REGULARITY FOR THE NAVIER-STOKES-FOURIER SYSTEM

L. CONSIGLIERI

Abstract. We prove that exist strong 2-dimensional solutions for two Cauchy-Dirichlet problems to the Navier-Stokes-Fourier system which characterizes the Newtonian fluids under heat-conducting effects. The regularity of solutions to the problems under study is proved through compactness methods and fixed point arguments, instead assuming the existence of weak solutions to the problems.

1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a bounded open domain sufficiently regular and $T > 0$. Let us consider the Cauchy-Dirichlet problem in the following form:

$$\begin{gathered}\partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathrm{div}(\mu(\theta) D\mathbf{u}) = \mathbf{f} - \nabla p \quad \text{in } Q_T := \Omega\times]0,T[; \\ \mathrm{div}\ \mathbf{u} = 0 \quad \text{in } Q_T;\end{gathered} \tag{1}$$

$$\partial_t \theta + \mathbf{u}\cdot\nabla\theta - \mathrm{div}(k(\theta)\nabla\theta) = \mu(\theta)|D\mathbf{u}|^2 + g \quad \text{in } Q_T; \tag{2}$$

$$\mathbf{u}\big|_{t=0} = \mathbf{u}_0, \qquad \theta\big|_{t=0} = \theta_0, \quad \text{in } \Omega; \tag{3}$$

$$\mathbf{u} = \bar{\mathbf{u}}, \qquad \theta = \bar{\theta}, \quad \text{on } \partial\Omega\times]0,T[; \tag{4}$$

where p denotes the pressure, μ the viscosity, θ the temperature, $\mathbf{u}$ the velocity of the fluid and $D\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T)$, $\mathbf{f}$ denotes the given external body forces and g the heat source. The Kirchoff transformation allows us to assume constant the thermal conductivity, i.e.

$$k(\theta) \equiv k.$$

In the present work the product of two tensors is given by $D : \tau = D_{ij}\tau_{ij}$ and the norm by $|D|^2 = D : D$.

The Navier-Stokes-Fourier system arises from fluid thermomechanics. In fact, it is constituted by momentum and energy equations when the constitutive relations for the Cauchy stress and heat flux are assumed linear. The density is constant and assumed equal to one. The initial conditions are given in (3) and we assume Dirichlet boundary conditions in (4). For the sake of clarity we found convenient that the boundary conditions are taken to be homogeneous,

$$\bar{\mathbf{u}} \equiv \mathbf{0} \quad \text{and} \quad \bar{\theta} \equiv 0. \tag{5}$$

2000 *Mathematics Subject Classification.* Primary 35Q30, 80A20; Secondary 35K55, 76D03.

The abstract mathematical viscosity can be illustrated with Arrhenius law: $\mu(\theta) = \mu_0 \exp[E_0(1/\theta - 1/\theta_r)/R]$, where μ_0, E_0, θ_r are constants of reference and R is the gas constant.

In the seventies, Lauder and Spalding proposed the $k - \varepsilon$ model (it consists of two equations for the turbulence kinetic energy k and the rate of dissipation ε of the turbulent energy) to describe the mean of a turbulence flow. Unfortunately, the turbulence is essentially a three dimensional phenomenon and it is not clear that this model produces physically relevant results (a positive energy, for example). Despite the fact that the validity of $k - \varepsilon$ model is not universal, it presents a good compromise between simplicity and generality (see [17]). In this context, the equations (1) represent the averaged Navier-Stokes equation in which u, π and f are the mean values of velocity, pressure and external forces, respectively, the viscosity is the eddy viscosity, and the equation (2) represents the $k-\varepsilon$ model, that is, θ denotes the mean turbulent kinetic energy and

$$g = -\theta|\theta|^{1/2}$$

denoting the Navier-Stokes turbulence. More physical motivation can be found in [1, 3] for instance.

Several authors proved existence of solutions to similar mathematical problems in fluid thermomechanics (see for example [4, 8, 13, 14, 18, 21] and the references therein). The existence of at least a weak solution is given in [6] for different constitutive relations in the Cauchy stress and in the Fourier heat flux. We refer to [7] for the existence of strong and classical solutions to the stationary coupled system under general constitutive relations.

Although the continuity of the coefficients, to prove the regularity of solution to the coupled system, additional terms appear which invalidate the direct application of known regularity results ([9, 11, 12, 16] between others). Notice that if the velocity $\mathbf{u}$ is a weak solution to (1) the Joule effect term $\mu(\theta)|D\mathbf{u}|^2$ belongs to $L^1(Q_T)$ and the existence of a solution of the energy equation (2) requires L^1-theory (see [5, 6] and the references therein). We wish to emphasize that at the present work we do not show regularity for every weak solution, we prove existence results under smallness restrictions only on the ratio between the derivative of the viscosity and the lower bound of the viscosity function and the thermal conductivity constant (cf. (11)). Indeed, here we prove that the equation (which is satisfied by the solution) is valid almost everywhere in Q_T which means that the strong solutions coincide by the uniqueness result with the weak solutions in a smaller space.

The outline of the paper is as follows. In next section we present the appropriate functional framework and we state two main existence results. First existence result is established under a known function g and the second one is given for $g = -\theta|\theta|^{1/2}$. In section 3, we prove some technical results. Sections 4 and 5 are devoted to the proofs of the solvability of the problems under study. The uniqueness results are proved in section 6 without any additional assumption on the data.

2. Assumptions and main results

Here we assume that $\Omega \subset \mathbb{R}^n$ is a bounded open set with sufficiently smooth boundary $\partial\Omega$. In the framework of Lebesgue and Sobolev spaces, we introduce for $q > 1$ [11]

$$J^{1,q}(\Omega) = \{\mathbf{u} \in \mathbf{W}^{1,q}(\Omega) : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega\},$$

where the vector spaces of vector-valued or tensor-valued functions are denoted by bold. It is known that $J_0^{1,q}(\Omega) = J^{1,q}(\Omega) \cap \mathbf{W}_0^{1,q}(\Omega)$ with norm

$$\| \cdot \|_{(1),q,\Omega} = \|\nabla \cdot \|_{q,\Omega}.$$

We will use the following Banach spaces, for $1 \leq q, r \leq \infty$, [12]

$$\begin{aligned} L^{q,r}(Q_T) &= L^r(0,T; L^q(\Omega)) \\ W_q^{1,0}(Q_T) &= L^q(0,T; W^{1,q}(\Omega)) \\ W_q^{1,1}(Q_T) &= L^q(0,T; W^{1,q}(\Omega)) \cap W^{1,q}(0,T; L^q(\Omega)) \\ W_q^{2,1}(Q_T) &= L^q(0,T; W^{2,q}(\Omega)) \cap W^{1,q}(0,T; L^q(\Omega)). \end{aligned}$$

We recall that the following continuous inclusion $W_q^{2,1}(Q_T) \hookrightarrow C^{k,\alpha}(\bar{Q}_T)$ only occurs if $q > (n+2)/(2-k)$ and $0 \leq \alpha < 2 - k - (n+2)/q$. This means for $k = 0$ that $q > n/2 + 1 \geq 2$ $(n = 2, 3)$, i.e., the Banach space $W_2^{2,1}(Q_T)$ is not embedded in the Banach space of Hölder continuous functions with exponent α in the x-variables and $\alpha/2$ in the t-variable. Note that $u(0)$ makes sense for all $u \in W_q^{1,1}(Q_T)$ since $W_q^{1,1}(Q_T) \hookrightarrow C([0,T]; L^q(\Omega))$.

The following assertions on data are assumed as well as the following assumptions on the physical parameters appearing in the equations are established.

- $\mathbf{f} : Q_T \to \mathbb{R}^n$ is given such that $\mathbf{f} \in \mathbf{L}^2(Q_T)$ and

$$\partial_t \mathbf{f} \in \mathbf{L}^2(Q_T); \tag{6}$$

- $\mu : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that

$$0 < \mu_0 \leq \mu(s) \leq \mu_1, \quad |\mu'(s)| \leq \mu_2, \qquad \forall s \in \mathbb{R}; \tag{7}$$

- $\mathbf{u}_0 \in J_0^{1,2}(\Omega) \cap \mathbf{H}^2(\Omega)$ and $\theta_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ satisfy the following compatibility conditions

$$\nabla \mathbf{u}_0 \cdot \mathbf{n} = \mathbf{0}, \quad \nabla \theta_0 \cdot \mathbf{n} = 0, \qquad \text{on } \partial\Omega, \tag{8}$$

  where $\mathbf{n}$ denotes the unit outward normal to the boundary $\partial\Omega$.

