VANISHING SOLUTIONS OF ANISOTROPIC PARABOLIC EQUATIONS WITH VARIABLE NONLINEARITY

S. ANTONTSEV AND S. SHMAREV

Abstract. We study the property of finite time vanishing of solutions of the homogeneous Dirichlet problem for the anisotropic parabolic equations

$$u_t - \sum_{i=1}^{n} D_i \left(a_i(x,t,u)|D_i u|^{p_i(x,t)-2} D_i u \right) + c(x,t)|u|^{\sigma(x,t)-2}u = f(x,t)$$

with variable exponents of nonlinearity $p_i(x,t)$, $\sigma(x,t) \in (1,\infty)$. We show that the solutions of this problem may vanish in a finite time even if the equation combines the directions of slow and fast diffusion and estimate the extinction moment in terms of the data. If the solution does not identically vanish in a finite time, we estimate the rate of vanishing of the solution as $t \to \infty$. We establish conditions on the nonlinearity exponents which guarantee vanishing of the solution at a finite instant even if the equation eventually transforms into the linear one.

1. Introduction

We study the behavior of solutions to the Dirichlet problem for the anisotropic parabolic equations with variable nonlinearity

$$\left\{ \begin{array}{l} u_t - \displaystyle\sum_{i=1}^{n} D_i \left(a_i |D_i u|^{p_i(z)-2} D_i u \right) + c|u|^{\sigma(z)-2}u = f(z) \quad \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \qquad u(x,0) = u_0(x) \text{ in } \Omega, \end{array} \right. \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a domain with Lipschitz-continuous boundary Γ, $\Gamma_T = \Gamma \times (0,T)$, $Q_T = \Omega \times (0,T)$, $z = (x,t)$.

Equations of the type (1.1) appear in various applications such as the mathematical modelling of heat and mass transfer in nonhomegeneous media, in description of the filtration processes, in the processes of recovery of digital images (see [4, 8, 9] and the references therein for an account of such models in the stationary case). For the sake of presentation, we will regard problem (1.1) as the mathematical model of a diffusion process.

The questions we address in this paper are already studied for the evolutional p-Laplacian equation

$$u_t = \Delta_p u \equiv \operatorname{div} \left(|\nabla u|^{p-2} \nabla u \right), \quad p \in (1,\infty). \tag{1.2}$$

Date: December 01, 2008.

The first author was partially supported by the research project POCI/MAT/61576/2004, FCT/MCES (Portugal).

The second author acknowledges the support of the research grant MTM2007-65088 (Spain).

If $p \in (1,2)$, this equation describes processes of fast diffusion, the case $p > 2$ corresponds to slow diffusion. It is well-known that the solutions of the nonlinear equation (1.2) with $p \neq 2$ possess some properties not displayed by the solutions of the linear equation. The solutions of the linear equation obey the strong maximum principle which prevents them from attaining the maximum and minimum values in the interior of the problem domain. Unlike the linear case, for $p \neq 2$ the solutions of the Dirichlet problem for equation (1.2) are localized either in space, or in time. More precisely, the following alternative holds: if u is a solution of the Dirichlet problem for equation (1.2) with $p \neq 2$, then either

$$2 > p > 1 \text{ (fast diffusion)} \;\Rightarrow\; \exists\, T_1 :\; u \equiv 0 \text{ for all } t \geq T_1,$$

or

$$\left.\begin{array}{l} p > 2 \text{ (slow diffusion)} \\ u_0 \equiv 0 \text{ in } B_r(x_0) \subset \Omega \end{array}\right\} \;\Rightarrow\; \exists\, t_*(x_0) :\; u(x_0,t) \equiv 0 \text{ for } t \in [0, t_*(x_0)],$$

where $B_s(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < s\}$. These properties complement each other: the former is called *extinction in a finite time*, the latter is usually referred to as *finite speed of propagation* of disturbances from the data. If $p > 2$ and if the support of the initial function u_0 is compact in Ω, then the support of the solution is expanding with time and eventually covers the whole of Ω.

It is proved in [10, 11] that the same behavior is intrinsic for the solutions of equation (1.1) (anisotropic and variable diffusion) in the cases when the diffusion is either fast, or slow in every space direction: $p_i(z) \geq p^- > 2$ or $1 < p^- \leq p_i(z) \leq p^+ < 2$ for every $i = 1, \ldots, n$ and all $z \in Q_T$. At the same time, it is known that solutions of anisotropic equations may display specific localization properties caused by the anisotropy and impossible in the isotropic cases: it is shown in [8, 9, 17] that in the stationary diffusion processes the anisotropy of the diffusion operator acts like the strong absorption and may cause localization of the solution. This fact makes feasible the hypothesis that the solutions to problem (1.1) possess some specific properties of time localization.

In the present paper we study the influence of the anisotropy and the variable nonlinearity on the possibility of vanishing in a finite time of solutions to problem (1.1). We show that the solutions may vanish in a finite time even in the case when the diffusion operator in (1.1) combines the directions of locally slow diffusion ($p_i(z) > 2$) with the directions of locally fast diffusion ($p_i(z) \in (1,2)$) or linear diffusion ($p_i(z) = 2$). Our results apply to *energy solutions* of problem (1.1) (the rigorous definition is given in Section 2).

Let us illustrate the further results by the example of the model equation with two independent space variables:

$$(1.3)\quad \begin{cases} u_t = \left(|u_x|^{p(z)-2} u_x\right)_x + \left(|u_y|^{q(z)-2} u_y\right)_y + c_0 |u|^{\sigma(z)-2} u + f(z) \quad \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \\ u(x,0) = u_0(x) \text{ in } \Omega = (0,a) \times (0,a). \end{cases}$$

For the sake of definiteness we assume that $c_0 > 0$. By agreement, here and throughout the paper we use the notations

$$\phi^+(t) = \sup_\Omega \phi(\cdot,t), \qquad \phi^-(t) = \inf_\Omega \phi(\cdot,t) \quad \text{for } \phi \in C^0(Q_T).$$

• *Vanishing in a finite time.* Let $f(z) \equiv 0$ for all $t \geq t_f$. Then every (energy) solution of problem (1.3) vanishes in a finite time $t_* \geq t_f$ if

$$\frac{1}{\nu^+(t)} \equiv \frac{1}{\sigma^+(t)} + \frac{1}{2}\left(\frac{1}{p^+(t)} + \frac{1}{q^+(t)}\right) > 1 \quad \text{in } Q_T$$

and the oscillation of the variable exponents of nonlinearity in Ω is appropriately small: for every $t \in (0,T)$

$$\frac{1}{\nu^-(t)} \equiv \frac{1}{\sigma^-(t)} + \frac{1}{2}\left(\frac{1}{p^-(t)} + \frac{1}{q^-(t)}\right) \leq \frac{3}{2}.$$

• *Vanishing at a prescribed moment.* Let us additionally assume that in the above conditions

$$\int_\Omega |f(x,y,t)|^{\sigma(x,y,t)}\, dxdy \begin{cases} \equiv 0 & \text{as } t \geq t_f, \\ \leq\ C\,(t_f - t)^\mu & \text{for } t \in (0,t_f), \end{cases} \qquad C,\ \mu = const > 0,$$

with a suitably big exponent μ. Then every (energy) solution of problem (1.1) vanishes at the instant $t = t_f$, provided that C and $\|u_0\|_{2,\Omega}$ sufficiently small.

• *Vanishing of solutions of eventually linear equations.* Although the effect of finite time vanishing is never displayed by the solutions of the linear parabolic equations, it may happen that equation (1.1) with variable nonlinearity transforms into the linear one as $t \to \infty$ and nonetheless possesses localized in time solutions. A condition sufficient for such an effect can be formulated as a restriction on the rate of vanishing of $\nu^+(t) - 1$ as $t \to \infty$: if $f \equiv 0$ and $\|u_0\|_{2,\Omega} \leq 1$, then every solution of problem (1.3) vanishes at a finite moment, provided that

$$\int_0^\infty \|u_0\|_{2,\Omega}^{\nu^+(t)-1}\, dt = \infty \qquad \text{and} \qquad \int_0^\infty \frac{dt}{\mathrm{e}^{t(1-\nu^+(t))}} < \infty.$$

Notice that these conditions are surely fulfilled if $\nu^+(t) \leq \nu_0 < 1$.

• *Large time behavior.* In case that the sufficient conditions of time localization are not fulfilled, we study the behavior of the norm $\|u(\cdot,t)\|_{2,\Omega}$ as $t \to \infty$. In dependence on the properties of the data, we establish the conditions of power or exponential decreasing of the $L^2(\Omega)$-norm of the solution.

Equations (1.1) with constant and/or isotropic nonlinearity fall into the scope of our analysis as partial cases. In these cases the conditions of time localization coincide with the already known in the literature – see, e.g., [11] and [5, Ch. 2] for a survey of relevant results.

Existence, uniqueness and boundedness of solutions to anisotropic parabolic equations of the type (1.1) have been studied by many authors under various conditions on the data and with different methods - see, e.g., [1, 2, 3, 13, 15, 16, 18, 19, 20, 26, 27, 30, 31] and the further references therein. In the present paper we are interested in the energy solutions of problem (1.1) - see Definition 2.1 below.

The localization properties of solutions of problem (1.1) with anisotropic diffusion operator and constant nonlinearity are studied in [14]. Now we extend the analysis of the phenomenon of time localization to equations with variable nonlinearity which may include include the absorption terms. The study is based on the analysis of the energy functions associated with the solution. This method allows us to carry out the analysis in the situation when the equation under study need not admit explicit sub/super solutions.

The paper is organized as follows. In Section 2 we collect some facts from the theory of variable exponent Orlicz-Sobolev spaces used in the rest of the paper, define the weak (energy) solution of problem (1.1) and recall the results about its solvability. Section 3 contains results about the embedding of anisotropic Orlicz-Sobolev spaces the energy solutions of problem (1.1) belong to. Sections 4 and 5 are devoted to derivation of the ordinary nonlinear differential inequalities for the energy functions associated with the solution of problem (1.1). The proof of the main results is based on the study of the differential inequalities and is given in Sections 6-9.

2. The function spaces. Preliminaries.

The definitions of the function spaces used throughout the paper and a brief description of their properties follow [21, 22, 24, 28]. The further references can be found in the survey papers [23, 29].

2.1. **Spaces $L^{p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$.** Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $\partial\Omega$ be Lipschitz-continuous, and let the function $p(x)$ be continuous with the logarithmic module of continuity:

$$\forall\, z,\, \zeta \in Q_T,\ |z-\zeta|<1, \qquad \sum_i |p(z)-p(\zeta)| \leq \omega(|z-\zeta|), \tag{2.1}$$

where

$$\overline{\lim_{\tau\to 0^+}}\, \omega(\tau)\ln\frac{1}{\tau} = C < +\infty.$$

By $L^{p(x)}(\Omega)$ we denote the space of measurable functions $f(x)$ on Ω such that

$$A_{p(\cdot)}(f) = \int_\Omega |f(x)|^{p(x)}\,dx < \infty.$$

The space $L^{p(x)}(\Omega)$ equipped with the norm

$$\|f\|_{p(\cdot),\Omega} \equiv \|f\|_{L^{p(x)}(\Omega)} = \inf\left\{\lambda>0:\ A_{p(\cdot)}\left(f/\lambda\right)\leq 1\right\}$$

becomes a Banach space. The Banach space $W_0^{1,\,p(x)}(\Omega)$ with $p(x)\in[p^-,p^+]\subset(1,\infty)$ is defined by

$$\begin{cases} W_0^{1,\,p(x)}(\Omega) = \left\{f\in L^{p(x)}(\Omega):\, |\nabla f|\in L^{p(x)}(\Omega),\ u=0 \text{ on } \partial\Omega\right\}, \\ \|u\|_{W_0^{1,p(x)}(\Omega)} = \sum_i \|D_i u\|_{p(\cdot),\Omega}. \end{cases} \tag{2.2}$$

It is known [33] that condition (2.1) is sufficient and necessary for the density of smooth functions in $L^{p(x)}(\Omega)$.

