The non-incompressibility on heat-conducting fluids. The stationary case.

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Abstract

This work deals with generalized viscous flows that can only undergo isochoric motions in isothermal processes, but can sustain motions that are not necessarily isochoric in processes that are not isothermal. The heat-conducting Stokes and Bingham fluids appear as a direct application. The method used here is a combination of a fixed point argument, the Uzawa-type algorithm and the optimization theory. The pressure is found as a limit of a sequence such that satisfies a constraint condition.

1 Introduction

Heat-conducting viscous flows are thermally compressible (see [11, 16] and the references therein):

$$\det \mathbb{F} = f(\theta) \Leftrightarrow \nabla \cdot \mathbf{u} = \alpha(\theta)\mathbf{u} \cdot \nabla\theta, \tag{1}$$

where $\mathbb{F}$ is the deformation gradient, f is a given function on the temperature θ, $\mathbf{u}$ denotes the fluid velocity vector, $\alpha = f'(\theta)/f(\theta)$ represents the coefficient of thermal expansion, the prime denotes the derivative with respect to the argument, and the steady-state material derivative is $d/dt = \mathbf{u} \cdot \nabla$ when the Eulerian description is taken care of.

In [16, pp. 278] the numerical analysis of a perfect gas characterized by low-Mach-number speed is studied. The model known as *weakly compressible flow* has a hydrodynamically incompressible behavior and the pressure is slip into hydrodynamic and thermodynamic parts

$$p(x,t) = p_H(x,t) + p_T(t),$$

with the relations

$$\int_\Omega p_H(x,t)dx = 0 \Leftrightarrow p_T(t) = \frac{1}{|\Omega|}\int_\Omega p(x,t)dx \quad \text{and} \quad \rho = \frac{p_T}{R\theta},$$

where Ω represents a domain, $x \in \Omega$ and t is the time variable, ρ denotes the density and R is the gas constant. This weakly compressible flow model corresponds to the particular case $\alpha(\theta) = 1/\theta$ in (1) or equivalently $f \equiv id$.

Using the method of vanishing viscosity, different concepts (entropy [5], viscosity [13] and renormalized [8]) are introduced to justify the sense of convergence and to establish uniqueness and stability.

Here we consider the existence of weak solutions for a stationary problem describing low-speed flows. The Stokes approximation corresponds to the limiting situation of vanishingly small Reynolds number, that is, the stress due to viscosity is predominant on that due to inertia. This also happens when the characteristic velocity is small (slow motion). Also the influence of Cauchy stress in the energy equation can be neglected. The study with the dissipative term $\tau : D\mathbf{u}$ responsible for the irreversible transfer of the mechanical energy into heat is already known (see [4] and the references therein) if the incompressibility is assumed, but it remains an open problem if the non-incompressibility is taken into account.

The outline of the present work is as follows. Next section we formulate the problem under study. In section 3 the assumptions and the main results are presented. The sections 4, 5, 6 and 7 are devoted to the proofs of the presented results.

2 The formulation of the problem

Let Ω be an open bounded set of $\mathbb{R}^n$ $(n > 1)$ with a sufficiently smooth boundary $\partial\Omega$. The equations governing the heat transport in compressible viscous fluids at steady-state consist of

$$\nabla \cdot (\rho \mathbf{u}) = 0 \quad in \quad \Omega; \tag{2}$$
$$\nabla \cdot \sigma + \rho \mathbf{f} = 0 \quad in \quad \Omega; \tag{3}$$
$$\rho \mathbf{u} \cdot \nabla e + \nabla \cdot \mathbf{q} = \rho g \quad in \quad \Omega. \tag{4}$$

In the continuity equation (2), ρ and $\mathbf{u}$ are the unknown functions (denoting the density and the fluid velocity vector, respectively). In the motion equation (3), $\mathbf{f}$ represents the external force and $\sigma = -pI + \tau$ denotes the Cauchy stress tensor, where p represents the pressure and I is the identity matrix.

Here we consider a generalized viscous fluid, that is, the constitutive law for the viscous stress tensor τ is

$$\tau - (\mu(\theta) D\mathbf{u} + \lambda(\theta)\mathrm{tr}(D\mathbf{u})I) \in \gamma(\theta)\partial(J \circ \Lambda)(D\mathbf{u}), \tag{5}$$

where θ is the third unknown independent state variable representing the temperature, $\mu/2$ and λ are the viscosity coefficients in accordance with the second law of thermodynamics

$$\mu(\theta) \geq \mu_{\#} > 0, \quad n\lambda(\theta) + \mu(\theta) \geq 0, \tag{6}$$

$D = (\nabla + \nabla^T)/2$ the symmetrized gradient, ∂ denotes the subdifferential of $J \circ \Lambda$ at the point $D\mathbf{u}$ with J a convex functional known as superpotential [15, pp. 68, 99], Λ will be defined below and γ denotes the yield limit.

The Newtonian fluid (linearly viscous fluid) corresponds to ($J \equiv 0$)

$$\tau = \mu(\theta) D\mathbf{u} + \lambda(\theta)\mathrm{tr}(D\mathbf{u})I, \qquad \mathrm{tr}(D\mathbf{u}) = I : D\mathbf{u} = \nabla \cdot \mathbf{u}.$$

The Bingham materials have a rigid viscoplastic behavior with the yield limit γ:

$$\tau = \left(1 + \frac{\gamma(\theta)}{\varphi(\Lambda(D\mathbf{u}))}\right)\Lambda(D\mathbf{u}), \qquad \Lambda(D\mathbf{u}) = \mu(\theta) D\mathbf{u} + \lambda(\theta)\mathrm{tr}(D\mathbf{u})I,$$

where φ is a nonnegative function invariant under a change of reference frame (see [1] and the references therein for the description of isothermal compressible Bingham fluids).

In the energy equation (4), e is the specific (i.e., per unit mass) internal energy, g represents the heat source, and $\mathbf{q} = -k(\theta)\nabla\theta$ is the Fourier heat flux with k denoting the conductivity. Notice that we can assume k as a constant because the thermal conductivity k is a function only dependent on temperature and not on space and then the Kirchoff argument can be applied to the Fourier law inducing the heat flux in the form $\mathbf{q} = -k\nabla\Theta$.