The system (1)-(5) has the variational formulation

$$
(9)\quad \begin{cases} \int_Q \partial_t \mathbf{u}\cdot\mathbf{v} + \int_Q \Big(\mu(\theta) D\mathbf{u} : D\mathbf{v} + (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}\Big) = \int_Q \mathbf{f}\cdot\mathbf{v} \\ \qquad \forall \mathbf{v}\in L^2(0,T; J_0^{1,2}(\Omega)), \qquad \mathbf{u}\big|_{t=0} = \mathbf{u}_0 \text{ in } \Omega; \\ \int_Q (\partial_t\theta)\eta + \int_Q \Big(k\nabla\theta\cdot\nabla\eta + \mathbf{u}\cdot\nabla\theta\eta\Big) = \int_Q \Big(\mu(\theta)|D\mathbf{u}|^2 + g\Big)\eta \\ \qquad \forall \eta\in L^2(0,T; W_0^{1,2}(\Omega)), \qquad \theta\big|_{t=0} = \theta_0 \text{ in } \Omega. \end{cases}
$$

Remark 2.1. *For all $\mathbf{u},\mathbf{v}\in J_0^{1,2}(\Omega)$, the convective term verifies*

$$
\int_\Omega (\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v} = \int_\Omega \nabla\mathbf{u} : \mathbf{v}\otimes\mathbf{u} = \int_\Omega (\nabla\mathbf{u})^T : \mathbf{u}\otimes\mathbf{v} = 2\int_\Omega D\mathbf{u} : \mathbf{u}\otimes\mathbf{v}.
$$

Theorem 2.1. *Suppose that (6)-(8) be fulfilled and $n = 2$. Let $g : Q_T \to \mathbb{R}$ be such that $g\in L^2(Q_T)$ and*

$$
(10)\qquad \partial_t g \in L^2(Q_T).
$$

Under the assumption that

$$
(11)\qquad \mu_2^4/(2\mu_0) \quad \text{and} \quad \mu_2^4/(2k) \text{ are sufficiently small,}
$$

then the problem (9) admits, at least, one solution $(\mathbf{u},\theta)\in L^2(0,T; J_0^{1,2}(\Omega))\times L^2(0,T; H_0^1(\Omega))$ which is strong, i.e.

$$
(\mathbf{u},\theta)\in \mathbf{W}_2^{2,1}(Q_T)\times W_2^{2,1}(Q_T).
$$

Moreover, such solution is Hölder continuous, $(\mathbf{u},\theta)\in C^{0,\alpha}(\bar{Q}_T)\times C^{0,\alpha}(\bar{Q}_T)$, for some $\alpha > 0$.

It is known that the pressure is recovered as a distribution from the variational formulation thanks to the De Rham Theorem [15]. Using Theorem 2.1 we can rewrite (1) as

$$
\nabla p = \mathbf{f} - \partial_t\mathbf{u} - (\mathbf{u}\cdot\nabla)\mathbf{u} + \mu'(\theta)\nabla\theta D\mathbf{u} + \mu(\theta)\Delta\mathbf{u} \in \mathbf{L}^2(Q_T).
$$

Theorem 2.2. *Suppose that (6)-(8) and (11) be fulfilled. Under $g = -\theta|\theta|^{1/2}$, the problem (9) admits, at least, one strong solution. Moreover, such solution is Hölder continuous, and $\theta\in W_r^{2,1}(Q_T)$, for all $r < 3$.*

Henceforth we denote by C every positive constant depending on the data, but not on the unknown functions $\mathbf{u}$, p or θ.

3. A priori estimates

The main result theorems 2.1 and 2.2 are proved using the following fixed point argument. We fix $\xi\in W_4^{1,1}(Q_T)$ and we solve the following auxiliary problems

$$\int_\Omega \partial_t \mathbf{u}\cdot\mathbf{v} + \int_\Omega \Big(\mu(\xi)D\mathbf{u} : D\mathbf{v} + \mathbf{v}\otimes\mathbf{u} : \nabla\mathbf{u}\Big) = \int_\Omega \mathbf{f}\cdot\mathbf{v} \tag{12}$$
$$\text{a.e. } t\in[0,T] \quad \forall \mathbf{v}\in J_0^{1,2}(\Omega), \qquad \mathbf{u}\big|_{t=0} = \mathbf{u}_0 \text{ in } \Omega;$$

$$\int_\Omega (\partial_t\theta)\eta + \int_\Omega \Big(k\nabla\theta\cdot\nabla\eta + \mathbf{u}\cdot\nabla\theta\eta\Big) = \int_\Omega \Big(\mu(\xi)|D\mathbf{u}|^2 + g\Big)\eta, \tag{13}$$
$$\text{a.e. } t\in[0,T] \quad \forall \eta\in W_0^{1,2}(\Omega), \qquad \theta\big|_{t=0} = \theta_0 \text{ in } \Omega.$$

Let us begin to prove a crucial embedding proposition.

Proposition 3.1. *Assuming $v \in L^{2(q-1)}(Q_T)$, for any $q \geq 2$, and $\partial_t v \in L^2(Q_T)$ then v belongs to a bounded set of $L^\infty(0,T;L^q(\Omega))$.*

Proof. Let us argue as in [10] writing

$$|v(\cdot,t)|^q = \int_0^t \frac{d}{ds}|v(\cdot,s)|^q = q\int_0^t |v(\cdot,s)|^{q-2}v(\cdot,s)\partial_s v(\cdot,s).$$

Integrating with respect to the space variable and using Schwarz inequality, it follows

$$\begin{aligned}\|v\|^q_{q,\Omega}(t) &\leq q\int_0^t \||v|^{q-1}\|_{2,\Omega}\|\partial_s v\|_{2,\Omega}\\ &\leq q\left(\int_0^t \|v\|^{2(q-1)}_{2(q-1),\Omega}\right)^{1/2}\left(\int_0^t \|\partial_s v\|^2_{2,\Omega}\right)^{1/2}.\end{aligned}$$

Then we can conclude

$$\sup_{0\leq t\leq T}\|v\|^q_{q,\Omega}(t) \leq q\|v\|^{(q-1)}_{2(q-1),Q_T}\|\partial_t v\|_{2,Q_T}.$$

□

Lemma 3.1 (Interpolative inequalities [20])**.** *The interpolative inequalities hold*

$$\forall v\in H^2(\Omega), \quad \|\nabla v\|_{q,\Omega} \leq \|v\|_{2,\Omega}^{\frac{(2-n)q+2n}{4q}}\|\nabla^2 v\|_{2,\Omega}^{\frac{(2+n)q-2n}{4q}} \tag{14}$$

$$\forall v\in H_0^1(\Omega), \quad \|v\|_{q,\Omega} \leq \|v\|_{r,\Omega}^{\frac{r(2q+(2-q)n)}{q(2r+(2-r)n)}}\|\nabla v\|_{2,\Omega}^{\frac{\frac{1}{r}-\frac{1}{q}}{\frac{1}{r}-\frac{n-2}{2n}}}.$$

In particular, $L^{r,\infty}(Q_T)\cap L^2(0,T;H^1(\Omega)) \hookrightarrow L^{2(n+r)/n}(Q_T)$ and

$$\forall v\in H_0^1(\Omega), \quad \|v\|_{q,\Omega} \leq \|v\|_{2,\Omega}^{\frac{2q+(2-q)n}{2q}}\|\nabla v\|_{2,\Omega}^{\left(\frac{1}{2}-\frac{1}{q}\right)n}, \tag{15}$$

$$\forall v\in H^1(\Omega), \quad \|v\|_{2(n+2)/n,\Omega} \leq \|v\|_{2,\Omega}^{\frac{2}{n+2}}\|\nabla v\|_{2,\Omega}^{\frac{n}{n+2}}. \tag{16}$$

Taking $n=2$ in Lemma 3.1 we obtain $L^{2,\infty}(Q_T)\cap L^2(0,T;H^1(\Omega)) \hookrightarrow L^4(Q_T)$, and if we take $n=3$ we have $L^{2,\infty}(Q_T)\cap L^2(0,T;H^1(\Omega)) \hookrightarrow L^{10/3}(Q_T)$. Let us prove the following regularity result.

Proposition 3.2 ($n \geq 2$)**.** *Assuming* $v \in L^{2,\infty}(Q_T) \cap L^2(0,T;H^1(\Omega))$ *and* $\partial_t v \in L^2(Q_T)$ *then* v *belongs to a bounded set of* $L^q(Q_T)$*, for any* $q < 2(n+1)/(n-1)$*. Moreover,* v *belongs to a bounded set of* $L^{r,\infty}(Q_T)$*, for any* $r < 2n/(n-1)$*.*

Proof. From Lemma 3.1, we have $v \in L^{2(n+2)/n}(Q_T)$ and we apply Proposition 3.1 with $2(q-1) = 2(n+2)/n$, i.e., $q = 1 + (n+2)/n$. Next, using Lemma 3.1 with $v \in L^{1+(n+2)/n,\infty}(Q_T) \cap L^2(0,T;H^1(\Omega))$ we obtain $v \in L^{2(n+1+(n+2)/n)/n}(Q_T)$. Define

$$q_0 = \frac{n+2}{n}, \qquad q_1 = \frac{n+1+q_0}{n}$$

and arguing by iteration, we apply Proposition 3.1 with $2(q-1) = 2q_k$, i.e., $q = 1 + q_k$. Now, using Lemma 3.1 with $v \in L^{1+q_k,\infty}(Q_T) \cap L^2(0,T;H^1(\Omega))$ we obtain $v \in L^{2\frac{n+1+q_k}{n}}(Q_T)$.

Thus defining by recurrence

$$q_{k+1} = \frac{n+1+q_k}{n}, \quad k \in \mathbb{N},$$

this sequence is monotone increasing, bounded onto $]0,(n+1)/(n-1)[$ and its limit is $q = (n+1)/(n-1)$, which concludes the first statement of the proof of Proposition 3.2.

