Non-existence of global solutions for a quasilinear Benney system

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Abstract

D.J. Benney introduced in 1977 (cf. [1]) a general strategy for deriving systems of nonlinear PDEs describing the interaction between long and short waves. In [7] we have studied the local existence and uniqueness of solutions to a quasilinear version of these systems. In the present paper we prove that in some important cases global strong solutions do not exist.

Mathematics Subject Classification (2000): 35L70, 35L65, 76B15.
Keywords: Capillary/Gravity Waves; Mixed Problem; Blow-up.

1 Introduction

The Benney equations

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial x^2} = m_1 |u|^2 u + m_2 uv \\ \dfrac{\partial v}{\partial t} + m_3 \dfrac{\partial u}{\partial x} = m_4 \dfrac{\partial}{\partial x}(|u|^2), \qquad x \in \mathbb{R},\ t \geq 0,\ u = u(x,t) \in \mathbb{C},\ v = v(x,t) \in \mathbb{R} \end{cases} \tag{1}$$

appear in several physical contexts as a model for the interaction between unidimensional short and long waves in dispersive media. In general water waves theory, applications of this model include the interaction of capillary and gravity waves or internal and surface waves (see for instance [3],[8]). In plasma physics, the system (1) is a useful model to describe the interaction between Alfvén/Langmuir waves and magneto-acoustic waves ([4],[13],[16]). Also, in [9], this system appears in the context of long-wave short-wave

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resonance in geometric optics. Several mathematical aspects of (1) such as well-posedness of the Cauchy problem, stablity of solitary waves and existence of an adiabatic limit have been extensively studied by many authors (see [10], [11], [12], [14], [15]).

In the case where the amplitudes of the short and long waves are of the same order, long waves become considerably weaker than the short ones. As pointed out in [1] (see Eqs. (3.27) and (3.28)), the following system should then be considered:

$$\begin{cases} i\dfrac{\partial u}{\partial t} + \dfrac{\partial^2 u}{\partial x^2} = |u|^2 u + buv & (2.a) \\ \dfrac{\partial v}{\partial t} + \dfrac{\partial}{\partial x}(f(v)) = \alpha \dfrac{\partial}{\partial x}(|u|^2) & (2.b). \end{cases} \tag{2}$$

Here $x \in \mathbb{R}$, $t \geq 0$, $f(v) = av^2$ and a, b, α are real constants. Contrarely to system (1), very few results concerning system (2) are present in the literature. In this paper we will consider this system in the half-line $x \in]0, +\infty[$.

The local existence theory for the Initial Value-Problem associated to (2) in the real line $x \in]-\infty, +\infty[$ was adressed in [7]. In Section 2 we wil state the analogous result in the half-line $]0, +\infty[$ with a null boundary condition at $x = 0$. This result can be deduced from the proof stated in [7] by replacing $H^1(\mathbb{R})$ by $H^1_o(\mathbb{R}_+)$. Using essencially the same proofs, the conservation laws for (2) derived in [5] also remain valid in the half-line.

In Section 3 and 4 we derive two new results concerning the non-existence of global strong solutions:

The first one is an extension of the classical blow-up result for the Nonlinear Schrödinger Equation (see for example [2]) in spatial dimension $N \geq 2$ concerning the behaviour of the function

$$t \to \int |x|^2 |u(x,t)|^2 dx.$$

In our case, this function will be replaced by

$$t \to \int_{\mathbb{R}_+} x^2(|u|^2 + bv)(x,t)dx$$

for suitable initial data and constants $b = \frac{1}{a}$, $\alpha = -\frac{1}{a}$ for large values of $a > 0$. In dimension 1 it is known that the I.V.P associated with the Cubic Nonlinear Schrödinger is globally well-posed for initial data in $L^2(\mathbb{R})$. Here, the blow-up will be somehow imposed by the long wave v, which satisfies a Burgers-like equation forced by the term $\dfrac{\partial}{\partial x}(|u|^2)$.

