The $(p-q)$ coupled fluid-energy systems

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Abstract We investigate the nonlinear coupled system of elliptic partial differential equations which describes the fluid motion and the energy transfer what we call *the* $(p-q)$ *coupled fluid-energy system* due to p and q coercivity parameters correlated to the motion and heat fluxes, respectively. Due to the simultaneous action of the convective-radiation effects on a part of the boundary, such system leads to a boundary value problem. We present existence results of weak solutions under different constitutive laws for the Cauchy stress tensor with $p > 3n/(n+2)$, in a n-dimensional space. If the Joule effect is neglected in the energy equation, the existence result is stated for a broader class of fluids such that $p > 2n/(n+1)$, and related q-coercivity parameter to the heat flux.

1 The formulation of the problem

Let Ω be an open bounded set of $\mathbb{R}^n$ $(n > 1)$ with a sufficiently smooth boundary $\partial\Omega$. The equations governing the heat transport in incompressible viscous fluids at steady-state consist of

$$(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\tau = -\nabla\pi + \mathbf{f} \quad \text{in} \quad \Omega \tag{1}$$

$$\nabla\cdot\mathbf{u} = \sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i} = 0 \quad \text{in} \quad \Omega \tag{2}$$

$$\mathbf{u}\cdot\nabla e - \nabla\cdot(\chi(\cdot,e)\mathbf{a}(\nabla e)) = \tau : D\mathbf{u} + g \quad \text{in} \quad \Omega. \tag{3}$$

Here $\mathbf{u}$ and e are unknown functions denoting the fluid velocity vector and the specific (i.e., per unit mass) internal energy, respectively, the density ρ is assumed equal

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to one, $\mathbf{f}$ and g represent the external and the heat forces, respectively, π denotes the pressure, and $D = (\nabla + \nabla^T)/2$ denotes the symmetrized velocity gradient. We suppose the generalized Fourier law for heat flux, where χ denotes the diffusity,

$$\mathbf{q} = -\chi(\cdot, e)\mathbf{a}(\nabla e)$$

to be consistent with the following constitutive relations between the deviator stress tensor τ and the kinematical and thermal quantities [14].

The differentiable flux. The Cauchy stress tensor σ for the class of non-Newtonian fluids considered here as dependent on the temperature is defined by

$$\sigma = -\pi I + \nu(\cdot, \theta)\tau(D\mathbf{u}), \tag{4}$$

where I is the identity matrix, ν the viscosity, θ the temperature, and $\tau : \mathbb{M}^{n\times n}_{\mathrm{sym}} \to \mathbb{M}^{n\times n}_{\mathrm{sym}}$ is a continuous function which satisfies the conditions of p-coercivity

$$\exists \tau_1 > 0 \ \exists \varphi_1 \in L^1(\Omega): \quad \tau(\varkappa) : \varkappa \geq \tau_1 |\varkappa|^p - \varphi_1, \tag{5}$$

of polynomial growth of the power $p-1$

$$\exists \tau_2 > 0 \ \exists \varphi_2 \in L^{p/(p+1)}(\Omega): \quad |\tau(\varkappa)| \leq \tau_2 |\varkappa|^{p-1} + \varphi_2, \tag{6}$$

and of strict monotonicity

$$\Big(\tau(\zeta) - \tau(\varkappa)\Big) : (\zeta - \varkappa) > 0, \tag{7}$$

for any symmetric matrix $\zeta, \varkappa \in \mathbb{M}^{n\times n}_{\mathrm{sym}}$, $\zeta \neq \varkappa$, and taking into account the convention on implicit summation over repeated indices $\zeta : \varkappa = \zeta_{ij}\varkappa_{ij}$.

The non-differentiable flux. The deviator stress tensor τ is defined as a subgradient, i.e.,

$$\tau \in \nu(\cdot, \theta)\partial F(D\mathbf{u}), \tag{8}$$

where ∂ denotes the subdifferential of F at the point $D\mathbf{u}$ with F a convex functional known as superpotential. Indeed $F : \mathbb{M}^{n\times n}_{\mathrm{sym}} \to \mathbb{R}^+_0$ is a continuous and strictly convex function such that $F(0) = 0$, and for some $p > 1$

$$\exists \tau_1, \tau_2 > 0: \ \tau_1 |\zeta|^p \leq F(\zeta) \leq \tau_2(|\zeta|^p + 1), \ \forall \zeta \in \mathbb{M}^{n\times n}_{\mathrm{sym}}, \tag{9}$$

with $|\zeta| = (\zeta : \zeta)^{1/2}$.

Considering that the specific heat capacity at constant volume is

$$c_v(\theta)=\frac{de}{d\theta}(\theta)>0,$$

we assume invertible the nonlinear relation between the specific internal energy e and the temperature θ

$$e=e(\theta)\Leftrightarrow\theta=\theta(e).$$

Then the viscosity is given by $\mu(\cdot,e)=\nu(\cdot,\theta(e))$.

