The obstacle problem for nonlinear elliptic equations with variable growth and L^1-data

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Abstract

The aim of this paper is twofold: to prove, for L^1-data, the existence and uniqueness of an entropy solution to the obstacle problem for nonlinear elliptic equations with variable growth, and to show some convergence and stability properties of the corresponding coincidence set. The latter follow from extending the Lewy–Stampacchia inequalities to the general framework of L^1.

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1 Introduction

Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be a bounded domain. The purpose of this paper is the study of the obstacle problem associated with nonlinear elliptic equations

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with data $f \in L^1(\Omega)$ and principal part modeled on the $p(\cdot)-$Laplacian with variable exponent

$$\Delta_{p(x)} u := \text{div } |\nabla u|^{p(x)-2} \nabla u.$$

These obstacle problems fall into the framework of the model problem

$$\begin{cases} -\Delta_{p(\cdot)} u + \beta(\cdot, u) & = & f & \text{in} & \Omega, \\ u & = & 0 & \text{on} & \partial\Omega, \end{cases} \tag{1.1}$$

for a certain function β, related to a maximal monotone graph. For instance, in the case of the zero obstacle problem, when $u \geq 0$ a.e. in Ω, it can be shown that β is a.e. given by the nonlinear discontinuity

$$\beta(x, u) = \begin{cases} 0 & \text{if} & u(x) > 0, \\ -f^-(x) & \text{if} & u(x) = 0, \end{cases} \tag{1.2}$$

where f^- is the negative part of the decomposition $f = f^+ - f^-$. Problems of the type (1.1) have been solved by Brézis and Strauss [12] for linear elliptic operators ($p(\cdot) \equiv 2$) and general maximal monotone graphs β. An L^1-theory for the case of $p-$Laplacian type operators (with p constant) has been proposed in [8] by Bénilan *et als.*, *via* the introduction of the notion of entropy solution. This framework has been extended to unilateral problems with constant p in [10], [9], and [27]. The interesting cases are those of $1 < p \leq N$, since for $p > N$ the variational methods of Leray–Lions (see, for instance, [25]) easily apply, the solution being bounded and with gradient in $L^p(\Omega)$. Recently, the obstacle problem with more general data, namely with f only a measure, has been considered by several authors (see, *e.g.*, [15, 23, 24, 11]). In particular, Brézis and Ponce show in [11], still in the case $p = 2$ and for a constant obstacle, that $f^- \in L^1(\Omega) + H^{-1}(\Omega)$ is a necessary and sufficient condition for the existence (and the uniqueness) of a solution to (1.1).

On the other hand, for the case of a variable exponent, the existence and uniqueness of an entropy solution to (1.1), with $\beta \equiv 0$ and $f \in L^1$, has been recently obtained by two of the authors in [31]. The result builds upon [8] and [4], assumes the exponent to be log-Hölder continuous, and relies on *a priori* estimates in Marcinkiewicz spaces with variable exponent. A primary aim of this paper is to extend this theory to obstacle problems ($u \geq \psi$ in Ω), for admissible general obstacles $\psi = \psi(x)$ and nonlinear operators with variable growth.

The natural framework to solve problem (1.1) is that of Sobolev spaces with variable exponent. Recent applications in elasticity [32], non–Newtonian fluid mechanics [33, 30, 5], or image processing [13], gave rise to a revival of the interest in these spaces, the origins of which can be traced back to the work of Orlicz in the 1930's. An account of recent advances, some open problems, and an extensive list of references can be found in the interesting surveys by Diening *et als.* [17] and Antontsev *et al.* [6] (*cf.* also the work of Kováčik and Rákosník [22], where many of the basic properties of these spaces are established). A brief introduction to the subject, which is pertinent to the present paper can be found in [31]; we will refer the reader to this paper, when appropriate, to avoid an unnecessary duplication of arguments.

For quasilinear operators in divergence form of $p(\cdot)-$Laplacian type

$$\mathcal{A}u := -\text{div } a(x, \nabla u),$$

the classical obstacle problem can be formulated, using the duality between $W_0^{1,p(\cdot)}(\Omega)$ and $W^{-1,p'(\cdot)}(\Omega)$, in terms of the variational inequality

$$u \in \mathcal{K}_\psi \ : \ \int_\Omega a(x, \nabla u) \cdot \nabla(v - u)\, dx \geq \langle f, v - u \rangle \ , \quad \forall v \in \mathcal{K}_\psi, \tag{1.3}$$

whenever $f \in W^{-1,p'(\cdot)}(\Omega)$ and the convex subset

$$\mathcal{K}_\psi = \left\{ v \in W_0^{1,p(\cdot)}(\Omega) \ : \ v \geq \psi \ \text{ a.e. in } \Omega \right\} \tag{1.4}$$

is nonempty. The former holds in the case $f \in L^1(\Omega)$ and $p(\cdot) > N$ (since then, by Sobolev's embedding, $W_0^{1,p(\cdot)}(\Omega) \subset L^\infty(\Omega)$) or if $f \in L^{r(\cdot)}(\Omega)$, with $N/p(\cdot) < r(\cdot)$, for $1 < p(\cdot) < N$. The theory of monotone operators then applies to (1.3) (see [25, 21]), with

$$\langle f, v - u \rangle = \int_\Omega f(v - u)\, dx.$$

As in the case of a constant p, for $f \in L^1(\Omega)$ and $1 < p(\cdot) < N$, both sides of inequality (1.3) may have no meaning, so we are led, following [9] (*cf.* also [10] and [27]), to extend the formulation of the unilateral problem by replacing $v - u$ by its truncation $T_t(u - v)$, for every level $t > 0$, where T_t is defined by

$$T_t(s) := \max \{-t, \min\{t, s\}\} , \quad s \in \mathbb{R}.$$

The resulting notion of entropy solution for the obstacle problem is made precise in the following definition.

Definition 1.1. *An entropy solution of the obstacle problem for* $\{f, \psi\}$ *is a measurable function* u *such that* $u \geq \psi$ *a.e. in* Ω*, and, for every* $t > 0$, $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$ *and*

$$\int_\Omega a(x, \nabla u) \cdot \nabla T_t(\varphi - u)\ dx \geq \int_\Omega f\, T_t(\varphi - u)\ dx, \qquad (1.5)_{f,\psi}$$

for all $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$.

This entropic formulation is adequate since we are able to show the existence and uniqueness of a solution. In general, entropic solutions do not belong to $\mathcal{K}_\psi$, since they do not have an integrable distributional gradient; if $1 < p(\cdot) < 2 - 1/N$, they may not even be L^1-functions. However, they belong to $W_0^{1,1}(\Omega)$ if $p(\cdot) > 2 - 1/N$.

The framework is also adequate in order to obtain the continuous dependence of the solution with respect to variations of the obstacle in $W^{1,p(\cdot)}(\Omega)$ and of the nonhomogeneous term in $L^1(\Omega)$, extending the results of [14] concerning the constant exponent case.

For constant p, and certain assumptions on f and $\mathcal{A}\psi$, implying that $\mathcal{A}u \in L^1(\Omega)$, it has been observed in [29] that the variational solution to (1.3) actually satisfies, a.e. in Ω, an equation with a nonlinear discontinuity and, in particular, that

$$\mathcal{A}u = f\ , \quad \text{a.e. in } \{u > \psi\}, \qquad (1.6)$$

where $\{u > \psi\} = \Omega \setminus \{u = \psi\}$ is the complement of the coincidence set $\{u = \psi\} := \{x \in \Omega\ :\ u(x) = \psi(x)\}$. In fact, in the free boundary domain $\{u > \psi\}$, equation (1.6) can be obtained as a consequence of the well-known Lewy–Stampacchia inequalities

$$f \leq \mathcal{A}u \leq f + (\mathcal{A}\psi - f)^+\ , \quad \text{a.e. in } \Omega. \qquad (1.7)$$

A second main result we obtain in this paper is the extension of these assertions to the general framework of entropy solutions of equations involving variable exponents. In particular, for the obstacle problem with an admissible obstacle ψ such that $(\mathcal{A}\psi - f)^+ \in L^1(\Omega)$, we show, still in the L^1-framework, that in (1.1),

$$\beta(\cdot, u) = -(\mathcal{A}\psi - f)^+ \chi_{\{u=\psi\}}\ , \quad \text{a.e. in } \Omega,$$

where χ_S denotes the characteristic function of the set S. In the special case $\psi \equiv 0$, we obtain (1.2).

