is a simplified version of the estimate proved in [11, Theorem 6.1], which is why we limit ourselves by giving a sketch of the arguments and skip the details. The proof of existence of a weak solution to problem (5.1) (in a more general setting) is performed with the Galerkin-Faedo method. The solution is obtained as the limit of the sequence of functions $u^{(k)} = \sum_1^k c_i(t)\psi_i(x)$, $\{\psi_i\}$ is the orthogonal basis of the function space $L^{p^+}(\Omega)$, which is dense in $L^{p(x)}(\Omega)$. In this approach estimates on the limit function result from the uniform in k estimates for the approximate solutions $u^{(k)}$. Let u be a sufficiently regular solution of problem (5.1) (or the approximate solution $u^{(k)}$). Multiplying the equation by u_t, integrating by parts, and using the obvious relations

$$\partial_t \left(a \frac{|\nabla u\ |^p}{p} \right) = a_t \frac{|\nabla u\ |^p}{p} + a \left(|\nabla u|^{p-2} \nabla u \ \nabla u_t \right),$$
$$\partial_t \left(\frac{b}{\sigma} |u|^\sigma \right) = b_t \frac{|u|^\sigma}{\sigma} + b \left(|u|^{\sigma-2} u \ u_t \right),$$

we have:

$$E'(t)=\frac{d}{dt}\int_\Omega\left(\frac{a}{p}\left|\nabla u\right|^p-\frac{b}{\sigma}\left|u\right|^\sigma\right)=-\int_\Omega u_t^2\,dx+\Lambda_1(t)+\Lambda_2(t)\leq-\int_\Omega u_t^2\,dx,$$

because

$$\Lambda_1(t)=\int_\Omega a_t\frac{|\nabla u\ |^p}{p}\,dx\leq 0,\qquad \Lambda_2(t)=-\int_\Omega b_t\frac{|u|^\sigma}{\sigma}\,dx\leq 0$$

by assumption. Inequality (5.7) follows after integration in t. □

Remark 5.1. *In the case that a, p, b, σ are independent of t, the resulting energy relation simplifies and takes on the form*

$$E(t)+\int_0^t\int_\Omega u_t^2\,dxdt=E(0).$$

5.3. **Ordinary differential inequality for $f(t)$.** Let us consider the function $f(t)$ defined in (5.2). Under the conditions of Lemma 5.1, for every solution of problem (5.1) and for a.e. $t>0$

$$\begin{aligned}&f'(t)=\frac{1}{2}\int_\Omega\left|u(\cdot,t)\right|^2\,dx\geq 0,\\ &f''(t)=\int_\Omega u\,u_t\,dx=\int_\Omega\left(-a\left|\nabla u\right|^p+b\left|u\right|^\sigma\right)\,dx.\end{aligned}\tag{5.8}$$

Equalities (5.8) follow directly from the definition of weak solution. Gathering (5.8) with (5.6), we find that

$$0\leq f'(t)=\frac{1}{2}\int_\Omega\left|u_0\right|^2+\int_0^t\int_\Omega\left(-a\left|\nabla u\right|^p+b\left|u\right|^\sigma\right)\,dxdt=\frac{1}{2}\int_\Omega\left|u_0\right|^2dx+f''(t),$$

Let us take a constant $\lambda>0$ such that

$$\frac{1}{\sigma^-}<\lambda<\frac{1}{p^+}.$$

Multiplying the second equality of (5.8) by λ and adding the result to (5.7), we obtain the inequality

$$E(t)+\lambda\int_\Omega\left(-a\left|\nabla u\right|^p+b\left|u\right|^\sigma\right)\,dx+\int_0^t\int_\Omega u_t^2ds\leq\lambda f''(t)+E(0),\tag{5.9}$$

which leads to the following one: for $E(0)\leq 0$

$$\begin{aligned}&\left(\frac{1}{p^+}-\lambda\right)a^-\int_\Omega\left|\nabla u\right|^p\,dx+\left(\lambda-\frac{1}{\sigma^-}\right)b^-\int_\Omega\left|u\right|^\sigma\,dx\\ &\qquad+\int_0^t\int_\Omega u_t^2\,dx\leq\lambda f''(t).\end{aligned}\tag{5.10}$$

5.4. **Lower estimates on the growth of $f(t)$. Proof of Theorem** 5.1**.** We proceed to study the behavior of the function $f(t)$ and consider separately the two possibilities: either $p^+>2$, or $p^+\leq 2$.