Definition 1. We say that (1)-(3) is a $(p-q)$ *coupled fluid-energy system* if τ obeys (4)-(7) or (8)-(9) and $\mathbf{a}$ corresponds to a nonlinear generalization related with the $p-$growth of the constitutive law for the flow, that is, $\mathbf{a}:\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function which satisfies the conditions

$$\exists\alpha_1>0\ \exists\psi_1\in L^1(\Omega):\quad \mathbf{a}(\varkappa)\cdot\varkappa\geq\alpha_1|\varkappa|^q-\psi_1 \tag{10}$$

$$\exists\alpha_2>0\ \exists\psi_2\in L^{q/(q+1)}(\Omega):\quad |\mathbf{a}(\varkappa)|\leq\alpha_2|\varkappa|^{q-1}+\psi_2,\quad\forall\varkappa\in\mathbb{R}^n \tag{11}$$

$$\Big(\mathbf{a}(\zeta)-\mathbf{a}(\varkappa)\Big)\cdot(\zeta-\varkappa)>0,\quad\forall\zeta,\varkappa\in\mathbb{R}^n,\ \zeta\neq\varkappa. \tag{12}$$

The application of the developed theory allows to avoid the study of the free boundaries between the different flow regions (e.g. rigid/plastic zones). The $(p-q)$ coupled fluid-energy system includes the Navier-Stokes fluid coupled with Fourier law $(p=q)$, namely when $\tau\equiv id$ $(p=2)$, $\mathbf{a}\equiv\mathbf{id}$ $(q=2)$ and the thermal conductivity given by

$$k(\cdot,\theta)=\chi(\cdot,e(\theta))c_v(\theta).$$

This model also includes the generalized Newtonian fluids (see, for instance, [1, 2, 13, 16] and the references therein) in particular the power-law fluid and its variants describing as the shear thinning and shear thickening behaviors, the modified Navier-Stokes system introduced by Ladyzenskaya [11], and the p-Laplacian. Otherwise the Bingham viscoplastic fluid does not flow at all unless acted on by at least some critical shear stress (the plasticity threshold) dependent on the temperature θ which can be modeled by (8).

The Lipschitz continuous boundary $\partial\Omega$ is assumed to consist of two disjoint open parts Γ_0 and Γ such that $\partial\Omega=\bar{\Gamma}_0\cup\bar{\Gamma}$ and $\mathrm{meas}(\Gamma_0)>0$. For p and $q>1$ correlated one each other, we present a solution $(\mathbf{u},e)\in\mathbf{W}^{1,p}(\Omega)\times W^{1,q}(\Omega)$ satisfying what we call *the* $(p-q)$ *coupled fluid-energy system* under the following boundary conditions

$$\mathbf{u}=\mathbf{0}\text{ on }\partial\Omega \tag{13}$$

$$e=e_0\text{ on }\Gamma_0\quad \chi(\cdot,e)\mathbf{a}(\nabla e)\cdot\mathbf{n}+\gamma(\cdot,e)=h\text{ on }\Gamma:=\partial\Omega\setminus\bar{\Gamma}_0, \tag{14}$$

where $\mathbf{n}=(n_i)$ denotes the unit outward normal to Γ. For the sake of clarity we assume that $e_0\equiv0$.

Traditionally the radiation effects have been described by Stefan-Boltzmann law, which represents 'radiation-to-infinity'. The radiative heat flux is given by $q_r=\varepsilon\sigma(\theta^4-\theta_0^4)$, where ε represents the emissivity coefficient, σ the Stefan-Boltzmann constant, θ the temperature which satisfies $\theta=\theta(e)$, and θ_0^4 the effective external radiation temperature. Hence, γ and h denote the outgoing radiation, $\gamma(e)=\varepsilon\sigma\theta^4(e)$, and the incoming radiation, $h=\varepsilon\sigma\theta_0^4$, respectively. This emission reflects the radiative flux on a convex Γ [10].

However industrial devices such as furnaces with surfaces which permit emitted as well as incident radiation require convective-radiation effect coupled with a nonlocal boundary condition. Thus, when the surface Γ is not convex, the outgoing radiation is a combination of emission and reflected fraction of incoming radiation which receives radiation from other parts of itself (see [5, 6] and the references therein). Moreover, since Γ is not an enclosure, then $\gamma:\Gamma\times\mathbb{R}\to\mathbb{R}$ is a Carathéodory function such that

$$\exists l\geq 1,\ \exists\gamma_1>0,\qquad \gamma(x,e)\,\mathrm{sign}\,e\geq\gamma_1|e|^l, \tag{15}$$

$$\exists\gamma_2>0,\qquad |\gamma(x,e)|\leq\gamma_2(1+|e|)^l, \tag{16}$$

$$(\gamma(x,e)-\gamma(x,\xi))\mathrm{sign}(e-\xi)>0,\ \text{ a.e. } x\in\Gamma,\ \forall e,\xi\in\mathbb{R}, \tag{17}$$

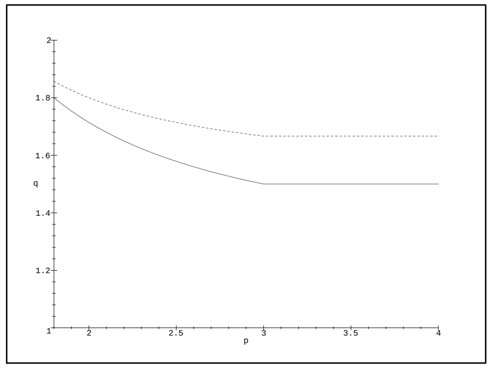
for limited physical values of the emissivity, $0\leq\varepsilon(x)\leq 1(\varepsilon\not\equiv 0)$. When $l=1$, (14) corresponds to a convection condition. In the case $l=4$, it corresponds to the radiative heat transfer condition.