Again applying Proposition 3.1 with $2(r-1) = q < 2(n+1)/(n-1)$, i.e., $r = q/2 + 1 < 2n/(n-1)$, we get $v \in L^{q/2+1,\infty}(Q_T)$ if $q \geq 2$. $\square$

Lemma 3.2 ([19, Lemma 4])**.** *Let* $\delta > 0$*,* $\alpha > 0$ *and* $q > 1$*. For any* $v \in L^\infty(0,T;C^{0,\alpha}(\Omega))$ *verifying* $\partial_t v \in L^q(Q_T)$*, there exists a constant* $C > 0$ *such that*

$$|v(x,t_1) - v(x,t_2)| \leq C(\|v\|_{L^\infty(0,T;C^{0,\alpha}(\Omega))} + \|\partial_t v\|_{q,Q_T})|t_1 - t_2|^\beta,$$

for every $x \in B_\delta$ *and every* $t_1, t_2 \in]0,T[$*, where* $\beta = \alpha(q-1)/(\alpha q + n(q-1))$*. In particular,* v *is Hölder continuous in* Q_T*.*

Proposition 3.3. *Under the assumptions* $\mathbf{f} \in \mathbf{L}^2(Q_T)$*, (7) and* $\mathbf{u}_0 \in J_0^{1,2}(\Omega)$*, if* $\nabla\xi \in \mathbf{L}^4(Q)$ *then the solution* $\mathbf{u}$ *of (12) is such that* $\nabla\mathbf{u}$ *belongs to a bounded set of* $L^2(0,T;J_0^{1,2}(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$*. Moreover, it satisfies*

$$\|\nabla\mathbf{u}\|^4_{4,Q_T} \leq \frac{4}{\mu_0}(T\|\nabla\mathbf{u}_0\|^2_{2,\Omega} + \frac{4}{\mu_0}\|\mathbf{f}\|^2_{2,Q_T})^2\mathcal{F}(\frac{\mu_2^4}{2\mu_0}\|\nabla\xi\|^4_{4,Q_T}), \tag{17}$$

with $\mathcal{F}$ *a positive strictly increasing function on its argument.*

Proof. We choose $\mathbf{v} = \Delta\mathbf{u}$ as a test function in (12), then

$$\int_\Omega \partial_t(\nabla\mathbf{u})\cdot\nabla\mathbf{u} + \int_\Omega \{\mu'(\xi)\nabla\xi \otimes D\mathbf{u} + \mu(\xi)\nabla^2\mathbf{u}\} : \nabla^2\mathbf{u} =$$

$$= -\int_\Omega \Delta\mathbf{u} \otimes \mathbf{u} : \nabla\mathbf{u} - \int_\Omega \mathbf{f}\cdot\Delta\mathbf{u}.$$

Here $\nabla^2\mathbf{u} = (\partial_k D_{ij})$ is a third order tensor, with $D_{ij} = ((\partial_j u_i + \partial_i u_j)/2)$, and $\nabla\xi \otimes D\mathbf{u} : \nabla^2\mathbf{u} = \partial_k\xi D_{ij}\partial_k D_{ij}$ under the Einstein convention.

Using the assumption (7) and the property of the convective term vanishes in the two dimensional space (cf. [16], for instance) we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mathbf{u}\|_{2,\Omega}^2 + \mu_0\int_\Omega |\nabla^2\mathbf{u}|^2 \leq \mu_2\int_\Omega |\nabla\xi\otimes D\mathbf{u} : \nabla^2\mathbf{u}| + \int_\Omega |\mathbf{f}\cdot\Delta\mathbf{u}|.$$

Applying Hölder inequalities and integrating in time for each $t\in]0,T[$, it follows

$$\frac{1}{2}\|\nabla\mathbf{u}\|_{2,\Omega}^2(t) + \mu_0\int_0^t\int_\Omega |\nabla^2\mathbf{u}|^2 \leq \frac{1}{2}\|\nabla\mathbf{u}(0)\|_{2,\Omega}^2 +$$

$$+\mu_2\int_0^t \|\nabla\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\nabla^2\mathbf{u}\|_{2,\Omega} + \int_0^t \|\mathbf{f}\|_{2,\Omega}\|\nabla^2\mathbf{u}\|_{2,\Omega}.$$

Applying Lemma 3.1, (16) with $n=2$ and $v=\nabla\mathbf{u}$, to the second term on right hand side of the above inequality and successively using Young inequalities, we obtain

$$\frac{1}{2}\|\nabla\mathbf{u}\|_{2,\Omega}^2(t) + \mu_0\int_0^t \|\nabla^2\mathbf{u}\|_{2,\Omega}^2 \leq \frac{1}{2}\|\nabla\mathbf{u}(0)\|_{2,\Omega}^2 +$$

$$+\frac{\mu_2^4}{4\mu_0}\int_0^t \|\nabla\xi\|_{4,\Omega}^4\|\nabla\mathbf{u}\|_{2,\Omega}^2 + \frac{3\mu_0}{4}\int_0^t \|\nabla^2\mathbf{u}\|_{2,\Omega}^2 +$$

$$+\frac{2}{\mu_0}\int_0^t \|\mathbf{f}\|_{2,\Omega}^2 + \frac{\mu_0}{8}\int_0^t \|\nabla^2\mathbf{u}\|_{2,\Omega}^2.$$

Thus, we deduce

$$\frac{1}{2}\|\nabla\mathbf{u}\|_{2,\Omega}^2(t) + \frac{\mu_0}{8}\int_0^t \|\nabla^2\mathbf{u}\|_{2,\Omega}^2 \leq \frac{1}{2}\|\nabla\mathbf{u}_0\|_{2,\Omega}^2 + \frac{2}{\mu_0}\int_0^t \|\mathbf{f}\|_{2,\Omega}^2 +$$

$$+\frac{\mu_2^4}{4\mu_0}\int_0^t \|\nabla\xi\|_{4,\Omega}^4\|\nabla\mathbf{u}\|_{2,\Omega}^2.$$

Thus we conclude the estimate in $L^\infty(0,T;L^2(\Omega))$ with help of Gronwall Lemma

$$\operatorname{ess}\sup_{t\in[0,T]} \|\nabla\mathbf{u}\|_{2,\Omega}^2 \leq (T\|\nabla\mathbf{u}_0\|_{2,\Omega}^2 + \frac{4}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2)\exp\left[\mathcal{G}(\xi)\right] \tag{18}$$

$$\text{where } \mathcal{G}(\xi) = \frac{\mu_2^4}{2\mu_0}\|\nabla\xi\|_{4,Q_T}^4.$$

Next the estimate in $L^2(0,T;H_0^1(\Omega))$ follows

$$\|\nabla^2\mathbf{u}\|_{2,Q_T}^2 \leq \frac{4}{\mu_0}(T\|\nabla\mathbf{u}_0\|_{2,\Omega}^2 + \frac{4}{\mu_0}\|\mathbf{f}\|_{2,Q_T}^2)\left(1+\mathcal{G}(\xi)\exp\left[\mathcal{G}(\xi)\right]\right).$$

Finally applying Lemma 3.1, (16) with $n=2$ and $v=\nabla\mathbf{u}$, $\nabla\mathbf{u}$ belonging to a bounded set of $L^2(0,T;\mathbf{H}_0^1(\Omega)\cap L^{2,\infty}(Q_T)$ implies (17) with $\mathcal{F}(d)=\exp[d](1+d\exp[d])$. $\square$

Lemma 3.3. *Under the assumptions of Proposition 3.3, $\theta_0 \in L^2(\Omega)$ and $g \in L^2(Q_T)$, then the solution $(\mathbf{u},\theta)$ of (12)-(13) satisfies*

$$\|\mathbf{u}\|^4_{4,Q_T} \leq \frac{1}{\mu_0}(T\|\mathbf{u}_0\|^2_{2,\Omega} + \frac{1}{\mu_0}\|\mathbf{f}\|^2_{2,Q_T})^2; \tag{19}$$

$$\|\theta\|^4_{4,Q_T} \leq \frac{1}{k}\Big(T\|\theta_0\|^2_{2,\Omega} + \tag{20}$$

$$+\frac{2}{k}\left(\frac{4\mu_1^2}{\mu_0}(T\|\nabla\mathbf{u}_0\|^2_{2,\Omega} + \frac{4}{\mu_0}\|\mathbf{f}\|^2_{2,Q_T})^2\mathcal{F}(\frac{\mu_2^4}{2\mu_0}\|\nabla\xi\|^4_{4,Q_T}) + \|g\|^2_{2,Q_T}\Big)\right)^2.$$

Proof. The energy inequalities hold

$$\|\mathbf{u}\|^2_{2,\Omega}(t) + \mu_0\int_0^t \|\nabla\mathbf{u}\|^2_{2,\Omega} \leq \|\mathbf{u}_0\|^2_{2,\Omega} + \frac{1}{\mu_0}\int_0^t \|\mathbf{f}\|^2_{2,\Omega};$$

$$\|\theta\|^2_{2,\Omega}(t) + k\int_0^t \|\nabla\theta\|^2_{2,\Omega} \leq \|\theta_0\|^2_{2,\Omega} + \frac{1}{k}\int_0^t \left(\mu_1\|\nabla\mathbf{u}\|^2_{4,\Omega} + \|g\|_{2,\Omega}\right)^2.$$

To prove the estimates in $L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H^1_0(\Omega))$ we do simultaneously as in standard manner. Using (16) and (17) we obtain (19)-(20). □

Remark 3.1. *If $g = -\theta|\theta|^{1/2}$ the energy inequality on θ given in the proof of Lemma 3.3 can be simplified since*

$$\int_\Omega g\theta = -\int_\Omega |\theta|^{1/2}\theta^2 \leq 0.$$

Consequently (20) reads

$$\|\theta\|^4_{4,Q_T} \leq \frac{1}{k}\Big(T\|\theta_0\|^2_{2,\Omega} +$$

$$+\frac{4\mu_1^2}{k\mu_0}(T\|\nabla\mathbf{u}_0\|^2_{2,\Omega} + \frac{4}{\mu_0}\|\mathbf{f}\|^2_{2,Q_T})^2\mathcal{F}(\frac{\mu_2^4}{2\mu_0}\|\nabla\xi\|^4_{4,Q_T})\Big)^2.$$

Proposition 3.4. *Under the assumptions of Lemma 3.3, (7) and $\theta_0 \in H^1_0(\Omega)$, if $\nabla\xi \in \mathbf{L}^4(Q)$ then the solution θ of (13) is such that $\nabla\theta$ belongs to a bounded set of $L^2(0,T;H^1_0(\Omega)) \cap L^\infty(0,T;L^2(\Omega))$. Moreover, it satisfies*

$$\|\nabla\theta\|^4_{4,Q_T} \leq C(1 + \mathcal{F}(\frac{\mu_2^4}{2\mu_0}\|\nabla\xi\|^4_{4,Q_T})), \tag{21}$$

with $\mathcal{F}$ given as in Proposition 3.3 and C depending only on Ω, T, μ_0, μ_1, k, $\|\mathbf{u}_0\|_{2,\Omega}$, $\|\nabla\mathbf{u}_0\|_{2,\Omega}$, $\|\mathbf{f}\|_{2,Q_T}$, $\|\nabla\theta_0\|_{2,\Omega}$ and $\|g\|_{2,Q_T}$.