An important consequence of inequalities (1.7) is the reduction of the regularity issue for the solutions of the obstacle problem to that of the solutions of the corresponding equations. In particular, we conclude that the boundedness of f and $(\mathcal{A}\psi - f)^+$ are sufficient to guarantee the local Hölder continuity of the solution and its gradient for the $p(\cdot)-$obstacle problem, in accordance with the case of equations (see [3] and [19]) or that of functionals with non–standard growth conditions ([1]).

We also extend, for a fixed admissible obstacle ψ, the L^1-contraction property of Brézis and Strauss [12] for the map $f \longmapsto \beta_f$. The property was obtained by one of the authors for quasilinear obstacle problems (see [28, 29]), with the aim of estimating the stability of two coincidence sets $\{u_1 = \psi\}$ and $\{u_2 = \psi\}$ with respect to the L^1-norm of the difference $f_1 - f_2$ of the corresponding variational data. The extension of these results to entropy solutions, in the context of data merely in L^1, places the stability theory of the coincidence sets (with respect to the variation of non-degenerate data) in its natural and more general framework.

The paper is organized as follows. In section 2, we introduce the assumptions and state the main results. In section 3, we prove *a priori* estimates for an entropy solution of the obstacle problem. Section 4 deals with the existence and uniqueness of an entropy solution and its continuous dependence with respect to the data. In section 5, we extend Lewy–Stampacchia inequalities to the context of entropy solutions and analyze their consequences, namely the characterization of the obstacle problem in L^1 in terms of an equation with a nonlinear discontinuity, and the stability of the coincidence sets.

2 Main results

Let $a : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be a Carathéodory function (*i.e.*, $a(\cdot, \xi)$ is measurable on Ω, for every $\xi \in \mathbb{R}^N$, and $a(x, \cdot)$ is continuous on $\mathbb{R}^N$, a.e. $x \in \Omega$), such that the following assumptions hold:

$$a(x,\xi) \cdot \xi \geq \alpha |\xi|^{p(x)}, \tag{2.1}$$

a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where α is a positive constant;

$$|a(x,\xi)| \leq \gamma \left(j(x) + |\xi|^{p(x)-1} \right), \tag{2.2}$$

a.e. $x \in \Omega$, for every $\xi \in \mathbb{R}^N$, where j is a nonnegative function in $L^{p'(\cdot)}(\Omega)$ and $\gamma > 0$;

$$(a(x,\xi) - a(x,\xi')) \cdot (\xi - \xi') > 0, \tag{2.3}$$

a.e. $x \in \Omega$, for every $\xi, \xi' \in \mathbb{R}^N$, with $\xi \neq \xi'$.

These are standard assumptions when dealing with monotone operators in divergence form, the novelty being the fact that the exponent $p(\cdot)$, appearing in (2.1) and (2.2), does not need to be constant but may depend on the variable x. Throughout the paper, the following notation for a measurable function $q(\cdot) : \Omega \to \mathbb{R}$ will be used:

$$\underline{q} := \operatorname*{ess\,inf}_{x\in\Omega} \ q(x) \qquad \text{and} \qquad \overline{q} := \operatorname*{ess\,sup}_{x\in\Omega} \ q(x).$$

The exponent is assumed here to be a measurable function $p(\cdot) : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \exists C > 0 \ : \ |p(x) - p(y)| \leq \frac{C}{-\ln|x-y|}, \quad \text{for } |x-y| < \frac{1}{2}; \\ \\ 1 < \underline{p} \leq \overline{p} < N. \end{cases} \tag{2.4}$$

The first condition says that $p(\cdot)$ is a log-Hölder continuous function. On the other hand, the second assumption in (2.4) is quite natural if one wants to define an appropriate functional setting. Assumption (2.4) puts us in the framework of reflexive Sobolev spaces with variable exponent and allows us to exploit their properties, like the crucial Poincaré and Sobolev inequalities. These generalized Sobolev-Orlicz spaces consist of measurable functions $v : \Omega \to \mathbb{R}$, such that v and its distributional gradient ∇v are in $L^{p(\cdot)}(\Omega)$, the space of functions with finite modular

$$\varrho_{p(\cdot)}(v) = \int_\Omega |v(x)|^{p(x)}\, dx,$$

normed by

$$\|v\|_{p(\cdot)} = \inf\left\{\lambda > 0 \ : \ \varrho_{p(\cdot)}(v/\lambda) \leq 1\right\}.$$

Under assumption (2.4), the variable exponent Lebesgue spaces have properties similar to those of the classical Lebesgue spaces, being reflexive and separable Banach spaces, and satisfying the continuous embedding $L^{p(\cdot)}(\Omega) \hookrightarrow L^{q(\cdot)}(\Omega)$, for Ω bounded and $p(x) \geq q(x)$. These spaces are not invariant to translations (see [22]) although a Hölder type inequality holds. For Sobolev spaces with variable exponent, we can define $W^{-1,p'(\cdot)}(\Omega)$ as the dual space of $W_0^{1,p(\cdot)}(\Omega)$, where Poincaré's inequality is also valid. Besides, the Sobolev embedding

$$W_0^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p^*(\cdot)}(\Omega) \ , \qquad p^*(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)}$$

still holds (see [18, 16, 20]). Let us finally introduce the following notation: given two bounded measurable functions $p(\cdot), q(\cdot) : \Omega \to \mathbb{R}$, we write

$$q(\cdot) \ll p(\cdot) \qquad \text{if} \qquad \operatorname*{ess\,inf}_{x\in\Omega} \, (p(x) - q(x)) > 0.$$

Concerning the right-hand side of $(1.5)_{f,\psi}$ and the obstacle ψ we make the following assumptions:

$$f \in L^1(\Omega), \quad \psi \in W^{1,p(\cdot)}(\Omega), \quad \text{and} \quad \psi^+ \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega). \tag{2.5}$$

In particular, the last assumption guarantees that $\mathcal{K}_\psi \cap L^\infty(\Omega) \neq \emptyset$.

Our first result concerns the existence and uniqueness of an entropy solution, in the sense of Definition 1.1, to the obstacle problem; we also obtain regularity results for the solution and its weak gradient. We recall from [31] that it is still possible, as in the case of a constant p (cf. [8]), to define the weak gradient of a measurable function u such that $T_t(u) \in W_0^{1,p(\cdot)}(\Omega)$, for all $t > 0$. In fact, there exists a unique measurable vector field $\mathbf{v} : \Omega \to \mathbb{R}^N$ such that

$$\mathbf{v}\chi_{\{|u|<t\}} = \nabla T_t(u), \quad \text{a.e. in } \Omega, \quad \text{for all } t > 0.$$

Moreover, if $u \in W_0^{1,1}(\Omega)$ then $\mathbf{v}$ coincides with ∇u, the standard distributional gradient of u.

Assuming (2.4), the critical Sobolev exponent and the conjugate of $p(\cdot)$ are, respectively,

$$p^*(\cdot) = \frac{Np(\cdot)}{N - p(\cdot)} \quad \text{and} \quad p'(\cdot) = \frac{p(\cdot)}{p(\cdot) - 1}.$$

The following is one of our main results.