We refer to the work [3] where the class of non-Newtonian fluids has a non-differentiable velocity-stress flux but a Fourier law of heat conduction to the heat flux, under a convective boundary condition, that is, $\gamma(\cdot,e)=h(\cdot,e)e$ with a bounded Carathéodory function h. In the work [5] the Navier-Stokes-Fourier fluid is studied under a convective-radiative boundary condition, but the exponent l is restricted undirectly by the exponents couple $(p-q)$. The reader can find in [7] the study for the coupled fluid-energy system when slip boundary conditions are also taken into account. The existence of solution to the coupled system for $2n/(n+1)<p\leq 3n/(n+2)$ based on the $L^\infty-$ truncation method can be found in [6]. The unsteady-state case on the $(p-q)$ coupled fluid-energy systems has been studied since [4].

The outline of the paper is as follows: in next section we establish the appropriate functional framework and we state the main results. Sect. 3 is devoted to the proofs of the existence of a solution for the coupled fluid-energy system when the Joule effect is neglected, in Sect. 3.1 for $q>2n/(n+1)$ if $p\geq n$, and $q>2np/(p(n+2)-n)$ if $3n/(n+2)<p<n$ (see Fig. 1), in Sect. 3.2 for $q>np/(p(n+1)-n)$ if $2n/(n+1)<p\leq 3n/(n+2)$, and in Sect. 3.3 when the external forces involve the energy. In Sect. 4 we prove the existence of a weak solution provided that $q>2-1/n$ if $p\geq n$, and $q>n(2p-1)/(p(n+1)-n)$ if $3n/(n+2)<p<n$ (see Fig. 1), when the Joule effect is taken into account. We apply different fixed point arguments.

The problem of finding a weak solution for the larger range to the coercivity parameter $2n/(n+2)<p\leq 2n/(n+1)$ is still an open problem for coupled fluid-energy systems.

Fig. 1 At 3D, the relations $(p-q)$: $q=6p/(5p-3)$ if $9/5<p<3$, $q=3/2$ if $p\geq 3$ (solid line) and $q=(6p-3)/(4p-3)$ if $9/5<p<3$, $q=5/3$ if $p\geq 3$ (dashed line)



2 Existence results

In the framework of Lebesgue and Sobolev spaces, we introduce the Banach spaces, for $p,q>1$, $l\geq 1$ and $p'=p/(p-1)$,

$$\begin{aligned} \mathscr{V} &= \{\mathbf{v}\in \mathbf{C}_0^\infty(\Omega):\ \nabla\cdot\mathbf{v}=0 \text{ in } \Omega\} \\ H_p=\overline{\mathscr{V}}^{\|\cdot\|_{p,\Omega}} &= \{\mathbf{v}\in \mathbf{L}^p(\Omega):\ \nabla\cdot\mathbf{v}=0 \text{ in } \Omega,\ v_N=0 \text{ on } \partial\Omega\} \\ V_p=\overline{\mathscr{V}}^{\|\cdot\|_{1,p,\Omega}} &= \{\mathbf{v}\in \mathbf{W}_0^{1,p}(\Omega):\ \nabla\cdot\mathbf{v}=0 \text{ in } \Omega\} \\ Y_{p'} &= \{\tau=(\tau_{ij}):\ \tau_{ij}=\tau_{ij}\in L^{p'}(\Omega),\ \nabla\cdot\tau\in (V_p)'\} \\ X_{q,l} &= \{e\in W^{1,q}(\Omega)\cap L^l(\Gamma):\ e=0 \text{ on } \Gamma_0\}, \end{aligned}$$

where X' means the dual space of the Banach space X and it is implicit that the symbol $\cap$ represents the function and its trace. Applying the Trace Theorem, it follows

$$X_{q,l}\equiv W_q := \{e\in W^{1,q}(\Omega):\ e=0 \text{ on } \Gamma_0\}$$

if $q\geq nl/(n+l-1)$ or equivalently $l\leq q(n-1)/(n-q)$ for $q<n$. Considering the Poincaré inequality, we can endow the above spaces with the standard norms

$$\|\mathbf{v}\|_{V_p}=\|D\mathbf{v}\|_{p,\Omega}, \quad \|e\|_{X_{q,l}} := \|\nabla e\|_{q,\Omega}+\|e\|_{l,\Gamma}, \quad \|e\|_{W_q}=\|\nabla e\|_{q,\Omega}.$$