Considering the pressure $p = P(\theta, \rho)$, from the Maxwell equation

$$\rho^2 \frac{\partial e}{\partial \rho}(\theta, \rho) = P(\theta, \rho) - \theta \frac{\partial P}{\partial \theta}(\theta, \rho),$$

it results

$$e(\theta, \rho) = \int_1^\theta c_v(z)dz + \int_1^\rho \frac{P_E(z)}{z^2}dz, \tag{7}$$

where c_v denotes the specific heat capacity at constant volume

$$c_v(\theta) = \frac{\partial e}{\partial \theta}(\theta, \rho) > 0,$$

and P_E is a known function such that the relation holds

$$P(\theta, \rho) - \theta \frac{\partial P}{\partial \theta}(\theta, \rho) = P_E(\rho).$$

The relation (7) means that the thermal part related to the random translational motion of the molecules contributes separately and in an additive way of the elastic part related to the action of intermolecular forces. Then the energy equation (4) reads

$$\begin{aligned} \rho c_v(\theta)\mathbf{u} \cdot \nabla\theta - k\triangle\theta + \frac{P_E(\rho)}{\rho}[\mathbf{u} \cdot \nabla\rho + \rho\nabla \cdot \mathbf{u}] = \\ = P_E(\rho)\nabla \cdot \mathbf{u} + \rho g. \end{aligned} \tag{8}$$

Using (1), the method of characteristics gives the explicit solution (see [14] for instance)

$$\rho(x,t) = \rho_0(x) \exp\left[-\ln(f(\theta(x,t))) + \ln(f(\theta_0(x)))\right] = \frac{\rho_0(x) f(\theta_0(x))}{f(\theta(x,t))} > 0.$$

At steady-state, applying (1) and (2) in (8), it results the heat equation

$$\begin{aligned} \beta(\theta)\mathbf{u} \cdot \nabla\theta - k\triangle\theta &= \rho g, \\ \text{with } \beta(\theta) &= \rho(\theta)c_v(\theta) - P_E(\rho(\theta))\alpha(\theta). \end{aligned} \tag{9}$$

In conclusion, the problem under study is

$(\mathbf{P})$: Find $(\mathbf{u}, \tau, p, \theta)$ such that

$$\begin{cases} -\nabla \cdot \tau = -\nabla p + \rho(\theta)\mathbf{f}, \quad \text{with (5)}, \\ \beta(\theta)\mathbf{u} \cdot \nabla\theta - k\triangle\theta = \rho(\theta)g \\ \nabla \cdot \mathbf{u} = \alpha(\theta)\mathbf{u} \cdot \nabla\theta \qquad \text{in } \mathcal{D}'(\Omega), \end{cases}$$

with the following Dirichlet conditions on the boundary

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad \theta = 0. \tag{10}$$

Here we assume Dirichlet conditions but the presented method is valid for slip boundary conditions [4, 15].

3 Existence results

We assume that $\mu, \lambda, \gamma, \rho, \alpha, \beta : \Omega \times \mathbb{R} \to \mathbb{R}^+$ are Carathéodory functions, that is, measurable with respect to $x \in \Omega$ for every $e \in \mathbb{R}$, and continuous with respect to $e \in \mathbb{R}$ almost every $x \in \Omega$ such that verify (6) and

$$\exists\mu^\# > 0 : \quad \mu(\cdot, e) \leq \mu^\#, \quad \forall e \in \mathbb{R}, \text{ a.e. in } \Omega; \tag{11}$$

$$\exists\lambda^\# > 0 : \quad \lambda(\cdot, e) \leq \lambda^\#, \quad \forall e \in \mathbb{R}, \text{ a.e. in } \Omega; \tag{12}$$

$$\exists\gamma^\# > 0 : \quad 0 \leq \gamma(\cdot, e) \leq \gamma^\#, \quad \forall e \in \mathbb{R}, \text{ a.e. in } \Omega; \tag{13}$$

$$\exists\rho^\# > 0 : \quad |\rho(\cdot, e)| \leq \rho^\#, \quad \forall e \in \mathbb{R}, \text{ a.e. in } \Omega; \tag{14}$$

$$\exists\alpha^\# > 0 : \quad |\alpha(\cdot, e)| \leq \alpha^\#, \quad \forall e \in \mathbb{R}, \text{ a.e. in } \Omega; \tag{15}$$

$$\exists\beta^\# > 0 : \quad |\beta(\cdot, e)| \leq \beta^\#, \quad \forall e \in \mathbb{R}, \text{ a.e. in } \Omega; \tag{16}$$

the function $f \in C^1(\mathbb{R})$ is positive and convex and the function $J : \mathbb{M}^{n\times n}_{\text{sym}} \to \mathbb{R}^+_0$ is continuous and convex such that $J(0) = 0$, and for some $1 \leq r < 2$

$$\exists\upsilon^\# > 0 : \ 0 \leq J(\zeta) \leq \upsilon^\#(|\zeta|^r + 1), \ \forall\zeta \in \mathbb{M}^{n\times n}_{\text{sym}}, \tag{17}$$

where $\mathbb{M}^{n\times n}_{\text{sym}}$ is the set of symmetric matrices of the type $n \times n$. Noting that $\Lambda(\zeta) = \mu(\theta)\zeta + \lambda(\theta)\text{tr}(\zeta)I$, for some given $\theta : \Omega \to \mathbb{R}$, we define

$$\mathcal{J}(\theta, \zeta) := J(\Lambda(\zeta)).$$

Finally we assume that

$$\mathbf{f} \in \mathbf{L}^2(\Omega) \quad \text{and} \quad g \in L^2(\Omega). \tag{18}$$

Notice that the existence result is still valid for broader Lebesgue spaces in (18) with exponents $s > 2n/(n + 2)$, $n = 2, 3$.

Let us state the main existence result for a general J.

Theorem 3.1 *For some $0 < \varkappa < 1$, assume that*

$$\rho^{\#}\|g\|_{2,\Omega} \leq \frac{C_S^2 k}{\alpha^{\#}}\varkappa, \tag{19}$$

with C_S denoting the Poincaré-Sobolev constant due to $H_0^1(\Omega) \hookrightarrow L^4(\Omega)$. Under the assumptions (6), (11)-(18), the problem $(\mathbf{P})$ *has a weak solution* $(\mathbf{u}, p, \theta) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega) \times [H_0^1(\Omega) \cap H^2(\Omega)]$ *in the following sense*

$$\int_\Omega \mu(\theta) D\mathbf{u} : D(\mathbf{v}-\mathbf{u})dx + \int_\Omega \lambda(\theta)\nabla\cdot\mathbf{u}\nabla\cdot(\mathbf{v}-\mathbf{u})dx + \\ + \int_\Omega \gamma(\theta)\Big(\mathcal{J}(\theta, D\mathbf{v}) - \mathcal{J}(\theta, D\mathbf{u})\Big)dx \geq \\ \geq \int_\Omega p\nabla\cdot(\mathbf{v}-\mathbf{u})dx + \int_\Omega \rho(\theta)\mathbf{f}\cdot(\mathbf{v}-\mathbf{u})dx, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega); \tag{20}$$

$$\int_\Omega \beta(\theta)\mathbf{u}\cdot\nabla\theta\phi dx + k\int_\Omega \nabla\theta\cdot\nabla\phi dx = \int_\Omega \rho(\theta) g\phi dx, \quad \forall \phi \in H_0^1(\Omega); \tag{21}$$

$$\int_\Omega \nabla\cdot\mathbf{u} q dx = \int_\Omega \alpha(\theta)\mathbf{u}\cdot\nabla\theta q dx, \quad \forall q \in L^2(\Omega). \tag{22}$$

Here we denote $\zeta : \varsigma = \zeta_{ij}\varsigma_{ij}$ taking into account the convention on implicit summation over repeated indices.