Theorem 2.1. *Assume* (2.1)–(2.5). *There exists a unique entropy solution u to the obstacle problem $(1.5)_{f,\psi}$. Moreover, $|u|^{q(\cdot)} \in L^1(\Omega)$, for all $0 \ll q(\cdot) \ll q_0(\cdot)$, and $|\nabla u|^{q(\cdot)} \in L^1(\Omega)$, for all $0 \ll q(\cdot) \ll q_1(\cdot)$, where*

$$q_0(\cdot) := \frac{p^*(\cdot)}{\overline{p'}} \quad \textit{and} \quad q_1(\cdot) := \frac{q_0(\cdot)}{q_0(\cdot) + 1} p(\cdot). \tag{2.6}$$

In particular, if $2 - 1/N \ll p(\cdot)$ *then*

$$u \in W_0^{1,q(\cdot)}(\Omega), \quad \textit{for all} \;\; 1 \leq q(\cdot) \ll q_1(\cdot).$$

Remark 2.2. Among other results, Boccardo and Cirmi prove in [9] an analogous of Theorem 2.1, for constant $p(\cdot) \equiv p > 2 - 1/N$, and under the assumption that $\psi \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$. Under our assumptions, since $\psi^+ \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, ψ is bounded above but not necessarily bounded below.

Remark 2.3. Similar results of existence of entropy solutions for L^1-data could be obtained for more general elliptic operators with variable growth in the form

$$\mathcal{A}u = -\mathrm{div}\, a(x, u, \nabla u) + H(x, u, \nabla u),$$

where H has the natural growth with respect to the gradient; these would follow as an extension of the recent results obtained in [2] for constant p.

We now consider a sequence $\{f_n, \psi_n\}_n$ and the corresponding obstacle problems $(1.5)_{f_n,\psi_n}$. The next result states that, under adequate assumptions, the limit of entropy solutions u_n of $(1.5)_{f_n,\psi_n}$ is the solution of the limit obstacle problem $(1.5)_{f,\psi}$.

Theorem 2.4. *Let $\{f_n, \psi_n\}_n$ be a sequence in $L^1(\Omega) \times W^{1,p(\cdot)}(\Omega)$. Assume (2.1)–(2.5) and that ${\psi_n}^+ \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, for all n. Let u_n be the entropy solution of the obstacle problem $(1.5)_{f_n,\psi_n}$. If*

$$f_n \longrightarrow f \quad \text{in } L^1(\Omega) \qquad \text{and} \qquad \psi_n \longrightarrow \psi \quad \text{in } W^{1,p(\cdot)}(\Omega), \tag{2.7}$$

then

$$u_n \longrightarrow u \quad \text{in measure},$$

where u is the unique entropy solution of the obstacle problem $(1.5)_{f,\psi}$. If $2 - 1/N \ll p(\cdot)$ then

$$u_n \rightharpoonup u \quad \text{in} \quad W_0^{1,q(\cdot)}(\Omega), \quad \text{for all } 1 \leq q(\cdot) \ll q_1(\cdot).$$

We also establish the so-called Lewy–Stampacchia inequalities and deduce from them a few interesting properties. These results require the extra assumption $p(\cdot) - 1 \ll q_1(\cdot)$, which is necessary in order to pass to the limit in a sequence of approximated obstacle problems (see Proposition 5.1 and Remark 5.2).

Theorem 2.5. *Assume (2.1)–(2.5), $\mathcal{A}\psi \in L^1(\Omega)$, and $p(\cdot) - 1 \ll q_1(\cdot)$. Let u be the entropy solution of the obstacle problem $(1.5)_{f,\psi}$. Then $\mathcal{A}u \in L^1(\Omega)$ and the following Lewy–Stampacchia inequalities hold*

$$f \leq \mathcal{A}u \leq f + (\mathcal{A}\psi - f)^+, \quad \text{a.e. in } \Omega. \tag{2.8}$$

The most immediate consequences of the Lewy–Stampacchia inequalities concern the regularity of solutions. If f, $\mathcal{A}\psi \in L^{m(\cdot)}(\Omega)$, with $m(\cdot) = (p^*(\cdot))'$, then the entropy solution u of the obstacle problem $(1.5)_{f,\psi}$ is the variational solution $u \in W_0^{1,p(\cdot)}(\Omega)$ of (1.3), for which the following regularity assertions hold.

Proposition 2.6. *Assume* (2.1)–(2.5) *and* $p(\cdot) - 1 \ll q_1(\cdot)$. *If* f, $\mathcal{A}\psi \in L^\infty(\Omega)$ *then the solution* u *of* (1.3) *is such that* $u \in L^\infty(\Omega) \cap C^{0,\alpha}(\Omega)$. *If, in addition,* $\partial\Omega \in C^{0,1}$ *then* $u \in C^{0,\alpha}(\overline{\Omega})$.

Moreover, in the case that $\mathcal{A} \equiv \Delta_{p(\cdot)}$, *we further have* $u \in C^{1,\alpha'}_{\mathrm{loc}}(\Omega)$.

The first part is a straightforward consequence of [19, Theorems 4.2–4.4], where the Hölder continuity of weak solutions of quasilinear elliptic equations with variable growth is obtained; the second part follows from [1, Theorem 2.2], that concerns the Hölder continuity of the gradient of local minimizers of functionals with non-standard growth.

Using the Lewy–Stampacchia inequalities and showing that $\mathcal{A}u = f$, a.e. in $\{u > \psi\}$, we prove that the entropy solution of $(1.5)_{f,\psi}$ satisfies an equation involving the coincidence set $\{u = \psi\}$.

Theorem 2.7. *Assume* (2.1)–(2.5), $\mathcal{A}\psi \in L^1(\Omega)$, *and* $p(\cdot) - 1 \ll q_1(\cdot)$. *The entropy solution* u *of the obstacle problem* $(1.5)_{f,\psi}$ *satisfies the equation*

$$\mathcal{A}u - (\mathcal{A}\psi - f)\chi_{\{u=\psi\}} = f \ , \quad a.e. \ in \ \Omega. \tag{2.9}$$

We note that (2.8) and (2.9) imply, in particular,

$$(\mathcal{A}\psi - f)\chi_{\{u=\psi\}} = (\mathcal{A}\psi - f)^+\chi_{\{u=\psi\}} \ , \quad \text{a.e. in } \Omega.$$

The next result establishes the convergence of the coincidence set of a sequence of entropy solutions to the limit coincidence set.

Theorem 2.8. *Under the assumptions of Theorem* 2.4, *assume that* $p(\cdot) - 1 \ll q_1(\cdot)$,

$$\mathcal{A}\psi_n \longrightarrow \mathcal{A}\psi \quad in \ L^1(\Omega) \qquad and \qquad \mathcal{A}\psi \neq f, \quad a.e. \ in \ \Omega.$$

Then

$$\chi_{\{u_n=\psi_n\}} \longrightarrow \chi_{\{u=\psi\}} \quad in \ L^q(\Omega), \tag{2.10}$$

for all $1 \leq q < +\infty$.

Finally, we obtain an L^1-contraction property for the obstacle problem and an estimate for the stability of two coincidence sets I_1 and I_2 in terms of their symmetric difference

$$I_1 \div I_2 := (I_1 \setminus I_2) \cup (I_2 \setminus I_1).$$

The results were known for more regular solutions (*cf.* [28, 29]) but meet their natural and more general formulation in the context of entropy solutions for data precisely in $L^1(\Omega)$.

Theorem 2.9. *Assume* (2.1)–(2.4) *and* $p(\cdot) - 1 \ll q_1(\cdot)$*. Let* $f_1, f_2 \in L^1(\Omega)$ *and let* ψ *satisfy* (2.5) *and* $\mathcal{A}\psi \in L^1(\Omega)$*. Let* u_1 *and* u_2 *be the entropy solutions of the obstacle problems* $(1.5)_{f_1,\psi}$ *and* $(1.5)_{f_2,\psi}$*, respectively. If* $\xi_i := f_i - \mathcal{A}u_i$*,* $i = 1, 2$*, then*

$$\|\xi_1 - \xi_2\|_1 \leq \|f_1 - f_2\|_1. \tag{2.11}$$

If, in addition, the non-degeneracy condition

$$f_i - \mathcal{A}\psi \leq -\lambda < 0, \quad \textit{a.e. in } D, \ i = 1, 2, \tag{2.12}$$

holds in a measurable subset $D \subset \Omega$*, then, for* $I_i := \{u_i = \psi\}$*,*

$$\operatorname{meas}\left((I_1 \div I_2) \cap D\right) \leq \frac{1}{\lambda}\|f_1 - f_2\|_1. \tag{2.13}$$

3 A priori estimates

The main purpose of this section is to obtain *a priori* estimates in Marcinkiewicz spaces with variable exponent for an entropy solution of the obstacle problem $(1.5)_{f,\psi}$. In face of the embedding results of [31], we then derive *a priori* estimates in Lebesgue spaces with variable exponent. We recall the definition of Macinkiewicz spaces with variable exponent introduced in [31].