2.1 The differentiable flux

If we neglect the source term in the energy equation due to the Joule effect, the coupling behavior is due to the fact that physical parameters depend on the temperature. Indeed, these parameters depend not only on the temperature but on the position as well and this prevents us from using Kirchoff transformation to eliminate the nonlinearity in the conductive term of the energy equation. We assume that $\mu,\chi : \Omega\times\mathbb{R}\to\mathbb{R}^+$ are Carathéodory functions, that is, measurable with respect to $x\in\Omega$ for every $e\in\mathbb{R}$, and continuous with respect to $e\in\mathbb{R}$ for almost every $x\in\Omega$ such that

$$\exists\mu_1,\mu_2>0:\ \ \mu_1\leq\mu(\cdot,e)\leq\mu_2,\quad \forall e\in\mathbb{R},\ \text{a.e. in }\Omega; \tag{18}$$

$$\exists\chi_1,\chi_2>0:\ \ \chi_1\leq\chi(\cdot,e)\leq\chi_2,\quad \forall e\in\mathbb{R},\ \text{a.e. in }\Omega. \tag{19}$$

Let us state the following results.

Theorem 1. *Under the assumptions (5)-(7), (10)-(12), (15)-(16) and (18)-(19), for*

$$q>\frac{2np}{p(n+2)-n}\quad \textit{and}\quad \frac{3n}{n+2}<p<n\ , \tag{20}$$

or $q>2n/(n+1)$ and $p\geq n$, let

$$\mathbf{f}\in(V_p)',\quad g\in\left(W^{1,q}(\Omega)\right)'\quad \textit{and } h\in L^{(l+1)/l}(\Gamma). \tag{21}$$

Then the (p-q) coupled fluid-energy system has a weak solution $(\mathbf{u},e)\in V_p\times X_{q,l+1}$ in the following sense

$$\int_\Omega \mu(\cdot,e)\tau(D\mathbf{u}):D\mathbf{v}dx+\int_\Omega \mathbf{u}\otimes\mathbf{v}:D\mathbf{u}dx=\langle\mathbf{f},\mathbf{v}\rangle,\quad \forall\mathbf{v}\in V_p, \tag{22}$$

$$\int_\Omega \chi(\cdot,e)\mathbf{a}(\nabla e)\cdot\nabla\phi dx+\int_\Omega \phi\mathbf{u}\cdot\nabla e dx+\int_\Gamma\gamma(\cdot,e)\phi d\Gamma=$$
$$=\langle g,\phi\rangle+\int_\Gamma h\phi d\Gamma,\quad \forall\phi\in X_{q,l+1}, \tag{23}$$

where the symbol $\langle\cdot,\cdot\rangle$ denotes a generic duality pairing, not distinguished between scalar and vector fields.

Remark 1. The convective term $\int_\Omega \mathbf{w}\otimes\mathbf{v}:D\mathbf{u}dx$ is well defined for $\mathbf{u},\mathbf{v},\ \mathbf{w}\in V_p$ if $p\geq 3n/(n+2)$ [11, 12]. For $q>2n/(n+1)$ arbitrary, the convective term $\int_\Omega\phi\mathbf{u}\cdot\nabla e dx$ is well defined for $\mathbf{u}\in H_t$ and $e,\phi\in W^{1,q}(\Omega)$, if

$$\begin{cases} t\geq nq/(q(n+1)-2n)\\ q<n\end{cases}\ \text{or}\ \begin{cases} t>n'=n/(n-1)\\ q=n\end{cases}\ \text{or}\ \begin{cases} t\geq q'\\ q>n.\end{cases} \tag{24}$$

Moreover, both terms satisfy the anti-symmetry property due to the incompressibility condition (2) and the Dirichlet condition (13). Notice that the restriction $p>3n/(n+2)$ is a sufficient condition to the existence of $t\in]pn/(p(n+1)-2n),pn/(n-p)[$. The restriction $q>2pn/(p(n+2)-n)$ is a sufficient condition to the existence of $t<pn/(n-p)$ verifying (24).

Theorem 2. *Under all assumptions in Theorem 1 except (20) consider*

$$\frac{2n}{n+1}<p\leq\frac{3n}{n+2}\quad \textit{and}\quad q>\frac{np}{p(n+1)-n}\ . \tag{25}$$

Then there exists a solution to the problem (22)-(23), for $\mathbf{v} \in \mathscr{V}$ *and* $\phi \in C_0^\infty(\Omega)$.

Remark 2. The convective terms $\int_\Omega \mathbf{w}\otimes\mathbf{v} : D\mathbf{u}dx$ and $\int_\Omega \phi\mathbf{u}\cdot\nabla e dx$ are still meaningful for $\mathbf{v} \in \mathscr{V}$ and $\mathbf{w},\mathbf{u} \in V_p$ for $p \geq 2n/(n+1)$; and for $\phi \in L^\infty(\Omega)$, $e \in W^{1,q}(\Omega)$, considering $q > np/(p(n+1)-n)$ which means $nq/(n-q) > p'$ or equivalently $np/(n-p) > q'$. However the anti-symmetry properties are no more valid.