It follows directly from the definition that

$$\min\left(\|f\|_{p(\cdot)}^{p^-},\|f\|_{p(\cdot)}^{p^+}\right)\le A_{p(\cdot)}(f)\le\max\left(\|f\|_{p(\cdot)}^{p^-},\|f\|_{p(\cdot)}^{p^+}\right). \tag{2.3}$$

The functions from the space $L^{p(x)}(\Omega)$ satisfy the Hölder inequality: for all $f\in L^{p(x)}(\Omega)$, $g\in L^{p'(x)}(\Omega)$ with $p(x)\in(1,\infty)$, $p'(x)=\dfrac{p(x)}{p(x)-1}$ (Hölder's conjugate),

$$\int\limits_{\Omega}|f\,g|\,dx\le 2\,\|f\|_{p(\cdot)}\,\|g\|_{p'(\cdot)}. \tag{2.4}$$

In particular, for every constant $q\in(1,p^-)$

$$\|f\|_{q,\Omega}\le C\,\|f\|_{p(\cdot),\Omega}\quad\text{with the constant}\quad C=2\,\|1\|_{\frac{p(\cdot)}{p(\cdot)-q},\Omega}.$$

2.2. **Anisotropic spaces $\mathbf{V}(\Omega)$ and $\mathbf{W}(Q_T)$.** For every fixed $t\in[0,T]$ we introduce the Banach space

$$\begin{cases}\mathbf{V}(\Omega)=\left\{u(x):\ u(x)\in L^2(\Omega),\ |D_iu|^{p_i(x,t)}\in L^1(\Omega),\ u=0\text{ on }\Gamma\right\},\\[1ex] \|u\|_{\mathbf{V}(\Omega)}=\|u\|_{2,\Omega}+\displaystyle\sum_{i=1}^n\|D_iu\|_{p_i(\cdot,t),\Omega}\end{cases}$$

and denote by $\mathbf{W}(Q_T)$ the Banach space

$$\mathbf{W}(Q_T)=\Big\{u:[0,T]\mapsto\mathbf{V}(\Omega):\ u\in L^2(Q_T),\ |D_iu|^{p_i(z)}\in L^1(Q_T),\ u=0\text{ on }\Gamma_T\Big\}$$

with the norm

$$\|u\|_{\mathbf{W}(Q_T)}=\|u\|_{2,Q_T}+\sum_{i=1}^n\|D_iu\|_{p_i(\cdot),Q_T}.$$

If the exponents $p_i(z)\in[p^-,p^+]\subset(1,\infty)$ and satisfy the log-continuity condition (2.1), then the space $\mathbf{W}(Q_T)$ is separable and every $v\in\mathbf{W}(Q_T)$ can be represented in the form $v=\sum_{i=1}^\infty c_i(t)\psi_i(x)$, where $c_i(t)\in C^1[0,T]$ and $\{\psi_i\}$ is an orthonormal basis of the space

$$\mathbf{V}_+(\Omega)=\Big\{u(x)|\ u\in L^2(\Omega)\cap W_0^{1,1}(\Omega),\ |\nabla u|\in L^{p^+}(\Omega)\Big\}$$

(see [13, Sec. 2]). By $\mathbf{W}'(Q_T)$ we denote the dual of $\mathbf{W}(Q_T)$ (the space of linear functionals over $\mathbf{W}(Q_T)$):

$$w\in\mathbf{W}'(Q_T)\iff\begin{cases}w=w_0+\displaystyle\sum_{i=1}^n D_iw_i,\quad w_0\in L^2(Q_T),\quad w_i\in L^{p_i'(z)}(Q_T),\\[1ex] \forall\phi\in\mathbf{W}(Q_T)\quad\langle\langle w,\phi\rangle\rangle=\displaystyle\int_{Q_T}\left(w_0\phi+\sum_i w_iD_i\phi\right)dz.\end{cases}$$

The norm in $\mathbf{W}'(Q_T)$ is defined by

$$\|v\|_{\mathbf{W}'(Q_T)} = \sup\left\{\langle\langle v, \phi\rangle\rangle |\, \phi \in \mathbf{W}(Q_T),\ \|\phi\|_{\mathbf{W}(Q_T)} \leq 1\right\}.$$

Let ρ be the Friedrics mollifying kernel

$$\rho(s) = \begin{cases} \kappa\, \exp\left(-\frac{1}{1-|s|^2}\right) & \text{if } |s| < 1, \\ 0 & \text{if } |s| > 1, \end{cases} \qquad \kappa = const: \quad \int_{\mathbb{R}^{n+1}} \rho(z)\, dz = 1.$$

Given a function $v \in \mathbf{W}(Q_T)$, we extend it by zero to the whole $\mathbb{R}^{n+1}$ (keeping the same notation for the continued function) and then define

$$v_h(z) = \int_{\mathbb{R}^{n+1}} v(s)\rho_h(z-s)\, ds \qquad \text{with } \rho_h(s) = \frac{1}{h^{n+1}}\rho\left(\frac{s}{h}\right), \quad h > 0.$$

Proposition 2.1. *If $u \in \mathbf{W}(Q_T)$ with the exponents $p_i(z)$ satisfying* (2.1)*, then*

$$\|u_h\|_{\mathbf{W}(Q_T)} \leq C\left(\|u\|_{W_0^{1,1}(Q_T)} + \|u\|_{\mathbf{W}(Q_T)}\right) \quad \textit{and} \quad \|u_h - u\|_{\mathbf{W}(Q_T)} \to 0 \ \textit{as}\ h \to 0.$$

Proposition 2.1 is an immediate byproduct of [33, Theorem 2.1].

Proposition 2.2. *Let in the conditions of Proposition* 2.1 *$u_t \in \mathbf{W}'(Q_T)$. Then $(u_h)_t \in \mathbf{W}'(Q_T)$, and for every $\psi \in \mathbf{W}(Q_T)$*

$$\langle\langle (u_h)_t\, \psi\rangle\rangle \to \langle\langle u_t\, \psi\rangle\rangle \quad \textit{as } h \to 0.$$

Proof. By the definition of $\mathbf{W}'(Q_T)$ there exist $\phi_0 \in L^2(Q_T)$, $\phi_i \in L^{p_i'(\cdot)}(Q_T)$ such that $\langle\langle u_t, \psi\rangle\rangle = (\phi_0, \psi)_{2,Q_T} + \sum_i (\phi_i, D_i\psi)_{2,Q_T}$ $\forall\, \psi \in \mathbf{W}(Q_T)$. It follows that

$$\begin{aligned} \langle\langle (u_h)_t, \psi\rangle\rangle &= \int_{Q_T} (u_t)_h \psi\, dz = \int_{Q_T} u_t \psi_h\, dz = \int_{Q_T} \left(\phi_0\psi_h + \sum_i \phi_i\, D_i\psi_h\right) dz \\ &= \int_{Q_T} \left((\phi_0)_h\psi + \sum_i (\phi_i)_h\, D_i\psi\right) dz \to \langle\langle u_t, \psi\rangle\rangle \quad \text{as } h \to 0 \end{aligned}$$

by virtue of Proposition 2.1. □

2.3. **Solvability of problem (1.1).** In the rest of the paper we refer to the existence and uniqueness theorems for problem (1.1) proved in [13].

Definition 2.1. *A function $u \in \mathbf{W}(Q_T)$ is called weak (energy) solution of problem* (1.1) *if $u_t \in \mathbf{W}'(Q_T)$ and for every t_1, $t_2 \in [0, T]$ and every test–function $\phi \in \mathbf{W}(Q_T)$ such that $\phi_t \in \mathbf{W}'(Q_T)$*

$$\begin{aligned} \int_\Omega u\phi\, dx\Big|_{t=t_1}^{t=t_2} - \int_{Q_{t_2}\setminus Q_{t_1}} \left(u\,\phi_t - \sum_{i=1}^n a_i|D_iu|^{p_i-2}D_iu\, D_i\phi + c\,|u|^{\sigma-2}u\phi\right) dz \\ = \int_{Q_{t_2}\setminus Q_{t_1}} f\,\phi\, dz. \end{aligned} \tag{2.5}$$

It is assumed that the coefficients $a_i(z,u)$, $c(z,u)$ depend on $z=(x,t)$, $u(z)$ and are Carathéodory functions (measurable in z for every $r\in\mathbb{R}$, continuous in r for a.a. $z\in Q_T$), such that

$$
(2.6)\quad \forall\,(z,r)\in\overline{Q}_T\times\mathbb{R}\qquad \begin{array}{l} 0<a_0\le a_i(z,r)\le A_0<\infty,\\ 0\le c_0\le c(z,r)\le A_0 \end{array}\qquad a_0,\,c_0,\,A_0=const.
$$

Theorem 2.1 ([13])**.** *Let conditions* (2.6) *be fulfilled and let the exponents* $p_i(z)$ *and* $\sigma(z)$ *satisfy condition* (2.1) *of log–continuity. For every* $u_0\in L^2(\Omega)$ *and* $f\in L^2(Q_T)$ *problem* (1.1) *has a unique energy solution* $u\in\mathbf{W}(Q_T)$ *such that* $u_t\in\mathbf{W}'(Q_T)$*. Moreover, if* $c_0>0$ *and* $f\in L^{\sigma'(z)}(Q_T)$*, then* $u\in\mathbf{W}(Q_T)\cap L^{\sigma(z)}(Q_T)$*.*

A weak solution of problem (1.1) is constructed in [13] as the limit of a sequence of Galerkin's approximations. Such a procedure is possible if the variable exponents of nonlinearity satisfy the log–continuity condition (2.1). The proof relies on the monotonicity of the elliptic part of equation (1.1) (see [**?**, Ch. 2, Sec. 1.2]) and the formula of integration by parts for the products $v\,w_t$ with $v,w\in\mathbf{W}(Q_T)$, $v_t,w_t\in\mathbf{W}'(Q_T)\cap L^1(Q_T)$ (cf. with Lemma 4.3 below).