Definition 3.1. *Let* $q(\cdot)$ *be a measurable function such that* $\underline{q} > 0$*. We say that a measurable function* u *belongs to the Marcinkiewicz space* $M^{q(\cdot)}(\Omega)$ *if there exists a positive constant* M *such that*

$$\int_{\{|u|>t\}} t^{q(x)}\, dx \leq M, \quad \textit{for all } \ t > 0.$$

The following result is instrumental in obtaining *a priori* estimates for the obstacle problem.

Lemma 3.2. *Assume* (2.1)–(2.5) *and let* $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$. *If* u *is an entropy solution of the variational inequality* $(1.5)_{f,\psi}$ *then*

$$\int_{\{|u|\le t\}} |\nabla u|^{p(x)} dx \le C\biggl((t + \|\varphi\|_\infty) \|f\|_1 + \int_\Omega \left(|\nabla\varphi|^{p(x)} + j(x)^{p'(x)}\right) dx\biggr),$$

for all $t > 0$, *where* C *is a constant depending only on* α, γ *and* $p(\cdot)$.

Proof. Take $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$ in the variational inequality $(1.5)_{f,\psi}$ to obtain

$$\int_{\{|u-\varphi|\le t\}} a(x,\nabla u)\cdot\nabla(u-\varphi)\ dx \le \int_\Omega f\, T_t(u-\varphi)\ dx \le \|f\|_1 t, \qquad (3.1)$$

for all $t > 0$. On the other hand, using assumptions (2.1)–(2.2) and Young's inequality, we have, for all $t > 0$,

$$\begin{aligned}
&\int_{\{|u-\varphi|\le t\}} a(x,\nabla u)\cdot\nabla(u-\varphi)\ dx\\
&\qquad \ge \alpha \int_{\{|u-\varphi|\le t\}} |\nabla u|^{p(x)}\ dx - \gamma \int_{\{|u-\varphi|\le t\}} (j(x) + |\nabla u|^{p(x)-1})|\nabla\varphi|\ dx\\
&\qquad \ge \frac{\alpha}{2} \int_{\{|u-\varphi|\le t\}} |\nabla u|^{p(x)}\ dx - C\int_\Omega \left(|\nabla\varphi|^{p(x)} + j(x)^{p'(x)}\right)\ dx, \qquad (3.2)
\end{aligned}$$

where C, here and in the rest of the proof, is a constant depending only on α, γ, and $p(\cdot)$. Now, from (3.1) and (3.2), we obtain

$$\int_{\{|u-\varphi|\le t\}} |\nabla u|^{p(x)}\ dx \le \frac{2\|f\|_1 t}{\alpha} + C\int_\Omega \left(|\nabla\varphi|^{p(x)} + |j(x)|^{p'(x)}\right)\ dx,$$

for all $t > 0$. Replacing t with $t + \|\varphi\|_\infty$ in the last inequality, we get

$$\begin{aligned}
\int_{\{|u|\le t\}} |\nabla u|^{p(x)}\ dx \ &\le\ \int_{\{|u-\varphi|\le t+\|\varphi\|_\infty\}} |\nabla u|^{p(x)}\ dx\\
&\le\ C\left((t + \|\varphi\|_\infty)\|f\|_1 + \int_\Omega \left(|\nabla\varphi|^{p(x)} + j(x)^{p'(x)}\right)\ dx\right),
\end{aligned}$$

for all $t > 0$. □

In the next result we prove *a priori* estimates for an entropy solution of $(1.5)_{f,\psi}$ in Marcinkiewicz spaces with variable exponent. The proof is based on Lemma 3.2 and Sobolev inequality.

Proposition 3.3. *Assume* (2.1)–(2.5) *and let* $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$*. If* u *is an entropy solution of the variational inequality* $(1.5)_{f,\psi}$ *then the following assertions hold:*

(i) *There exists a positive constant* M*, depending only on* α*,* γ*,* N*,* $p(\cdot)$*, and* Ω*, such that*

$$\int_{\{|u|>t\}} t^{p^*(x)/\overline{p'}} dx$$
$$\leq M\left((1+\|\varphi\|_\infty)\|f\|_1 + \int_\Omega \left(|j(x)|^{p'(x)} + |\nabla\varphi|^{p(x)}\right) dx + 1\right)^{\overline{p^*}/\underline{p'}},$$

for all $t > 0$*.*

(ii) *If there exists a positive constant* M *such that*

$$\int_{\{|u|>t\}} t^{q(x)}\ dx \leq M, \quad \textit{for all } t > 0, \tag{3.3}$$

then $|\nabla u|^{r(\cdot)} \in M^{q(\cdot)}(\Omega)$*, where* $r(\cdot) := p(\cdot)/(q(\cdot)+1)$*. Moreover, there exists a constant* C*, depending only on* α*,* γ*, and* $p(\cdot)$*, such that*

$$\int_{\{|\nabla u|^{r(\cdot)}>t\}} t^{q(x)}\ dx$$
$$\leq C\left((1+\|\varphi\|_\infty)\|f\|_1 + \int_\Omega \left(|\nabla\varphi|^{p(x)} + |j(x)|^{p'(x)}\right) dx\right) + M + |\Omega|,$$

for all $t > 0$*.*

Proof. (i) We proceed as in the proof of Proposition 3.2 in [31], sketching here only the main steps (we refer to [31] for a complete account of the details). From Lemma 3.2, we have

$$\frac{1}{t}\int_\Omega |\nabla T_t(u)|^{p(x)}\ dx \leq M_1 + \frac{M_2}{t}, \tag{3.4}$$

for all $t > 0$, where $M_1 := C_1\|f\|_1$ and

$$M_2 := C_1\left(\|f\|_1\|\varphi\|_\infty + \int_\Omega \left(|\nabla\varphi|^{p(x)} + j(x)^{p'(x)}\right)\ dx\right),$$

for a constant C_1, depending only on α, γ and $p(\cdot)$. On the other hand, using Lemma 2.3 in [31] and Sobolev's inequality, we estimate

$$\int_{\{|u|>t\}} t^{p^*(x)/\overline{p'}}\ dx \leq C_2 \left(\int_\Omega t^{p(x)/\overline{p'}+1-p(x)} \frac{1}{t} |\nabla T_t(u)|^{p(x)}\ dx + 1 \right)^{\overline{p^*}/\underline{p}}, \quad (3.5)$$

where C_2 is a constant depending only on N, $p(\cdot)$, and Ω.

Noting that $t^{p(x)/\overline{p'}+1-p(x)} \leq 1$ for all $t \geq 1$, and using (3.4), we obtain

$$\int_{\{|u|>t\}} t^{p^*(x)/\overline{p'}}\ dx \leq C_2 \left(M_1 + M_2 + 1\right)^{\overline{p^*}/\underline{p}},$$

for all $t \geq 1$. For $0 < t < 1$, we have

$$\int_{\{|u|>t\}} t^{p^*(x)/\overline{p'}}\ dx \leq |\Omega|.$$

Combining both estimates and using the definition of M_1 and M_2, we prove the result after simple estimates.