When the external body forces are dependent on the specific internal energy in the form $\mathbf{f} = (e-\bar{e})\mathbf{b}$, for some given functions $\bar{e}$ and $\mathbf{b}$, the coupled system is motivated by the buoyancy driven flow, also known as free or natural convection flow. The motion of a viscous fluid driven by buoyancy forces is in fact generated by density gradients which are not aligned with the gravitational acceleration vector $\mathbf{g}$ and the variation of density is usually provoked by some external heat source, that means, $\mathbf{b} = -\beta\mathbf{g}$ with β denoting the volumetric factor of thermal expansion. Assuming that the variation in density is negligible which corresponds to the constraint of incompressibility (2), we can state the following result.

Theorem 3. *There exists a solution in the conditions of Theorems 1 and 2, if* $\mathbf{f} = \mathbf{b}e$, *with* $\mathbf{b} \in \mathbf{L}^\infty(\Omega)$.

2.2 The nondifferentiable flux

The minimization problem related to (1)-(2) and (8) in its weak formulation (29) follows from the property of subdifferentiability

$$\partial F(D\mathbf{u}) = (-\nabla\cdot)\partial(F\circ D)(\mathbf{u}), \tag{26}$$

for a continuous convex functional F. Indeed, using the definition of subdifferential ∂ in (8) and considering (26) and (1), we find

$$\langle \mathbf{f} - (\mathbf{u}\cdot\nabla)\mathbf{u}, \mathbf{v}-\mathbf{u}\rangle \leq J(e,D\mathbf{v}) - J(e,D\mathbf{u}), \quad \forall \mathbf{v} \in V_p,$$

which corresponds to (29), where $J : L^1(\Omega)\times V_p \to \mathbb{R}_0^+$ is such that

$$J(e,\mathbf{v}) = \int_\Omega \mu(e)F(D\mathbf{v})dx.$$

Finally we assume that

$$\mathbf{f} \in (V_p)', \quad g \in L^1(\Omega) \quad \text{and} \quad h \in L^1(\Gamma). \tag{27}$$

Theorem 4. *Suppose that the assumptions (9)-(12), (15)-(19) and (27) are fulfilled. For*

$$q > \frac{n(2p-1)}{p(n+1)-n} \quad \textit{and} \quad \frac{3n}{n+2} < p < n \tag{28}$$

or $q > 2-1/n$ and $p \geq n$, then the (p-q) coupled fluid-energy system has a weak solution $(\mathbf{u},\tau,e) \in V_p \times Y_{p'} \times X_{r,l}$, for all $1 < r < (q-1)n/(n-1)$, in the following sense

$$J(e,\mathbf{v}) - J(e,\mathbf{u}) - \int_\Omega (\mathbf{u}\otimes\mathbf{u}) : D(\mathbf{v}-\mathbf{u})dx \geq \langle \mathbf{f}, \mathbf{v}-\mathbf{u}\rangle, \quad \forall \mathbf{v} \in V_p; \tag{29}$$

$$\int_\Omega (\chi(e)\mathbf{a}(\nabla e) - e\mathbf{u})\cdot\nabla\phi dx + \int_\Gamma \gamma(\cdot,e)\phi d\Gamma =$$
$$= \int_\Omega (\tau : D\mathbf{u} + g)\phi dx + \int_\Gamma h\phi d\Gamma, \quad \forall \phi \in W_{r/(r-q+1)}, \tag{30}$$

and (8) is satisfied.

The convective term in (30) is meaningful for $\mathbf{u} \in H_t$, $e \in W^{1,r}(\Omega)$ and $\phi \in W^{1,r/(r-q+1)}(\Omega)$ if $t \geq r'$. Thus the requirement $\max(pn/(p(n+1)-2n), r') \leq t < pn/(n-p)$ leads to the restriction (28). Then all the terms on (30) have sense, since $\phi \in W^{1,r/(r-q+1)}(\Omega) \hookrightarrow C(\bar{\Omega})$ for $r/(r-q+1) > n$, that is, $r < (q-1)n/(n-1)$.

3 The differentiable flux

Let us recall the Tychonoff extension to weak topologies of the Schauder fixed point theorem [17, pp. 452].

Theorem 5. *Let K be a nonempty closed bounded convex subset of a reflexive separable Banach space X. Let $\mathscr{L} : K \to K$ be a weakly sequential continuous operator. Then $\mathscr{L}$ has at least one fixed point.*

3.1 Proof of Theorem 1

The proof of Theorem 1 is based on the following fixed point argument (cf. Theorem 5): defining

$$K := \{(\mathbf{w},\xi) \in V_p \times X_{q,l+1} : \|\mathbf{w}\|_{V_p} \leq R_1, \|\xi\|_{X_{q,l+1}} \leq R_2\},$$

with R_1 and R_2 given as in (32) and (34), respectively, and the operator $\mathscr{L}$ by $\mathscr{L}(\mathbf{w},\xi) = (\mathbf{u},e)$, where $\mathbf{u}$ and e are the following fluid velocity and energy solutions, respectively.