5.4.1. *Case 1:* $p^+ > 2$. Notice that the first two terms on the left-hand side of (5.10) are nonnegative due to assumption (5.3). Dropping these terms, we have that $f''(t)$ satisfies the inequality

$$\int_0^t \int_\Omega u_t^2 \, dx \leq \lambda f''(t),$$

whence $f''(t) \geq 0$. Let us denote by t^* the time of existence of the solution u:

$$t^* = \sup\{t > 0 : \|u\|_{\infty,\Omega} < \infty \text{ for } t < t^*\}.$$

Using Hölder's inequality, we obtain the chain of relations

$$\begin{aligned}(f'(t) - f'(0))^2 &= \left(\int_0^t \frac{d}{dt} \left(\frac{1}{2} \int_\Omega u^2 \, dx \right) dt \right)^2 = \left(\int_0^t \int_\Omega u u_t \, dx \right)^2 \\ &\leq \left(\int_0^t \|u_t\|_{2,\Omega} \|u\|_{2,\Omega} \, dt \right)^2 \leq \|u_t\|^2_{2,\Omega\times(0,t)} \|u\|^2_{2,\Omega\times(0,t)} \\ &\leq \lambda f''(t) \int_0^t \int_\Omega u^2 \, dx dt = 2\lambda f''(t) f(t),\end{aligned}$$

which gives the second-order nonlinear ordinary differential inequality for the function $f(t)$:

$$(f'(t) - f'(0))^2 \leq 2\lambda f''(t) f(t) < \frac{2}{p^+} f''(t) f(t) \tag{5.11}$$

We want to prove that the function $f(t)$ becomes unbounded at a finite moment. Let us argue by contradiction and assume that $t^* = \infty$. Since f, f', f'' are nonnegative, it is necessary that $f(t) \nearrow \infty$ as $t \to \infty$. It follows that there exists a moment t_0 and a constant $1 < \nu < \frac{p^+}{2}$ such that $(f'(t) - f'(0))^2 \geq \frac{2\nu}{p^+}(f'(t))^2$ for $t \geq t_0$. This allows one to continue (5.11) as follows:

$$\nu(f'(t))^2 \leq \frac{p^+}{2} (f'(t) - f'(0))^2 \leq f''(t) f(t) \qquad \text{for } t \geq t_0.$$

Integration of this inequality gives

$$f^{\nu-1}(t) \geq \frac{f^{\nu-1}(t_0)}{1 - t(\nu-1)\frac{(f'(t_0))^{\nu-1}}{f(t_0)}} \to \infty \quad \text{as } t \to T = \frac{f(t_0)}{(\nu-1)\,(f'(t_0))^{\nu-1}},$$

which contradicts the assumption $t^* = \infty$ because

$$\infty > \frac{1}{2} |\Omega| T \, \|u\|^2_{\infty,\Omega} \geq \frac{1}{2} \int_0^t |u|^2 \, dx dt \equiv f(t) \to \infty \quad \text{as } t \nearrow T.$$

This completes the proof of Theorem 5.1 in the case $p^+ > 2$.