3. Auxiliary propositions.

Theorem 3.1. *Let* $\Omega=\{x\in\mathbb{R}^n:\ x_i\in(0,a)\}$ *and let* $p_i=const>1$*. Then*

$$
\|v\|_{r,\Omega}\le C(a,n)\left(\prod_{i=1}^{n}\|D_iv\|_{p_i,\Omega}\right)^{\frac{1}{n}},
$$

with

$$
r=\begin{cases}\dfrac{n\,p}{n-p} & \text{if } p<n,\\ \text{any number from } [1,\infty) & \text{if } p\ge n,\end{cases}\qquad \frac{1}{p}=\frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}.
$$

The assertion of Theorem 3.1 is a byproduct of results of [27, 32]. Let us introduce the functions

$$
(3.1)\qquad \begin{aligned}
\Theta(t)&=\int_\Omega u^2(z)\,dx,\qquad z=(x,t)\in Q_T,\\
\Lambda(t)&=\int_\Omega\sum_{i=1}^{n}|D_iu(z)|^{p_i(z)}\,dx,\\
\Lambda_\sigma(t)&=\int_\Omega\left(\sum_{i=1}^{n}|D_iu(z)|^{p_i(z)}+|u(z)|^{\sigma(z)}\right)dx.
\end{aligned}
$$

Lemma 3.1. *If* $u\in\mathbf{W}(Q_T)$*, then the functions* $\Theta(t)$*,* $\Lambda(t)$ *exist for a.e.* $t\in(0,T)$*. If* $u\in\mathbf{W}(Q_T)\cap L^{\sigma(x,t)}(Q_T)$*, then for a.e.* $t\in(0,T)$ *there exist* $\Theta(t)$ *and* $\Lambda_\sigma(t)$*.*

Proof. The functions $\Theta(t)$, $\Lambda(t)$ are nonnegative and, by the definition of $\mathbf{W}(Q_T)$, $\Theta(t)$, $\Lambda(t)\in L^1(0,T)$. It follows that the functions

$$
\int_0^t\Theta(\tau)\,d\tau,\quad \int_0^t\Lambda(\tau)\,d\tau
$$

are differentiable for a.e. $t \in (0,T)$. In the case when $u \in L^{\sigma(z)}(Q_T)$ the same is true for the function $\displaystyle\int_0^t |u|^{\sigma(z)}\,d\tau$. □

Lemma 3.2. *Let $u \in \mathbf{W}(Q_T)$, and let p_i, σ be constant. If $p_i > 1$ and*

$$\frac{1}{p} \equiv \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i} \leq \frac{1}{2} + \frac{1}{n},$$

then for a.e. $t \in (0,T)$

$$\Theta(t) \leq C\,\Lambda^{\frac{1}{\nu}}(t) \quad \textit{with} \quad \nu = \frac{p}{2}. \tag{3.2}$$

Let $u \in \mathbf{W}(Q_T)\cap L^{\sigma}(Q_T)$. If $p_i > 1$ and

$$\begin{cases} \frac{1}{\sigma} + \frac{1}{p} \geq 1 + \frac{1}{n} & \textit{if } n > p, \\ \sigma > 1 & \textit{if } n \leq p, \end{cases}$$

then for a.e. $t \in (0,T)$

$$\Theta(t) \leq C\,\Lambda_{\sigma}(t)^{\frac{1}{\nu}} \quad \textit{with} \quad \frac{1}{\nu} = \frac{1}{\sigma} + \frac{1}{p}. \tag{3.3}$$

Proof. We start by proving (3.2). By Hölder's inequality and due to Theorem 3.1 with

$$r = \frac{np}{n-p} \geq 2 \quad \Longleftrightarrow \quad p \geq \frac{2n}{n+2}$$

we have that

$$\Theta(t) \equiv \int_{\Omega} u^2\,dx \leq \|u\|^2_{r,\Omega} \leq C\prod_{i=1}^{n}\|D_iu\|^{\frac{2}{n}}_{p_i,\Omega} \leq C\,\Lambda(t)^{\frac{1}{\nu}}. \tag{3.4}$$

To prove (3.3) we use Hölder's inequality and Theorem 3.1 with

$$\sigma' = \frac{\sigma}{\sigma-1} \leq r = \frac{np}{n-p} \quad \text{if } n > p \quad \Longleftrightarrow \quad \frac{1}{\sigma} + \frac{1}{p} \geq 1 + \frac{1}{n},$$

which gives:

$$\begin{aligned}\Theta(t) = \int_{\Omega} u^2\,dx \leq \|u\|_{\sigma,\Omega}\|u\|_{\sigma',\Omega} \leq C\,\|u\|_{\sigma,\Omega}\left(\sum_{i=1}^{n}\int_{\Omega}|D_iu|^{p_i}\,dx\right)^{\frac{1}{p}} \\ \leq C\,\Lambda_{\sigma}^{\frac{1}{\sigma}+\frac{1}{p}}(t) = C\Lambda_{\sigma}^{\frac{1}{\nu}}(t).\end{aligned}$$

□

Let us now assume that the exponents p_i, σ are functions defined on Q_T and subject to the following conditions:

$$\left\{\begin{array}{l} 1 < \underline{p} \leq p_i^-(t) \leq p_i(z) \leq p_i^+ \leq (t) \leq \overline{p} < \infty, \\ 1 < \underline{\sigma} \leq \sigma^-(t) \leq \sigma(z) \leq \sigma^+(t) \leq \overline{\sigma} < \infty \end{array}\right. \tag{3.5}$$

with

$$
\begin{aligned}
p_i^+(t) &= \sup_\Omega p_i(x,t), \qquad & p_i^-(t) &= \inf_\Omega \, p_i(x,t),\\
\sigma^+(t) &= \sup_\Omega \sigma(x,t), \qquad & \sigma^-(t) &= \inf_\Omega \, \sigma(x,t),
\end{aligned}
$$

and $\underline{p}$, $\overline{p}$ are constants chosen so that $1 < \underline{p} \leq p_i^-(t) \leq p^+(t) \leq \overline{p} < \infty$ for every $i = 1, \ldots, n$. We will use the notations

$$\frac{1}{p^\pm(t)} = \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i^\pm(t)}.$$

Lemma 3.3. *Let conditions* (3.5) *be fulfilled and*

$$\frac{1}{\nu^-(t)} = \frac{1}{\sigma^-(t)} + \frac{1}{p^-(t)} \leq \frac{n+1}{n} \quad \textit{for } \frac{1}{n} < \frac{1}{p^-(t)}. \tag{3.6}$$

Then for every $u \in \mathbf{W}(Q_T) \cap L^{\sigma(z)}(Q_T)$ *and a.e.* $t \in (0,T)$

$$\min\left\{\Theta^{\nu^+(t)}(t),\, \Theta^{\nu^-(t)}\right\} \leq C\,\Lambda_\sigma(t) \quad \textit{with} \quad \frac{1}{\nu^+(t)} = \frac{1}{\sigma^+(t)} + \frac{1}{p^+(t)}. \tag{3.7}$$

Proof. By Lemma 3.2 for every fixed $t \in (0,T)$

$$\int_\Omega u^2\,dx \leq C\,\|u\|_{\sigma^-(t),\Omega} \|u\|_{(\sigma^-(t))',\Omega} \leq C\,\|u\|_{\sigma^-(t),\Omega}\left(\prod_{i=1}^{n}\|D_i u\|_{p_i^-(t),\Omega}\right)^{\frac{1}{n}}, \tag{3.8}$$

provided that (3.6) is fulfilled. Using (2.4) we find that

$$
\begin{aligned}
\|u\|_{\sigma^-(t),\Omega} &= \left(\int_\Omega |u|^{\sigma^-(t)}\,dx\right)^{\frac{1}{\sigma^-(t)}} \leq C\,\left\| |u|^{\sigma^-(t)}\right\|^{\frac{1}{\sigma^-(t)}}_{\frac{\sigma(\cdot,t)}{\sigma^-(t)},\Omega}\\
&\leq C' \max\left\{\left(\int_\Omega |u|^{\sigma(z)}\,dx\right)^{\frac{1}{\sigma^+(t)}},\ \left(\int_\Omega |u|^{\sigma(z)}\,dx\right)^{\frac{1}{\sigma^-(t)}}\right\}
\end{aligned}
$$

or, equivalently,

$$C'\int_\Omega |u|^{\sigma(z)}\,dx \geq \begin{cases} \|u\|^{\sigma^+(t)}_{\sigma^-(t),\Omega} & \text{if } \|u\|_{\sigma^-(t),\Omega} \leq 1,\\ \|u\|^{\sigma^-(t)}_{\sigma^-(t),\Omega} & \text{if } \|u\|_{\sigma^-(t),\Omega} > 1.\end{cases}$$

Next, for every $i = 1, 2, \ldots, n$

$$C''\int_\Omega |D_i u|^{p_i(z)}\,dx \geq \begin{cases} \|D_i u\|^{p_i^+(t)}_{p_i^-(t),\Omega} & \text{if } \|D_i u\|_{p_i^-(t),\Omega} \leq 1,\\ \|D_i u\|^{p_i^-(t)}_{p_i^-(t),\Omega} & \text{if } \|D_i u\|_{p_i^-(t),\Omega} > 1.\end{cases}$$

It follows that

$$\|u\|_{\sigma^-(t),\Omega}\left(\prod_{i=1}^{n}\|D_i u\|_{p_i^-(t),\Omega}\right)^{\frac{1}{n}} \leq C\,\Lambda_\sigma^{\frac{1}{\mu(t)}}(t),$$

with the exponents

$$\frac{1}{\mu(t)} = \frac{1}{\widetilde{\sigma}(t)} + \frac{1}{n}\sum_{i=1}^{n}\frac{1}{\widetilde{p}_i(t)} \in \left[\frac{1}{\nu^+(t)}, \frac{1}{\nu^-(t)}\right],$$

and

$$\widetilde{p}_i(t) = \begin{cases} p_i^-(t) & \text{if } \|D_i u\|_{p_i^-(t),\Omega} > 1, \\ p_i^+(t) & \text{if } \|D_i u\|_{p_i^-(t),\Omega} \le 1, \end{cases} \qquad \widetilde{\sigma}(t) = \begin{cases} \sigma^-(t) & \text{if } \|u\|_{\sigma^-(t),\Omega} > 1, \\ \sigma^+(t) & \text{if } \|u\|_{\sigma^-(t),\Omega} \le 1. \end{cases}$$

Plugging these formulas to (3.8) we find that

$$\Theta^{\mu(t)}(t) \le C\,\Lambda_\sigma(t).$$

The assertion now follows because

$$\Theta^{\mu(t)}(t) \ge \min\left\{\Theta^{\nu^+(t)}(t),\, \Theta^{\nu^-(t)}(t)\right\} = \begin{cases} \Theta^{\nu^+(t)} & \text{if } \Theta(t) \le 1, \\ \Theta^{\nu^-(t)} & \text{if } \Theta(t) \ge 1. \end{cases}$$

□

Lemma 3.4. *Let $u \in \mathbf{W}(Q_T)$. If*

$$p^-(t) \ge \frac{2n}{n+2}, \tag{3.9}$$

then for a.e. $t \in (0,T)$

$$\min\left\{\Theta^{\nu^+(t)}(t),\, \Theta^{\nu^-(t)}(t)\right\} \le C\,\Lambda(t), \qquad \frac{1}{\nu^\pm(t)} = \frac{2}{p^\pm(t)}. \tag{3.10}$$

with a constant $C = C\left(\Omega,\, \underline{p},\, \overline{p},\, n\right)$.