FLUID VELOCITY SOLUTION. For a fixed $\mathbf{w} \in H_t$, for t verifying (24), and $\xi \in L^1(\Omega)$ the existence of the solution $\mathbf{u} = \mathbf{u}(\mathbf{w},\xi)$ for the problem

$$\int_\Omega (\mu(\cdot,\xi)\tau(D\mathbf{u}) - \mathbf{w}\otimes\mathbf{u}) : D\mathbf{v}dx = \langle \mathbf{f}, \mathbf{v}\rangle, \quad \forall \mathbf{v} \in V_p\ , \tag{31}$$

holds in V_p for all $p \geq 3n/(n+2)$. Choosing $\mathbf{v} = \mathbf{u} \in V_p$ as a test function in (31), we can state the following result.

Lemma 1. *Under the assumptions (5), (18) and (21), the solution* $\mathbf{u}$ *of (31) is such that:*

$$\|\mathbf{u}\|_{V_p} \leq \left(\frac{\|\mathbf{f}\|_{(V_p)'}^{p'}}{(\mu_1 \tau_1)^{p'}} + \frac{p' \mu_2}{\mu_1 \tau_1} \|\varphi_1\|_{1,\Omega} \right)^{1/p} := R_1. \tag{32}$$

ENERGY SOLUTION. The existence of the solution $e = e(\mathbf{u}, \xi)$ for the problem

$$\int_\Omega (\chi(\cdot,\xi)\mathbf{a}(\nabla e) - e\mathbf{u}) \cdot \nabla \phi dx + \int_\Gamma \gamma(\cdot, e) \phi d\Gamma = \\ = \langle g, \phi \rangle + \int_\Gamma h \phi d\Gamma, \quad \forall \phi \in X_{q,l+1}, \tag{33}$$

holds in $X_{q,l+1}$ for q satisfying (20). Choosing $\phi = e \in X_{q,l+1}$ as a test function in (33), we obtain the following result.

Lemma 2. *Under the conditions (10), (15), (19) and (21), the solution e of (33) satisfies the estimate*

$$\|e\|_{X_{q,l+1}} \leq C \left[\|g\|_{(W^{1,q}(\Omega))'}^{q'} + \|h\|_{(l+1)/l,\Gamma}^{(l+1)/l} + \chi_2 \|\psi_1\|_{1,\Omega} \right]^\lambda := R_2, \tag{34}$$

where $C = C(q, l, \Omega, \chi_1 \alpha_1, \gamma_1)$ *and* $\lambda = \lambda(q, l)$.

Henceforth we denote by C every positive constant depending on n, p, q, l, the domain Ω, and the constants $\tau_i, \alpha_i, \mu_i, \chi_i, \gamma_i$ $(i = 1, 2)$.

The operator $\mathscr{L}$ is well defined, since from classical theory of monotone operators (see for example [11, 12]), there exist $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi) \in V_p$ a unique solution to the system (31) as well as $e = e(\mathbf{u}, \xi) \in X_{q,l+1}$ a unique solution to the problem (33). Lemmas 1 and 2 guarantee that $\mathscr{L}$ maps K into itself. Since that, for $(p-q)$ verifying (20), $\mathbf{w}_m \rightharpoonup \mathbf{w}$ in V_p and $\xi_m \rightharpoonup \xi$ in $X_{q,l+1}$ imply that $\mathbf{u}_m \rightharpoonup \mathbf{u}$ in V_p and $e_m \to e$ in $W^{1,q}(\Omega)-$weak, $L^1(\Omega)-$strong and $L^l(\Gamma)-$strong, the proof of Theorem 1 holds (for details see [6]).

3.2 *Proof of Theorem 2*

The proof is based on a fixed point argument for the following approximate problem (cf. [6]). For $M \in \mathbb{N}$ and

$$t > \max\left(\frac{2p}{p-1}, \frac{nq}{q(n+1)-2n} \right),$$

find $(\mathbf{u}_M, e_M)$ satisfying

$$\int_\Omega \mu(\cdot,e_M)\tau(D\mathbf{u}_M):D\mathbf{v}dx+\int_\Omega \mathbf{u}_M\otimes\mathbf{v}:D\mathbf{u}_M dx+$$
$$+\frac{1}{M}\int_\Omega |\mathbf{u}_M|^{t-2}\mathbf{u}_M\cdot\mathbf{v}dx=\langle\mathbf{f},\mathbf{v}\rangle,\quad \forall\mathbf{v}\in\mathscr{V}; \tag{35}$$
$$\int_\Omega \chi(\cdot,e_M)\mathbf{a}(\nabla e_M)\cdot\nabla\phi dx+\int_\Omega \phi\mathbf{u}_M\cdot\nabla e_M dx+\int_\Gamma \gamma(\cdot,e_M)\phi d\Gamma=$$
$$=\langle g,\phi\rangle+\int_\Gamma h\phi d\Gamma,\quad \forall\phi\in C_0^\infty(\Omega). \tag{36}$$

