Remarks on a second order non-autonomous problem on the half-line

Ricardo Enguiça[*] and Luís Sanchez[†]

[*]*Área Científica de Matemática, Instituto Superior de Engenharia de Lisboa,*
Rua Conselheiro Emídio Navarro, 1 - 1950-062 Lisboa, Portugal.
E-mail : rroque@dec.isel.ipl.pt
[†]*Faculdade de Ciências da Universidade de Lisboa,*
Avenida Professor Gama Pinto 2, 1649-003 Lisboa, Portugal.
E-mail : sanchez@ptmat.fc.ul.pt

Abstract. We study the existence of positive solutions for the differential equation

$$u''(x) = a(x)u(x) - g(u(x)),$$

with $u'(0) = u(+\infty) = 0$, where a is a positive function and g satisfies some growth hypotheses. The main motivation is to check that some well known results concerning the existence of homoclinics for the autonomous case (where a is constant) extend to the nonautonomous equation.

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1. INTRODUCTION

The study of existence of positive homoclinics of the ordinary differential equation

$$u''(x) = a(x)u(x) - g(u(x)) \tag{1}$$

where $g(0) = 0$ is partially motivated by a problem in higher dimensions: the search for special stationary states of the Klein-Gordon type equation

$$\Phi_{tt} - \Delta\Phi + a^2\Phi = f(\Phi),$$

where $\Phi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a complex function, $a \in \mathbb{R}$ and $f(\rho\, e^{i\theta}) = f(\rho)e^{i\theta}$. Looking for a "standing wave" solution $\Phi(t,x) = e^{i\omega t}u(x)$, one is lead to the equation

$$-\Delta u + \left(a^2 - \omega^2\right) u = f(u). \tag{2}$$

The corresponding Euler-Lagrange functional is

$$\frac{1}{2}\int_{\mathbb{R}^N} \left(|\nabla u|^2 + \left(a^2 - \omega^2\right) u^2 - 2F(u)\right) dx,$$

where $F(u) = \int_0^u f(s)\,ds$, and for this integral to be well-defined, $|u|$ needs to vanish at $+\infty$. Our problem is somehow the corresponding in dimension one (the case of radial solutions of (2) when $N \geq 2$ yields a different kind of ODE), but working with a non-autonomous term $a(x)u$ instead. In [3], G. Cerami surveys the $\mathbb{R}^N$ non-autonomous case for $\mathbb{N} \geq 3$, under several conditions concerning the non-autonomous term. There, an overall picture is given of what is known in the cases where some symmetry properties in the domain and in the solution are required, in particular the case of radial solutions. Non symmetric problems are addressed as well.

Equations of type (1) have been studied in the last two decades, especially in the case where $g(u)$ is a superlinear power. In this note we are interested in superlinear functions $g(u)$ but we remark that with some changes we can consider the case where $g(u)$ is bounded. P. Korman and A. Lazer gave a variational approach for the cases $g(u) = u^3$ in [5] and $g(u) = u^p$, where $p > 1$, in [6]. In these papers, the coefficient $a(x)$ is increasing in $[0,+\infty[$. In this note we partially generalize some of those results by allowing a to have a different behaviour, although we confine ourselves to the case where a is even, thus reducing our problem to the half line $[0,+\infty[$. We shall solve a sequence of boundary value problems in $[0,T]$ and if we consider an appropriate sequence of T's tending to $+\infty$, a nontrivial solution of the infinite interval problem will be found as the limit of the corresponding solutions u_T. M. Grossinho, F. Minhós and S. Tersian also gave a similar variational approach for this problem in [4], but working with two simultaneous powers in the nonlinear term.

The autonomous problem has been completely solved by H. Berestycki and P. Lions [1] as they gave a necessary and sufficient condition for the problem

$$\begin{cases} -u'' = f(u) \\ u(\pm\infty) = 0 \end{cases}$$

to have a unique positive homoclinic (up to translation), and gave some important results concerning the shape of that solution, which will be used in Section 2. Some of the hypotheses used in this note are reminiscent of those used by Berestycki and Lions.

We treat a case where $a(x)$ is positive and has a behaviour, which, as far as we know, has deserved less attention in the literature: we mean the case where $a(x)$, while having a limit at $+\infty$, does not approach its limit in an increasing, or even monotonic way. The arguments that we will use to deal with equation (1) are also valid for the more general equation

$$u'' = (a(x) + \varepsilon\, b(x))\, u - c(x)\, g(u)\,,$$

where $a(x)$ is as above, ε is small enough, and $b(x)$ and $0 < \delta \leq c(x)$ are bounded functions.

The assumptions concerning $a(x)$ will be

(A_1) there exists $0 < a < A$ such that $0 < a(x) \leq A \;\forall x \geq 0$ and $\lim_{x\to+\infty} a(x) = a$,

(A_2) $J_A{}^* < 2J_a{}^*$.

Here $J_a{}^* = 2\int_0^{u_a(0)} u\sqrt{a - \frac{2}{p+1}u^{p-1}}\,du$ is the value of the Euler-Lagrange functional associated with the autonomous problem

$$\begin{cases} u'' = au - u^p \\ u'(0) = u(+\infty) = 0 \end{cases}$$

computed at its nontrivial solution u_a (see Section 2).