Remark 3.1 *The minimization problem related to (3) and (5) in its weak formulation (20) follows from the subdifferentiability property (cf. [6])*

$$\partial\mathcal{J}(D\mathbf{u}) = (-\nabla\cdot)\partial(\mathcal{J}\circ D)(\mathbf{u}), \tag{23}$$

for any continuous convex functional $\mathcal{J}$. In particular, for $\mathcal{J} = J \circ \Lambda$, using the definition of subdifferential in (5) and considering (23) and (3), we find

$$\langle \rho\mathbf{f} - \nabla p + \nabla\cdot\Lambda(D\mathbf{u}), \mathbf{v}-\mathbf{u}\rangle \leq \gamma\Big((J\circ\Lambda\circ D)(\mathbf{v}) - (J\circ\Lambda\circ D)(\mathbf{u})\Big), \quad \forall \mathbf{v},$$

which delivers (20). The set of subgradients is nonsingular when a non-differentiable convex functional is taken into account.

When J is differentiable or simply $J \equiv 0$ (Stokes-Fourier case) we obtain the following result.

Theorem 3.2 *Suppose that the assumptions in Theorem 3.1 are fulfilled. Then there exists a weak solution* $(\mathbf{u}, p, \theta) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega) \times [H_0^1(\Omega) \cap H^2(\Omega)]$ *in the following sense*

$$\int_\Omega \mu(\theta) D\mathbf{u} : D\mathbf{v} dx + \int_\Omega \lambda(\theta) \nabla \cdot \mathbf{u} \nabla \cdot \mathbf{v} dx + \int_\Omega \gamma(\theta) \frac{\partial \mathcal{J}}{\partial \zeta}(\theta, D\mathbf{u}) : D\mathbf{v} dx = \\ = \int_\Omega p \nabla \cdot \mathbf{v} dx + \int_\Omega \rho(\theta) \mathbf{f} \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega), \tag{24}$$

and (21) and (22) are satisfied. Moreover, the viscous stress tensor τ *is given by*

$$\tau = \mu(\theta) D\mathbf{u} + \lambda(\theta) \nabla \cdot \mathbf{u} I + \gamma(\theta) \frac{\partial \mathcal{J}}{\partial \zeta}(\theta, D\mathbf{u}).$$

A numerical procedure will be used. The proofs of above main results are based on the known Uzawa and Arrow-Hurwicz algorithms (see [6, pp. 172-179] or [18, pp. 138-144]). However in the iterative algorithm presented here the pressure is found as a Lagrange multiplier in (25) and is not given as in the mentioned algorithms.

We will begin by proving the iterative existence result.

Theorem 3.3 *Suppose that the assumptions (6) and (11)-(18) are fulfilled. Let* $\mathbf{w} \in \mathbf{L}^4(\Omega)$ *and* $\xi \in W^{1,4}(\Omega)$ *be given. For each* $i \in \mathbb{N}$ *if* p^i *is an element of* $L^2(\Omega)$*, there exists* $(\mathbf{u}, \theta) \in V := \mathbf{H}_0^1(\Omega) \times H_0^1(\Omega)$ *such that satisfies*

$$\int_\Omega \Big(\mu(\xi) D\mathbf{u} : D(\mathbf{v} - \mathbf{u}) + \lambda(\xi) \nabla \cdot \mathbf{u} \nabla \cdot (\mathbf{v} - \mathbf{u}) \Big) dx \\ + \int_\Omega \gamma(\xi) \Big(\mathcal{J}(\xi, D\mathbf{v}) - \mathcal{J}(\xi, D\mathbf{u}) \Big) dx \geq \\ \geq \int_\Omega p \nabla \cdot (\mathbf{v} - \mathbf{u}) dx + \int_\Omega \rho(\xi) \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) dx, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega); \tag{25}$$

$$k \int_\Omega \nabla \theta \cdot \nabla \phi dx = \int_\Omega (\rho(\xi) g - \beta(\xi) \mathbf{w} \cdot \nabla \xi) \phi dx, \quad \forall \phi \in H_0^1(\Omega); \tag{26}$$

where the pressure $p \in L^2(\Omega)$ *is a Lagrangian multiplier given by the following algorithmic formulation*

$$\int_\Omega (p - p^i) q dx = \int_\Omega (\alpha(\xi) \mathbf{w} \cdot \nabla \xi - \nabla \cdot \mathbf{u}) q dx, \quad \forall q \in L^2(\Omega). \tag{27}$$

Moreover, $(\mathbf{u}, p, \theta)$ *is unique.*

Next we will prove the following auxiliar result wherein its proof is based on a fixed point argument for multivalued mappings.

Theorem 3.4 *Suppose that the assumptions (6) and (11)-(18) are fulfilled. Then there exists at least one triple* $(\mathbf{u}, \bar{p}, \theta) \in \mathbf{H}^1_0(\Omega) \times L^2(\Omega) \times [H^1_0(\Omega) \cap H^2(\Omega)]$ *such that satisfies (20)-(21) with* p *replaced by* $\bar{p}$ *and*

$$\int_\Omega (\nabla \cdot \mathbf{u} + A) q dx = \int_\Omega \alpha(\theta) \mathbf{u} \cdot \nabla\theta q dx, \quad \forall q \in L^2(\Omega), \tag{28}$$

where $A = (\text{meas}(\Omega))^{-1} \int_\Omega \alpha(\theta)\mathbf{u} \cdot \nabla\theta dx$.

4 The existence of a saddle-point

Proof of Theorem 3.3.

In order to apply the Lagrangian multipliers theory, let us introduce $\mathcal{F} : \mathbf{H}^1_0(\Omega) \to \mathbb{R}$ and $\mathcal{G} : H^1_0(\Omega) \to \mathbb{R}$,

$$\begin{aligned} \mathcal{F}(\mathbf{v}) &= \int_\Omega \left(\mu(\xi)\frac{|D\mathbf{v}|^2}{2} + \lambda(\xi)\frac{|\nabla \cdot \mathbf{v}|^2}{2} + \gamma(\xi)\mathcal{J}(\xi, D\mathbf{v}) - \rho(\xi)\mathbf{f} \cdot \mathbf{v} \right) dx \\ \mathcal{G}(\phi) &= \int_\Omega \beta(\xi)\mathbf{w} \cdot \nabla\xi\phi dx + k\int_\Omega \frac{|\nabla\phi|^2}{2} dx - \int_\Omega \rho(\xi) g\phi dx, \end{aligned}$$

the energy functionals associated to the motion system and heat equation, respectively:

$$\begin{aligned} \rho(\xi)\mathbf{f} - \nabla p + \nabla \cdot (\mu(\xi)D\mathbf{u} + \lambda(\xi)\text{tr}(D\mathbf{u})I) \in \gamma(\xi)\partial(\mathcal{J} \circ D)(\xi, \mathbf{u}), \\ -k\triangle\theta = \rho(\xi)g - \beta(\xi)\mathbf{w} \cdot \nabla\xi. \end{aligned} \tag{29}$$

Let us recall the definition of a saddle-point and its existence [6, prop.2.4 page 164].