In order to apply Theorem 5, let $\mathscr{L}:K\to K$ be the operator defined by

$$\mathscr{L}:(\mathbf{w},\xi)\in V_p\times X_{q,l+1}\mapsto(\mathbf{u}_M,e_M),$$

where $\mathbf{u}_M=\mathbf{u}_M(\mathbf{w},\xi)\in V_p\cap H_t$ is the solution to the system

$$\int_\Omega (\mu(\cdot,\xi)\tau(D\mathbf{u}_M)-\mathbf{w}\otimes\mathbf{u}_M):D\mathbf{v}dx+\frac{1}{M}\int_\Omega |\mathbf{u}_M|^{t-2}\mathbf{u}_M\cdot\mathbf{v}dx=$$
$$=\langle\mathbf{f},\mathbf{v}\rangle,\quad \forall\mathbf{v}\in V_p \tag{37}$$

and $e_M=e_M(\mathbf{u}_M,\xi)\in X_{q,l+1}$ is the solution to the equation

$$\int_\Omega (\chi(\cdot,\xi)\mathbf{a}(\nabla e_M)-e_M\mathbf{u}_M)\cdot\nabla\phi dx+\int_\Gamma \gamma(\cdot,e_M)\phi d\Gamma=$$
$$=\langle g,\phi\rangle+\int_\Gamma h\phi d\Gamma,\quad \forall\phi\in X_{q,l+1}. \tag{38}$$

The existence and uniqueness of solutions to (37) and (38) arises as in the proof of Theorem 1. Note that $2p/(p-1)\geq pn/(n-p)$ (the critical value of Sobolev-Rellich imbedding) for $p\leq 3n/(n+2)$. Moreover, the following uniform estimate holds

$$\|D\mathbf{u}_M\|_{p,\Omega}^p+\frac{1}{M}\|\mathbf{u}_M\|_{t,\Omega}^t\leq R_1^{1/p}$$

as well as (34). Thus $\mathscr{L}$ is a well defined operator that maps K into itself, and it is weakly sequential continuous. Hence Theorem 5 guarantees the existence of a weak solution $(\mathbf{u}_M,e_M)$ to the approximate problem (35)-(36). Passing to the limit as M tends to infinity we conclude the proof of Theorem 2 (for details see [6]).

3.3 Proof of Theorem 3

The proof is similar to the proofs of Theorems 1 and 2 freezing the term $\mathbf{f}=\mathbf{b}\xi\in L^{qn/(n-q)}(\Omega)$ in (31) and in (35), respectively, and defining R_2 by (34) and

$$R_1 := \left(\frac{R_2 \|\mathbf{b}\|_{\infty,\Omega}^{p'}}{(\mu_1\tau_1)^{p'}} + \frac{p'\mu_2}{\mu_1\tau_1}\|\varphi_1\|_{1,\Omega} \right)^{1/p}.$$

For these suitable $R_1, R_2 > 0$, the operator $\mathscr{L}$ is well defined. Its weakly sequential continuity holds by compactness arguments.

4 The nondifferentiable flux

PROOF OF THEOREM 4

First we recall the fixed point result for multivalued mappings due to Ky Fan and Glicksberg [17, pp. 452, Generalized Theorem of Kakutani].

Theorem 6. *Let K be a nonempty compact convex subset of a locally convex linear topological vector space X. Let $\mathscr{L} : K \to \mathscr{P}(K)$ be a multivalued upper semicontinuous operator such that the set $\mathscr{L}(z)$ is nonempty closed and convex for all $z \in K$. Then $\mathscr{L}$ has at least one fixed point.*

The proof of Theorem 4 is based on the following fixed point argument. Let us consider the space $X := V_p \times L^1(\Omega) \times X_{r,l}$ endowed with the product of weak topologies. Thus X becomes a locally convex Hausdorff topological vector space, and the ball

$$K = \{(\mathbf{w}, \upsilon, \xi) \in X : \|\mathbf{w}\|_{V_p} \leq R_1, \|\upsilon\|_{1,\Omega} \leq R_2, \|\xi\|_{X_{r,l}} \leq R_3\}$$

is a nonempty convex compact set in X, considering R_i $(i = 1,2,3)$ the chosen below positive constants. The operator $\mathscr{L}$ defined by

$$\mathscr{L}(\mathbf{w}, \upsilon, \xi) = \{(\mathbf{u}, \tau : D\mathbf{u}, e)\} \subset \mathscr{P}(K)$$

verifies the fixed point argument (cf. Theorem 6). This proof is slip in the following steps. For details, see [3] and [7].