(ii) Using (3.3), the definition of $r(\cdot)$, and (3.4), we have

$$\begin{aligned}
\int_{\{|\nabla u|^{r(x)}>t\}} t^{q(x)}\ dx \ &\leq \int_{\{|\nabla u|^{r(x)}>t\}\cap\{|u|\leq t\}} t^{q(x)}\ dx + \int_{\{|u|>t\}} t^{q(x)}\ dx \\
&\leq \int_{\{|u|\leq t\}} t^{q(x)} \left(\frac{|\nabla u|^{r(x)}}{t} \right)^{p(x)/r(x)} dx + M \\
&= \frac{1}{t} \int_{\{|u|\leq t\}} |\nabla T_t(u)|^{p(x)}\ dx + M \\
&\leq C \left((1 + \|\varphi\|_\infty)\|f\|_1 + \int_\Omega \left(|\nabla\varphi|^{p(x)} + |j(x)|^{p'(x)} \right) dx \right) + M,
\end{aligned}$$

for all $t \geq 1$, where C is a constant depending only on α and $p(\cdot)$. Noting that

$$\int_{\{|\nabla u|^{r(x)}>t\}} t^{q(x)}\ dx \leq |\Omega|, \qquad \text{for all } t \leq 1,$$

we conclude the proof. $\square$

Using Proposition 3.3 and Proposition 2.5 in [31] one obtains the following result (see the proofs of Corollaries 3.5 and 3.7 in [31]).

Corollary 3.4. *Assume* (2.1)–(2.5). *Let*

$$q_0(\cdot) = \frac{p^*(\cdot)}{\overline{p'}} \quad and \quad q_1(\cdot) = \frac{q_0(\cdot)}{q_0(\cdot)+1} p(\cdot). \tag{3.6}$$

If u is an entropy solution of the variational inequality $(1.5)_{f,\psi}$, then there exists a constant C, which is independent of u, such that

$$\int_\Omega |u|^{q(x)} \, dx \le C, \quad for\ all\ 0 \ll q(\cdot) \ll q_0(\cdot), \tag{3.7}$$

and

$$\int_\Omega |\nabla u|^{q(x)} \, dx \le C, \quad for\ all\ 0 \ll q(\cdot) \ll q_1(\cdot). \tag{3.8}$$

In particular, $|u|^{q(\cdot)} \in L^1(\Omega)$, for all $q(\cdot)$ such that $0 \ll q(\cdot) \ll q_0(\cdot)$, and $|\nabla u|^{q(\cdot)} \in L^1(\Omega)$, for all $q(\cdot)$ such that $0 \ll q(\cdot) \ll q_1(\cdot)$.

4 Existence and uniqueness of entropy solutions

In this section we prove the existence and uniqueness of an entropy solution to the obstacle problem $(1.5)_{f,\psi}$. We also prove the continuous dependence of the solution with respect to the right–hand side f and the obstacle ψ.

We start by proving that a sequence $\{u_n\}_n$ of entropy solutions of the obstacle problems $(1.5)_{f_n,\psi_n}$ converges in measure to a measurable function u. We also show that the sequence of weak gradients $\{\nabla u_n\}_n$ converges in measure to ∇u, the weak gradient of u. Finally, we prove some regularity properties using Proposition 3.3 and Corollary 3.4.

Proposition 4.1. *Let $\{f_n, \psi_n\}_n$ be a sequence in $L^1(\Omega) \times W^{1,p(\cdot)}(\Omega)$. Assume* (2.1)–(2.5) *and that ${\psi_n}^+ \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$, for all n. Let u_n be an entropy solution of the obstacle problem $(1.5)_{f_n,\psi_n}$. If*

$$f_n \longrightarrow f \quad in\ L^1(\Omega) \qquad and \qquad \psi_n \longrightarrow \psi \quad in\ W^{1,p(\cdot)}(\Omega), \tag{4.1}$$

then the following assertions hold:

(i) *There exists a measurable function u such that $u_n \to u$ in measure.*

(ii) *∇u_n converges in measure to ∇u, the weak gradient of u.*

Proof. Let $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$, *e.g.* $\varphi = \psi^+$, and note that $\varphi_n := \varphi + (\psi_n - \varphi)^+ \in L^\infty(\Omega)$ since $\varphi \in L^\infty(\Omega)$ and ψ_n is bounded above (see Remark 2.2). In particular, $\varphi_n \in \mathcal{K}_{\psi_n} \cap L^\infty(\Omega)$. Moreover, by (4.1), there exists a constant C, independent of n, such that

$$\|f_n\|_1 \le C(\|f\|_1 + 1), \qquad \|\varphi_n\|_\infty \le C\left(\|\varphi\|_\infty + 1\right), \tag{4.2}$$

and

$$\int_\Omega |\nabla \varphi_n|^{p(x)}\, dx \le C\left(\int_\Omega |\nabla \varphi|^{p(x)}\, dx + 1\right), \quad \text{for all } n. \tag{4.3}$$

(i) Let s, t, and ε be positive numbers. Noting that

$$\begin{aligned} \text{meas}\,\{|u_n - u_m| > s\} \;&\le\; \text{meas}\,\{|u_n| > t\} + \text{meas}\,\{|u_m| > t\} \\ &+\; \text{meas}\,\{|T_t(u_n) - T_t(u_m)| > s\}, \end{aligned} \tag{4.4}$$

from Proposition 3.3(i) and (4.2)–(4.3), we can choose $t = t(\varepsilon)$ such that $\text{meas}\,\{|u_n| > t\} < \varepsilon/3$ and $\text{meas}\,\{|u_m| > t\} < \varepsilon/3$. On the other hand, from Lemma 3.2 applied to u_n and (4.2)–(4.3), we obtain

$$\begin{aligned} \int_\Omega |\nabla T_t(u_n)|^{p(x)} dx \;&\le C\Big((t + \|\varphi\|_\infty + 1)(\|f\|_1 + 1) \\ &+ \int_\Omega \left(|\nabla \varphi|^{p(x)} + j(x)^{p'(x)}\right) dx + 1\Big), \end{aligned}$$

for all $t > 0$, where C is a constant depending only on α, γ and $p(\cdot)$. Therefore, we can assume, by Sobolev embedding, that $\{T_t(u_n)\}_n$ is a Cauchy sequence in $L^{q(\cdot)}(\Omega)$, for all $1 \le q(\cdot) \ll p^*(\cdot)$. Consequently, there exists a measurable function u such that

$$T_t(u_n) \longrightarrow T_t(u), \quad \text{in } L^{q(\cdot)}(\Omega) \text{ and a.e. in } \Omega.$$

Thus,

$$\text{meas}\,\{|T_t(u_n) - T_t(u_m)| > s\} \le \int_\Omega \left(\frac{|T_t(u_n) - T_t(u_m)|}{s}\right)^{q(x)} dx < \frac{\epsilon}{3}$$

for all $n, m \ge n_0(s, \epsilon)$. Finally, from (4.4), we obtain

$$\text{meas}\,\{|u_n - u_m| > s\} < \epsilon, \quad \text{for all } n, m \ge n_0(s, \epsilon),$$

i.e., $\{u_n\}_n$ is a Cauchy sequence in measure. The assertion follows.

The proof of (ii) is entirely similar to the corresponding one in Proposition 5.3 of [31]. We omit the details. $\square$

At this point, we prove Theorem 2.4 using Proposition 4.1.

Proof of Theorem 2.4. Let $\varphi \in \mathcal{K}_\psi \cap L^\infty(\Omega)$ and define $\varphi_n := \varphi + (\psi_n - \varphi)^+$. Note that $\varphi_n \in \mathcal{K}_{\psi_n} \cap L^\infty(\Omega)$ and that φ_n converges strongly to φ in $W_0^{1,p(\cdot)}(\Omega)$, due to (2.7). Taking φ_n as a test function in $(1.5)_{f_n,\psi_n}$, we obtain

$$\int_\Omega a(x, \nabla u_n) \cdot \nabla T_t(u_n - \varphi_n)\, dx \le \int_\Omega f_n(x) T_t(u_n - \varphi_n)\, dx.$$

Next, we pass to the limit in the previous inequality.