Definition 4.1 *Let* V *and* Z *be two Banach spaces and* $L : V \times Z \to \mathbb{R}$. *We say that* (u, p) *is a saddle-point of* L *if it satisfies one of the following equivalent relations*

1. *for all* $v \in V$ *and* $q \in Z$

$$L(u, q) \leq L(u, p) \leq L(v, p);$$

2. $L(u,p) = \max_{q\in Z}\inf_{v\in V} L(v,q) = \min_{v\in V}\sup_{q\in Z} L(v,q)$;

3. if L *is differentiable,*

$$\langle \frac{\partial L}{\partial v}(u,p), v\rangle = 0, \quad \forall v \in V;$$
$$\langle \frac{\partial L}{\partial q}(u,p), q\rangle = 0, \quad \forall q \in Z,$$

where the symbol $\langle\cdot,\cdot\rangle$ *denotes a generic duality pairing, not distinguished between scalar and vector fields.*

Theorem 4.1 *Let* V *and* Z *be two reflexive Banach spaces and* $L : V \times Z \to \mathbb{R}$ *be the Lagrangian functional such that*

$$\forall p \in Z, \quad u \mapsto L(u,p) \text{ is convex and lower semicontinuous on } V; \tag{30}$$
$$\exists p_0 \in Z, \quad L(u,p_0) \to +\infty \text{ as } \|u\| \to +\infty \text{ on } V; \tag{31}$$
$$\forall u \in V, \quad p \mapsto L(u,p) \text{ is concave and upper semicontinuous on } Z; \tag{32}$$
$$\exists u_0 \in V, \quad L(u_0,p) \to -\infty \text{ as } \|p\| \to +\infty \text{ on } Z. \tag{33}$$

Then L *has a saddle-point.*

Let us verify the conditions stated at Theorem 4.1. Set $V = \mathbf{H}^1_0(\Omega) \times H^1_0(\Omega)$ and $Z = L^2(\Omega)$ and define the Lagrangian functional $L : V \times Z \to \mathbb{R}$ as

$$L((\mathbf{v},\phi),p) = \mathcal{F}(\mathbf{v}) + \mathcal{G}(\phi) + \int_\Omega p\mathcal{Q}(\mathbf{v})dx - \int_\Omega \frac{(p-p^i)^2}{2}dx,$$

with

$$\mathcal{Q}(\mathbf{v}) = \alpha(\xi)\mathbf{w}\cdot\nabla\xi - \nabla\cdot\mathbf{v}.$$

The condition (30) is a consequence of $\mathcal{F}$ and $\mathcal{G}$ being continuous and convex functions. For $p_0 = 0 \in Z$, the condition (31) is satisfied. The condition (32) is verified since $p \mapsto L((\mathbf{v},\phi),p)$ is the sum of the linear continuous function $p \mapsto \int_\Omega p\mathcal{Q}(\mathbf{v})dx$ with a continuous concave one. Finally in order to check (33), it suffices to choose $u_0 = (\mathbf{u},\theta) \in V$ such that $\nabla\cdot\mathbf{u} = 0$ and $\nabla\theta = \mathbf{0}$ in Ω.

Then Theorem 4.1 guarantees the existence of a saddle-point $((\mathbf{u},\theta),p)$ to L. In accordance to Definition 4.1 we have

$$\int_\Omega q\mathcal{Q}(\mathbf{u})dx = \int_\Omega q(p-p^i)dx.$$

In order to conclude that the saddle-point is the required solution to Theorem 3.3, it remains to show that we can rewrite the problem (25)-(26) as

$$\begin{aligned} \mathcal{F}(\mathbf{v})-\mathcal{F}(\mathbf{u}) &\geq \langle -p\mathcal{Q}'(\mathbf{u}), \mathbf{v}-\mathbf{u}\rangle, \quad \forall \mathbf{v}\in \mathbf{H}^1_0(\Omega); &(34)\\ \langle \mathcal{G}'(\theta),\phi\rangle &= 0, \quad \forall \phi \in H^1_0(\Omega). &(35)\end{aligned}$$

The following theorem asserts the existence of the Fréchet derivative of a given Nemytskii operator [17, Appendix C].

Theorem 4.2 *Suppose $G : \Omega\times\mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ is measurable in $x\in\Omega$, continuously differentiable in $v\in\mathbb{R}$ and $\eta\in\mathbb{R}^n$, and satisfies the following growth conditions:*

$$|G(x,v,\eta)| \leq C(1+|v|^{s_1}+|\eta|^2), \qquad \textit{with } s_1\leq 2n/(n-2) \textit{ if } n\geq 3;$$

$$|\frac{\partial G}{\partial v}(x,v,\eta)| \leq C(1+|v|^{s_2}+|\eta|^r), \qquad \textit{with} \begin{cases} r\leq 2, \textit{ if } n=2\\ s_2\leq (n+2)/(n-2),\\ r\leq (n+2)/n, \textit{ if } n\geq 3;\end{cases}$$

$$|\frac{\partial G}{\partial \eta}(x,v,\eta)| \leq C(1+|v|^{s_3}+|\eta|), \qquad \textit{with } s_3\leq n/(n-2) \textit{ if } n\geq 3;$$

for a.e. $x\in\Omega$ and for all $v\in\mathbb{R}$ and $\eta\in\mathbb{R}^n$. Then the functional $\mathcal{G} : H^1(\Omega)\to\mathbb{R}$ given by $\mathcal{G}(v)=\int_\Omega G(x,v(x),\nabla v(x))dx$ is of class C^1. Its directional derivative at u in direction v exists and is given by

$$\langle \mathcal{G}'(u),v\rangle = \int_\Omega \left(\frac{\partial G}{\partial v}(x,u,\nabla u)v + \frac{\partial G}{\partial \eta}(x,u,\nabla u)\cdot\nabla v\right)dx.$$

In particular, if $u\in H^1_0(\Omega)$ is a minimizer of $\mathcal{G}$, the Euler-Lagrange equations are weakly satisfied in the sense that

$$\int_\Omega \left(\frac{\partial G}{\partial v}(x,u,\nabla u)v + \frac{\partial G}{\partial \eta}(x,u,\nabla u)\cdot\nabla v\right)dx = 0, \quad \forall v\in H^1_0(\Omega).$$

In order to apply Theorem 4.2, define $F : \Omega\times\mathbb{R}^n\times\mathbb{M}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}$, $G : \Omega\times\mathbb{R}\times\mathbb{R}^n\to\mathbb{R}$ and $Q : \Omega\times\mathbb{R}^n\to\mathbb{R}$ by

$$\begin{aligned} F(\cdot,\mathbf{v},\zeta) &= \mu(\xi)\frac{|\zeta|^2}{2}+\lambda(\xi)\frac{(I:\zeta)^2}{2}-\rho(\xi)\mathbf{f}\cdot\mathbf{v};\\ G(\cdot,\phi,\eta) &= \beta(\xi)\mathbf{w}\cdot\nabla\xi\phi + k\frac{|\eta|^2}{2}-\rho(\xi)g\phi;\\ Q(\cdot,\eta) &= \alpha(\xi)\mathbf{w}\cdot\nabla\xi - [1\cdots 1]^T\cdot\eta.\end{aligned}$$