The second result concerns the blow-up of the function

$$t \to \int_{\mathbb{R}_+} e^{-x} v(x,t)dx = \int_{\mathbb{R}_+} e^{-x} \frac{\partial v}{\partial x}(x,t)dx$$

and implies, under certain asumptions and if $v_o \in L^1(\mathbb{R}_+)$ (respectively $v_o \in W^{1,1}(\mathbb{R}_+)$) the blow up of the norm $\|v(t)\|_{L^1}$ (respectively $\|v(.,t)\|_{L^1}$ and $\|\frac{\partial v}{\partial x}(.,t)\|_{L^1}$).

An interesting open problem is to extend the results obtained in this paper to the more general quasilinear systems studied in [6].

2 Local existence and conservation laws

The next two theorems are variants of the Theorem 1.2 in [7] and Lemma 1.2 in [5], and the proofs will be omitted.

Theorem 2.1 *Let $(u_o, v_o) \in (H_o^1(\mathbb{R}_+) \cap H^3(\mathbb{R})) \times (H_o^1(\mathbb{R}_+) \cap H^2(\mathbb{R}))$.*
Then there exists a unique strong solution (u, v) to the I.V.P associated to (2), with

$$(u, v) \in \mathcal{C}([0, T]; H_o^1(\mathbb{R}_+) \cap H^3(\mathbb{R}_+)) \times \mathcal{C}([0, T]; H_o^1(\mathbb{R}_+) \cap H^2(\mathbb{R}_+)),$$

$$\left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}\right) \in \mathcal{C}([0, T]; H^1(\mathbb{R})) \times \mathcal{C}([0, T]; H^1(\mathbb{R})).$$

Here, the life-span $T > 0$ depends exclusively on the initial data (u_o, v_o) and on the constants a, b, α.

Theorem 2.2 *Under the hypothesis of Theorem 2.1, the following conservation laws hold: for all $t \in [0, T[$,*

$$\frac{d}{dt}\int_{\mathbb{R}_+} |u(x,t)|^2 dx = 0; \tag{3}$$

For $\alpha \neq 0$,

$$\frac{d}{dt}\int_{\mathbb{R}_+} \left(\frac{1}{2}|\frac{\partial u}{\partial x}(x,t)|^2 + \frac{b}{2}v|u(x,t)|^2(x,t) + \frac{1}{4}|u(x,t)|^4 - \frac{ab}{6\alpha}v^3(x,t)\right) dx = 0; \tag{4}$$

$$\frac{d}{dt}\int_{\mathbb{R}_+} \left(\frac{b}{\alpha}v(x,t)^2 - 2Im\int \overline{u(x,t)}u_x(x,t)\right) dx = 0. \tag{5}$$

3 Non-existence of global strong solutions

In this section we denote by (u, v) the unique strong solution of the Cauchy problem for the system (2) with initial data (u_o, v_o) verifying the assumptions of Theorem 2.1 and by $[0, T[$ the maximal interval of existence of the solution.

We will need some auxiliary lemmas:

Lemma 3.1 *Let $\alpha \neq 0$.*
If $xu_o, xu_{ox} \in L^2(\mathbb{R}_+)$ and $xv_o^2 \in L^1(\mathbb{R}_+)$ then

$$xu, xu_x \in \mathcal{C}([0,T[; L^2(\mathbb{R}_+)) \quad and \quad xv^2 \in \mathcal{C}([0,T[; L^1(\mathbb{R}_+)).$$

Furthermore, for all $t \in [0,T[$,

$$\frac{d}{dt}\int_{\mathbb{R}_+} x^2|u(x,t)|^2dx = 4Im\int_{\mathbb{R}_+}\frac{\partial u}{\partial x}(x,t)x\overline{u}(x,t)dx \tag{6}$$