Concerning the right-hand side, the convergence is obvious since f_n converges to f, strongly in $L^1(\Omega)$, and $T_t(u_n - \varphi_n)$ converges to $T_t(u - \varphi)$, weakly$-*$ in L^∞ and a.e. in Ω. To deal with the left-hand side we write it as

$$\int_{\{|u_n - \varphi_n| \le t\}} a(x, \nabla u_n) \cdot \nabla u_n\, dx - \int_{\{|u_n - \varphi_n| \le t\}} a(x, \nabla u_n) \cdot \nabla \varphi_n\, dx \tag{4.5}$$

and note that $\{|u_n - \varphi_n| \le t\}$ is a subset of $\{|u_n| \le t + C(\|\varphi\|_\infty + 1)\}$, where C is a constant that does not depend on n (see (4.2)). Hence, taking $s = t + C(\|\varphi\|_\infty + 1)$, we rewrite the second integral in (4.5) as

$$\int_{\{|u_n - \varphi_n| \le t\}} a(x, \nabla T_s(u_n)) \cdot \nabla \varphi_n\, dx.$$

Since $a(x, \nabla T_s(u_n))$ is uniformly bounded in $(L^{p'(\cdot)}(\Omega))^N$ (by assumption (2.2) and Lemma 3.2), it converges weakly to $a(x, \nabla T_s(u))$ in $(L^{p'(\cdot)}(\Omega))^N$, due to Proposition 4.1$(ii)$. Therefore the last integral converges to

$$\int_{\{|u - \varphi| \le t\}} a(x, \nabla u)) \cdot \nabla \varphi\, dx.$$

The first integral in (4.5) is nonnegative, by (2.1), and it converges a.e. by Proposition 4.1. It follows from Fatou's lemma that

$$\int_{\{|u - \varphi| \le t\}} a(x, \nabla u) \cdot \nabla u\, dx \le \liminf_{n \to +\infty} \int_{\{|u_n - \varphi_n| \le t\}} a(x, \nabla u_n) \cdot \nabla u_n\, dx.$$

Gathering results, we obtain

$$\int_\Omega a(x, \nabla u) \cdot \nabla T_t(u - \varphi)\, dx \le \int_\Omega f\, T_t(u - \varphi)\, dx,$$

i.e., u is an entropy solution of $(1.5)_{f,\psi}$. □

Finally, we prove Theorem 2.1, as an application of Theorem 2.4.

Proof of Theorem 2.1. Let us consider the sequence of approximated obstacle problems $(1.5)_{f_n,\psi}$, where $\{f_n\}_n$ is a sequence of bounded functions strongly converging to f in $L^1(\Omega)$. It is straightforward, from classical results (see [25, 21]), to prove the existence of a unique solution $u_n \in W_0^{1,p(\cdot)}(\Omega)$ of the obstacle problem $(1.5)_{f_n,\psi}$. Noting that a weak energy solution is also an entropy solution, we may apply Theorem 2.4 to obtain that u_n converges to a measurable function u which is an entropy solution of the limit obstacle problem $(1.5)_{f,\psi}$. Now, the regularity stated in the theorem follows immediately from Corollary 3.4.

Finally, we prove the uniqueness. Let u and v be entropy solutions of $(1.5)_{f,\psi}$. Since $\psi^+ \in W_0^{1,p(\cdot)}(\Omega) \cap L^\infty(\Omega)$ and $\psi \le \|\psi^+\|_\infty$, $T_h u$ and $T_h v$ belong to the convex set $\mathcal{K}_\psi$ for $h > 0$ large enough. Now, we proceed as in the proof of Theorem 4.1 in [31]. We write the variational inequality $(1.5)_{f,\psi}$ corresponding to the solution u, with $T_h v$ as test function, and to the solution v, with $T_h u$ as test function. Upon addition, we get

$$\int_{\{|u-T_h v|\le t\}} a(x,\nabla u)\cdot\nabla(u-T_h v)\ dx + \int_{\{|v-T_h u|\le t\}} a(x,\nabla v)\cdot\nabla(v-T_h u)\ dx$$
$$\le \int_\Omega f\ \Big(T_t(u-T_h v)+T_t(v-T_h u)\Big)\ dx.$$

We let h go to infinity in this inequality. By Proposition 3.3(i), it is easy to prove that the right-hand side tends to zero. Moreover, using assumptions (2.1)–(2.2), Hölder's inequality, and Proposition 3.3(ii) to study the left-hand side, we obtain

$$\int_{\{|u-v|\le t\}} (a(x,\nabla u)-a(x,\nabla v))\cdot\nabla(u-v)\ dx \le 0, \quad \text{for all } t>0.$$

By assumption (2.3), we conclude that $\nabla u = \nabla v$, a.e. in Ω, and hence, from Poincaré's inequality, it follows that $u = v$, a.e. in Ω. ☐

5 Lewy–Stampacchia inequalities and stability of the coincidence set

The aim of this section is to prove the Lewy–Stampacchia inequalities and the resulting properties stated in Section 2.

In order to prove Theorem 2.5, we consider a sequence of approximated obstacle problems for which the abstract theory developed in [26, 7] applies. Once we have the Lewy–Stampacchia inequalities for the approximated problems, we may pass to the limit using the following proposition.

Proposition 5.1. *Assume $p(\cdot) - 1 \ll q_1(\cdot)$. Under the assumptions of Proposition* 4.1 *the following assertions hold:*

(i) $a(x, \nabla u_n)$ *converges to* $a(x, \nabla u)$*, strongly in* $L^1(\Omega)$*.*

(ii) $a(x, \nabla u) \in L^{q(\cdot)}(\Omega)$*, for some* $1 \leq q(\cdot)$*.*

(iii) u *and* ∇u *satisfy* (3.7) *and* (3.8)*.*

Proof. We omit the proof, since it is analogous to the proof of Proposition 5.5 in [31]. □

Remark 5.2. As pointed out in [31], assumption $p(\cdot) - 1 \ll q_1(\cdot)$, which is obviously satisfied for p constant, is equivalent to the condition

$$\frac{Np'(\cdot)}{N - p(\cdot)} \gg \overline{p'}. \tag{5.1}$$

The analysis of the behaviour of the function on the left-hand side of this inequality leads to the following conclusions:

(i) if $\overline{p} < \sqrt{N}$ then (5.1) is satisfied for any function $p(\cdot)$ such that

$$\frac{1}{\underline{p}} - \frac{1}{\overline{p}} < \frac{\overline{p} - 1}{N};$$

(ii) if $\underline{p} \leq \sqrt{N} \leq \overline{p}$ then (5.1) is satisfied for any function $p(\cdot)$ such that

$$\underline{p} > \frac{N}{2\sqrt{N} - 1};$$

(iii) if $\underline{p} > \sqrt{N}$ then (5.1) is satisfied for any function $p(\cdot)$.

The condition in case (i) only holds if $\underline{p}$ is *close* to $\overline{p}$, so it forces a modest variation in the field of values of $p(\cdot)$.

Now, we are able to prove Theorem 2.5.

Proof of Theorem 2.5. Consider a sequence $\{f_n\}_n$ of $L^\infty(\Omega)$ functions such that $f_n \to f$ in $L^1(\Omega)$. Let $u_n \in W_0^{1,p(\cdot)}(\Omega)$ be the unique weak energy solution of the obstacle problem

$$u_n \in \mathcal{K}_\psi \ : \ \langle \mathcal{A}u_n - f_n, v - u_n \rangle \geq 0, \quad \forall v \in \mathcal{K}_\psi.$$

Since $V := W_0^{1,p(\cdot)}(\Omega)$ is a reflexive Banach space and $\mathcal{A} : V \to V'$ is strictly T-monotone, it follows from the abstract theory developed in [26] that

$$f_n \leq \mathcal{A}u_n \leq f_n + (\mathcal{A}\psi - f_n)^+ \quad \text{in } V'.$$

In particular, these inequalities hold in the sense of distributions.