Next we calculate

$$\frac{\partial F}{\partial \mathbf{v}} = -\rho(\xi)\mathbf{f}; \qquad \frac{\partial F}{\partial \zeta} = \mu(\xi)\zeta + \lambda(\xi)(I:\zeta)I;$$
$$\frac{\partial Q}{\partial \eta} = -[1\cdots 1]^T;$$
$$\frac{\partial G}{\partial \phi} = \beta(\xi)\mathbf{w}\cdot\nabla\xi - \rho(\xi)g; \qquad \frac{\partial G}{\partial \eta} = k\eta$$

and then we conclude

$$\begin{aligned}
\mathcal{J}(\xi, D\mathbf{v}) - \mathcal{J}(\xi, D\mathbf{u}) &\geq \left(\frac{\partial F}{\partial \mathbf{v}}(\mathbf{u}, D\mathbf{u}), \mathbf{v}-\mathbf{u}\right) + \\
&\quad + \left(\frac{\partial F}{\partial \zeta}(\mathbf{u}, D\mathbf{u}), D(\mathbf{v}-\mathbf{u})\right) + (p, \nabla\cdot(\mathbf{v}-\mathbf{u})) \\
\left\langle \frac{\partial L}{\partial \theta}((\mathbf{u},\theta),p), \phi\right\rangle &= \langle \mathcal{G}'(\theta), \phi\rangle = 0,
\end{aligned}$$

which corresponds to (34)-(35).

Uniqueness. Let us prove the uniqueness of the solution to the problem (25)-(27). Applying the uniqueness argument to the solutions $(\mathbf{u}_1, p_1, \theta_1)$ and $(\mathbf{u}_2, p_2, \theta_2)$ into (26) we obtain

$$\|\theta_1 - \theta_2\|^2_{\mathbf{H}^1_0(\Omega)} \leq 0.$$

Then we have $\theta_1 = \theta_2$.

Next choosing $\mathbf{v} = \mathbf{u}_1$ and $\mathbf{v} = \mathbf{u}_2$ as a test function in (25) for the solutions $\mathbf{u}_2$ and $\mathbf{u}_1$, respectively, we get

$$\int_\Omega \left(\mu(\theta)|D(\mathbf{u}_1-\mathbf{u}_2)|^2 + \lambda(\theta)(\nabla\cdot(\mathbf{u}_1-\mathbf{u}_2))^2\right) dx \leq \int_\Omega \nabla\cdot(\mathbf{u}_1-\mathbf{u}_2)(p_1-p_2)dx.$$

Since $n\lambda(\theta) + \mu(\theta) \geq 0$ we have

$$\int_\Omega \mu(\theta)\left(|D(\mathbf{u}_1-\mathbf{u}_2)|^2 - \frac{1}{n}(\nabla\cdot(\mathbf{u}_1-\mathbf{u}_2))^2\right) dx \leq \int_\Omega \nabla\cdot(\mathbf{u}_1-\mathbf{u}_2)(p_1-p_2)dx.$$

Using the fact that $(\nabla\cdot\mathbf{u})^2 \leq |D\mathbf{u}|^2$ and (6), it follows

$$\frac{(n-1)}{n}\mu_\#\|D(\mathbf{u}_1-\mathbf{u}_2)\|^2_{2,\Omega} \leq \int_\Omega \nabla\cdot(\mathbf{u}_1-\mathbf{u}_2)(p_1-p_2)dx.$$

The Poincaré inequality implies that

$$\nu \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{H}^1_0(\Omega)} \leq \int_\Omega \nabla \cdot (\mathbf{u}_1 - \mathbf{u}_2)(p_1 - p_2) dx. \tag{36}$$

with $\nu = C(n-1)\mu_\#/n$, $n > 1$ and C the Poincaré constant.

Finally using (27) it follows

$$\int_\Omega ((p_1 - p^i) - (p_2 - p^i))(p_1 - p_2) dx = -\int_\Omega \nabla \cdot (\mathbf{u}_1 - \mathbf{u}_2)(p_1 - p_2) dx.$$

Considering the monotone property of the left hand side, the last equality implies that (36) reads

$$\nu \|\mathbf{u}_1 - \mathbf{u}_2\|^2_{\mathbf{H}^1_0(\Omega)} \leq 0.$$

Hence we conclude that $\mathbf{u}_1 = \mathbf{u}_2$ and consequently

$$\int_\Omega (p_1 - p_2)^2 dx = \int_\Omega ((p_1 - p^i) - (p_2 - p^i))(p_1 - p_2) dx \leq 0.$$

Then we obtain $p_1 = p_2$ and the proof of Theorem 3.3 is concluded.

5 The existence of fixed points

Proof of Theorem 3.4.

First we recall the fixed point result for multivalued mappings [10].

Theorem 5.1 *Let K be a nonempty compact convex subset of a locally convex Hausdorff topological vector space X. Let $\mathcal{L} : K \to \mathcal{P}(K)$ be a multivalued upper semi-continuous operator such that the set $\mathcal{L}(z)$ is nonempty closed and convex for all $z \in K$. Then $\mathcal{L}$ has at least one fixed point, i.e. $z \in \mathcal{L}(z)$ for some $z \in K$.*

If X is endowed with the weak topology, thus X becomes a locally convex Hausdorff topological vector space, and the Tychonoff extension to weak topologies is valid [7] for a nonempty weakly sequential compact convex subset K and an operator $\mathcal{L}$ with its graph being weakly sequential closed in K. Thus a ball K is weakly sequential compact provided that the Banach space $X = \mathbf{H}^1_0(\Omega) \times W^{1,4}_0(\Omega)$ is endowed with the product of weak topologies.

Let K be the ball

$$K = \{(\mathbf{w}, \xi) \in X : \ \|\nabla \mathbf{w}\|_{2,\Omega} \leq \frac{C_S^2 k}{\beta^\#}(1 - \varkappa), \ \|\nabla \xi\|_{4,\Omega} \leq \frac{C_S}{\alpha^\#}\}. \tag{37}$$

For each $(\mathbf{w}, \xi) \in X$, let us show the solvability of the problem constituted by (25)-(26) with p replaced by $\bar{p}$ and

$$\int_\Omega \nabla \cdot \mathbf{u} q dx = \int_\Omega \left(\alpha(\xi)\mathbf{w} \cdot \nabla \xi - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \alpha(\xi)\mathbf{w} \cdot \nabla \xi dx \right) q dx, \tag{38}$$

for all $q \in L^2(\Omega)$. The problem under study can be writen in the following sucint form in $\mathcal{D}'(\Omega)$, (29) and

$$\nabla \cdot \mathbf{u} = \alpha(\xi)\mathbf{w} \cdot \nabla \xi - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \alpha(\xi)\mathbf{w} \cdot \nabla \xi dx, \tag{39}$$

$$\tau \in (\mu(\xi) D\mathbf{u} + \lambda(\xi)\mathrm{tr}(D\mathbf{u})I) + \gamma(\xi)\partial \mathcal{J}(\xi, D\mathbf{u}), \tag{40}$$

$$\nabla \bar{p} = \nabla \cdot \tau + \rho(\xi)\mathbf{f}. \tag{41}$$

Since the problem constituted by (29) and (39)-(41) is uncoupled we obtain sucessively