5.4.2. *Case 2:* $1 < p^+ \le 2$. In this case we reduce (5.10) to the inequality

$$\left(\lambda - \frac{1}{\sigma^-}\right) b^- \int_\Omega |u|^\sigma \le \lambda f''(t). \tag{5.12}$$

Using Hölder's inequality for the functions from the Lebesgue-Orlicz spaces $L^{q(\cdot)}(\Omega)$, we have:

$$\begin{aligned}&\min\left\{(f')^{\frac{\sigma^+}{2}}, (f')^{\frac{\sigma^-}{2}}\right\}\\&\qquad = \min\left\{\left(\frac{1}{2}\|u\|^2_{2,\Omega}\right)^{\frac{\sigma^+}{2}}, \left(\frac{1}{2}\|u\|^2_{2,\Omega}\right)^{\frac{\sigma^-}{2}}\right\} \le C\int_\Omega |u|^{\sigma(x)}\, dx \le Cf''(t).\end{aligned}$$

Arguing like in the Case 1, we assume for contradiction that $t^* = \infty$ and conclude that $f(t) \to \infty$ as $t \to \infty$. It follows that $f'(t) > 1$ for all t beginning with some t_0 and, thus, the previous inequality transforms into

$$(f'(t))^{\frac{\sigma^-}{2}} \le C\, f''(t) \quad \text{for } t \ge t_0.$$

It follows by straightforward integration that for $t \ge t_0$

$$f'(t) \ge \frac{f'(t_0)}{\left(1 - \frac{\sigma^- - 2}{2C}(t - t_0)(f'(t_0))^{\frac{\sigma^-}{2} - 1}\right)^{\frac{2}{\sigma^- - 2}}} \to \infty$$

$$\text{as } t \to t' = t_0 + \frac{2C}{\sigma^- - 2}(f'(t_0))^{1 - \frac{\sigma^-}{2}}.$$

Since $f''(t) \ge 0$ for all $t > 0$, the proof of Theorem 5.1 is completed.

References

[1] R. Aboulaicha, D. Meskinea, and A. Souissia, *New diffusion models in image processing*, Computers and Mathematics with Applications, 56 (2008), pp. 874–882.

[2] E. Acerbi and G. Mingione, *Regularity results for stationary electro-rheological fluids*, Arch. Ration. Mech. Anal., 164 (2002), pp. 213–259.

[3] E. Acerbi, G. Mingione, and G. Seregin, *Regularity results for paraboliqc systems related to a class of non-Newtonian fluids*, Ann. Inst. Henri Poincaré, Anal. Non Linéaire, 21(01) (2004), pp. 25–60.

[4] Y. Alkhutov, S. N. Antontsev, and V. Zhikov, *Parabolic equations with variable order of nonlinearity*, Collection of works of Institute of Mathematics NAS of Ukraine, 6 (2009), pp. 23–50.

[5] S. Antontsev, M. Chipot, and Y. Xie, *Uniqueness results for equations of the $p(x)$-Laplacian type*, Adv. Math. Sci. Appl., 17 (2007), pp. 287–304.

[6] S. Antontsev and V. Zhikov, *Higher integrability for parabolic equations of $p(x,t)$-Laplacian type*, Adv. Differential Equations, 10 (2005), pp. 1053–1080.

[7] S. N. Antontsev and J. F. Rodrigues, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara, Sez.,VII. Sci. Mat., 52 (2006), pp. 19–36.

[8] S. N. Antontsev and S. I. Shmarev, *Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions*, Journal Nonlinear Analysis, 65 (2006), pp. 722–755.

[9] ———, *Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions*, Elsevier, 2006. Handbook of Differential Equations. Stationary Partial Differential Equations, Elsevier, Vol. 3, Chapter 1, pp.1-100.