Proof. By Theorem 3.1

$$\Theta(t) \le C\left(\prod_{i=1}^{n}\|D_i u\|_{p_i^-(t),\Omega}\right)^{\frac{2}{n}}$$

and the conclusion follows like in Lemma 3.3. □

Corollary 3.1. *Let $\Theta(t) \le 1$. Since $\nu^+(t) \ge \nu^-(t)$, in this case the left-hand sides of inequalities* (3.10) *and* (3.7) *can be changed to $\Theta^{\nu^+(t)}(t)$.*

4. The energy relations

Let us introduce the notations

$$\Omega(s) = \Omega \cap \{x_1 > s\}, \qquad \omega(s) = \Omega \cap \{x_1 = s\}.$$

4.1. **Formulas of integration by parts.** Let ρ be the Friedrics mollifying kernel

$$\rho(s)=\begin{cases}\kappa\,\exp\left(-\frac{1}{1-|s|^2}\right) & \text{if } |s|<1,\\ 0 & \text{if } |s|>1,\end{cases}\qquad \kappa=const:\quad \int_{\mathbb{R}^{n+1}}\rho(z)\,dz=1.$$

Given a function $v\in\mathbf{W}(Q_T)$, we extend it by zero to the whole $\mathbb{R}^{n+1}$ (keeping the same notation for the continued function) and then define

$$v_h(z)=\int_{\mathbb{R}^{n+1}}v(s)\rho_h(z-s)\,ds\qquad\text{with } \rho_h(s)=\frac{1}{h^{n+1}}\rho\left(\frac{s}{h}\right),\quad h>0.$$

Lemma 4.1. *If $u\in\mathbf{W}(Q_T)$ with the exponents $p_i(z)$ satisfying* (2.1)*, then*

$$\|u_h\|_{\mathbf{W}(Q_T)}\le C\left(\|u\|_{W^{1,1}_0(Q_T)}+\|u\|_{\mathbf{W}(Q_T)}\right)\quad\text{and}\quad \|u_h-u\|_{\mathbf{W}(Q_T)}\to 0\ \text{as } h\to 0.$$

Lemma 4.1 is an immediate byproduct of [33, Theorem 2.1].

Lemma 4.2. *Let in the conditions of Proposition* 2.1 *$u_t\in\mathbf{W}'(Q_T)\cap L^1(Q_T)$. Then $(u_h)_t\in\mathbf{W}'(Q_T)$, and for every $\psi\in\mathbf{W}(Q_T)$*

$$\langle\langle (u_h)_t\,\psi\rangle\rangle\to\langle\langle u_t\,\psi\rangle\rangle\quad\text{as } h\to 0.$$

Proof. By the definition of $\mathbf{W}'(Q_T)$ there exist $\phi_0\in L^2(Q_T)$, $\phi_i\in L^{p_i'(\cdot)}(Q_T)$ such that $\langle\langle u_t,\psi\rangle\rangle=(\phi_0,\psi)_{2,Q_T}+\sum_i(\phi_i,D_i\psi)_{2,Q_T}$ $\forall\,\psi\in\mathbf{W}(Q_T)$. It follows that

$$\begin{aligned}\langle\langle (u_h)_t,\psi\rangle\rangle&=\int_{Q_T}(u_t)_h\psi\,dz=\int_{Q_T}u_t\psi_h\,dz=\int_{Q_T}\left(\phi_0\psi_h+\sum_i\phi_i\,D_i\psi_h\right)dz\\&=\int_{Q_T}\left((\phi_0)_h\psi+\sum_i(\phi_i)_h\,D_i\psi\right)dz\to\langle\langle u_t,\psi\rangle\rangle\quad\text{as } h\to 0\end{aligned}$$

by virtue of Lemma 4.1. $\square$

Lemma 4.3. *Let $u\in\mathbf{W}(Q_T)$ and $u_t\in\mathbf{W}'(Q_T)\cap L^1(Q_T)$ with the exponents $p_i(z)$ satisfying* (2.1)*. Then*

$$\forall\,\text{a.e. } t_1,t_2\in(0,T]\qquad \int_{t_1}^{t_2}\int_\Omega u\,u_t\,dz=\frac{1}{2}\|u(\cdot,t_2)\|^2_{2,\Omega}-\frac{1}{2}\|u(\cdot,t_1)\|^2_{2,\Omega}.$$

Proof. Let $t_1<t_2$. Take

$$\chi_k(t)=\begin{cases}0 & \text{for } t\le t_1,\\ k(t-t_1) & \text{for } t_1\le t\le t_1+\frac{1}{k},\\ 1 & \text{for } t_1+\frac{1}{k}\le t\le t_2-\frac{1}{k},\\ k(t_2-t) & \text{for } t_2-\frac{1}{k}\le t\le t_2,\\ 0 & \text{for } t\ge t_2.\end{cases}$$

Integrating by parts we have that for every sufficiently large $k\in\mathbb{N}$

$$2\int_{Q_T}u_h(u_h)_t\chi_k(t)\,dz=k\int_{t_2-\frac{1}{k}}^{t_2}\int_\Omega u_h^2\,dz-k\int_{t_1}^{t_1+\frac{1}{k}}\int_\Omega u_h^2\,dz.$$

The integrals on the right-hand side exist because $u_h \in L^2(Q_T)$. Letting $h \to 0$, we obtain the equality

$$2 \lim_{h\to 0} \int_{Q_T} u_h\,(u_h)_t\,\chi_k(t)\,dz = k\int_{t_2-\frac{1}{k}}^{t_2}\int_\Omega u^2\,dz - k\int_{t_1}^{t_1+\frac{1}{k}}\int_\Omega u^2\,dz.$$

According to Propositions 2.1, 2.2 $u_h \to u$ in $\mathbf{W}(Q_T)$, $(u_h)_t = (u_t)_h \to u_t$ weakly in $\mathbf{W}'(Q_T)$ as $h \to 0$, and $\|u\|_{\mathbf{W}}$, $\|(u_h)_t\|_{\mathbf{W}'}$ are uniformly bounded. It follows that

$$\begin{aligned}\lim_{h\to 0}\int_{Q_T} u_h\,(u_h)_t\,\chi_k(t)\,dz &= \lim_{h\to 0}\int_{Q_T}(u_h-u)(u_h)_t\chi_k(t)\,dz\\ &+\lim_{h\to 0}\int_{Q_T} u\,((u_h)_t-u_t)\chi_k(t)\,dz+\int_{Q_T} u\,u_t\,\chi_k(t)\,dz=\int_{Q_T} u\,u_t\chi_k(t)\,dz.\end{aligned}$$

By the Lebesgue differentiation theorem

$$\forall\,\text{a.e.}\,\theta>0 \qquad \lim_{k\to 0} k\int_{\theta-\frac{1}{k}}^{\theta}\left(\int_\Omega u^2(x,t)\,dx\right)dt = \|u(\cdot,\theta)\|^2_{2,\Omega},$$

whence for almost every $t_1,\, t_2 \in [0,T]$

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega u\,u_t\,dz &= \lim_{k\to\infty}\int_{Q_T} u\,u_t\chi_k(t)\,dz\\ &= \lim_{k\to\infty} k\int_{t_2-\frac{1}{k}}^{t_2}\int_\Omega u^2\,dz - \lim_{k\to\infty} k\int_{t_1}^{t_1+\frac{1}{k}}\int_\Omega u^2\,dz\\ &= \frac{1}{2}\|u(\cdot,t_2)\|^2_{2,\Omega}-\frac{1}{2}\|u(\cdot,t_1)\|^2_{2,\Omega}.\end{aligned}$$

□

Lemma 4.4. *Let conditions* 2.1 *and* (2.6) *be fulfilled. For every solution* $u \in \mathbf{W}(Q_T)$ *of problem* (1.1) *and a.e.* s *such that* $\Omega \cap \{x_1 = s\} \neq \emptyset$

$$\begin{aligned}&\frac{1}{2}\int_{\Omega(s)} u^2(x,\tau)dx\Big|_{\tau=0}^{\tau=t}+\sum_{i=1}^n\int_{Q(s,t)} a_i(z,u)|D_iu|^{p_i(z)}\,dz\\ &+\int_{Q(s,t)} c(z,u)|u|^{\sigma(z)}\,dz+\int_0^t dt\int_{\omega(s)} a_1(z,u)\,u\,|D_1u|^{p_1-2}D_1u\,dx'\\ &=\int_{Q(s,t)} u\,f\,dz.\end{aligned}\tag{4.1}$$

Proof. The energy solution can be taken for the test–function in identity (2.5). Let us set

$$\phi_k(x_1,x',s)=\begin{cases}1 & \text{for } x_1 > s+\frac{1}{k},\\ k(x_1-s) & \text{for } x_1\in\left[s,s+\frac{1}{k}\right],\\ 0 & \text{for } x_1<s, \qquad k\in\mathbb{N},\end{cases}$$

and choose $u(x,t)\phi_k(x_1,x',s)$. The resulting identity has the form

$$
\begin{aligned}
\sum_{j=1}^{5} I_j(k,s) \equiv & \sum_{i=1}^{n} \int_{Q(s,t)} a_i(z,u)|D_i u|^{p_i(z)}\, \phi_k\, dz + \int_{Q(s,t)} c(z,u)|u|^{\sigma(z)} \phi_k\, dz \\
& + k \int_0^t dt \int_{\Omega(s+1/k)\setminus\Omega(s)} a_1(z,u)\, u\, |D_1 u|^{p_1(z)-2} D_1 u\, dx \\
& + \frac{1}{2} \int_{\Omega(s)} \phi_k u^2\, dx \bigg|_{\tau=0}^{\tau=t} - \int_{Q(s,t)} u\, f\, \phi_k\, dz = 0.
\end{aligned} \tag{4.2}
$$

By the definition of $\mathbf{W}(Q_T)$

$$
|u|^{\sigma}\phi_k,\ |D_i u|^{p_i}\, \phi_k,\ u\phi_k \in L^1(Q_T), \qquad u^2\phi_k \in L^1(\Omega) \quad \text{for a.e. } t \in (0,T),
$$

which allows us to pass to the limit as $k \to \infty$ in I_1, I_2 I_4 and I_5:

$$
\begin{aligned}
& \lim_{k\to\infty} I_1 = \sum_{i=1}^{n} \int_{Q(s,t)} a_i(z,u)\, |D_i u|^{p_i}\, dz, \\
& \lim_{k\to\infty} I_2 = \int_{Q(s,t)} c(z,u)|u|^{\sigma(z)}\, dz \\
& \lim_{k\to\infty} I_4 = \frac{1}{2} \int_{\Omega(s)} u^2\, dx - \frac{1}{2} \int_{\Omega(s)} u_0^2\, dx, \\
& \lim_{k\to\infty} I_5 = \int_{Q(s,t)} u\, f\, dz.
\end{aligned}
$$

By virtue of (4.2) I_3 is bounded uniformly with respect to k, provided that so are the integrals I_1, I_2 and I_4. Writing I_3 in the form

$$
I_3 = k \int_s^{s+1/k} \left(\int_0^t \left(\int_{\omega(s)} a_i(z,u)\, u\, |D_1 u|^{p_1-2} D_1 u\, dx' \right) dt \right) dx_1
$$

and applying the Lebesgue differentiation theorem we conclude that there exists

$$
\lim_{k\to\infty} I_3(k,s) = \int_0^t dt \int_{\omega(s)} a_1(z,u)\, u\, |D_1 u|^{p_1-2} D_1 u\, dx'.
$$

Equality (4.2) transforms into (4.1) as $k \to \infty$. □

Remark 4.1. *Let*

$$
\Omega \subset \Omega_a \equiv \{x \in \mathbb{R}^n :\, x_i \in (0,a),\, i = 1,\dots,n\}
$$

and let $u \in \mathbf{W}(Q_T)$ *be a solution of problem* (1.1). *Set*

$$
u^*(x,t) = \begin{cases} u(x,t) & \text{in } Q_T, \\ 0 & \text{in } Q_T^{(a)} \equiv (\Omega_a \setminus \Omega) \times (0,T). \end{cases}
$$

The function $u^*(x,t)$ *belongs to* $\mathbf{W}(Q_T^{(a)})$ *and formally satisfies identity* (4.1) *in the cylinder* $Q_T^{(a)}$. *If* $u^*(x,t)$ *possesses the property of time localization (i.e. vanishes at a finite instant), so does the function* $u(x,t)$, *which is why in what follows we* *study the localization properties of the energy solutions of problem* (1.1) *formulated in the domain* $\Omega \equiv \{x \in \mathbb{R}^n : x_i \in (0,a)\}$.