and

$$2\frac{d}{dt}Im\int_{\mathbb{R}_+}\frac{\partial u}{\partial x}(x,t)x\overline{u}(x,t)dx = \frac{b}{\alpha}\frac{d}{dt}\int_{\mathbb{R}_+} xv^2(x,t)dx + 4\int_{\mathbb{R}_+}\left|\frac{\partial u}{\partial t}(x,t)\right|^2dx + \int_{\mathbb{R}_+}|u(x,t)|^4dx + 2b\int_{\mathbb{R}_+}|u(x,t)|^2v(x,t)dx - \frac{4ab}{3\alpha}\int_{\mathbb{R}_+}v^3(x,t)dx, \tag{7}$$

the functions

$$t \to \int_{\mathbb{R}_+} x^2|u(x,t)|^2dx \quad and \quad t \to \int_{\mathbb{R}_+} x|v(x,t)|^2dx$$

belonging to $\mathcal{C}^2([0,T[)$ and $\mathcal{C}^1([0,T[)$ respectively.

Proof:

The proof follows closely the proof of Proposition 6.5.1 in [2], replacing $\int_{\mathbb{R}_+} x^2|u|^2dx$ by $\int_{\mathbb{R}_+} e^{-\epsilon x}x^2|u|^2dx$ and $\int_{\mathbb{R}_+} xv^2dx$ by $\int_{\mathbb{R}_+} e^{-\epsilon x}xv^2dx$, $\epsilon > 0$, making some technical computations and the letting $\epsilon \to 0$. For simplicity we will do these computations formally, that is with $\epsilon = 0$:

Multiplying (2.a) by $x^2\overline{u}$ and integrating the imaginary part leads to

$$Re\int_{\mathbb{R}_+}\frac{\partial u}{\partial t}x^2\overline{u}dx + Im\int_{\mathbb{R}_+}\frac{\partial^2 u}{\partial x^2}x^2\overline{u}dx = 0. \tag{8}$$

By integrating by parts,

$$Im\int_{\mathbb{R}_+}\frac{\partial^2 u}{\partial x^2}x^2\overline{u}dx = -Im\int_{\mathbb{R}_+}\frac{\partial u}{\partial x}2x\overline{u}dx - Im\int_{\mathbb{R}_+}\frac{\partial u}{\partial x}x^2\frac{\partial \overline{u}}{\partial x}dx$$

and (6) follows.

In addition, multiplying (2.a) by $x\frac{\partial \overline{u}}{\partial x}$ and integrating the real part:

$$- 2Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial t} x \frac{\partial \overline{u}}{\partial x} dx + 2Re\int_{\mathbb{R}_+} \frac{\partial^2 u}{\partial x^2} x \frac{\partial \overline{u}}{\partial x} dx = 2Re\int_{\mathbb{R}_+} |u|^2 ux \frac{\partial \overline{u}}{\partial x} dx + 2bRe\int_{\mathbb{R}_+} vxu\frac{\partial \overline{u}}{\partial x} dx. \quad (9)$$

Integrating by parts,

$$2Re\int_{\mathbb{R}_+} \frac{\partial^2 u}{\partial x^2} x \frac{\partial \overline{u}}{\partial x} dx = -2Re\int_{\mathbb{R}_+} \frac{\partial u}{\partial x}\frac{\partial \overline{u}}{\partial x} dx - 2Re\int_{\mathbb{R}_+} \frac{\partial u}{\partial x} x \frac{\partial^2 \overline{u}}{\partial x^2} dx,$$

that is,

$$2Re\int_{\mathbb{R}_+} \frac{\partial^2 u}{\partial x^2} x \frac{\partial \overline{u}}{\partial x} dx = -\int_{\mathbb{R}_+} \left|\frac{\partial u}{\partial x}\right|^2 dx. \quad (10)$$