FLUID VELOCITY SOLUTION. For every $\mathbf{w} \in H_t$, for t verifying (24), and $\xi \in L^1(\Omega)$ there exists a unique solution $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi)$ to the variational inequality

$$J(\xi, \mathbf{v}) - J(\xi, \mathbf{u}) - \int_\Omega (\mathbf{w} \otimes \mathbf{u}) : D(\mathbf{v} - \mathbf{u})dx \geq \langle \mathbf{f}, \mathbf{v} - \mathbf{u} \rangle, \quad \forall \mathbf{v} \in V_p. \tag{39}$$

The existence and uniqueness of a solution are consequences of classical results (see [12], for instance) on variational inequalities with convex continuous functionals. Choosing $\mathbf{v} = \mathbf{0}$ as a test function in (39) the following estimate holds

$$\|\mathbf{u}\|_{V_p} \leq \left(\frac{\|\mathbf{f}\|_{(V_p)'}}{\mu_1\tau_1} \right)^{1/(p-1)} := R_1.$$

STRESS SUBGRADIENT SOLUTIONS. Let $\mathbf{u}$ be the fluid velocity solution. From the duality theory of convex analysis [9, pp. 50-52], there exists a Lagrange multiplier $\varsigma = \varsigma(\xi, \mathbf{u}) \in Y_{p'}$ and from the Rham Theorem (see, for instance, [12]) there exists a pressure $\pi \in L^{p'}(\Omega)$ such that

$$\langle -\varsigma, D\mathbf{u}\rangle = J(\xi, D\mathbf{u}) + \int_\Omega \mu(\xi) F^* \left(\left| \frac{\varsigma}{\mu(\xi)} \right| \right) \text{ in } \Omega$$
$$(\mathbf{w}\cdot\nabla)\mathbf{u} - \nabla\pi + \nabla\cdot\varsigma = \mathbf{f} \text{ in } \Omega,$$

where F^* represents the conjugate functional of F. The estimates

$$\|\varsigma\|_{p',\Omega} \leq C(\|D\mathbf{u}\|_{p,\Omega}^{p-1} + 1)$$
$$\|\varsigma : D\mathbf{u}\|_{1,\Omega} \leq C(\|D\mathbf{u}\|_{p,\Omega}^{p-1} + 1)\|D\mathbf{u}\|_{p,\Omega} \leq C(R_1^p + R_1) := R_2$$

hold. We set the stress subgradient $\tau = -\varsigma$. Note that we have no uniqueness of τ. However it is known that the subdifferential of the convex functional F (the set of the subgradient solutions) is a convex set.

ENERGY SOLUTION. For each $M \in \mathbb{N}$, define

$$g_M = \frac{M(\upsilon + g)}{M + |\upsilon + g|} \in L^\infty(\Omega) \quad \text{and} \quad h_M = \frac{Mh}{M + |h|} \in L^\infty(\Gamma).$$

From classical elliptic theory, there exists a unique solution $e_M \in X_{q,l+1}$ to the following variational equality

$$\int_\Omega (\chi(\xi)\mathbf{a}(\nabla e_M) - e_M \mathbf{w}) \cdot \nabla\phi dx + \int_\Gamma \gamma(e_M)\phi d\Gamma = \int_\Omega g_M \phi dx + \int_\Gamma h_M \phi d\Gamma,$$

for all $\phi \in X_{q,l+1}$. Using L^1-data theory [8, 15], a unique solution $e = e(\mathbf{w}, \upsilon, \xi) \in X_{r,l}$ is obtained as the limit approximation of e_M and the estimate holds, independently on $\mathbf{w}$ and ξ,

$$\|e\|_{X_{r,l}} \leq C(\|\upsilon + g\|_{1,\Omega} + \|h\|_{1,\Gamma})^\lambda \leq C(R_2 + \|g\|_{1,\Omega} + \|h\|_{1,\Gamma})^\lambda := R_3,$$

with $\lambda = \lambda(n, r, q)$ some positive constant.

CONTINUOUS DEPENDENCE. Let $\{(\mathbf{w}_m, \upsilon_m, \xi_m)\}$ be a sequence in K and $\mathbf{u}_m = \mathbf{u}(\mathbf{w}_m, \xi_m)$, $\tau_m = \tau(\xi_m, \mathbf{u}_m)$ and $e_m = e(\mathbf{w}_m, \upsilon_m, \xi_m)$ be solutions given by the above steps, respectively. Then, there exists $(\mathbf{u}, \tau, e)$ solution to (29)-(30) and (8) verifying

$$\mathbf{w}_m \rightharpoonup \mathbf{w},\ \mathbf{u}_m \rightharpoonup \mathbf{u} \text{ in } V_p \hookrightarrow\hookrightarrow H_t \cap \mathbf{L}^p(\Gamma);$$
$$\upsilon_m \rightharpoonup \upsilon,\ \tau_m : D\mathbf{u}_m \rightharpoonup \tau : D\mathbf{u} \text{ in } L^1(\Omega);$$
$$\xi_m \rightharpoonup \xi,\ e_m \rightharpoonup e \text{ in } X_{r,l} \hookrightarrow\hookrightarrow L^1(\Omega) \cap L^1(\Gamma).$$

We conclude that the conditions of Theorem 6 are fulfilled and then Theorem 4 holds. For details see [7].

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