Proof. Proceeding as in the proof of Proposition 3.3, we choose $\eta = \Delta\theta$ as a test function in (13), then we have

$$\frac{1}{2}\|\nabla\theta\|^2_{2,\Omega}(t) + k\int_0^t \|\nabla^2\theta\|^2_{2,\Omega} \leq \frac{1}{2}\|\nabla\theta(0)\|^2_{2,\Omega} + \\ + \int_0^t \|\mathbf{u}\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\nabla^2\theta\|_{2,\Omega} + \mu_1\int_0^t \|\nabla\mathbf{u}\|^2_{4,\Omega}\|\nabla^2\theta\|_{2,\Omega} + \\ + \int_0^t \|g\|_{2,\Omega}\|\nabla^2\theta\|_{2,\Omega},$$

denoting by ∇^2 the tensor (∂_{ij}) of second order spatial derivatives.

Applying interpolation (16) and Young inequalities, we deduce

$$\frac{1}{2}\|\nabla\theta\|^2_{2,\Omega}(t) + k\int_0^t \|\nabla^2\theta\|^2_{2,\Omega} \leq \frac{1}{2}\|\nabla\theta_0\|^2_{2,\Omega} + \frac{1}{4k}\int_0^t \|\mathbf{u}\|^4_{4,\Omega}\|\nabla\theta\|^2_{2,\Omega} + \\ + \frac{3k}{4}\int_0^t \|\nabla^2\theta\|^2_{2,\Omega} + \frac{2}{k}\int_0^t \{\mu_1^2\|\nabla\mathbf{u}\|^4_{4,\Omega} + \|g\|^2_{2,\Omega}\} + \frac{k}{8}\int_0^t \|\nabla^2\theta\|^2_{2,\Omega}.$$

Using (17), from Gronwall Lemma we conclude

$$\operatorname*{ess\,sup}_{t\in[0,T]} \|\nabla\theta\|^2_{2,\Omega} \leq C(1 + \mathcal{F}(\frac{\mu_2^4}{2\mu_0}\|\nabla\xi\|^4_{4,Q_T}))\exp\left[\|\mathbf{u}\|^4_{4,Q_T}/4\right],$$

with C depending only on Ω, T, μ_0, μ_1, $\|\nabla\mathbf{u}_0\|_{2,\Omega}$, $\|\mathbf{f}\|_{2,Q_T}$, $\|\nabla\theta_0\|_{2,\Omega}$ and $\|g\|_{2,Q_T}$. Notice that the estimate (19) on $\mathbf{u}$ is only dependent on the data. Then we get $\|\nabla^2\theta\|^2_{2,Q_T} \leq C(1 + \mathcal{F}(\mu_2^4\|\nabla\xi\|^4_{4,Q_T}/(2\mu_0)))$ and also (21). ☐

Remark 3.2. *For $g = -\theta|\theta|^{1/2}$, the proof of Proposition 3.4 is still valid if we take into account that*

$$\int_\Omega g\Delta\theta \leq \|\theta\|_{4,\Omega}\|\theta\|^{1/2}_{2,\Omega}\|\Delta\theta\|_{2,\Omega}.$$

Consequently, using (20) we conclude (21) with C depending only on Ω, T, μ_0, μ_1, k, $\|\mathbf{u}_0\|_{2,\Omega}$, $\|\nabla\mathbf{u}_0\|_{2,\Omega}$, $\|\mathbf{f}\|_{2,Q_T}$, $\|\nabla\theta_0\|_{2,\Omega}$ and $\|\theta_0\|_{2,\Omega}$.

Proposition 3.5. *Under the assumptions (6)-(8), if $\xi \in W^{1,1}_4(Q)$ such that $\nabla\xi(0) = \nabla\theta_0$ then the solution $\mathbf{u}$ of (12) is such that $\partial_t\mathbf{u}$ belongs to a bounded set of $L^2(0,T;\mathbf{H}^1_0(\Omega)) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$.*

Proof. Differentiating (1) with respect to time (cf. Remark 3.3), multiplying by $\mathbf{v} = \partial_t\mathbf{u}$, integrating over the space variables and using the orthogonality property of the convective term (cf. [11, pp. 128]), we obtain

$$\frac{1}{2}\frac{d}{dt}\|\partial_t\mathbf{u}\|^2_{2,\Omega}+\int_\Omega[\mu'(\xi)\partial_t\xi D\mathbf{u}+\mu(\xi)\partial_t D\mathbf{u}]:D\partial_t\mathbf{u}=$$
$$=\int_\Omega\partial_t^2\mathbf{u}\partial_t\mathbf{u}+\int_\Omega\partial_t[\mu(\xi)D\mathbf{u}]:D\partial_t\mathbf{u}=$$
$$=-\int_\Omega\partial_t\mathbf{u}\otimes\partial_t\mathbf{u}:\nabla\mathbf{u}+\int_\Omega\partial_t\mathbf{f}\cdot\partial_t\mathbf{u}.$$

Using the assumption (7), Hölder inequalities and integrating in time for each $t\in]0,T[$, it follows

$$\frac{1}{2}\|\partial_t\mathbf{u}\|^2_{2,\Omega}+\mu_0\int_0^t\int_\Omega|\nabla\partial_t\mathbf{u}|^2\leq\frac{1}{2}\|\partial_t\mathbf{u}(0)\|^2_{2,\Omega}+\int_0^t\|\partial_t\mathbf{f}\|_{2,\Omega}\|\partial_t\mathbf{u}\|_{2,\Omega}+$$
$$+\mu_2\int_0^t\|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}+\int_0^t\|\partial_t\mathbf{u}\|^2_{4,\Omega}\|\nabla\mathbf{u}\|_{2,\Omega}. \tag{22}$$

Let us study separately each term of the right hand side of the above inequality.
First term. To estimate $\|\partial_t\mathbf{u}(0)\|^2_{2,\Omega}$ we choose $\mathbf{v}=\partial_t\mathbf{u}(0)$ as a test function in (12) for the particular case $t=0$. Thus, we observe that

$$\|\mathbf{u}'(0)\|^2_{2,\Omega}+\int_\Omega\mathbf{u}'(0)\otimes\mathbf{u}_0:\nabla\mathbf{u}_0-\int_\Omega\mathbf{f}(0)\cdot\mathbf{u}'(0)=$$
$$=-\int_\Omega\mu(\xi(0))D\mathbf{u}_0:D\mathbf{u}'(0).$$

Using the Green formula and (8) it follows

$$\|\mathbf{u}'(0)\|^2_{2,\Omega}+\int_\Omega\mathbf{u}'(0)\otimes\mathbf{u}_0:\nabla\mathbf{u}_0-\int_\Omega\mathbf{f}(0)\cdot\mathbf{u}'(0)=$$
$$=\int_\Omega\mu'(\xi(0))D\mathbf{u}_0:\nabla\xi(0)\otimes\mathbf{u}'(0)+\int_\Omega\mu(\xi(0))\Delta\mathbf{u}_0\cdot\mathbf{u}'(0).$$

Applying Hölder inequalities, we get

$$\|\mathbf{u}'(0)\|_{2,\Omega}\leq\mu_2\|\nabla\theta_0\|_{4,\Omega}\|\nabla\mathbf{u}_0\|_{4,\Omega}+\mu_1\|\nabla\mathbf{u}_0\|_{(1),2,\Omega}+$$
$$+\|\mathbf{u}_0\|_{4,\Omega}\|\nabla\mathbf{u}_0\|_{4,\Omega}+\|\mathbf{f}(0)\|_{2,\Omega}:=C.$$

Note that $\mathbf{f}\in\mathbf{L}^2(Q_T)$ and $\partial_t\mathbf{f}\in\mathbf{L}^2(Q_T)$ then $\mathbf{f}\in C([0,T];L^2(\Omega))$.
Second term. It is sufficient the use of Young inequality.
Third term. Applying Young inequalities, it follows:

$$\mu_2\|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\leq\frac{\mu_2^2}{2\mu_0}\{\|\partial_t\xi\|^4_{4,\Omega}+\|\nabla\mathbf{u}\|^4_{4,\Omega}\}+$$
$$+\frac{\mu_0}{4}\|\nabla\partial_t\mathbf{u}\|^2_{2,\Omega}.$$