Moreover,

$$2Re\int_{\mathbb{R}_+} |u|^2 ux \frac{\partial \overline{u}}{\partial x} dx = -\frac{1}{2}\int_{\mathbb{R}_+} |u|^4 dx \quad (11)$$

and, using equation (2.b),

$$2Re\, b\int_{\mathbb{R}_+} vxu\frac{\partial \overline{u}}{\partial x} = b\int_{\mathbb{R}_+} vx\frac{\partial}{\partial x}(|u|^2)dx = \frac{b}{\alpha}\int_{\mathbb{R}_+} vx(\frac{\partial v}{\partial t} + 2av\frac{\partial v}{\partial x})dx$$

$$= \frac{b}{2\alpha}\frac{d}{dt}\int_{\mathbb{R}_+} xv^2 dx + \frac{ba}{\alpha}\int_{\mathbb{R}_+} x\frac{\partial}{\partial x}\left(\frac{2}{3}v^3\right)dx = \frac{b}{2\alpha}\frac{d}{dt}\int_{\mathbb{R}_+} xv^2 dx - \frac{2ab}{3\alpha}\int_{\mathbb{R}+} v^3 dx. \quad (12)$$

Futhermore,

$$-2Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial t} x \frac{\partial \overline{u}}{\partial x} dx = 2Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial t}\overline{u} dx + 2Im\int_{\mathbb{R}_+} \frac{\partial^2 u}{\partial t \partial x} x\overline{u} dx$$

$$= 2Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial t}\overline{u} dx + 2\frac{d}{dt}Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial x} x\overline{u} dx - 2Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial x} x\frac{\partial \overline{u}}{\partial t} dx,$$

hence

$$-2Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial t} x \frac{\partial \overline{u}}{\partial x} dx = Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial t}\overline{u} dx + \frac{d}{dt}Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial x} x\overline{u} dx. \quad (13)$$

In addition, by (2.a),

$$Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial t}\overline{u} dx = -Re\int_{\mathbb{R}_+} i\frac{\partial u}{\partial t}\overline{u} dx$$

$$= Re\int_{\mathbb{R}_+} \frac{\partial^2 u}{\partial x^2}\overline{u} dx - \int_{\mathbb{R}_+} |u|^4 dx - b\int_{\mathbb{R}_+} |u|^2 v dx = -Re\int_{\mathbb{R}_+} \left|\frac{\partial u}{\partial x}\right|^2 dx - \int_{\mathbb{R}_+} |u|^4 dx - b\int_{\mathbb{R}_+} |u|^2 v dx.$$

From (13), we obtain

$$-2Im\int_{\mathbb{R}_+}\frac{\partial u}{\partial x}x\frac{\partial \overline{u}}{\partial x}dx=-\int_{\mathbb{R}_+}\left|\frac{\partial u}{\partial x}\right|^2dx-\int_{\mathbb{R}_+}|u|^4dx-b\int_{\mathbb{R}_+}|u|^2vdx+\frac{d}{dt}Im\int_{\mathbb{R}_+}\frac{\partial u}{\partial x}x\overline{u}dx. \tag{14}$$

Now, by (14),(9),(10),(11) and (12) we deduce (7).
For the continuity properties we refer the reader to the proof of Proposition 6.5.1 in [2].■

Lemma 3.2 *Let $a\alpha\neq 0$.*
If $xu_o, xu_{ox}\in L^2(\mathbb{R}^+)$ and $x^2v_o\in L^1(\mathbb{R}_+)$ then for all $t\in[0,T[$,

$$\frac{d}{dt}\frac{1}{2}\int_{\mathbb{R}_+}x^2vdx=a\int_{\mathbb{R}_+}xv^2dx-\alpha\int_{\mathbb{R}_+}x|u|^2dx \tag{15}$$

and

$$\frac{d^2}{dt^2}\frac{b}{2a\alpha}\int_{\mathbb{R}_+}x^2vdx=\frac{b}{\alpha}\int_{\mathbb{R}_+}xv^2dx-\frac{2b}{a}Im\int_{\mathbb{R}_+}\overline{u}\frac{\partial u}{\partial x}dx, \tag{16}$$

the function $t\to\int_{\mathbb{R}_+}x^2v(x,t)dx$ belonging to $\mathcal{C}^2([0,T[)$.