4.2. **Estimates on the total energy.** In the further study of the behavior of the energy functions $\Theta(t)$, $\Lambda(t)$, $\Lambda_\sigma(t)$ we will need the following uniform in t estimates.

Lemma 4.5. *Let conditions* (2.1), (2.6) *be fulfilled,* $c_0 \geq 0$ *and* $u_0 \in L^2(\Omega)$.

(a) *If* $f \in L^2(Q_T)$, *then every solution* $u \in \mathbf{W}(Q_T)$ *of problem* (1.1) *satisfies the estimate*

$$\frac{1}{4}\operatorname{ess}\sup_{(0,T)} \Theta(t) + a_0 \int_0^T \Lambda(t)\,dt \leq \frac{1}{2}\Theta(0) + 4\left(\int_0^T \|f(\cdot,t)\|_{2,\Omega}\,dt\right)^2. \tag{4.3}$$

(b) *If*

$$p^-(t) \geq \frac{2n}{n+2} \quad \text{and} \quad f \in L^{(p^+)'(t)}(0,T;\, L^2(\Omega)) \cap L^{(p^-)'(t)}(0,T;\, L^2(\Omega)),$$

then

$$\begin{aligned} &\frac{1}{4}\operatorname{ess}\sup_{(0,T)} \Theta(t) + \frac{a_0}{2}\int_0^T \Lambda(t)\,dt \\ &\quad \leq \frac{1}{2}\Theta(0) + C\int_0^T \left(\|f(\cdot,t)\|_{2,\Omega}^{(p^+)'(t)} + \|f(\cdot,t)\|_{2,\Omega}^{(p^-)'(t)}\right)dt \end{aligned} \tag{4.4}$$

with a constant $C \equiv C(a_0, n, |\Omega|)$.

Proof. (a) It follows from the energy relation (4.1) with $s = 0$ that

$$\begin{aligned} &\frac{1}{2}\operatorname{ess}\sup_{t\in(0,T)} \int_{\Omega(s)} u^2(x,t)dx + a_0\sum_{i=1}^n \int_{Q_T} |D_i u|^{p_i(z)}\,dz \\ &\qquad + c_0\int_{Q_T} |u|^{\sigma(z)}\,dz = \int_{Q_T} u\,f\,dz + \frac{1}{2}\int_\Omega u_0^2\,dx. \end{aligned} \tag{4.5}$$

Using Hölder's and Young's inequalities, we estimate

$$\begin{aligned} \int_{Q_T} u\,f\,dz &\leq \int_0^T \|u(\cdot,t)\|_{2,\Omega}\|f(\cdot,t)\|_{2,\Omega}\,dt \\ &\leq \operatorname{ess}\sup_{t\in(0,T)} \|u\|_{2,\Omega}\int_0^T \|f(\cdot,t)\|_{2,\Omega}\,dt \\ &\leq \frac{1}{4}\operatorname{ess}\sup_{t\in(0,T)} \|u\|_{2,\Omega}^2 + 4\left(\int_0^T \|f(\cdot,t)\|_{2,\Omega}\,dt\right)^2, \end{aligned}$$

and (4.3) follows.

(b) We apply Lemma 3.4 and Young's inequality

$$\begin{aligned}\int_{Q_T} u\, f\, dz &\leq \int_0^T \|u(\cdot,t)\|_{2,\Omega}\|f(\cdot,t)\|_{2,\Omega}\, dt\\ &\leq C\int_0^T \max\left\{\Lambda^{\frac{1}{p^+(t)}}(t),\, \Lambda^{\frac{1}{p^-(t)}}(t)\right\}\, \|f(\cdot,t)\|_{2,\Omega}\, dt\\ &\leq \frac{a_0}{2}\int_0^T \Lambda(t)\, dt + C'\int_0^T \max\left\{\|f(\cdot,t)\|_{2,\Omega}^{(p^+)'(t)},\, \|f(\cdot,t)\|_{2,\Omega}^{(p^-)'(t)}\right\}\, dt.\end{aligned}$$

□

Lemma 4.6. *Let conditions* (2.1), (2.6) *be fulfilled. If* $c_0 > 0$ *and* $f \in L^{\sigma'(z)}(Q_T)$, *then the solution of problem* (1.1) $u \in \mathbf{W}(Q_T) \cap L^{\sigma(z)}(Q_T)$ *satisfies the estimate*

$$\begin{aligned}&\frac{1}{2}\operatorname{ess}\sup_{t\in(0,T)} \Theta(t) + \frac{1}{2}\min\{a_0,\, c_0\}\int_0^T \Lambda_\sigma(t)\, dt\\ &\qquad \leq \frac{1}{2}\,\Theta(0) + \left(\frac{2}{c_0}\right)^{\frac{1}{\sigma^- -1}}\int_{Q_T} |f|^{\sigma'(z)}\, dz.\end{aligned} \tag{4.6}$$

Proof. The assertion follows from (4.5) and Young's inequality:

$$\int_{Q_T} u\, f\, dz \leq \frac{c_0}{2}\int_{Q_T} |u|^{\sigma(z)}\, dz + \frac{1}{\sigma^-}\frac{1}{\left(\frac{c_0}{2}\sigma^-\right)^{\frac{1}{\sigma^- -1}}}\int_{Q_T} |f|^{\sigma'(z)}\, dz.$$

□

5. Differential inequality for the energy function

Let $u(z) \in \mathbf{W}(Q_T)$ be a solution of problem (1.1). Letting in (4.1) $s = 0$, we find that for every $t,\, t + \Delta\, t \in [0,T]$ the solution satisfies the identity

$$\begin{aligned}&\frac{1}{2}\int_\Omega u^2(z)\Big|_{\tau=t}^{\tau=t+\Delta\, t} dx + \int_t^{t+\Delta\, t}\int_\Omega \left(\sum_i a_i|D_i u|^{p_i(z)} + c\,|u|^{\sigma(z)}\right) dz\\ &\qquad = \int_t^{t+\Delta\, t}\int_\Omega f\, u\, dz.\end{aligned} \tag{5.1}$$

Lemma 5.1. *Let in the conditions of Lemma* 4.4 $c_0 > 0$. *Then the functions* $\Theta(t)$, $\Lambda(t)$ *satisfy the differential inequality*

$$\frac{1}{2}\Theta'(t) + \min\{a_0, c_0\}\,\Lambda_\sigma(t) \leq \int_\Omega |f\, u|\, dx \quad \forall\, a.e.\, t \in (0,T).$$

Proof. For every $t,\, t + \Delta t \in [0,T]$ equality (5.1) gives the inequality

$$\begin{aligned}\frac{1}{2|\Delta t|}\Theta(\tau)\Big|_{\tau=t}^{\tau=t+\Delta t} &\leq -\frac{1}{|\Delta t|}\min\{a_0,\, c_0\}\int_t^{t+\Delta t} \Lambda_\sigma(\tau)\, d\tau\\ &\quad + \frac{1}{|\Delta t|}\int_t^{t+\Delta t}\int_\Omega |f\, u|\, dz.\end{aligned} \tag{5.2}$$

For $u \in \mathbf{W}(Q_T)$

$$\int_\Omega |D_i u|^{p_i(z)}\,dx,\ \int_\Omega |u|^{\sigma(z)}\,dx,\ \int_\Omega |u\,f|\,dx \in L^1(0,T),$$

whence for a.e. $t\in(0,T)$ every term on the right-hand side of (5.2) has a limit as $\Delta\,t\to 0$. It follows that there exists a limit of the left-hand side as $\Delta t\to 0$, whence

$$\frac{1}{2}\Theta'(t)+\min\{a_0,c_0\}\int_\Omega\left(\sum_{i=1}^n |D_i u|^{p_i(z)}+|u|^{\sigma(z)}\right)dx \le \int_\Omega |f\,u|\,dx.$$

□

Corollary 5.1. *Let in the conditions of Lemma* 5.1 $c_0\ge 0$. *Then*

$$\frac{1}{2}\Theta'(t)+a_0\,\Lambda(t)\le \int_\Omega |f\,u|\,dx \quad \forall\, a.e.\, t\in(0,T).$$

Lemma 5.2. *Let in the conditions of Lemma* 4.6

$$\frac{1}{\sigma^-(t)}+\frac{1}{p^-(t)}=\frac{1}{\nu^-(t)}\le\frac{n+1}{n}\quad \textit{for}\ \frac{1}{n}<\frac{1}{p^-(t)}\equiv\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i^-(t)}. \tag{5.3}$$

If

$$1<\frac{1}{\nu^+(t)}=\frac{1}{\sigma^+(t)}+\frac{1}{p^+(t)},\qquad \frac{1}{p^+(t)}\equiv\frac{1}{n}\sum_{i=1}^n\frac{1}{p_i^+(t)},$$

then the energy solution $u\in\mathbf{W}(Q_T)$ *of problem* (1.1) *for a.e.* $t\in(0,T)$ *satisfies the differential inequality*

$$\frac{1}{2}\Theta'(t)+K_1 L(t)\,\Theta^{\nu^+(t)}(t)\le K_2\int_\Omega |f|^{\sigma'(z)}\,dx \tag{5.4}$$

with the constants $M=\operatorname{ess\,sup}\Theta(t)$, $K_1=\frac{1}{2C}\min\{a_0,\,c_0\}$, $K_2=\left(\frac{2}{c_0}\right)^{\frac{1}{\sigma^--1}}$, C *from* (3.7), *and the coefficient*

$$L(t)=\begin{cases}1 & \textit{if}\ M\le 1,\\ M^{\nu^-(t)-\nu^+(t)} & \textit{otherwise.}\end{cases} \tag{5.5}$$

Proof. By Young's inequality

$$\int_\Omega |f\,u|\,dx\le\frac{c_0}{2}\int_\Omega |u|^{\sigma(z)}\,dx+\left(\frac{2}{c_0}\right)^{\frac{1}{\sigma^--1}}\int_\Omega |f|^{\sigma'(z)}$$

and by Lemma 5.1

$$\frac{1}{2}\Theta'(t)+\frac{1}{2}\min\{a_0,c_0\}\,\Lambda_\sigma(t)\le\left(\frac{2}{c_0}\right)^{\frac{1}{\sigma^--1}}\int_\Omega |f|^{\sigma'(z)}\,dx.$$

By Lemma 3.3

$$
\begin{aligned}
C\Lambda_\sigma(t) &\geq \min\left\{\Theta^{\nu^+(t)}(t),\, \Theta^{\nu^-(t)}(t)\right\}\\
&= \min\left\{M^{\nu^+(t)}\left(\frac{\Theta(t)}{M}\right)^{\nu^+(t)},\, M^{\nu^-(t)}\left(\frac{\Theta(t)}{M}\right)^{\nu^-(t)}\right\}\\
&\geq \left(\frac{\Theta(t)}{M}\right)^{\nu^+(t)} \min\left\{M^{\nu^+(t)}, M^{\nu^-(t)}\right\}\\
&= \Theta^{\nu^+(t)}(t)\min\left\{1,\, M^{\nu^-(t)-\nu^+(t)}\right\} \geq L(t)\,\Theta^{\nu^+(t)}(t).
\end{aligned}
$$

□

Remark 5.1. *Estimates on the "total energy" M are given in Lemmas* 4.5, 4.6.