Let $0 \leq \varphi \in \mathcal{D}(\Omega)$; then

$$\int_\Omega f_n \varphi \, dx \leq \int_\Omega a(x, \nabla u_n) \cdot \nabla \varphi \, dx \leq \int_\Omega \left[f_n + (\mathcal{A}\psi - f_n)^+ \right] \varphi \, dx.$$

We can pass to the limit in this expression using the facts that $f_n \to f$ in $L^1(\Omega)$ and $a(x, \nabla u_n) \to a(x, \nabla u)$ in $L^1(\Omega)$ (see Proposition 5.1(i)), and obtain

$$f \leq \mathcal{A}u \leq f + (\mathcal{A}\psi - f)^+ \quad \text{in } \mathcal{D}'(\Omega).$$

Finally, since f and $f + (\mathcal{A}\psi - f)^+$ are $L^1(\Omega)$ functions, we conclude that also $\mathcal{A}u \in L^1(\Omega)$ and (2.8) follows. □

In order to prove Theorem 2.7 we need two preliminary lemmas.

Lemma 5.3. *Let w_i be measurable functions such that $T_t(w_i) \in W_0^{1,p(\cdot)}(\Omega)$, for all $t > 0$, $a(x, \nabla w_i) \in \left[L^1(\Omega)\right]^N$, and $\mathcal{A}w_i \in L^1(\Omega)$, for $i = 1, 2$. Then*

$$\mathcal{A}w_1 = \mathcal{A}w_2 \quad \textit{a.e. in } \{w_1 = w_2\}. \tag{5.2}$$

Proof. Let

$$\boldsymbol{L}^1_\nabla(\Omega) = \left\{ \boldsymbol{\xi} \in \left[L^1(\Omega)\right]^N \ : \ \operatorname{div} \boldsymbol{\xi} \in L^1(\Omega) \right\}.$$

Since $\left[C^1(\overline{\Omega})\right]^N$ is dense in $\boldsymbol{L}^1_\nabla(\Omega)$ for the graph norm, it follows from the arguments in *Lemmata* A3 and A4 of [21, pages 52–53] that the following property holds in $\boldsymbol{L}^1_\nabla(\Omega)$:

$$\operatorname{div} \boldsymbol{\xi} = 0 \quad \text{a.e. in } \{\boldsymbol{\xi} = \boldsymbol{0}\}.$$

Due to the assumptions, $a(x, \nabla w_1) - a(x, \nabla w_2) \in \boldsymbol{L}^1_\nabla(\Omega)$, so we have

$$\mathcal{A}w_1 = \mathcal{A}w_2 \quad \text{a.e. in } \{a(x, \nabla w_1) = a(x, \nabla w_2)\}. \tag{5.3}$$

Finally, it is standard that

$$\nabla T_t(w_1) = \nabla T_t(w_2) \quad \text{a.e. in} \quad \{w_1 = w_2\}\,,$$

for any $t > 0$, so the weak gradients ∇w_1 and ∇w_2 coincide in $\{w_1 = w_2\}$ and the conclusion follows from (5.3). $\square$

The other lemma requires a definition of the coincidence set for the obstacle problem, which poses a difficulty in face of the available regularity for the solution and the obstacle. Indeed, if u and ψ are continuous functions, the coincidence set is defined as the closed subset of Ω

$$\{x \in \Omega \ : \ u(x) = \psi(x)\} = (u - \psi)^{-1}\left(\{0\}\right) \ ,$$

and this definition is unambiguous. But, in general, the entropy solution is not necessarily continuous, and we are not making that assumption for the obstacle either. So we need to interpret the coincidence set in a different and more elaborate sense.

We first define the *non–coincidence set* $\{u > \psi\}$. Since ψ is bounded above (*cf.* Remark 2.2), we can take $s > \sup_\Omega \psi$. The function $T_s(u)$ belongs to $W_0^{1,p(\cdot)}(\Omega)$, by the definition of entropy solution. Then

$$\{u > \psi\} := \left\{x \in \Omega \ : \ \left(T_s(u) - \psi\right)(x) > 0 \text{ in the sense of } W^{1,p(\cdot)}(\Omega)\right\} \ .$$

Given $w \in W^{1,p(\cdot)}(\Omega)$, we say that $w(x) > 0$ in the sense of $W^{1,p(\cdot)}(\Omega)$ if there exists a neighborhood of x, $N_x \subset \Omega$, and a nonnegative function $\zeta \in W^{1,\infty}(N_x)$, such that $\zeta(x) > 0$ and $w \geq \zeta$ a.e. in N_x. The definition is clearly independent of the choice of s and it turns out that $\{u > \psi\}$ is necessarily an open subset of Ω. We then define the *coincidence set* as

$$\{u = \psi\} := \Omega \setminus \{u > \psi\} \ .$$

Lemma 5.4. *Assume* (2.1)–(2.5) *and* $p(\cdot) - 1 \ll q_1(\cdot)$*. The entropy solution of the obstacle problem* $(1.5)_{f,\psi}$ *solves*

$$\mathcal{A}u = f \ , \quad \textit{a.e. in } \{u > \psi\} \ . \tag{5.4}$$

Proof. To simplify, let us denote $\Lambda = \{u > \psi\}$, which is an open subset of Ω. Let $\varphi \in \mathcal{D}(\Lambda)$. Let $h > \sup_\Omega \psi$ and choose $\varepsilon > 0$ small enough such that

$$v = T_h(u) \pm \varepsilon\varphi \ \in \ \mathcal{K}_\psi \cap L^\infty(\Omega).$$

Taking v as a test function in $(1.5)_{f,\psi}$, we obtain

$$\int_\Omega a(x,\nabla u)\cdot\nabla T_t\left(T_h(u)\pm\varepsilon\varphi-u\right)\,dx \geq \int_\Omega f\,T_t\left(T_h(u)\pm\varepsilon\varphi-u\right)\,dx.$$

From (2.1), it follows that

$$\pm\varepsilon\int_{\{\,|T_h(u)\pm\varepsilon\varphi-u|\leq t\}} a(x,\nabla u)\cdot\nabla\varphi\,dx \geq \int_\Omega f\,T_t\left(T_h(u)\pm\varepsilon\varphi-u\right)\,dx.$$

Choosing $t>\varepsilon\|\varphi\|_\infty$ and letting $h\to\infty$ (using Proposition 5.1 (i)), we obtain

$$\pm\varepsilon\int_\Lambda a(x,\nabla u)\cdot\nabla\varphi\,dx \geq \pm\varepsilon\int_\Lambda f\varphi\,dx,$$

and, hence, we conclude that

$$\mathcal{A}u = -\operatorname{div}\,a(x,\nabla u) = f \qquad \text{in} \quad \mathcal{D}'(\Lambda)$$

and the result follows. □

We prove Theorem 2.7 as a consequence of *Lemmata* 5.3 and 5.4.

Proof of Theorem 2.7. By the previous lemma, we have $\mathcal{A}u=f$, a.e. in $\{u>\psi\}$. The result follows from the fact that $\mathcal{A}u=\mathcal{A}\psi$, a.e. in $\{u=\psi\}$, which is a consequence of Lemma 5.3, since $\mathcal{A}u\in L^1(\Omega)$ by Theorem 2.5. □

Using Theorems 2.4 and 2.7 we prove the convergence of a sequence of coincidence sets to the coincidence set of the limit.

Proof of Theorem 2.8. Let u_n and u be the entropy solutions of the obstacle problems $(1.5)_{f_n,\psi_n}$ and $(1.5)_{f,\psi}$, respectively. By Theorem 2.4, u_n converges to u in measure, and hence, a.e. in Ω. Moreover, by Theorem 2.7, and denoting $\chi_n=\chi_{\{u_n=\psi_n\}}$, u_n satisfies

$$\mathcal{A}u_n-(\mathcal{A}\psi_n-f_n)\chi_n=f_n, \quad \text{a.e. in } \Omega, \text{ for all } n. \tag{5.5}$$

Since $0\leq\chi_n\leq 1$, there exists a subsequence (still denoted by χ_n) and a function $\chi\in L^\infty(\Omega)$, such that

$$\chi_n \rightharpoonup \chi \quad \text{weakly}-* \ \text{ in } L^\infty(\Omega).$$

Hence, since $\mathcal{A}\psi_n\to\mathcal{A}\psi$ and $f_n\to f$, strongly in $L^1(\Omega)$, taking the limit in (5.5) we obtain

$$\mathcal{A}u-(\mathcal{A}\psi-f)\chi=f, \quad \text{a.e. in } \Omega.$$

On the other hand, by Theorem 2.7, u also satisfies the previous identity with χ replaced by $\chi_{\{u=\psi\}}$. Therefore, using $\mathcal{A}\psi \neq f$, a.e. in Ω, the whole sequence χ_n converges to the characteristic function $\chi_{\{u=\psi\}}$ and satisfies (2.10). The theorem is proved. $\square$

Finally, we prove Theorem 2.9 using again Proposition 4.1 and the Lewy–Stampacchia inequalities.