Proof:
We point out that $xv_o^2\leq v_o^2+x^2v_o^2\in L^1(\mathbb{R}_+)$: $xv_o^2\in L^1(\mathbb{R}_+)$.
Now, let $\epsilon\in]0,1[$. From (2.a),

$$\frac{d}{dt}\int_{\mathbb{R}_+}e^{-\epsilon x}x^2|v|dx=\int_{\mathbb{R}_+}e^{-\epsilon x}x^2sgn(v)\frac{\partial v}{\partial t}dx$$

$$=-\int_{\mathbb{R}_+}e^{-\epsilon x}x^2sgn(v)a\frac{\partial}{\partial x}(v^2)dx+\alpha\int_{\mathbb{R}_+}e^{-\epsilon x}x^2sgn(v)\frac{\partial}{\partial x}(|u|^2)dx$$

and

$$-\int_{\mathbb{R}_+}e^{-\epsilon x}x^2sgn(v)\frac{\partial}{\partial x}(v^2)dx=-\int_{\mathbb{R}_+}e^{-\epsilon x}x^22|v|v_xdx$$

$$=-2\epsilon\int_{\mathbb{R}_+}e^{-\epsilon x}x^2|v|\frac{\partial v}{\partial x}dx+4\int_{\mathbb{R}_+}e^{-\epsilon x}xv|v|dx+2\int_{\mathbb{R}_+}e^{-\epsilon x}x^2v\frac{\partial}{\partial x}|v|dx.$$

Since $|v|_xv=sgn(v)v_xv=|v|v_x$, we deduce

$$\left(\frac{\epsilon}{2}-1\right)\int_{\mathbb{R}_+}e^{-\epsilon x}x^2sgn(v)a\frac{\partial}{\partial x}v^2dx=a\int_{\mathbb{R}_+}e^{-\epsilon x}|v|vdx$$

and so

$$\frac{d}{dt}\int_{\mathbb{R}_+}x^2|v|dx\leq\frac{2|a|}{2-\epsilon}\int_{\mathbb{R}_+}e^{-\epsilon x}xv^2dx+2|\alpha|\int_{\mathbb{R}_+}e^{-\epsilon x}|u||u_x|dx.$$

By integrating in t and letting $\epsilon \to 0$, we obtain

$$\int_{\mathbb{R}_+} x^2|v|dx \leq \int_{\mathbb{R}_+} x^2|v_o|dx + c(t), \quad \text{where } c \text{ is a continuous function.}$$

Furthermore, from (2.a),

$$\int_{\mathbb{R}_+} xv^2dx = -\frac{1}{2}\int_{\mathbb{R}_+} x^2\frac{\partial}{\partial x}(v^2)dx = \frac{1}{2a}\frac{d}{dt}\int_{\mathbb{R}_+} x^2vdx - \frac{\alpha}{2a}\int_{\mathbb{R}_+} x^2\frac{\partial}{\partial x}|u|^2dx$$
$$= \frac{1}{2a}\frac{d}{dt}\int_{\mathbb{R}_+} x^2vdx + \frac{\alpha}{a}\int_{\mathbb{R}_+} x|u|^2dx$$

and (15) is proved.

To obtain (16) from (15), we point out that

$$\int_{\mathbb{R}_+} x\frac{\partial}{\partial t}|u|^2dx = 2\int_{\mathbb{R}_+} xRe(\overline{u}\frac{\partial u}{\partial t})dx = 2\int_{\mathbb{R}_+} xIm(i\overline{u}\frac{\partial u}{\partial t})dx$$
$$= 2\int_{\mathbb{R}_+} xIm\left(\overline{u}\left(-\frac{\partial^2 u}{\partial x^2} + |u|^2u + bvu\right)\right)dx = -2Im\int_{\mathbb{R}_+} x\overline{u}\frac{\partial^2 u}{\partial x^2}dx = 2Im\int_{\mathbb{R}_+} \overline{u}\frac{\partial u}{\partial x}dx.$$

■

Remark 3.3 *By similar computations we easily deduce if* $v_o \in L^1(\mathbb{R}_+)$ *(respectively* $v_o \in W^{1,1}(\mathbb{R}_+)$*) that* $v \in \mathcal{C}([0,T[, L^1(\mathbb{R}_+))$ *(respectively* $v \in \mathcal{C}([0,T[, W^{1,1}(\mathbb{R}_+))$*.*