Lemma 5.3. *Let $c_0 \geq 0$ and let* $\operatorname{ess\,sup} \Theta(t) = M$. *If*

$$1 < \frac{2}{p^+(t)} \leq \frac{2}{p^-(t)} \leq 1 + \frac{2}{n},$$

then the solution of problem (1.1) *satisfies the differential inequality*

$$\Theta'(t) + a_0\, L(t)\, \Theta^{\nu^+(t)}(t) \leq F(t) \tag{5.6}$$

with the exponents and the right-hand side

$$\nu^\pm(t) = \frac{p^\pm(t)}{2} \in \left[\frac{n}{n+2}, 1\right), \quad F(t) = K_2\|f(\cdot,t)\|_{2,\Omega}^{\frac{2\nu^+(t)}{2\nu^+(t)-1}},$$

the coefficients K_1, $L(t)$ from Lemma 5.2 *and a finite constant $K_2 \equiv K_2(K_1, M, n)$.*

Proof. By Corollary 5.1

$$\Theta'(t) + a_0\, L(t)\, \Theta^{\nu^+(t)} \leq 2\int_\Omega |f\, u|\, dx.$$

Applying Hölder's and Young's inequalities we estimate the right–hand side

$$\int_\Omega |u|\,|f|\,dx \leq \sqrt{\Theta(t)}\ \|f(\cdot,t)\|_{2,\Omega} \leq \frac{a_0}{2}\, L(t)\Theta^{\nu^+(t)}(t) + K_2\,\|f(\cdot,t)\|_{2,\Omega}^{\frac{2\nu^+(t)}{2\nu^+(t)-1}},$$

whence the assertion. □

6. Vanishing in a finite time

6.1. **Sufficient conditions for the finite time extinction.**

Theorem 6.1. *Let $u_0 \in L^2(\Omega)$, $f \in L^2(Q_T) \cap L^{\sigma'(z)}(Q_T)$, $c_0 > 0$, and let the exponents $\sigma(z)$, $p_i(z)$ satisfy the conditions of Lemma 5.2. If $f \equiv 0$ for $t \geq t_f$, and if*

$$\frac{1}{\nu^+(t)} = \frac{1}{\sigma^+(t)} + \frac{1}{n}\sum_{i=1}^{n} \frac{1}{p_i^+(t)} \geq \frac{1}{\nu_0} > 1, \quad \nu_0 = const, \tag{6.1}$$

then there exists $t_* \geq t_f$ such that

$$u(x,t) \equiv 0 \quad \text{in } \Omega \text{ for all } t \geq t_*.$$

The extinction moment t_* depends on $\|u_0\|$, $\|f\|_{\sigma'(\cdot),Q_T}$, n, $\overline{p}$, $\underline{p}$, $\sup \nu(t)$ and $\inf \nu(t)$.

Proof. By Lemma 5.2 the energy function $\Theta(t)$ satisfies the ordinary differential inequality

$$\frac{1}{2}\,\Theta'(t) + K_1 L(t) \Theta^{\nu^+(t)}(t) \leq 0 \quad \text{in } (t_f, T). \tag{6.2}$$

By virtue of (4.6) $\Theta(t) \leq M$ with a constant M depending only on the data. Let us introduce the new energy function $Z(t) \equiv \Theta(t)/M$ and write (6.2) in the form

$$Z'(t) + K_1 L(t) M^{\nu^+(t)-1} Z^{\nu^+(t)}(t) \leq 0 \quad \text{in } (t_f, T), \qquad Z(t_f) \leq 1.$$

Since $Z^{\nu^+(t)} \geq Z^{\nu_0}(t)$, the last inequality leads to the differential inequality

$$Z'(t) + 2\, K_1 L(t) M^{\nu^+(t)-1} Z^{\nu_0}(t) \leq 0 \quad \text{in } (t_f, T), \qquad Z(t_f) \leq 1.$$

Integration of this inequality gives: for $t \geq t_f$

$$\begin{aligned} Z^{1-\nu_0}(t) &= 1 - 2\, K_1 (1-\nu_0) \int_{t_f}^{t} \min\left\{1, M^{\nu^-(\tau)-\nu^+(\tau)}\right\} M^{\nu^+(\tau)-1}\, d\tau \\ &\leq 1 - 2\, K_1(1-\nu_0) \max\{1, 1/M\}\,(t - t_f). \end{aligned}$$

The conclusion follows now because $Z(t) \geq 0$. $\square$

The assertion of Theorem 6.1 remains true in the case $c_0 = 0$, but under another conditions on the nonlinearity exponents.

Theorem 6.2. *Let $c_0 \geq 0$, $u_0 \in L^2(\Omega)$, and let f satisfy the conditions of Lemma 4.5. and let the exponents $p_i(z)$ satisfy the conditions of Lemma 5.3. If $f \equiv 0$ for $t \geq t_f$, and*

$$1 + \frac{2}{n} \geq \frac{2}{p^-(t)} \geq \frac{1}{\nu^+(t)} = \frac{2}{p^+(t)} \geq \frac{1}{\nu_0} > 1, \tag{6.3}$$

then there exists $t_* \geq t_f$ such that

$$u(x,t) \equiv 0 \quad \text{in } \Omega \text{ for all } t \geq t_*.$$

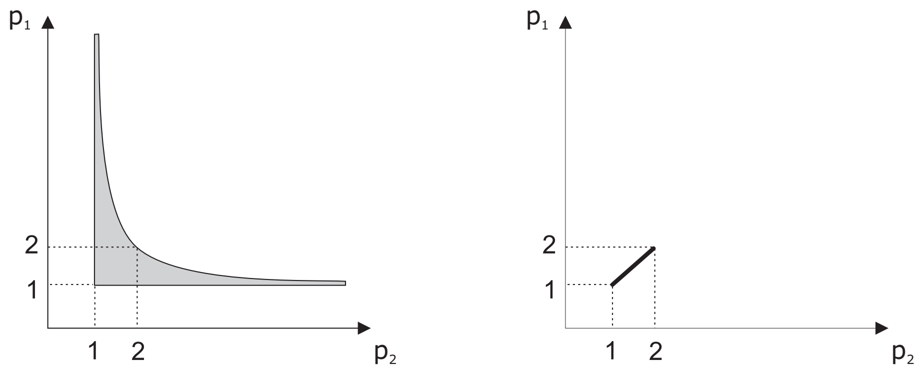

FIGURE 1. Admissible ranges of the exponents in the case $n = 2$

The proof is an imitation of the proof Theorem 6.1: the energy function $\Theta(t)$ satisfies the differential inequality (6.2) with the exponent $\nu^+(t)$ given in (6.3) and is globally bounded by virtue of (4.3).

6.2. **On the balance between slow and fast diffusion.** Let us briefly discuss the conditions on the nonlinearity exponents $p_i(z)$ which guarantee finite time vanishing of the solutions in the isotropic and anisotropic cases. For the diffusion equation

$$u_t = \sum_i D_i \left(|D_i u|^{p_i(z)-2} D_i u \right)$$

these conditions read are given in Theorem 6.2. In the special case of the isotropic diffusion and constant exponents of nonlinearity, $p_i \equiv p = const$, these conditions reduce to the well–known restrictions

$$2 > p \geq \frac{2n}{n+2}. \tag{6.4}$$

The former inequality of (6.4) guarantees extinction in a finite time (see [5, Ch. 2, Sec.2], while the latter provides the inclusion $W_0^{1,p}(\Omega) \subset L^2(\Omega)$. Moreover, it is known that in the case of constant isotropic diffusion and better regularity of the data this condition can be relaxed and transformed to the inequalities $2 > p > 1$ - see, e.g., [5, pp. 79-83]. We also refer to [6, 11] for the case of isotropic and variable nonlinearity.

Let us turn to the case of anisotropic and variable diffusion.

$\mathbf{n = 2}$. Condition (6.3) becomes

$$1 < p_1^+(t) < \frac{p_2^+(t)}{p_2^+(t) - 1}, \qquad 1 < p_2^+(t). \tag{6.5}$$

It follows that unlike the isotropic case, the solutions always extinct in a finite time, provided that the diffusion in the direction x_1 is fast in comparison with the diffusion in the direction x_2. In particular, even in the case of very slow diffusion in the direction x_2, i.e. when $p_2^+(t) \to \infty$, the localization effect still takes place if the diffusion in the direction x_1 is very fast.

Figure 1 illustrates the difference between the equations of anisotropic and isotropic diffusion in the case $n = 2$: the solutions of the equation of anisotropic diffusion vanish in a finite time if the point $(p_1^+(t), p_2^+(t))$ belongs to the shadowed zone for all t, while in the equation of isotropic diffusion the same effect takes place only if the point $(p_1(t), p_2(t))$ belongs to the open interval with the endpoints $(1, 1)$ and $(2, 2)$.

$\mathbf{n = 3}$. In this case the solutions of the equation of isotropic diffusion vanish in a finite time if

$$\frac{6}{5} < p^+(t) < 2.$$

Let us consider the situation when the diffusion is fast in the directions x_1, x_2. For the sake of simplicity we assume that $p_1^+(t) \equiv p_2^+(t)$. The solutions extinct in a finite if

$$\frac{3}{2} < \frac{1}{p_1^+(t)} + \frac{1}{p_2^+(t)} + \frac{1}{p_3^+(t)},$$

which leads to the following condition on the admissible rate of fast diffusion:

$$1 < p_1^+(t) = p_2^+(t) < \frac{4p_3^+(t)}{3p_3^+(t) - 2} \searrow \frac{4}{3} \quad \text{as } p_3^+(t) \nearrow \infty.$$

It follows that if the diffusion in the direction x_3 is very slow, the solutions vanish in a finite time, provided that the diffusion in the directions x_1, x_2 is suitably fast.

In case of fast diffusion in the direction x_1 ($p_1(z) \in (1, 2)$) and slow diffusion of the same rate in the directions x_2, x_3 ($p_2(z) = p_3(z) > 2$) we find that extinction takes place if

$$1 < p_1^+(t) < \frac{2p_2^+(t)}{3p_2^+(t) - 4} \quad \text{and} \quad p_2^+(t) = p_3^+(t) < 4.$$

In the limit case of fast diffusion in the direction x_1, $p_1^+(t) \searrow 1$, we arrive at the upper limit of the admissible rate of slow diffusion in the directions x_2, x_3: $p_2^+(t) = p_3^+(t) \nearrow 4$.

$\mathbf{n \geq 3}$. A similar analysis shows that in the case of higher space dimension the effect of finite time extinction in solutions of anisotropic equation takes place under stronger restrictions on the admissible rate of slow and fast diffusion in different directions.