FORTH TERM. Applying the interpolation inequality (16), with $n = 2$ and $v = \partial_t \mathbf{u}$, and using Young inequality, it follows

$$\|\partial_t \mathbf{u}\|_{4,\Omega}^2 \|\nabla \mathbf{u}\|_{2,\Omega} \leq \|\partial_t \mathbf{u}\|_{2,\Omega} \|\nabla \partial_t \mathbf{u}\|_{2,\Omega} \|\nabla \mathbf{u}\|_{2,\Omega} \leq$$
$$\leq \frac{1}{\mu_0} \|\nabla \mathbf{u}\|_{2,\Omega}^2 \|\partial_t \mathbf{u}\|_{2,\Omega}^2 + \frac{\mu_0}{4} \|\nabla \partial_t \mathbf{u}\|_{2,\Omega}^2.$$

Substituting each calculation in (22), we get

$$\|\partial_t \mathbf{u}\|_{2,\Omega}^2 + \mu_0 \int_0^t \|\nabla \partial_t \mathbf{u}\|_{2,\Omega}^2 \leq C + \int_0^t \|\partial_t \mathbf{f}\|_{2,\Omega}^2 +$$
$$+ \frac{\mu_2^2}{\mu_0} \{ \int_0^t \|\partial_t \xi\|_{4,\Omega}^4 + \int_0^t \|\nabla \mathbf{u}\|_{4,\Omega}^4 \} + \int_0^t \{1 + \frac{2}{\mu_0} \|\nabla \mathbf{u}\|_{2,\Omega}^2 \} \|\partial_t \mathbf{u}\|_{2,\Omega}^2.$$

We conclude that $\partial_t \mathbf{u}$ belongs in a bounded set of $L^{2,\infty}(Q_T)$ from Gronwall Lemma and using (17) and (18). Subsequently we obtain that $\partial_t \mathbf{u}$ belongs to bounded sets of $W_2^{1,0}(Q_T)$ and of $\mathbf{L}^4(Q_T)$. $\square$

REMARK 3.3. *To differentiate the system (1) with respect to the variable t and the choice of the test function have meaning in the following sense. We first form the difference ratio of the system with respect to t, multiply the relation by the difference ratio of the solution with respect to t, integrate the obtained equation over the domain Ω and then pass to the limit as $\triangle t \to 0$.*

Proposition 3.6. *Under the assumptions (6)-(8) and (10), if $\partial_t \xi \in L^4(Q)$ then the solution θ given at Proposition 3.4 is such that $\partial_t \theta$ belongs to a bounded set of $L^2(0,T; H_0^1(\Omega)) \cap L^\infty(0,T; L^2(\Omega))$. In particular, the following estimate holds*

$$\|\partial_t \theta\|_{4,Q_T}^4 \leq C(1 + \mathcal{B}(\frac{\mu_2^4}{2\mu_0} \|\nabla \xi\|_{4,Q_T}^4, \frac{\mu_2^4}{2k} \|\partial_t \xi\|_{4,Q}^4)), \tag{23}$$

with $\mathcal{B}$ a positive strictly increasing function on its arguments. Moreover, $\nabla \theta$ belongs to a bounded set of $\mathbf{L}^q(Q_T)$ for any $q < 6$.

Proof. We recall the equation

$$\partial_t \theta + \mathbf{u} \cdot \nabla \theta - k \Delta \theta = \mu(\xi) |D\mathbf{u}|^2 + g \text{ in } Q_T.$$

Differentiating the above equation with respect to time (cf. Remark 3.3), we deduce

$$\partial_t^2 \theta - k \partial_t \Delta \theta = -\partial_t \mathbf{u} \cdot \nabla \theta - \mathbf{u} \cdot \partial_t \nabla \theta + \tag{24}$$
$$+ \mu'(\xi) \partial_t \xi |D\mathbf{u}|^2 + \mu(\xi) 2D\mathbf{u} : \partial_t D\mathbf{u} + \partial_t g.$$

If multiply (24) by $\eta = \partial_t \theta$, using the orthogonality property of the convective term, after standard calculations it follows

$$\frac{1}{2} \int_0^t \frac{d}{dt} \|\partial_t \theta\|_{2,\Omega}^2 + k \int_0^t \int_\Omega |\nabla \partial_t \theta|^2 + \int_0^t \int_\Omega \partial_t \mathbf{u} \cdot \nabla \theta \partial_t \theta =$$
$$= \int_0^t \int_\Omega (\mu'(\xi) \partial_t \xi |D\mathbf{u}|^2 + 2\mu(\xi) \partial_t D\mathbf{u} : D\mathbf{u}) \partial_t \theta + \int_0^t \int_\Omega \partial_t g \partial_t \theta.$$

Using the assumption (7) and Hölder inequalities, we get

$$\frac{1}{2}\|\partial_t\theta\|^2_{2,\Omega}(t) + k\int_0^t \|\nabla\partial_t\theta\|^2_{2,\Omega} \leq \frac{1}{2}\|\partial_t\theta(0)\|^2_{2,\Omega} + \tag{25}$$

$$+\int_0^t \|\partial_t\mathbf{u}\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\partial_t\theta\|_{2,\Omega} + \mu_2\int_0^t \|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|^2_{4,\Omega}\|\partial_t\theta\|_{4,\Omega} +$$

$$+2\mu_1\int_0^t \|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\partial_t\theta\|_{4,\Omega} + \int_0^t \|\partial_t g\|_{2,\Omega}\|\partial_t\theta\|_{2,\Omega}.$$

Let us examine separately each term of the above inequality.

First term. Choosing $\eta = \partial_t\theta(0)$ as a test function in (13) for the particular case $t = 0$ and applying (8), we obtain

$$\|\theta'(0)\|^2_{2,\Omega} + \int_\Omega \mathbf{u}_0\cdot\nabla\theta_0\theta'(0) - \int_\Omega g(0)\theta'(0) - \int_\Omega \mu(\xi(0))|D\mathbf{u}_0|^2\theta'(0)$$
$$= -k\int_\Omega \nabla\theta_0\cdot\nabla\theta'(0) = k\int_\Omega \Delta\theta_0\theta'(0).$$

Consequently, we find

$$\|\theta'(0)\|_{2,\Omega} \leq k\|\nabla\theta_0\|_{(1),2,\Omega} + \|\mathbf{u}_0\|_{4,\Omega}\|\nabla\theta_0\|_{4,\Omega} + \|g(0)\|_{2,\Omega} +$$
$$+\mu_1\|\nabla\mathbf{u}_0\|^2_{4,\Omega} := C.$$

Second term. Applying Young inequalities, we have

$$\|\partial_t\mathbf{u}\|_{4,\Omega}\|\nabla\theta\|_{4,\Omega}\|\partial_t\theta\|_{2,\Omega} \leq \frac{1}{4}\{\|\partial_t\mathbf{u}\|^4_{4,\Omega} + \|\nabla\theta\|^4_{4,\Omega}\} + \frac{1}{2}\|\partial_t\theta\|^2_{2,\Omega}.$$

Third term. Using interpolation inequality (16) and Young inequalities we get

$$\mu_2\|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|^2_{4,\Omega}\|\partial_t\theta\|_{4,\Omega} \leq \mu_2\|\partial_t\xi\|_{4,\Omega}\|\nabla\mathbf{u}\|^2_{4,\Omega}\|\nabla\partial_t\theta\|^{1/2}_{2,\Omega}\|\partial_t\theta\|^{1/2}_{2,\Omega}$$
$$\leq \frac{1}{2}\|\nabla\mathbf{u}\|^4_{4,\Omega} + \frac{\mu_2^4}{4k}\|\partial_t\xi\|^4_{4,\Omega}\|\partial_t\theta\|^2_{2,\Omega} + \frac{k}{4}\|\nabla\partial_t\theta\|^2_{2,\Omega},$$

Forth term. Using interpolation inequality (16) and Young inequalities we get

$$\mu_1\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\partial_t\theta\|_{4,\Omega} \leq \mu_1\|\nabla\partial_t\mathbf{u}\|_{2,\Omega}\|\nabla\mathbf{u}\|_{4,\Omega}\|\nabla\partial_t\theta\|^{1/2}_{2,\Omega}\|\partial_t\theta\|^{1/2}_{2,\Omega}$$
$$\leq \frac{\mu_1^2}{2}\|\nabla\partial_t\mathbf{u}\|^2_{2,\Omega} + \frac{1}{4k}\|\nabla\mathbf{u}\|^4_{4,\Omega}\|\partial_t\theta\|^2_{2,\Omega} + \frac{k}{4}\|\nabla\partial_t\theta\|^2_{2,\Omega}.$$

Fifth term. It is sufficient the use of Young inequality.

Then, introducing all these terms in (25) we conclude

$$\|\partial_t\theta\|^2_{2,\Omega}(t) + k\int_0^t \|\nabla\partial_t\theta\|^2_{2,\Omega} \leq C + \frac{1}{2}\int_0^t \|\nabla\theta\|^4_{4,\Omega} + \frac{1}{2}\int_0^t \|\partial_t\mathbf{u}\|^4_{4,\Omega} +$$
$$+\int_0^t (2 + \frac{1}{2k}\|\nabla\mathbf{u}\|^4_{4,\Omega} + \frac{\mu_2^4}{2k}\|\partial_t\xi\|^4_{4,\Omega})\|\partial_t\theta\|^2_{2,\Omega} +$$
$$+\int_0^t \|\nabla\mathbf{u}\|^4_{4,\Omega} + \mu_1^2\int_0^t \|\nabla\partial_t\mathbf{u}\|^2_{2,\Omega} + \int_0^t \|\partial_t g\|^2_{2,\Omega}.$$

Recalling Propositions 3.3, 3.4 and 3.5, we use Gronwall Lemma to obtain

$$\text{ess}\sup_{t\in[0,T]}\|\partial_t\theta\|_{2,\Omega}^2 \le C(1+\mathcal{B}(\frac{\mu_2^4}{2\mu_0}\|\nabla\xi\|_{4,Q_T}^4,\frac{\mu_2^4}{2k}\|\partial_t\xi\|_{4,Q}^4)). \tag{26}$$

Applying this result in the last expression we conclude the desired result (23).