Lemma 3.4 *Let* $a\alpha = -1$ *and* $xu_o, xu_{ox} \in L^2(\mathbb{R}^+)$ *and* $x^2v_o \in L^1(\mathbb{R}_+)$*.*
Let

$$L(t) = \frac{1}{2}\int_{\mathbb{R}_+} x^2(|u|^2 + bv)dx, \ t \in [0,T[.$$

Then $L \in \mathcal{C}^2([0,T[)$ *and*

$$\frac{dL}{dt}(t) = 2Im\int_{\mathbb{R}_+} \frac{\partial u}{\partial x}x\overline{u}dx + ab\int_{\mathbb{R}_+} xv^2dx + \frac{b}{a}\int_{\mathbb{R}_+} x|u|^2dx \tag{17}$$

$$\frac{d^2L}{dt^2}(t) = 8E(0) - \frac{a}{b}M(0) - b^2\int_{\mathbb{R}_+} v^2dx - \int_{\mathbb{R}_+} |u|^4dx - 2b\int_{\mathbb{R}_+} |u|^2\frac{\partial v}{\partial x}dx, \tag{18}$$

where (cf. (4) and (5)),

$$E(t) = \frac{1}{4}\int_{\mathbb{R}_+} |u|^4dx + \frac{1}{2}\int_{\mathbb{R}_+} \left|\frac{\partial u}{\partial x}\right|^2 dx + \frac{b}{2}\int_{\mathbb{R}_+} v|u|^2dx + \frac{a^2b}{6}\int_{\mathbb{R}_+} v^3dx = E(0)$$

and

$$M(t) = -2Im\int_{\mathbb{R}_+}\overline{u}\frac{\partial u}{\partial x}dx - ab\int_{\mathbb{R}_+} v^2dx = M(0).$$

Proof: The equality (17) is a consequence of (6) and (15). Equality (17) follows from (6), (7), (16) and from (5) to compute $\frac{2b}{a}Im\int_{\mathbb{R}_+}\overline{u}u_xdx$.

Now, from (18),

$$\frac{d^2L}{dt^2}(t) \leq 8E(0) - \frac{b}{a}M(0). \tag{19}$$

If we choose $a > 0$ and $b = \frac{1}{a}$ we obtain

$$\frac{d^2L}{dt^2}(t) \leq \frac{1}{4}\int_{\mathbb{R}_+}|u_o|^4dx + \frac{1}{2}\int_{\mathbb{R}_+}\left|\frac{\partial u_o}{\partial x}\right|^2dx + \frac{1}{2a}\int_{\mathbb{R}_+}v_o|u_o|^2dx + \frac{a}{6}\int_{\mathbb{R}_+}v_o^3dx$$

$$+\frac{2}{a^2}Im\int_{\mathbb{R}_+}\overline{u_o}\frac{\partial u_o}{\partial x}dx + \frac{1}{a^2}\int_{\mathbb{R}_+}v_o^2dx.$$

Hence, if we choose v_o such that

$$\int_{\mathbb{R}_+} v_o^3 < 0 \tag{20}$$

we obtain for large $a > 0$

$$\frac{d^2L}{dt^2}(t) < 0, \text{ for all } t \in [0, T[. \tag{21}$$

Now, from (15),

$$\frac{d}{dt}\frac{1}{2a}\int_{\mathbb{R}_+}x^2vdx = \int_{\mathbb{R}_+}xv^2dx + \frac{1}{a^2}dx\int_{\mathbb{R}_+}x|u|^2dx \geq 0.$$

Hence, for all $t \in [0, T[$,

$$L(t) = \frac{1}{2}\int_{\mathbb{R}_+}x^2|u|^2dx + \frac{1}{2a}\int x^2vdx \geq \frac{1}{2a}\int_{\mathbb{R}_+}x^2v_odx. \tag{22}$$