7. Vanishing at a prescribed moment

Theorem 7.1. *Let the conditions of Theorem* 6.1 *be fulfilled. If the integral*

$$K(t_f) = S(0) \int_0^{t_f} \frac{F(\tau)}{S(\tau)}\, d\tau$$

with

$$S^{1-\nu_0}(t) = 2K_1(1 - \nu_0) \int_t^{t_f} L(\theta)\, d\theta, \qquad F(t) \equiv 2K_2 \int_\Omega |f|^{\sigma'(z)}\, dx,$$

the constants K_1, K_2 *from the conditions of Lemma* 5.2 *and* $L(t)$ *defined in* (5.5) *is convergent, and if*

$$G(t_f) \equiv \|u_0\|^2_{2,\Omega} - S(0) + K(t_f) \le 0,$$

then

$$u(x,t) \equiv 0 \ \text{in } \Omega \text{ for all } t \ge t_f.$$

Proof. By Lemma 5.2 the energy function $\Theta(t)$ satisfies the differential inequality

$$\Theta'(t) + 2K_1\, L(t)\, \Theta^{\nu_0}(t) \le F(t) \quad \text{in } (0,T)\ , \quad \Theta(0) = \|u_0\|^2_{2,\Omega}.$$

Let us denote by $S(t)$ the nonnegative solution of the problem

$$S'(t) + 2K_1\, L(t)\, S^{\nu_0}(t) = 0, \qquad S(0) = S_0 > 0.$$

The function $S(t)$ is given by the explicit formula

$$S(t) = \max\left\{0,\ \left(S(0)^{1-\nu_0} - 2K_1(1-\nu_0)\int_0^t L(\tau)\,d\tau\right)^{\frac{1}{1-\nu_0}}\right\}.$$

Let us fix the initial value $S(0)$ by the condition

$$\frac{S^{1-\nu_0}(0)}{2K_1(1-\nu_0)} = \int_0^{t_f} L(\tau)\,d\tau,$$

i.e., $S(t_f) = 0$. Notice that this choice of $S(0)$ transforms the function $S(t)$ to the form

$$S^{1-\nu_0}(t) = 2K_1(1-\nu_0)\int_t^{t_f} L(\tau)\,d\tau \quad \text{if } t \le t_f.$$

Let us consider the function $W(t) \equiv \Theta(t) - S(t)$ which satisfies on the interval $(0,t_f)$ the inequality

$$W'(t) + 2K_1\nu\, L(t)\int_0^1 \frac{d\lambda}{(\lambda\Theta(t) + (1-\lambda)S(t))^{1-\nu_0}}\, W(t) \le F(t).$$

Multiplying this inequality by

$$\exp\left(2K_1\nu_0 \int_0^t L(\tau)\int_0^1 \frac{d\lambda d\tau}{(\lambda\Theta(\tau) + (1-\lambda)S(\tau))^{1-\nu_0}}\right)$$

and then integrating over the interval $(0,t_f)$, we transform it to the form

$$\begin{aligned}\Theta(t_f) &\le \Theta(0) - S(0)\\ &\quad + \int_0^{t_f} F(\tau)\exp\left(2K_1\nu_0\int_0^\tau L(z)\int_0^1 \frac{d\lambda\, dz}{(\lambda\Theta(z) + (1-\lambda)S(z))^{1-\nu_0}}\right) d\tau.\end{aligned}$$

Let us notice that by virtue of the equation for $S(t)$

$$
\begin{aligned}
&2K_1\nu_0\int_0^{\tau} L(z)\int_0^1 \frac{d\lambda\,dz}{(\lambda\Theta(z)+(1-\lambda)S(z))^{1-\nu_0}}\\
&\qquad\leq 2K_1\int_0^{\tau}\frac{L(z)dz}{S^{1-\nu_0}(z)}\int_0^1\frac{\nu_0\,d\lambda}{(1-\lambda)^{1-\nu_0}}=-\int_0^{\tau}\frac{S'(z)}{S(z)}\,dz=-\ln\frac{S(\tau)}{S(0)}.
\end{aligned}
$$

It follows that

$$
\Theta(t_f)\leq\Theta(0)-S(0)\left(1-\int_0^{t_f}\frac{F(\tau)}{S(\tau)}d\tau\right)\equiv G(t_f)\leq 0.
$$

Since $\Theta(t)\geq 0$ for all $t>0$, it is necessary that $\Theta(t_f)=0$. Considering now the differential inequality for $\Theta(t)$ on the interval (t_f,T),

$$
\begin{cases}
\Theta'(t)+2K_1\,L(t)\,\Theta^{\nu_0}(t)\leq 0 & \text{for } t\geq t_f,\\
\Theta(t_f)=0, \qquad \Theta(t)\geq 0, &
\end{cases}
$$

we conclude that these conditions are satisfied only if $\Theta(t)\equiv 0$ for $t\geq t_f$, that is, if $\|u\|_{2,\Omega}(t)\equiv 0$. □

Remark 7.1. *It is easy to see that the equation $G(t_*)=0$ always has a solution if, given t_f,*

$$
\|u_0\|^2_{2,\Omega}+\int_{\Omega}|f|^{\sigma(z)}\,dx\leq\epsilon
$$

with a suitably small $\epsilon>0$.

The next theorem refers to the case $c_0\geq 0$ and is a byproduct of Theorem 7.1.

Theorem 7.2. *Let the conditions of Theorem 6.2 be fulfilled. If the integral $K(t_f)$ with*

$$
F(t)\equiv 2K_2\,\|f(\cdot,t)\|_{2,\Omega}^{\frac{2\nu^+(t)}{2\nu^+(t)-1}}
$$

is convergent and if $G(t_f)\leq 0$, then

$$
u(x,t)\equiv 0 \text{ in } \Omega \text{ for all } t\geq t_f.
$$

8. Large time behavior

According to Theorems 6.1, 6.2, every solution of problem (1.1) vanishes at a finite moment t_*, provided that $\nu^+(t)<1$ and $f(x,t)\equiv 0$ from some t_f on. Let us now study the behavior of $\|u\|_{2,\Omega}(t)$ in the cases when at least one of these conditions is violated.

Theorem 8.1. *Let the exponents $p_i(z)$ and $\sigma(z)$ satisfy the conditions of Lemma 5.2 if $c_0 \geq 0$, or Lemma 5.3 if $c_0 = 0$. Let $\Theta(t) \leq M$ for all $t > 0$. Denote*

$$\frac{1}{\nu^+(t)} = \begin{cases} \dfrac{1}{\sigma^+(t)} + \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} \dfrac{1}{p_i^+(t)} & \text{if } c_0 > 0, \\ \dfrac{2}{n}\displaystyle\sum_{i=1}^{n} \dfrac{1}{p_i^+(t)} & \text{if } c_0 = 0. \end{cases}$$

(a) *If $\nu^+(t) \leq 1$, then*

$$\|u\|^2_{2,\Omega}(t) \leq \mathrm{e}^{-2K_1 \min\{1,1/M\}\, t} \left(\|u_0\|^2_{2,\Omega} + \int_0^t F(\tau) \mathrm{e}^{2K_1 \min\{1,1/M\}\, \tau}\, d\tau \right)$$

with

$$F(t) = \begin{cases} 2K_2 \displaystyle\int_\Omega |f|^{\sigma'(z)}\, dx & \text{if } c_0 > 0, \\ 2K_2 \, \|f\|_{2,\Omega}^{\frac{2\nu^+(t)}{2\nu^+(t)-1}}(t) & \text{if } c_0 = 0 \end{cases}$$

and the constant K_2 from (5.4) if $c_0 > 0$, or from (5.6) if $c_0 = 0$.

(b) *If $\nu_0 \geq \nu^+(t) > 1$ and*

$$F(t) \leq f_0 (1+t)^{-\frac{\nu_0}{\nu_0 - 1}}$$

from some t_0 on with a positive constant f_0, then there exists $C = const$ such that

$$\|u\|^2_{2,\Omega}(t) \leq C(1+t)^{-\frac{\nu_0}{\nu_0-1}} \quad \text{for } t \geq t_0.$$

Remark 8.1. *Gathering assertions (a) and (b), we have that for $f \equiv 0$ for $t \geq t_*$ and $\nu^{\pm}(t) = 1$*

$$\|u\|^2_{2,\Omega} \leq \mathrm{e}^{-C\,t} \|u_0\|^2_{2,\Omega} \quad \text{as } t \to \infty,\ C = const.$$

Proof. (a) Since $\Theta(t) \leq M$, using (5.5) we continue inequalities (5.4), (5.6) as follows:

$$\begin{aligned} \Theta'(t) + 2K_1 \min\{1,\, 1/M\}\, \Theta(t) &\leq \Theta'(t) + 2K_1 L(t) M^{\nu^+(t)-1} \Theta(t) \\ &\leq \Theta'(t) + 2K_1 L(t) M^{\nu^+(t)} \left(\frac{\Theta(t)}{M} \right)^{\nu^+(t)} \leq F(t). \end{aligned}$$

The conclusion follows by Gronwall's inequality.

(b) Without loss of generality we may assume that $t_0 = 0$. The function $\Theta(t)$ satisfies the differential inequality

$$\begin{aligned}
&\Theta'(t) + 2K_1(1+M)^{-\nu_0}\Theta^{\nu_0}(t) \\
&\qquad \le \Theta'(t) + 2K_1 L(t) M^{\nu^+(t)} \left(\frac{\Theta(t)}{M}\right)^{\nu^+(t)} \\
&\qquad \le \Theta'(t) + 2K_1 L(t)\Theta(t)^{\nu^+(t)} \le f_0(1+t)^{-\frac{\nu_0}{\nu_0-1}} \quad \text{for } t \ge t_0.
\end{aligned}$$

Let $Y(t)$ be a solution of the equation

$$Y'(t) + 2K_1(1+M)^{-\nu_0}Y^{\nu_0}(t) = f_*(1+t)^{-\frac{\nu_0}{\nu_0-1}}$$

with the constant $f_* > 0$ to be defined. A solution of this equation is given by the explicit formula

$$Y(t) = A(1+t)^{-\frac{1}{\nu_0-1}}$$

with the parameter A chosen from the condition

$$G(A) \equiv -\frac{A}{\nu_0 - 1} + 2K_1(1+M)^{-\nu_0}A^{\nu_0} - f_* = 0.$$

This algebraic equation always has a solution $A^* > 0$ because $G(0) = -f_* < 0$, $G(\infty) = \infty$. Moreover, the solution A^* is estimated from below

$$A^* = \left[\frac{(1+M)^{\nu_0}}{2K_1}\left(\frac{A^*}{\nu_0-1} + f_*\right)\right]^{\frac{1}{\nu_0}} \ge \left[\frac{(1+M)^{\nu_0}}{2K_1} f_*\right]^{\frac{1}{\nu_0}}.$$

If we claim that $f_* \ge \max(f_0, 2K_1(1+M)^{-\nu_0}\Theta^{\nu_0}(0))$, then

$$\Theta(0) \le Y(0) = A^* \quad \text{and} \quad f_* \ge f_0.$$

Let us introduce the function $W(t) = \Theta(t) - Y(t)$. The function $W(t)$ satisfies the linear differential inequality

$$W'(t) + D\,W(t) \le (f_0 - f_*)(1+t)^{-\frac{\nu_0}{\nu_0-1}} \le 0 \quad \text{for } t > 0, \qquad W(0) \le 0,$$

with the coefficient

$$D = \frac{2K_1}{(1+M)^{\nu_0}}\frac{\Theta^{\nu_0}(t) - Y^{\nu_0}}{\Theta(t) - Y} \equiv \nu\int_0^1 (\lambda\Theta(t) + (1-\lambda)Y(t))^{\nu_0-1}\,d\lambda \ge 0.$$

It follows that $W(t) \le 0$ for all $t > 0$, i.e.,

$$0 \le \Theta(t) \le Y(t) = A^*(1+t)^{-\frac{1}{\nu_0-1}}.$$

□

9. Limit cases

Let us now turn to the situation when in inequalities (5.4), (5.6)

$$\nu^{+}(t) \nearrow 1 \quad \text{as } t \to \infty.$$

If this happens, the arguments based on comparison with solutions of equations with constant exponents of nonlinearity are no longer valid because the nonlinear differential inequality (5.4) eventually transforms into the linear one. We will rely on the following assertion.