1. the existence of a unique $\theta = \theta(\mathbf{w}, \xi) \in H_0^1(\Omega)$ verifying (29) due to the existence and uniqueness theory for elliptic equations (see [12], for instance) since $\rho(\xi)g - \beta(\xi)\mathbf{w} \cdot \nabla \xi \in L^2(\Omega)$. Moreover, taking $\phi = \nabla^2 \theta$ as a test function in (26) we get

$$C_S k \|\nabla \theta\|_{4,\Omega} \leq k \|\nabla^2 \theta\|_{2,\Omega} \leq \beta^\# \|\mathbf{w}\|_{4,\Omega} \|\nabla \xi\|_{4,\Omega} + \rho^\# \|g\|_{2,\Omega}; \tag{42}$$

2. the existence of (a non unique) $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi) \in \mathbf{H}_0^1(\Omega)$ verifying (39) since $\alpha(\xi)\mathbf{w} \cdot \nabla \xi \in L^2(\Omega)$ and the compatibility condition is verified

$$\int_\Omega \left(\alpha(\xi)\mathbf{w} \cdot \nabla \xi - \frac{1}{\mathrm{meas}(\Omega)} \int_\Omega \alpha(\xi)\mathbf{w} \cdot \nabla \xi dx \right) dx = 0.$$

   Moreover the following estimate holds (cf. [9, pp. 121]):

$$\|\mathbf{u}\|_{\mathbf{H}_0^1(\Omega)} \leq \alpha^\# \|\mathbf{w}\|_{4,\Omega} \|\nabla \xi\|_{4,\Omega} \leq \frac{\alpha^\#}{C_S} \|\nabla \mathbf{w}\|_{2,\Omega} \|\nabla \xi\|_{4,\Omega}; \tag{43}$$

3. the existence of $\tau = \tau(\mathbf{u}, \xi) \in \mathbf{L}^2(\Omega)$ verifying (40) since the set of the subgradients for any continuous convex real function is nonempty (cf. [6, pp. 22]). Indeed, the solution is $\tau = \Lambda(D\mathbf{u}) - \varsigma$, where ς satisfies the known extremality relation

$$\int_\Omega \gamma(\xi)\mathcal{J}(\xi, D\mathbf{u})dx + \int_\Omega \gamma(\xi)\mathcal{J}^*\left(\xi, -\frac{\varsigma}{\gamma(\xi)}\right)dx = \langle -\varsigma, D\mathbf{u}\rangle, \qquad (44)$$

if $\gamma > 0$ (for details, see [2, 3]).

4. the existence of a unique $\bar{p} = p(\tau, \xi) \in L^2(\Omega)/\mathbb{R}$ which is explicitly given by (41). Thus the estimate holds

$$\|\bar{p}\|_{2,\Omega} \leq \|\tau\|_{2,\Omega} + \rho^\# \|\mathbf{f}\|_{2,\Omega}; \qquad (45)$$

and the variational formulation (25), with p replaced by $\bar{p}$, can be recovered as in Remark 3.1.

Let $\mathcal{L}$ be the mapping defined as

$$\mathcal{L}((\mathbf{w}, \xi)) = \{(\mathbf{u}, \theta)\},$$

where $(\mathbf{u}, \theta)$ is a solution in accordance to the above items 1, 2, 3 and 4.

In order to apply Theorem 5.1, let us show that $\mathcal{L}$ has a weakly sequential closed graph and for each $(\mathbf{w}, \xi) \in K$ we have $\mathcal{L}((\mathbf{w}, \xi)) \subset K$. The properties of $\mathcal{L}((\mathbf{w}, \xi))$ being closed and convex are consequence of the above assertions.

Considering that $(\mathbf{w}, \xi) \in K$, since (19) is assumed then (42) implies

$$\|\nabla\theta\|_{4,\Omega} \leq (1-\varkappa)\frac{C_S}{\alpha^\#} + \frac{\rho^\#}{C_S k}\|g\|_{2,\Omega} \leq \frac{C_S}{\alpha^\#},$$

and from (43) we find

$$\|\nabla\mathbf{u}\|_{2,\Omega} \leq \|\nabla\mathbf{w}\|_{2,\Omega} \leq \frac{(1-\varkappa)C_S k}{\beta^\#}.$$

Thus we obtain $(\mathbf{u}, \theta) \in K$, for every $(\mathbf{u}, \theta) \in \mathcal{L}((\mathbf{w}, \xi))$.

It remains to prove that the graph of $\mathcal{L}$ is weakly sequential closed. We begin by establishing some additional estimates.

Lemma 5.1 *The following estimates hold*

$$\begin{aligned} k\|\nabla\theta\|_{2,\Omega} &\leq \beta^{\#}\|\mathbf{w}\|_{4,\Omega}\|\nabla\xi\|_{4,\Omega}+\rho^{\#}\|g\|_{2,\Omega}; & (46)\\ \|\tau\|_{2,\Omega} &\leq (\mu^{\#}+n\lambda^{\#})\|D\mathbf{u}\|_{2,\Omega}+ \\ &+\gamma^{\#}\upsilon^{\#}C(\mu^{\#}+n\lambda^{\#})\left(1+(\mu^{\#}+n\lambda^{\#})^{r-1}\|D\mathbf{u}\|_{2,\Omega}^{r-1}\right), & (47)\end{aligned}$$

with C *denoting a constant only depending on* r *and* $\text{meas}(\Omega)$*.*

Proof. The estimate (46) easily holds taking $\phi=\nabla\theta$ as a test function in (26). Let us show the proof of the estimate for $\tau=\Lambda(D\mathbf{u})-\varsigma$. From the definition of conjugate function and using (17), (11) and (12) we calculate

$$\begin{aligned}\mathcal{J}^*(\xi,\upsilon) &:= \sup_\zeta\{\upsilon:\zeta-J(\mu(\xi)\zeta+\lambda(\xi)(I:\zeta)I)\}\\ &\geq \sup_\zeta\{\upsilon:\zeta-\upsilon^{\#}(\mu^{\#}|\zeta|+n\lambda^{\#}|I:\zeta|)^r\}-\upsilon^{\#}\\ &\geq \sup_\zeta\{\upsilon:\zeta-2\upsilon^{\#}(\mu^{\#}+n\lambda^{\#})^r|\zeta|^r\}-\upsilon^{\#}.\end{aligned}$$

Then we have the conjugate inequality

$$\mathcal{J}^*(\xi,\upsilon)\geq(2\upsilon^{\#})^{1-r'}\frac{r-1}{((\mu^{\#}+n\lambda^{\#})r)^{r'}}|\upsilon|^{r'}-\upsilon^{\#}. \qquad (48)$$

Inserting (48) into (44) it follows

$$(2\gamma^{\#}\upsilon^{\#})^{1-r'}\frac{r-1}{((\mu^{\#}+n\lambda^{\#})r)^{r'}}\|\varsigma\|_{r',\Omega}^{r'}\leq\gamma^{\#}\upsilon^{\#}\text{meas}(\Omega)+\|\varsigma\|_{2,\Omega}\|D\mathbf{u}\|_{2,\Omega}.$$

Since $r'\geq 2$ after some calculations we obtain

$$\|\varsigma\|_{2,\Omega}\leq\gamma^{\#}\upsilon^{\#}C\left((\mu^{\#}+n\lambda^{\#})+(\mu^{\#}+n\lambda^{\#})^r\|D\mathbf{u}\|_{2,\Omega}^{r-1}\right),$$

and then we conclude that (47).