Moreover, from (17),

$$\frac{d}{dt}L(t)|_{t=0} = 2Im\int_{\mathbb{R}_+}\frac{\partial u_o}{\partial x}x\overline{u_o}dx + \int_{\mathbb{R}_+}xv_o^2dx + \frac{1}{a^2}\int x|u_o|^2dx,$$

and so, if we assume in addition

$$2Im\int_{\mathbb{R}_+}\frac{\partial u_o}{\partial x}x\overline{u_o}dx + \int_{\mathbb{R}_+}xv_o^2dx < 0 \tag{23}$$

then for large $a > 0$ we obtain (21) and

$$L'(0) = \frac{d}{dt} L(t)|_{t=0} < 0. \tag{24}$$

Hence, if $T = +\infty$, by (22) we arrive to a contradiction and it is easy to obtain geometrically an estimate for the blow-up time T:

$$T \leq \left(\frac{1}{2} \int_{\mathbb{R}_+} x^2 |u_o|^2 dx \right) \left(-L'(0) \right)^{-1}. \tag{25}$$

We can summarize the previous results as follows:

Theorem 3.5 *Under the hypothesis of Lemma 3.4 choose $b = -\alpha = \dfrac{1}{a}$ and assume (20) and (23). Taking $a > 0$ large enough so that (21) and (24) hold, then the maximal interval $[0, T[$ of existence of the unique strong solution (u, v) to the Cauchy problem for the system (2) is finite and T verifyies the estimate (25).*

4 Blow-up of v in $L^1(\mathbb{R}_+)$

Let (u, v) the local strong solution to the I.V.P. associated to (2) given by Theorem 2.1. We choose $v_o \in L^1(\mathbb{R})$. By Remark 3.3, for all $t \in [0, T[$, $v(., t) \in L^1(\mathbb{R}_+)$.
In the rest of this section we will assume that $a > 0$.

Let $\phi \in L^\infty(\mathbb{R}_+)$ such that $\phi' \in L^\infty(\mathbb{R}_+)$. By multiplying equation (2.b) by ϕ, integrating with respect to x and using integration by parts, it is straightforward to prove that

$$\frac{d}{dt} \int_{\mathbb{R}_+} \phi v dx = \int_{\mathbb{R}_+} \phi' (a v^2 - \alpha |u|^2) dx. \tag{26}$$

Let us introduce the function

$$Y : t \to -\int_{\mathbb{R}_+} e^{-x} v(x, t) dx.$$

Then the following proposition holds:

Proposition 4.1 *Let $\alpha < 0$ and $v_o \in L^1(\mathbb{R}_+)$ such that $Y(0) = -\int_{\mathbb{R}_+} e^{-x} v_o(x) dx > 0$. Then, for all $t \in [0, T[$,*

$$Y(t) \geq \frac{Y(0)}{1 - a t Y(0)}. \tag{27}$$

In particular the function Y is unbounded on the finite time interval $[0, t^*[$, where $t^* = \dfrac{1}{a Y(0)}$.

Proof:

Setting $\phi(x) = -e^{-x}$ we obtain by (26)

$$\frac{dY}{dt}(t) = \frac{d}{dt}\left(-\int_{\mathbb{R}_+} e^{-x} v(x,t)dx\right) = \int_{\mathbb{R}_+} e^{-x}(|\alpha||u|^2 + av^2)dx, \tag{28}$$

hence $Y(t) \geq 0$ for all times. Furthermore, by Cauchy-Schwarz,

$$Y(t) = \left|\int_{\mathbb{R}_+} e^{-x} v(x,t)dx\right| = \left|\int_{\mathbb{R}_+} e^{-\frac{x}{2}} e^{-\frac{x}{2}} v(x,t)dx\right| \leq \left(\int_{\mathbb{R}_+} e^{-x}dx\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}_+} e^{-x}v^2 dx\right)^{\frac{1}{2}}$$

and

$$Y(t) \leq \left(\int_{\mathbb{R}_+} e^{-x} v^2 dx\right)^{\frac{1}{2}}. \tag{29}$$

Replacing this inequality in (28), we obtain $\frac{dY}{dt}(t) \geq aY^2(t)$, which by integration leads to (27). ■

We now turn our attention to the case where $\alpha > 0$.