Proof of Theorem 2.9. First, we claim that

$$\int_\Omega (\mathcal{A}u_1 - \mathcal{A}u_2)\, \varphi \, dx \geq 0, \quad \forall\, \varphi \in \sigma(u_1(x) - u_2(x)). \tag{5.6}$$

Here σ denotes the maximal monotone graph associated to the sign function (*i.e.*, $\sigma = \partial r$, $r(t) = |t|$).

Indeed, let $\{f_i^n\}_n$ be a sequence of bounded functions strongly converging in $L^1(\Omega)$ to f_i $(i = 1, 2)$, and let $u_i^n \in W_0^{1,p(\cdot)}(\Omega)$ be the corresponding weak energy solutions of $(1.5)_{f_n^i,\psi}$. Let $(\sigma_\varepsilon)_{\varepsilon>0}$ be a sequence of smooth functions satisfying $\sigma_\varepsilon(0) = 0$, $|\sigma_\varepsilon(t)| \leq 1$ and $\sigma'_\varepsilon(t) \geq 0$, for all $t \in \mathbb{R}$, such that $\sigma_\varepsilon(t) \to \operatorname{sign}(t)$ as $\varepsilon \downarrow 0$. Integration by parts and the use of assumption (2.3) yield the inequality

$$\int_\Omega (\mathcal{A}u_1^n - \mathcal{A}u_2^n)\, \sigma_\varepsilon(u_1^n - u_2^n)\, dx$$

$$= \int_\Omega (a(x, \nabla u_1^n) - a(x, \nabla u_2^n)) \cdot \nabla(u_1^n - u_2^n)\, \sigma'_\varepsilon(u_1^n - u_2^n)\, dx \geq 0. \tag{5.7}$$

We now pass to the limit as $n \to \infty$. To start with, we have (for a subsequence, relabeled if need be)

$$\mathcal{A}u_1^n - \mathcal{A}u_2^n \rightharpoonup \mathcal{A}u_1 - \mathcal{A}u_2, \quad \text{weakly in } L^1(\Omega).$$

This follows from Dunford-Pettis Theorem (the hypothesis of which are satisfied due to the Lewy–Stampacchia inequalities), and the fact that the convergence holds in the sense of distributions since, by Proposition 5.1(i),

$$a(x, \nabla u_1^n) - a(x, \nabla u_2^n) \longrightarrow a(x, \nabla u_1) - a(x, \nabla u_2), \quad \text{in } L^1(\Omega).$$

On the other hand, by Proposition 4.1(i),

$$\sigma_\varepsilon(u_1^n - u_2^n) \longrightarrow \sigma_\varepsilon(u_1 - u_2), \quad \text{a.e in } \Omega.$$

Fix an arbitrary $\delta > 0$. Again from the Lewy–Stampacchia inequalities, we can find $\nu > 0$ such that, for all $A \subset \Omega$,

$$\operatorname{meas}(A) < \nu \Longrightarrow \int_A |\mathcal{A}u_1^n - \mathcal{A}u_2^n|\, dx < \frac{\delta}{4}\,, \quad \text{for all } n. \tag{5.8}$$

By Egorov's Theorem, there exists a measurable subset $\omega \subset \Omega$ such that

$$\operatorname{meas}(\Omega \setminus \omega) < \nu \tag{5.9}$$

and

$$\sigma_\varepsilon(u_1^n - u_2^n) \longrightarrow \sigma_\varepsilon(u_1 - u_2), \quad \text{uniformly in } \omega. \tag{5.10}$$

To lighten the notation, we put $F^n := \mathcal{A}u_1^n - \mathcal{A}u_2^n$ and $G_\varepsilon^n := \sigma_\varepsilon(u_1^n - u_2^n) - \sigma_\varepsilon(u_1 - u_2)$. Then,

$$\begin{aligned}
\left| \int_\Omega F^n(x)\, G_\varepsilon^n(x)\; dx \right| &\leq \left| \int_{\Omega\setminus\omega} F^n(x)\, G_\varepsilon^n(x)\; dx \right| + \left| \int_\omega F^n(x)\, G_\varepsilon^n(x)\; dx \right| \\
&\leq 2 \int_{\Omega\setminus\omega} |F^n(x)|\; dx + \int_\omega |F^n(x)|\; |G_\varepsilon^n(x)|\; dx \\
&\leq 2\, \frac{\delta}{4} + \kappa\, \frac{\delta}{2\kappa} \\
&= \delta\,, \qquad (5.11)
\end{aligned}$$

for all $n \geq n_0$, using (5.9) and (5.8) to bound the first term and (5.10) to bound the second. Here $\kappa > 0$ is a constant (which exists due to the Lewy–Stampacchia inequalities) such that

$$\int_\omega |F^n(x)|\; dx \leq \int_\Omega |\mathcal{A}u_1^n - \mathcal{A}u_2^n|\; dx \leq \kappa, \quad \forall\, n.$$

Since $\delta > 0$ is arbitrary, we conclude from (5.11) that

$$\int_\Omega (\mathcal{A}u_1^n - \mathcal{A}u_2^n)\; [\sigma_\varepsilon(u_1^n - u_2^n) - \sigma_\varepsilon(u_1 - u_2)]\; dx \longrightarrow 0$$

so we can pass to the limit in (5.7) to obtain

$$\int_\Omega (\mathcal{A}u_1 - \mathcal{A}u_2)\, \sigma_\varepsilon(u_1 - u_2)\; dx \geq 0.$$

Finally, letting $\varepsilon \downarrow 0$, we obtain (5.6) with $\varphi = \operatorname{sign}(u_1 - u_2)$. Since, by Lemma 5.3,

$$(\mathcal{A}u_1 - \mathcal{A}u_2)\, \varphi = (\mathcal{A}u_1 - \mathcal{A}u_2) \operatorname{sign}(u_1 - u_2), \quad \text{a.e. } x \in \Omega,$$

for all $\varphi \in \sigma(u_1 - u_2)$, the claim follows.

To conclude the proof, take $\varphi \in \sigma(u_1 - u_2)$, defined by

$$\varphi := \begin{cases} -1 & \text{in} \quad \{u_1 < u_2\} \cup \{\xi_1 < \xi_2\} \\ 0 & \text{on} \quad \{u_1 = u_2\} \cap \{\xi_1 = \xi_2\} \\ 1 & \text{in} \quad \{u_1 > u_2\} \cup \{\xi_1 > \xi_2\}. \end{cases}$$

Multiplying

$$\xi_1 - \xi_2 = (f_1 - f_2) - (\mathcal{A}u_1 - \mathcal{A}u_2)$$

by φ, integrating in Ω, and using (5.6), we obtain

$$\int_\Omega |\xi_1 - \xi_2|\, dx = \int_\Omega (\xi_1 - \xi_2)\, \varphi\, dx \le \int_\Omega (f_1 - f_2)\, \varphi\, dx \le \int_\Omega |f_1 - f_2|\, dx,$$

proving (2.11). Finally, by Theorem 2.7, we have $\xi_i = (f_i - \mathcal{A}\psi)\chi_{\{u_i=\psi\}}$, for $i = 1, 2$. Therefore

$$|\chi_{\{u_1=\psi\}} - \chi_{\{u_2=\psi\}}| \le \frac{1}{\lambda}|\xi_1 - \xi_2|, \quad \text{a.e. in } D,$$

due to assumption (2.12). The theorem follows by integrating over D. $\square$

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