The proof of Lemma 5.1 is concluded.

Let $\{(\mathbf{w}_m,\xi_m)\}_{m\in\mathbb{N}}$ be a sequence in K verifying

$$\mathbf{w}_m\rightharpoonup\mathbf{w}\text{ in }\mathbf{H}_0^1(\Omega),\qquad\nabla\xi_m\rightharpoonup\nabla\xi\text{ in }L^4(\Omega),$$

and $(\mathbf{u}_m,\theta_m)\in\mathcal{L}((\mathbf{w}_m,\xi_m))$ for each $m\in\mathbb{N}$. Applying the Sobolev-Kondrachoff embeddings $\mathbf{H}_0^1(\Omega)\hookrightarrow\hookrightarrow\mathbf{L}^4(\Omega)$ and $W_0^{1,4}(\Omega)\hookrightarrow\hookrightarrow L^1(\Omega)$ we get

$$\mathbf{w}_m\to\mathbf{w}\text{ in }\mathbf{L}^4(\Omega),\qquad\xi_m\to\xi\text{ a.e. in }\Omega.$$

From the estimates (43) and (46) we can extract a subsequence, still denoted by $(\mathbf{u}_m, \theta_m)$ such that

$$\mathbf{u}_m \rightharpoonup \mathbf{u} \text{ in } \mathbf{H}^1_0(\Omega), \qquad \theta_m \rightharpoonup \theta \text{ in } H^1_0(\Omega).$$

Thanks to the continuity of the Nemytskii operators ρ, β and α on their second argument, the variational equalities, for all $\phi \in H^1_0(\Omega)$ and $q \in L^2(\Omega)$,

$$k\int_\Omega \nabla\theta_m \cdot \nabla\phi dx = \int_\Omega (\rho(\xi_m)g - \beta(\xi_m)\mathbf{w}_m \cdot \nabla\xi_m)\phi dx,$$

$$\int_\Omega \nabla\cdot\mathbf{u}_m q dx = \int_\Omega \left(\alpha(\xi_m)\mathbf{w}_m\cdot\nabla\xi_m - \frac{1}{\text{meas}(\Omega)}\int_\Omega \alpha(\xi_m)\mathbf{w}_m\cdot\nabla\xi_m dx\right) q dx,$$

easily pass to the limit as m tends to infinity concluding (26) and (38), respectively.

Since the solutions $\tau_m = \tau(\mathbf{u}_m, \xi_m)$ and $\bar{p}_m = \bar{p}(\tau_m, \xi_m)$, given as in items 3 and 4, satisfy (47) and (45), respectively, then we can extract a subsequence, still denoted by $(\tau_m, \bar{p}_m)$ such that

$$\tau_m \rightharpoonup \tau \text{ in } \mathbf{L}^2(\Omega), \qquad \bar{p}_m \rightharpoonup \bar{p} \text{ in } L^2(\Omega).$$

To prove that $(\mathbf{u}, \theta) \in \mathcal{L}((\mathbf{w}, \xi))$, it still remains to verify that (25) holds for the limit function $\bar{p}$ (see Remark 3.1). Indeed we pass to the limit as m tends to infinity in $(25)_m$, that is,

$$\int_\Omega \Big(\mu(\xi)|D\mathbf{u}|^2 + \lambda(\xi)(\nabla\cdot\mathbf{u})^2 + \gamma(\xi)\mathcal{J}(\xi, D\mathbf{u})\Big)dx - \int_\Omega \bar{p}\nabla\cdot\mathbf{u}dx \leq$$
$$\leq \liminf \int_\Omega \Big(\mu(\xi_m)|D\mathbf{u}_m|^2 + \lambda(\xi_m)(\nabla\cdot\mathbf{u}_m)^2 + \gamma(\xi_m)\mathcal{J}(\xi_m, D\mathbf{u}_m)\Big)dx -$$
$$- \limsup \int_\Omega \bar{p}_m\nabla\cdot\mathbf{u}_m dx \leq$$
$$\leq \int_\Omega \mu(\xi)D\mathbf{u} : D\mathbf{v}dx + \int_\Omega \lambda(\xi)\nabla\cdot\mathbf{u}\nabla\cdot\mathbf{v}dx + \int_\Omega \gamma(\xi)\mathcal{J}(\xi, D\mathbf{v})dx -$$
$$- \int_\Omega p\nabla\cdot\mathbf{v}dx - \int_\Omega \rho(\xi)\mathbf{f}\cdot(\mathbf{v}-\mathbf{u})dx, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega).$$

Thus we conclude (25), with p replaced by $\bar{p}$.

In conclusion, Theorem 5.1 guarantees the existence of a fixed point $(\mathbf{u}, \theta) \in \mathcal{L}((\mathbf{u}, \theta))$, that is $(\mathbf{u}, \bar{p}, \theta)$ is in the conditions of Theorem 3.4.

6 The Uzawa-type algorithm and its convergence

Proof of Theorem 3.1.

Let $(\mathbf{u}, \bar{p}, \theta) \in \mathbf{H}_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega)$ be a solution in accordance to Theorem 3.4. Let $p^0 \in L^2(\Omega)$ be an arbitrary function. When p^i is known ($i \in \mathbb{N}$), we define $(\mathbf{u}^{i+1}, p^{i+1}, \theta^{i+1})$ the unique solution to the boundary problem (25)-(27) with $\mathbf{w} = \mathbf{u}$ and $\xi = \theta$, that is,

$$\int_\Omega \Big(\mu(\theta) D\mathbf{u}^{i+1} : D(\mathbf{v} - \mathbf{u}^{i+1}) + \lambda(\theta)\nabla\cdot\mathbf{u}^{i+1}\nabla\cdot(\mathbf{v} - \mathbf{u}^{i+1})\Big)dx +$$
$$+ \int_\Omega \gamma(\theta)\Big(\mathcal{J}(\theta, D\mathbf{v}) - \mathcal{J}(\theta, D\mathbf{u}^{i+1})\Big)dx \geq$$
$$\geq \int_\Omega p^{i+1}\nabla\cdot(\mathbf{v} - \mathbf{u}^{i+1})dx + \int_\Omega \rho(\theta)\mathbf{f}\cdot(\mathbf{v} - \mathbf{u}^{i+1})dx, \quad \forall \mathbf{v} \in \mathbf{H}_0^1(\Omega); \tag{49}$$
$$k\int_\Omega \nabla\theta^{i+1}\cdot\nabla\phi dx = \int_\Omega (\rho(\theta)g - \beta(\theta)\mathbf{u}\cdot\nabla\theta)\phi dx, \quad \forall\phi \in H_0^1(\Omega); \tag{50}$$
$$\int_\Omega (p^{i+1} - p^i)q dx = \int_\Omega (\alpha(\theta)\mathbf{u}\cdot\nabla\theta - \nabla\cdot\mathbf{u}^{i+1})q dx, \quad \forall q \in L^2(\Omega). \tag{51}$$

Notice that $\theta^{i+1} \equiv \theta$ by the uniqueness of the solution.