Lemma 9.1. *Let a nonnegative function $\Theta(t)$ satisfy the conditions*

$$\left\{\begin{array}{l} \Theta'(t) + C\,\Theta^{\mu(t)}(t) \leq 0 \quad \textit{for a.e. } t \geq 0 \textit{ with } \mu(t) \in (0,1), \\ \Theta(t) \leq \Theta(0) < \infty. \end{array}\right. \tag{9.1}$$

If the exponent $\mu(t)$ is monotone increasing, then $\Theta(t) \equiv 0$ for all $t \geq t_*$ with t_* defined from the equality

$$C\int_0^{t_*} \Theta^{\mu(s)-1}(0)\,ds = \int_0^{\infty} \frac{dz}{\mathrm{e}^{z(1-\mu(z))}}.$$

Proof. Let us consider the function $J(t) = \Theta(t)/\Theta(0)$, satisfying the conditions

$$\forall\,\text{a.e.}\, t > 0 \quad J'(t) + C\Theta^{\mu(t)-1}(0)J^{\mu(t)} \leq 0, \quad J(0) = 1, \quad J'(t) \leq 0. \tag{9.2}$$

Introducing the new independent variable $\tau = C\displaystyle\int_0^t \Theta^{\mu(t)-1}(0)\,dt$, the exponent $a(\tau) \equiv \mu(t)$ and the new thought function $I(\tau) \equiv J(t)$, we find that the function $I(\tau)$ satisfies the conditions

$$\forall\,\text{a.e.}\, \tau > 0 \quad I'(\tau) + J^{a(\tau)} \leq 0, \quad I(0) = 1, \quad I(\tau) \geq 0, \quad I'(\tau) \leq 0.$$

By monotonicity of $I(\tau)$ there is an interval $[0,\epsilon)$ where $I(\tau) > 0$, otherwise $I(\tau) \equiv 0$ for all $\tau \geq 0$. Since $I(\tau) \leq 1$ and $a(\tau) > 0$, then $I(\tau) \leq I^{a(\tau)}(\tau)$ which leads to the inequality

$$I'(\tau) + I(\tau) \leq I'(\tau) + I^{a(\tau)}(\tau) \leq 0. \tag{9.3}$$

This inequality yields $I(\tau) \leq e^{-\tau}$, whence

$$\tau \leq -\ln I(\tau) \quad \text{and} \quad a(\tau) \leq a\left(-\ln I(\tau)\right). \tag{9.4}$$

Combining this inequality with (9.2) we have that

$$I'(\tau) + I^{a(-\ln I(\tau))}(\tau) \leq 0. \tag{9.5}$$

The straightforward integration of this inequality over the interval $(0,\tau)$ gives:

$$\int_1^{I(\tau)} \frac{dr}{r^{a(-\ln r)}} \leq -\tau.$$

Introducing the new variable $z = -\ln r$, we rewrite it in the form

$$\int_0^{-\ln I(\tau)} \frac{dz}{\mathrm{e}^{z(1-a(z))}} \geq \tau \qquad \text{for } \tau \geq 0.$$

According to the choice of τ_*

$$\int_0^{-\ln I(\tau_*)} \frac{dz}{\mathrm{e}^{z(1-a(z))}} \geq \tau_* = \int_0^{\infty} \frac{dz}{\mathrm{e}^{z(1-a(z))}},$$

which is impossible unless $I(\tau_*) = \Theta(t_*)/\Theta(0) = 0$. □

Theorem 9.1. *Let in the conditions of Theorem* 8.1 $f \equiv 0$ *and* $\Theta(t) \leq M$. *Assume that* $\nu^+(t)$ *is monotone increasing and* $\nu^+(t) \nearrow 1$ *as* $t \to \infty$. *If the equation*

$$2K_1 \min\{1, 1/M\} \int_0^R \|u_0\|_{2,\Omega}^{2(\nu^+(s)-1)}\, ds = \int_0^{\infty} \frac{dz}{\mathrm{e}^{z(1-\nu^+(z))}}$$

has a root $R = t_*$, *then the solution of problem* (1.1) $u \in \mathbf{W}(Q)$ *vanishes at the instant* $t = t_*$:

$$u \equiv 0 \text{ in } \Omega \text{ for } t \geq t_*.$$

Proof. By Lemmas 5.2, 5.3 for every finite $T > 0$ the energy function $\Theta(t)$ satisfies the ordinary differential inequality (5.4) or (5.6), which can be written in the form

$$\Theta'(t) + 2K_1 \min\{1, 1/M\}\, \Theta^{\nu^+(t)}(t) \leq 0.$$

The assertion follows then from Lemma 9.1. □

References

[1] Yu.A. Alkhutov and S. Antontsev and V. Zhikov, *Parabolic equations with variable order of nonlinearity*, The collection of works of Institute of mathematics NAS of Ukraine, Vol. 5, Nu.3, (2008).

[2] S. Antontsev and M. Chipot, *Anisotropic equations: uniqueness and existence results*, Differential and Integral Equations, Vol.21, Numbers 5-6(2008), 401-419.

[3] S. Antontsev and M. Chipot and Y. Xie, *Uniqueness results for equations of the $p(x)$-Laplacian type*, Advances in Mathematical Sciences and Applications, Volume 17, Nu.1 (2007), pp. 287-304.

[4] S. Antontsev and L. Consiglieri, *Elliptic boundary value problems with nonstandard growth conditions*, Nonlinear analysis (2008), in press.

[5] S. N. Antontsev, J. I. Díaz, and S. Shmarev, *Energy methods for free boundary problems*, Progress in Nonlinear Differential Equations and their Applications, 48, Birkhäuser Boston Inc., Boston, MA, 2002. Applications to nonlinear PDEs and fluid mechanics.

[6] S. Antontsev and S. Shmarev, *A model porous medium equation with variables exponent of nonlinearity: existence, uniqueness and localization properties of solutions*, Nonlinear Analysis Serie A: Theory and Methods, 60 (2005), pp. 515–545.

[7] ———, *Existence and uniqueness theorems for parabolic equations with anisotropic non-standard growth condition*, Publicacions Matematiques de l'Universitat Autonoma de Barcelona(2008), in press.

[8] ———, *Elliptic equations and systems with nonstandard growth conditions: existence, uniqueness and localization properties of solutions*, Nonlinear Anal., 65 (2006), pp. 728–761.

[9] ———, *Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions*, in Handbook of Differential Equation, M. Chipot and P. Quittner, eds., vol. 3, Elsevier, 2006, ch. 1, pp. 1–100.

[10] ———, *Parabolic equations with anisotropic nonstandard growth conditions*, in Internat. Ser. Numer. Math., vol. 154, Birkhäuser, Verlag Basel/Switzerland, 2006, pp. 33–44.

[11] ———, *Extinction of solutions of parabolic equations with variable anisotropic nonlinearities*, in Proceedings of the Steklov Institute of Mathematics, vol. 261, Steklov Institute of Mathematics, 2008, pp. 11–21.

[12] ———, *Localized solutions of anisotropic parabolic equations*, Nonlinear Analysis (2008), in press.

[13] ———, *Anisotropic parabolic equations with variable nonlinearity*, Preprint Pre-2007-013, University of Lisboa, CMAF, 2007. Downloadable at: http://www.ptmat.fc.ul.pt/preprints/preprints.html.

[14] ———, *Localized solutions of anisotropic parabolic equations*, Preprint Pre-2008-007, University of Lisboa, CMAF, 2008. Downloadable at: http://www.ptmat.fc.ul.pt/preprints/preprints.html.

[15] ———, *Existence and uniqueness theorems for parabolic equations with anisotropic nonstandard growth condition*, Publicacions Matematiques de l'Universitat Autonoma de Barcelona(2008), in press.

[16] ———, *Localized solutions of anisotropic parabolic equations*, Nonlinear Analysis (2008), in press.

[17] ———, *On the localization of solutions of elliptic equations with nonhomogeneous anisotropic degeneration*, Sibirsk. Mat. Zh., 46 (2005), pp. 963–984.

[18] S. N. Antontsev and V. V. Zhikov, *Higher integrability for parabolic equations of $p(x,t)$-Laplacian type*, Adv. Differential Equations, Vol.10, Number 9(2005),1053–1080.

[19] M. Bendahmane and K. H. Karlsen, *Renormalized solutions of an anisotropic reaction-diffusion-advection system with L^1 data*, Commun. Pure Appl. Anal., 5 (2006), pp. 733–762.

[20] M. Bendahmane and M. Saad, *Entropy solution for anisotropic reaction-diffusion-advection systems with L^1 data*, Rev. Mat. Complut., 18 (2005), pp. 49–67.

[21] L. Diening, *Maximal function on generalized Lebesgue spaces $L^{p(\cdot)}$*, Math. Inequal. Appl., 7 (2004), pp. 245–253.

[22] D. Edmunds and J. Rákosnǐk, *Sobolev embeddings with variable exponent*, Studia Math., 143(3) (2000), pp. 267–293.

[23] P. Harjulehto and P. Hästö, *An overview of variable exponent Lebesgue and Sobolev spaces*, in Future trends in geometric function theory, vol. 92 of Rep. Univ. Jyväskylä Dep. Math. Stat., Univ. Jyväskylä, Jyväskylä, 2003, pp. 85–93.

[24] O. Kováčik and J. Rákosník, *On spaces $L^{p(x)}$ and $W^{k,p(x)}$*, Czechoslovak Math. J., 41(116) (1991), pp. 592–618.

[25] Feng-Quan, Li, *Anisotropic Elliptic Equations in L^m*, Journal of Convex Analysis, Vol. 8(2), (2001), pp. 417-422.

[26] F. Li, *Anisotropic parabolic equations with measure data. II*, Math. Nachr., 279 (2006), pp. 1585–1596.

[27] F. Li and H. Zhao, *Anisotropic parabolic equations with measure data*, J. Partial Differential Equations, 14 (2001), pp. 21–30.

[28] J. Musielak, *Orlicz spaces and modular spaces*, vol. 1034 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, 1983.

[29] S. Samko, *On a progress in the theory of Lebesgue spaces with variable exponent: maximal and singular operators*, Integral Transforms Spec. Funct., 16 (2005), pp. 461–482.

[30] M. Sango, *On a doubly degenerate quasilinear anisotropic parabolic equation*, Analysis (Munich), 23 (2003), pp. 249–260.

[31] K. N. Soltanov, *Some nonlinear equations of the nonstable filtration type and embedding theorems*, Nonlinear Anal., 65 (2006), pp. 2103–2134.

[32] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat., 18 (1969), pp. 3–24.

[33] V. V. Zhikov, *On the density of smooth functions in Sobolev-Orlich spaces*, Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI), 310 (2004), pp. 1–14.

CMAF, Universidade de Lisboa, Portugal
E-mail address: anton@ptmat.fc.ul.pt, antontsevsn@mail.ru

Departamento de Matemáticas, Universidad de Oviedo, España
E-mail address: shmarev@orion.ciencias.uniovi.es