Finally applying Proposition 3.4, $\nabla\theta$ belongs to a bounded set of $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))$. Since we also have $\partial_t\nabla\theta\in\mathbf{L}^2(Q_T)$ then Proposition 3.2 implies that $\nabla\theta$ belongs to a bounded set of $\mathbf{L}^q(Q_T)$ for any $q<6$. □

Remark 3.4. *For $g=-\theta|\theta|^{1/2}$, the result of Proposition 3.6 is still valid if in its proof we take into account that*

$$\int_\Omega \partial_t g\partial_t\theta = -\frac{3}{2}\int_\Omega|\theta|^{1/2}|\partial_t\theta|^2\le 0.$$

Proposition 3.7. *Under the assumptions (7) and (10), if θ is the solution given at Proposition 3.4, then $\partial_t^2\theta$ belongs to a bounded set of $L^2(0,T;H^{-1}(\Omega))$.*

Proof. Considering (24) and since we already proved that

$$\begin{aligned}
&\partial_t\mathbf{u}\text{ and }\nabla\theta\in\mathbf{L}^4(Q_T);\\
&\mathbf{u}\in L^\infty(0,T;\mathbf{L}^2(\Omega))\quad\text{and}\quad\partial_t\nabla\theta\in\mathbf{L}^2(Q_T);\\
&\nabla\mathbf{u}\in L^\infty(0,T;\mathbf{L}^2(\Omega))\quad\text{and}\quad\nabla\partial_t\mathbf{u}\in\mathbf{L}^2(Q_T);\\
&|\nabla\mathbf{u}|^2\in L^4(0,T;L^{4/3}(\Omega))
\end{aligned}$$

we have

$$\partial_t^2\theta-k\partial_t\Delta\theta\in L^2(0,T;L^1(\Omega).$$

Taking the operator div: $\mathbf{L}^2(\Omega)\to H^{-1}(\Omega)$, it permits to get

$$\partial_t\Delta\theta=\nabla\cdot(\partial_t\nabla\theta)\in L^2(0,T;H^{-1}(\Omega)),$$

then it results that $\partial_t^2\theta\in L^2(0,T;H^{-1}(\Omega))$. □

Proposition 3.8. *Under the assumptions of Propositions 3.4, 3.5 and 3.6, the solution θ of (13) is such that $\partial_t\theta$ belongs to a bounded set of $L^\infty(0,T;L^{2+\delta}(\Omega))$ for some $\delta>0$, and consequently to a bounded set of $L^{4+\delta}(Q_T)$.*

Proof. In consequence of Proposition 3.7, $\partial_t^2\theta$ does not belong in $L^2(Q_T)$ and we cannot apply the argument used in Proposition 3.6. Let us argue as in [2], multiplying (24) by $\eta=\partial_t\theta|\partial_t\theta|^\delta$ and integrating over the space variable, we obtain

$$\int_\Omega\partial_t^2\theta\partial_t\theta|\partial_t\theta|^\delta+k(1+\delta)\int_\Omega\partial_t\nabla\theta\cdot\nabla(\partial_t\theta|\partial_t\theta|^\delta)+$$
$$\int_\Omega\partial_t(\mathbf{u}\cdot\nabla\theta)\partial_t\theta|\partial_t\theta|^\delta=\int_\Omega\partial_t(\mu(\xi)|D\mathbf{u}|^2)\partial_t\theta|\partial_t\theta|^\delta+\int_\Omega\partial_t g\partial_t\theta|\partial_t\theta|^\delta.$$

Applying Hölder inequalities under the relations for the exponents

$$\frac{1}{4}+\frac{1-2\delta}{4}+\frac{1+\delta}{2}=1,\quad \frac{1}{2}+\frac{1}{5}+\frac{3-5\delta}{10}+\frac{\delta}{2}=1,\quad \delta<\frac{1}{2}\left(<\frac{3}{5}\right),$$

it follows

$$\frac{1}{2+\delta}\int_0^t \frac{d}{dt}\|\partial_t\theta\|^{2+\delta}_{2+\delta,\Omega} \quad + \quad k(1+\delta)\int_0^t\int_\Omega |\nabla\partial_t\theta|^2|\partial_t\theta|^\delta \leq$$
$$\leq I_1+I_2+I_3+I_4+\int_0^t\int_\Omega \partial_t g\partial_t\theta,$$

with

$$\begin{aligned}
I_1 &:= \int_0^t \|\partial_t\mathbf{u}\|_{4,\Omega}\|\nabla\theta\|_{4/(1-2\delta),\Omega}\|\partial_t\theta\|^{1+\delta}_{2,\Omega}\\
I_2 &:= \int_0^t \|\mathbf{u}\|_{5,\Omega}\|\nabla\partial_t\theta\|_{2,\Omega}\|\partial_t\theta\|_{10/(3-5\delta),\Omega}\|\partial_t\theta\|^{\delta}_{2,\Omega}\\
I_3 &:= \mu_2\int_0^t \|\partial_t\xi\|_{4,\Omega}\|D\mathbf{u}\|^2_{8/(1-2\delta),\Omega}\|\partial_t\theta\|^{1+\delta}_{2,\Omega}\\
I_4 &:= 2\mu_1\int_0^t \|\partial_t D\mathbf{u}\|_{2,\Omega}\|D\mathbf{u}\|_{5,\Omega}\|\partial_t\theta\|_{10/(3-5\delta),\Omega}\|\partial_t\theta\|^{\delta}_{2,\Omega}.
\end{aligned}$$

From Proposition 3.6, $\nabla\theta$ belongs to a bounded set of $\mathbf{L}^q(Q_T)\hookrightarrow \mathbf{L}^{4/(1-2\delta),4/3}(Q_T)$ for any $q<6$ and consequently we must choose $\delta<1/6$. Thus, using (26) we obtain

$$I_1\leq C(1+\mathcal{B}(\xi))^{(1+\delta)/2}\|\partial_t\mathbf{u}\|_{4,Q_T}\|\nabla\theta\|_{4/(1-2\delta),4/3,Q_T}.$$

Thanks to Proposition 3.3, we get $\mathbf{u}\in L^\infty(0,T;\mathbf{H}^1(\Omega))\hookrightarrow L^\infty(0,T;\mathbf{L}^5(\Omega))$, and thanks to Proposition 3.6, we get $\partial_t\theta$ belongs to a bounded set of $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\hookrightarrow L^{10/(3-5\delta),2}(Q_T)$ for any $\delta<3/5$. Then, we obtain

$$I_2\leq C(1+\mathcal{B}(\xi))^{\delta/2}\|\mathbf{u}\|_{5,\infty,Q_T}\|\nabla\partial_t\theta\|_{2,Q_T}\|\partial_t\theta\|_{10/(3-5\delta),2,Q_T}.$$

From Proposition 3.3, $\nabla\mathbf{u}$ belongs to a bounded set of $L^2(0,T;\mathbf{H}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^2(\Omega))$. From Proposition 3.5, $\partial_t\mathbf{u}$ belongs to a bounded set of $L^2(0,T;\mathbf{H}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^2(\Omega))$. In particular, applying Proposition 3.1 we have

$$\nabla\mathbf{u}\in L^{3,\infty}(Q_T). \tag{27}$$

Thanks to Proposition 3.2, $\nabla\mathbf{u}$ belongs to a bounded set of $L^2(0,T;\mathbf{H}^1(\Omega))\cap L^\infty(0,T;\mathbf{L}^3(\Omega))\hookrightarrow \mathbf{L}^{8/(1-2\delta),8/3}(Q_T))$ for any $\delta<1/6$. Then, it follows

$$I_3\leq C(1+\mathcal{B}(\xi))^{(1+\delta)/2}\|\partial_t\xi\|_{4,Q_T}\|D\mathbf{u}\|^2_{8/(1-2\delta),8/3,Q_T}.$$

From Proposition 3.6, $\partial_t\theta$ belongs to a bounded set of $L^2(0,T;H^1(\Omega))\cap L^\infty(0,T;L^2(\Omega))\hookrightarrow L^{10/(3-5\delta),!0/3}(Q_T)$ for any $\delta<1/5$. Then, we obtain

$$I_4\leq C(1+\mathcal{B}(\xi))^{\delta/2}\|D\mathbf{u}\|_{5,Q_T}\|\partial_t D\mathbf{u}\|_{2,Q_T}\|\partial_t\theta\|_{10/(3-5\delta),10/3,Q_T}.$$

Therefore we conclude an estimate in $L^{2+\delta,\infty}(Q_T)$ and applying Lemma 3.1 we obtain $L^{2+\delta,\infty}(Q_T) \cap L^2(0,T;H^1(\Omega)) \hookrightarrow L^{4+\delta}(Q_T)$. □

4. Proof of Theorem 2.1

In order to apply Schauder theorem, we build an operator $\mathcal{L}$ defined on $W_4^{1,1}(Q)$, which maps

$$\xi \mapsto \mathbf{u} = \mathbf{u}(\xi) \mapsto \theta \in W_4^{1,1}(Q);$$

where $\mathbf{u}$ and θ are the solutions to the problems (12) and (13), respectively.