[10] ———, *Parabolic equations with anisotropic nonstandard growth conditions*, in Internat. Ser. Numer. Math. 154, Birkhäuser, Verlag Basel/Switzerland, 2006, pp. 33–44.
[11] ———, *Anisotropic parabolic equations with variable nonlinearity*, Preprint 2007-13, CMAF, University of Lisbon, Portugal, downloaable at :http://www.ptmat.fc.ul.pt/preprints/pdf/2007, (2007), pp. 1–34.
[12] ———, *Extinction of solutions of parabolic equations with variable anisotropic nonlinearities*, Proceedings of the Steklov Institute of Mathematics, Moscow, Russia, 268 (2008), pp. 2289–2301.
[13] ———, *Vanishing solutions of anisotropic parabolic equationbs with variable nonlinearity*, Preprint 2008-021, CMAF, University of Lisbon, Portugal, downloaable at :http://www.ptmat.fc.ul.pt/preprints/pdf/2008, (2008), pp. 1–27.
[14] ———, *Localized solutions of anisotropic parabolic equations*, Nonlinear Anal., Theory, Methods, Applications, (2009, in press), pp. 1–18.
[15] Y. Chen, S. Levine, and M. Rao, *Variable exponent, linear growth functionals in image restoration*, SIAM J. Appl. Math, 66 (2006), pp. 1383–1406.
[16] H. Fujita, *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$, J. Fac. Sci. Univ. Tokyo Sect. I, 13 (1966), pp. 109–124 (1966).
[17] ———, *On some nonexistence and nonuniqueness theorems for nonlinear parabolic equations*, in Nonlinear Functional Analysis (Proc. Sympos. Pure Math., Vol. XVIII, Part 1, Chicago, Ill., 1968), Amer. Math. Soc., Providence, R.I., 1970, pp. 105–113.
[18] V. A. Galaktionov and S. I. Pohozaev, *Blow-up and critical exponents for parabolic equations with non-divergent operators: dual porous medium and thin film operators*, J. Evol. Equ., 6 (2006), pp. 45–69.
[19] V. A. Galaktionov and J. L. Vázquez, *A stability technique for evolution partial differential equations*, Progress in Nonlinear Differential Equations and their Applications, 56, Birkhäuser Boston Inc., (2004).
[20] S. Kaplan, *On the growth of the solutions of quasilinear parabolic equations*, Comm. Pure Appl. Math., 16 (1963), pp. 305–330.
[21] S. Levine, Y. Chen, and J. Stanich, *Image restoration via nonstandard diffusion*, Technical Report ♯ 04-01, Dept. of Mathematics and Computer Science, Duquesne University, 2004.
[22] E. Mitidieri and S. Pohozaev, *Nonexistence of weak solutions for some degenerate elliptic and parabolic problems on* $\mathbb{R}^n$, J.evol.equ., 1 (2001), pp. 189–220.
[23] J. P. Pinasco, *Blow-up for parabolic and hyperbolic problems with variable exponents*, J. Nonlinear Analysis: Theory, Methods and Applications, (2008).
[24] S. I. Pohozaev and A. Tesei, *Blow-up of nonnegative solutions to quasilinear parabolic inequalities*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl., 11 (2000), pp. 99–109.
[25] S. I. Pokhozhaev and A. Teseĭ, *Critical exponents for the absence of solutions for systems of quasilinear parabolic inequalities*, Differ. Uravn., 37 (2001), pp. 521–528, 575.
[26] K. Rajagopal and M. Růžička, *Mathematical modelling of electro-rheological fluids*, Cont. Mech. Therm., 13 (2001), pp. 59–78.
[27] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Springer, Berlin, 2000. Lecture Notes in Mathematics, 1748.
[28] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov, and A. P. Mikhailov, *Blow-up in quasilinear parabolic equations*, Walter de Gruyter & Co., Berlin, 1995. Translated from the 1987 Russian original by Michael Grinfeld and revised by the authors.
[29] M. Tsutsumi, *Existence and nonexistence of global solutions for nonlinear parabolic equations*, Publ. Res. Inst. Math. Sci., 8 (1972), pp. 211–229.
[30] ———, *Existence and nonexistence of global solutions of the first boundary value problem for a certain quasilinear parabolic equation*, Funkcial. Ekvac., 17 (1974), pp. 13–24.

CMAF, Universidade de Lisboa, Portugal
E-mail address: `anton@ptmat.fc.ul.pt, antontsevsn@mail.ru`

Departamento de Matemáticas, Universidad de Oviedo, España
E-mail address: `shmarev@orion.ciencias.uniovi.es`