Taking $\mathbf{v} = \mathbf{u} \in \mathbf{H}_0^1(\Omega)$ and $\mathbf{v} = \mathbf{u}^{i+1} \in \mathbf{H}_0^1(\Omega)$ as test functions in (49) and in (20) with p replaced by $\bar{p}$, respectively, summing the obtained inequalities and arguing as for (36) we get

$$\nu\|\mathbf{u}^{i+1} - \mathbf{u}\|^2_{\mathbf{H}_0^1(\Omega)} \leq \int_\Omega \nabla\cdot(\mathbf{u}^{i+1} - \mathbf{u})(p^{i+1} - \bar{p})dx. \tag{52}$$

Now taking $q = p^{i+1} - \bar{p}$ as a test function in (51) and in (28) it follows

$$\int_\Omega (p^{i+1} - \bar{p} - (p^i - \bar{p}))(p^{i+1} - \bar{p})dx = -\int_\Omega \nabla\cdot(\mathbf{u}^{i+1} - \mathbf{u})(p^{i+1} - \bar{p})dx.$$

Using the relation

$$2(a-b)a = a^2 + (a-b)^2 - b^2, \quad \forall a, b \in \mathbb{R},$$

we have

$$\|p^{i+1} - \bar{p}\|^2_{2,\Omega} + \|p^{i+1} - p^i\|^2_{2,\Omega} = \|p^i - \bar{p}\|^2_{2,\Omega} - $$
$$-2\int_\Omega \nabla\cdot(\mathbf{u}^{i+1} - \mathbf{u})(p^{i+1} - \bar{p})dx$$

Introducing the above equality in (52) and summing over $i = 0, 1, \cdots, N-1$ we find

$$2\nu \sum_{i=0}^{N-1} \|\mathbf{u}^{i+1} - \mathbf{u}\|^2_{\mathbf{H}^1_0(\Omega)} + \|p^N - p\|^2_{2,\Omega} + \sum_{i=0}^{N-1} \|p^{i+1} - p^i\|^2_{2,\Omega} \le \|p^0 - p\|^2_{2,\Omega}.$$

Then the convergence of $\mathbf{u}^N$ to $\mathbf{u}$ in $\mathbf{H}^1_0(\Omega)$ is thereby proved as N tends to infinity. Moreover p^N is a Cauchy sequence in the Banach space $L^2(\Omega)$, which implies the existence of $p \in L^2(\Omega)$ such that

$$p^N \to p \text{ in } L^2(\Omega).$$

Therefore the variational formulation (49) gives in the limit (20).

Next passing to the limit as $N \to \infty$ the equality

$$\int_\Omega (p^N - p^{N-1}) q dx = \int_\Omega (\alpha(\theta)\mathbf{u} \cdot \nabla\theta - \nabla \cdot \mathbf{u}^N) q dx, \quad \forall q \in L^2(\Omega),$$

we obtain (22), which concludes Theorem 3.1.

7 The Newtonian case

Proof of Theorem 3.2.

Arguing as in the proof of Theorem 3.4, there exist $(\mathbf{u}, \theta)$ verifying (28) and (21), a subgradient τ and $\bar{p}$ satisfying

$$\nabla\bar{p} = \nabla \cdot \tau + \rho(\theta)\mathbf{f}, \qquad \text{in } \mathcal{D}'(\Omega). \tag{53}$$

Indeed we can apply the fixed point argument, observing that the passage to the limit as m tends to infinity into

$$\int_\Omega \Big(\mu(\xi_m) D\mathbf{u}_m : D\mathbf{v} + \lambda(\xi_m)\nabla \cdot \mathbf{u}_m \nabla \cdot \mathbf{v} + \gamma(\xi_m)\frac{\partial \mathcal{J}}{\partial \zeta}(\xi_m, D\mathbf{u}_m) : D\mathbf{v}\Big) dx =$$
$$= \int_\Omega \bar{p}_m \nabla \cdot \mathbf{v} dx + \int_\Omega \rho(\xi_m)\mathbf{f} \cdot \mathbf{v} dx, \quad \forall \mathbf{v} \in \mathbf{H}^1_0(\Omega),$$

holds, since $\frac{\partial \mathcal{J}}{\partial \zeta}$ is a monotone operator (cf. [12] for the monotone technique).

As a particular case of a differentiable J, Theorem 3.3 guarantees the existence of $(\mathbf{u}^{i+1}, p^{i+1}, \theta^{i+1})$ verifying

$$\int_\Omega \Big(\mu(\theta) D\mathbf{u}^{i+1} : D\mathbf{v} + \lambda(\theta)\nabla\cdot\mathbf{u}^{i+1}\nabla\cdot\mathbf{v} + \gamma(\theta)\frac{\partial\mathcal{J}}{\partial\zeta}(\theta, D\mathbf{u}^{i+1}) : D\mathbf{v}\Big) dx$$

$$= \int_\Omega p^{i+1}\nabla\cdot\mathbf{v} dx + \int_\Omega \rho(\theta)\mathbf{f}\cdot\mathbf{v} dx, \quad \forall \mathbf{v}\in\mathbf{H}_0^1(\Omega); \qquad (54)$$

and (50)-(51).

Next the argument used in the proof of Theorem 3.1 follows, choosing $\mathbf{v} = \mathbf{u}^{i+1} - \mathbf{u}$ as a test function in (54) and (24), after taking the difference of the obtained relations and considering the monotone property

$$\Big(\frac{\partial\mathcal{J}}{\partial\zeta}(\theta, D\mathbf{u}^{i+1}) - \frac{\partial\mathcal{J}}{\partial\zeta}(\theta, D\mathbf{u})\Big) : D(\mathbf{u}^{i+1} - \mathbf{u}) \geq 0.$$

The differentiability of J implies that the set of the subgradients is singular (for instance, see [6, pp. 23]). The existence of a unique τ and of a unique $\bar{p}\in L^2(\Omega)/\mathbb{R}$ satisfying (53) and of a unique $p\in L^2(\Omega)/\mathbb{R}$ satisfying (24), yields then $p = \bar{p}$ up to a constant.

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