Step 1. From Propositions 3.4 and 3.6, $\mathcal{L}$ maps the convex closed set $K := \{\xi \in W_4^{1,1}(Q_T) : \ \nabla\xi(0) = \nabla\theta_0, \ \|\xi\| < R\}$ to itself, choosing $R > 0$ such that

$$R := C\left(1 + \mathcal{F}(\frac{\mu_2^4}{2\mu_0}R) + \mathcal{B}(\frac{\mu_2^4}{2\mu_0}R, \frac{\mu_2^4}{2k}R)\right)$$

under the assumption (11).

Step 2. Let us prove that $\mathcal{L}$ is a well defined mapping. For each $\xi \in W_4^{1,1}(Q)$, from the existence theory for the Navier-Stokes system there exists a unique 2-dimensional solution $\mathbf{u} \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;J_0^{1,2}(\Omega))$ (see for example [11] or [15]). Hence, thanks to Proposition 3.3 we have $\nabla\mathbf{u} \in \mathbf{L}^4(Q_T)$. Thus from the existence theory for the parabolic equations there exists $\theta \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;H_0^1(\Omega))$ the unique solution to the problem (13). Then, Propositions 3.4 and 3.6 guarantee the sufficient regularity to obtain $\theta \in W_4^{1,1}(Q)$.

Step 3. In order to prove that $\mathcal{L}$ is compact, we take a sequence ξ_m weakly convergent to ξ in $W_4^{1,1}(Q)$, and corresponding solutions $\mathbf{u}_m$ and θ_m to the problems (12) and (13), respectively. The estimates (21) and (23) infer that we can extract a subsequence, still denoted by θ_m, such that

$$\nabla\theta_m \rightharpoonup \nabla\theta \text{ in } \mathbf{L}^4(Q_T), \quad \partial_t\theta_m \rightharpoonup \partial_t\theta \text{ in } L^4(Q_T).$$

From Propositions 3.4 and 3.6, we have $\nabla\theta_m$ and $\partial_t\nabla\theta_m$ bounded in $L^2(0,T;\mathbf{H}_0^1(\Omega))$ and $\mathbf{L}^2(Q_T)$, respectively. Moreover $\partial_t\theta_m$ bounded in $L^2(0,T;H_0^1(\Omega))$ and from Proposition 3.7 we get $\partial_t^2\theta_m$ bounded in $L^2(0,T;H^{-1}(\Omega))$. Then by a compactness result (cf. [22, page 90]) we obtain

$$\begin{aligned} \nabla\theta_m &\to \nabla\theta \text{ in } L^2(0,T;\mathbf{L}^q(\Omega)), \quad q < \infty; \\ \partial_t\theta_m &\to \partial_t\theta \text{ in } L^2(0,T;L^q(\Omega)), \quad q < \infty. \end{aligned}$$

In order to apply these strong convergences we use Proposition 3.2 to $\nabla\theta_m$ obtaining that it is bounded in $\mathbf{L}^5(Q_T)$ and consequently

$$\nabla\theta_m \to \nabla\theta \text{ in } \mathbf{L}^4(Q_T).$$

Next using Proposition 3.8, it follows

$$\partial_t \theta_m \to \partial_t \theta \text{ in } L^4(Q_T).$$

Easily we conclude that θ is the limit solution to the problem (13).

In conclusion, Schauder Theorem guarantees the existence of at least one fixed point and accordingly there exists a strong solution $(\mathbf{u}, \theta)$ in the conditions of Theorem 2.1.

Finally let us prove that the strong solution is Hölder continuous. First we recall (27) and thus using Lemma 3.2 we conclude that $\mathbf{u}$ is Hölder continuous. Analogous for θ.

5. Proof of Theorem 2.2

We argue as in the proof of Theorem 2.1. Let $\mathcal{L}$ be the operator which maps

$$\xi \in W_4^{1,1}(Q) \mapsto \mathbf{u} = \mathbf{u}(\xi) \mapsto \theta \in W_4^{1,1}(Q);$$

where $\mathbf{u}$ and θ are the solutions to the problems (12) and (13) with g replaced by $-\theta|\theta|^{1/2}$, respectively.

Proceeding as in steps 1 and 2, the operator $\mathcal{L}$ is well defined considering Remarks 3.2 and 3.4. Consequently the argument of the proof of Theorem 2.1 can be followed *mutatis mutandis*.

Moreover, considering that for all $q < 6$ we have

$$|\nabla \mathbf{u}|^2 \in L^{q/2}(Q_T); \quad |\theta|^{1/2}\theta \in L^\infty(Q_T); \quad \mathbf{u} \cdot \nabla\theta \in L^q(Q_T),$$

applying the regularity theory for the heat equation

$$\partial_t \theta - k\Delta\theta = \mu(\theta)|D\mathbf{u}|^2 - |\theta|^{1/2}\theta - \mathbf{u}\cdot\nabla\theta \quad \text{in } Q_T;$$

we find that $\theta \in W^{2,1}_{q/2}(Q_T)$.

6. Uniqueness

In this section, we show that the solutions obtained in Theorem 2.1 and 2.2 are unique.

6.1. Uniqueness for Theorem 2.1.

We proceed in a classical manner. We suppose the existence of two solutions $(\mathbf{u}_1, \theta_1)$ and $(\mathbf{u}_2, \theta_2)$ and we define $\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$ and $\theta = \theta_1 - \theta_2$. So that $(\mathbf{u}, \theta)$ verifies the following variational formulation

(28)
$$\left\{\begin{array}{rl}
\int_\Omega \partial_t \mathbf{u}\cdot\mathbf{v} + \int_\Omega \mu(\theta_1) D\mathbf{u} : D\mathbf{v} = & \int_\Omega (\mu(\theta_2) - \mu(\theta_1)) D\mathbf{u}_2 : D\mathbf{v} \\
& - \int_\Omega \Big(\nabla \mathbf{u}_2 : \mathbf{v}\otimes\mathbf{u} + \nabla\mathbf{u} : \mathbf{v}\otimes\mathbf{u}_1\Big) \\
\text{a.e. } t\in[0,T] \quad \forall \mathbf{v}\in J_0^{1,2}(\Omega), & \quad \mathbf{u}\Big|_{t=0} = \mathbf{0} \text{ in } \Omega; \\
\int_\Omega (\partial_t\theta)\eta + k\int_\Omega \nabla\theta\cdot\nabla\eta = & -\int_\Omega \Big(\mathbf{u}\cdot\nabla\theta_2 + \mathbf{u}_1\cdot\nabla\theta\Big)\eta + \\
& + \int_\Omega (\mu(\theta_1)|D\mathbf{u}_1|^2 - \mu(\theta_2)|D\mathbf{u}_2|^2)\eta \\
\text{a.e. } t\in[0,T] \quad \forall \eta\in W_0^{1,2}(\Omega), & \quad \theta\Big|_{t=0} = 0 \text{ in } \Omega.
\end{array}\right.$$

Taking $\mathbf{v} = \mathbf{u}$ and $\eta = \theta$, using (7) and the orthogonality property to the convective terms, and summing the resulting relations we find

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|_{2,\Omega}^2 + \frac{1}{2}\frac{d}{dt}\|\theta\|_{2,\Omega}^2 + \mu_0\|D\mathbf{u}\|_{2,\Omega}^2 + k\|\nabla\theta\|_{2,\Omega}^2 \leq$$
$$\leq -\int_\Omega \nabla\mathbf{u}_2 : \mathbf{u}\otimes\mathbf{u} - \int_\Omega \mathbf{u}\cdot\nabla\theta_2\theta + \int_\Omega (\mu(\theta_2)-\mu(\theta_1))D\mathbf{u}_2 : D\mathbf{u}$$
$$+ \int_\Omega (\mu(\theta_1)-\mu(\theta_2))|D\mathbf{u}_1|^2\theta + \int_\Omega \mu(\theta_2)D\mathbf{u} : D(\mathbf{u}_1+\mathbf{u}_2)\theta.$$

Arguing as in [11, pp. 154] we get

$$I_1 := \left|\int_\Omega \nabla\mathbf{u}_2 : \mathbf{u}\otimes\mathbf{u}\right| \leq \|\nabla\mathbf{u}_2\|_{2,\Omega}\|\mathbf{u}\|_{4,\Omega}^2 \leq$$
$$\leq \frac{\mu_0}{8}\|\nabla\mathbf{u}\|_{2,\Omega}^2 + \frac{2}{\mu_0}\|\nabla\mathbf{u}_2\|_{2,\Omega}^2\|\mathbf{u}\|_{2,\Omega}^2.$$

Analogously we have

$$I_2 := \left|\int_\Omega \mathbf{u}\cdot\nabla\theta_2\theta\right| \leq \|\mathbf{u}\|_{4,\Omega}\|\nabla\theta_2\|_{2,\Omega}\|\theta\|_{4,\Omega} \leq$$
$$\leq \frac{\mu_0}{8}\|\nabla\mathbf{u}\|_{2,\Omega}^2 + \frac{2}{\mu_0}\|\nabla\theta_2\|_{2,\Omega}^2\|\mathbf{u}\|_{2,\Omega}^2 + \frac{k}{8}\|\nabla\theta\|_{2,\Omega}^2 + \frac{2}{k}\|\nabla\theta_2\|_{2,\Omega}^2\|\theta\|_{2,\Omega}^2.$$

Next using the Mean Value Theorem, for every $X = (x,t) \in Q_T$ there exists ψ_X between $\theta_1(X)$ and $\theta_2(X)$ such that

$$|\mu(\theta_1)-\mu(\theta_2)| = \mu'(\psi_X)|\theta|,$$

and successively applying (7) and Hölder inequality

$$I_3 := \int_\Omega |\mu(\theta_1)-\mu(\theta_2)||D\mathbf{u}_2 : D\mathbf{u}| \leq \mu_2\|\theta\|_{4,\Omega}\|D\mathbf{u}_2\|_{4,\Omega}\|D\mathbf{u}\|_{2,\Omega}.$$