Proposition 4.2 *Let $\alpha > 0$. Let $v_o \in \mathbb{R}_+$ such that*

$$Y(0) > A^2, \tag{30}$$

where

$$A = \frac{\alpha}{a} C_o, \quad C_o = \int_{\mathbb{R}_+} |u_o|^2(x)dx.$$

Then, for all $t \in [0, T[$,

$$Y(t) \geq \frac{A(1 + \lambda e^{2Aat})}{1 + \lambda e^{2Aat}}, \quad \lambda = \frac{Y(0) - A}{Y(0) + A} < 1. \tag{31}$$

In particular the function Y is unbounded on the finite time interval $[0, t^*[$, where $t^* = \dfrac{|\ln(\lambda)|}{2Aa}$.

Proof:

In this case, $\dfrac{dY}{dt}(t) = \displaystyle\int_{\mathbb{R}_+} e^{-x}(-\alpha|u|^2 + av^2)dx.$

Applying (29), $\dfrac{dY}{dt}(t) \geq aY^2(t) - \alpha C_o$, where we have used the invariant (3). Hence

$$\frac{dY}{d\tau}(\tau) \geq Y^2(\tau) - A^2, \quad \tau = at. \tag{32}$$

According to (30), there exists $T^* > 0$ such that for all $t \leq T^*$, $Y(t) > A$.
Rewriting (32) as

$$\frac{1}{2A}\left(\frac{1}{Y(\tau)-A} - \frac{1}{Y(\tau)+A}\right) dY \geq d\tau$$

and integrating this last inequality yields the result. ■

Remark 4.3 *In the case where $a < 0$, by choosing $Y(t) = \displaystyle\int_{\mathbb{R}_+} e^{-x} v(x,t)dt$ similar results can be proved:*
If $\alpha > 0$ we obtain a result analogous to Proposition 4.1 with $t^* = \dfrac{Y(0)}{1+taY(0)}$;
if $\alpha < 0$ we obtain a result analogous to Proposition 4.2 with $t^* = -\dfrac{|\ln(\lambda)|}{2Aa}$;

Finally, we can state our last result:

Theorem 4.4 *Let $a\alpha \neq 0$ and (u,v) the unique solution to the Cauchy problem of (2) given by Theorem 2.1. Then, for initial data $v_o \in L^1(\mathbb{R}_+)$ (respectively $v_o \in W^{1,1}(\mathbb{R}_+)$), the assumptions in Proposition 4.1, Proposition 4.2 or Remark 4.3 being verified, the norm $\|v(.,t\|_{L^1}$ (respectively $\|v(.,t)\|_{L^1}$ and $\|\frac{\partial v}{\partial x}(.,t)\|_{L^1}$) becomes unbounded in a finite interval of the form $[0,t^*[$, $t^* > 0$. Hence, by Remark 3.3, $T < +\infty$.*

Proof:

In view of Proposition 4.1 and Proposition 4.2, one only has to notice that

$$Y(t) = |Y(t)| = \left|\int_{\mathbb{R}_+} e^{-x} v(x,t)dx\right| \leq \int_{\mathbb{R}_+} |v(x,t)|dx$$

$$Y(t) = |Y(t)| = \left|\int_{\mathbb{R}_+} e^{-x} \frac{\partial v}{\partial x}(x,t)dx\right| \leq \int_{\mathbb{R}_+} |\frac{\partial v}{\partial x}(x,t)|dx$$

Remark 4.5 *Similar results can be proved replacing the L^1 norm by the L^∞ norm.*

Acknowledgements: The research of S. Antontsev, J.P. Dias and M. Figueira was partially supported by FCT, Financiamento Base 2008-ISFL-1-209 and the research of F.Oliveira was partially supported by FCT, Financiamento Base 2008-ISFL-1-297.

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