Applying (15) this term can be estimated as follows

$$I_3 \leq \mu_2\|\theta\|_{2,\Omega}^{1/2}\|\nabla\theta\|_{2,\Omega}^{1/2}\|D\mathbf{u}_2\|_{4,\Omega}\|D\mathbf{u}\|_{2,\Omega} \leq$$
$$\leq \frac{4\mu_2^4}{k\mu_0}\|\theta\|_{2,\Omega}^2\|D\mathbf{u}_2\|_{4,\Omega}^4 + \frac{k}{8}\|\nabla\theta\|_{2,\Omega}^2 + \frac{\mu_0}{8}\|\nabla\mathbf{u}\|_{2,\Omega}^2.$$

Analogously we get

$$I_4 := \int_\Omega (\mu(\theta_1)-\mu(\theta_2))|D\mathbf{u}_1|^2\theta \leq \mu_2\|\theta\|_{4,\Omega}^2\|D\mathbf{u}_1\|_{4,\Omega}^2 \leq$$
$$\leq \frac{2\mu_2^2}{k}\|\theta\|_{2,\Omega}^2\|D\mathbf{u}_1\|_{4,\Omega}^4 + \frac{k}{8}\|\nabla\theta\|_{2,\Omega}^2$$

and also

$$I_5 := \int_\Omega \mu(\theta_2)D\mathbf{u} : D(\mathbf{u}_1+\mathbf{u}_2)\theta \leq \mu_1\|\theta\|_{4,\Omega}\|D(\mathbf{u}_1+\mathbf{u}_2)\|_{4,\Omega}\|D\mathbf{u}\|_{2,\Omega}$$
$$\leq \frac{4\mu_1^4}{k\mu_0}\|\theta\|_{2,\Omega}^2\|D(\mathbf{u}_1+\mathbf{u}_2)\|_{4,\Omega}^4 + \frac{k}{8}\|\nabla\theta\|_{2,\Omega}^2 + \frac{\mu_0}{8}\|\nabla\mathbf{u}\|_{2,\Omega}^2.$$

Then these inequalities imply

$$\frac{d}{dt}\left(\|\mathbf{u}\|^2_{2,\Omega}+\|\theta\|^2_{2,\Omega}\right)\leq\frac{2}{\mu_0}\left(\|D\mathbf{u}_2\|^2_{2,\Omega}+\|\nabla\theta_2\|^2_{2,\Omega}\right)\|\mathbf{u}\|^2_{2,\Omega}+$$

$$+\frac{2}{k}\left(\|\nabla\theta_2\|^2_{2,\Omega}+\frac{2\mu_2^4+8\mu_1^4}{\mu_0}\|D\mathbf{u}_2\|^4_{4,\Omega}+(\mu_2^2+\frac{8\mu_1^4}{\mu_0})\|D\mathbf{u}_1\|^4_{4,\Omega}\right)\|\theta\|^2_{2,\Omega}.$$

Considering that $(\mathbf{u},\theta)|_{t=0}=(\mathbf{0},0)$, Gronwall Lemma allows us to conclude that $(\mathbf{u},\theta)=(\mathbf{0},0)$.

6.2. **Uniqueness for Theorem 2.2.** Proceeding as in Section 6.2 we find

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2_{2,\Omega}+\frac{1}{2}\frac{d}{dt}\|\theta\|^2_{2,\Omega}+\mu_0\|D\mathbf{u}\|^2_{2,\Omega}+k\|\nabla\theta\|^2_{2,\Omega}\leq$$

$$\leq I_1+I_2+I_3+I_4+I_5-\int_\Omega(|\theta_1|^{1/2}-|\theta_2|^{1/2})\theta_2\theta.$$

For every $t\in[0,T]$, define

$$[|\theta_1|^{1/2}+|\theta_2|^{1/2}>0]=\{x\in\Omega:\ |\theta_1(x,t)|^{1/2}+|\theta_2(x,t)|^{1/2}>0\}$$

then the last term in RHS of the above inequality reads

$$-\int_\Omega(|\theta_1|^{1/2}-|\theta_2|^{1/2})\theta_2\theta=-\int_{[|\theta_1|^{1/2}+|\theta_2|^{1/2}>0]}\frac{|\theta_1|-|\theta_2|}{|\theta_1|^{1/2}+|\theta_2|^{1/2}}\theta_2\theta\leq$$

$$\leq\int_{[|\theta_1|^{1/2}+|\theta_2|^{1/2}>0]}\frac{|\theta_2|}{|\theta_1|^{1/2}+|\theta_2|^{1/2}}\theta^2\leq\int_\Omega|\theta_2|^{1/2}\theta^2\leq\|\theta_2\|^{1/2}_{\infty,\Omega}\|\theta\|^2_{2,\Omega}.$$

Therefore we can conclude the desired uniqueness.

Acknowledgement. Partially supported by FEDER and FCT-Plurianual 2007.

References

[1] S.N. Antontsev, A.V. Kazhikhov and V.N. Monakhov. *Boundary value problems in mechanics of nonhomogeneous fluids*. Studies in Mathematics and its Applications **22**. North-Holland, 1990. Translation of Russian, Nauka, Novosibirsk, 1983.

[2] A. Arkhipova. Global solvability of the Cauchy-Dirichlet problem for nondiagonal parabolic systems with variational structure in the case of two spatial variables. *Probl. Mat. Anal.* **16**, S.Petersburg Univ. (1997), pp. 3-40; English transl.: *J. Math. Sci.* **92** :6 (1998), 4231-4255.

[3] A. Bermúdez de Castro. *Continuum Thermomechanics*. Series: Progress in Mathematical Physics , Vol. 43, Birkhäuser 2005. ISBN: 978-3-7643-7265-1

[4] T. Clopeau and A. Mikelić. Nonstationary flows with viscous heating effects. *ESAIM Proc.* **2** (1997), 55-63.

[5] L. Consiglieri. Weak solutions for a class of non-Newtonian fluids with energy transfer. *J. Math. Fluid Mechanics* **2** (2000), 267-293.

[6] L. Consiglieri. Friction boundary conditions on thermal incompressible viscous flows. *Annali di Matematica Pura ed Applicata* (2008), online:
$http://dx.doi.org/10.1007/s10231-007-0060-3$

[7] L. Consiglieri and T. Shilkin. Regularity to stationary weak solutions for generalized Newtonian fluids with energy transfer. *Zapiski Nauchnyh Seminarov POMI* **271** (2000), 122-150.

[8] J.I. Diaz and G. Galiano. Existence and uniqueness of solutions to the Boussinesq system with nonlinear thermal diffusion. Report Modelling, Analysis and Simulation (MAS) MAS-R9722 September 30, 1997.

[9] Y. Giga. Solutions for semilinear parabolic equations in L^p and regularity of weak solutions of the Navier-Stokes system. *J. Diff. Eq.* **61** (1986), 186–212.

[10] P. Kaplický, J. Málek and J. Stará. Global-in-time Hölder continuity of the velocity gradients for fluids with shear-dependent viscosities. *NoDEA. Nonlinear Differential Equations and Applications* **9** (2002), 175-195.

[11] O.A. Ladyzhenskaya. Mathematical problems in the dynamics of a viscous incompressible fluid. 2nd rev. aug. ed., "Nauka", Moscow, 1970: English transl. of 1st ed.,*The mathematical theory of viscous incompressible flow.* Gordon and Breach. New York 1969.

[12] O.A. Ladyzhenskaya, V.A. Solonnikov and N.N. Uraltseva. Linear and quasilinear equations of parabolic type. Translations of Mathematical Monographs, Vol. 23 American Mathematical Society, Providence, R.I., 1967.

[13] R. Lewandowski. The mathematical analysis of the coupling of a turbulent kinetic energy equation to the Navier-Stokes equation with an eddy viscosity. *Nonlinear Analysis TMA* **28**: 2 (1997), 393-417.

[14] S.A. Lorca and J.L. Boldrini. The initial value problem for a generalized Boussinesq model. *Nonlinear Analysis* **36** (1999), pp. 457–480.

[15] J.L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod et Gauthier-Villars, Paris, 1969.

[16] J. Málek, J. Nečas, M. Rokyta and M. Ružička. *Weak and Measure-valued solutions to evolutionary PDEs.* Chapman and Hall, London, 1996.

[17] B. Mohammadi and O. Pironneau. *Analysis of the K-Epsilon turbulence model.* Research in Applied Mathematics. Wiley/Masson, 1994.

[18] J. Naumann. On the existence of weak solutions to the equations of non-stationary motion of heat-conducting incompressible viscous fluids. *Math. Meth. Appl. Sci.* 29, 2006, 1883-1906.

[19] J. Nečas and V. Šverák. On regularity of solutions of nonlinear parabolic systems. *Annali della Scuola Norm. Sup. Pisa* **18** :1 (1991), 1-11.

[20] L. Nirenberg. On elliptic partial differential equations. *Annali della Scuola Norm. Sup. Pisa* **13** (1959), 115-162.

[21] T. Roubíček. On non-Newtonian fluids with energy transfer. *J. Math. Fluid Mech.* **9** (2007), 1-16.

[22] J. Simon. Compact sets in the space. *Annali Mat. Pura Appl.IV* **146** (1987), 65-96.

Dep. Mathematics/FCUL and CMAF, Univ. Lisboa, 1749-016 Lisboa, Portugal