The main result of this paper is the following

Theorem 1.1. *Under the assumptions* (A_1), (A_2) *the boundary value problem*

$$\begin{cases} u'' = a(x)\,u - u^p \\ u'(0) = 0, \quad u(+\infty) = 0 \end{cases} \tag{3}$$

has a positive solution.

We also present in the *Appendix* a simpler approach using the shooting method for the case where $a(x)$ is a nondecreasing function.

2. THE AUTONOMOUS PROBLEM

In this section we make some considerations for the autonomous differential equation

$$u'' = a\,u - g\,(u)\,, \tag{4}$$

with $a \in \mathbb{R}^+$ and $g > 0$. We will divide the section into two parts: first we deal with an easier case where $g(u)$ is a power, and then we deal with the more general case. These are classic results (some of them based in the mentioned paper of H.Berestycki and P.Lions [1]), which we will use in the next section to deal with the non-autonomous case.

Let $G(u) = \int_0^u g(s)\,ds$. In the remainning of this note we will assume the following hypotheses:

(H_1) There exists $q > 2$ such that

$$0 < q\,G(u) \le u\,g(u), \quad \forall u \in]0, +\infty[$$

(H_2) $g(u) = o\,(u)$ at $x = 0$.

Hypotheses (H_1) and (H_2) are well known sufficient conditions for the existence of a positive solution via the Mountain-pass Theorem for the equation (4) with boundary conditions $u'(0) = 0$ and $u(T) = 0$, for $1 \le T < +\infty$. In fact, for the associated Euler-Lagrange functional

$$J_{a,T}(u) = \int_0^T \left(u'(x)^2 + a\,u(x)^2 - 2G(u_+)\right)\,dx,$$

defined in the functional space $H_T^* \equiv \left\{H^1[0,T]\colon u(T) = 0\right\}$, we have $J_{a,T}(0) = 0$, and, for $\varepsilon > 0$ small enough, if $\|u\| = \varepsilon$, then $J_{a,T}(u) > \delta(\varepsilon) > 0$. The Palais-Smale condition is satisfied and, setting $u_\lambda = \lambda(1 - x^2)$, it is easy to see that $J_{a,T}(u_\lambda^+) < 0$ for $\lambda > 0$ large enough (independent of $T > 1$). Since the autonomous problem has a unique solution, the positive solution obtained via mountain-pass is the well-known phase plane solution.

Consider first the case where $g(u) = u^p$, for $p > 1$, where hypotheses (H_1) and (H_2) are obviously verified. In this case the computations are easier to follow.

We know that there exists a positive homoclinic $u_a(x)$ at $u=0$ passing through $(\zeta_0,0)$, where $\zeta_0=(\frac{a(p+1)}{2})^{\frac{1}{p-1}}$ (in the general case $g(u)$, ζ_0 will be the smallest positive value u such that $au^2-2G(u)=0$).

Multiplying the differential equation by u' and integrating, we get

$$u'^2-au^2+\frac{2}{p+1}u^{p+1}=C, \quad C\in\mathbb{R}. \tag{5}$$

The homoclinic u_a corresponds to the constant $C=0$, and therefore we have

$$u_a'=-u_a\sqrt{a-\frac{2}{p+1}u_a^{p-1}}, \quad u_a(0)=\zeta_0.$$

Considering the formally associated Euler-Lagrange functional

$$J_a(u)=\int_0^{+\infty}\left(u'(x)^2+a\,u(x)^2-\frac{2}{p+1}u_+(x)^{p+1}\right)dx,$$

in the functional space $H^1[0,+\infty[$, the "critical level" of u_a satisfies

$$J_a(u_a)=\int_0^{+\infty}\left(u_a'^2+a\,u_a^2-\frac{2}{p+1}u_a^{p+1}\right)dx=2\int_0^{u_a(0)}u\sqrt{a-\frac{2}{p+1}u^{p-1}}du\equiv J_a{}^*.$$

Let us now consider the boundary value problem

$$\begin{cases} u''(x)=a\,u(x)-u(x)^p \\ u'(0)=0, \quad u(T)=0, \end{cases} \tag{6}$$

and the associated functional

$$J_{a,T}(u)=\int_0^T\left(u'^2+a\,u^2-\frac{2}{p+1}u_+{}^{p+1}\right)dx,$$

for $u\in H_T^*$. As we have seen, via the Mountain-pass Theorem, the boundary value problem (6) has a nontrivial positive solution $u_{a,T}$, which we identify also in the phase plane.

Proposition 2.1. *The solution $u_{a,T}$ of* (6) *satisfies* $\lim_{T\to+\infty}u_{a,T}(0)=u_a(0)$.

Proof. Using (5) we have ${u'_{a,T}}^2=a\,{u_{a,T}}^2-\frac{2}{p+1}{u_{a,T}}^{p+1}+C_T$ for some constant C_T, and consequently $u'_{a,T}(T)^2=C_T>0$. If there exists a sequence of $T's$ tending to $+\infty$ such that $u'_{a,T}(T)\to c<0$, then, by a phase plane analysis, knowing that the trajectories cannot cross each other, we could easily see that trajectories that cross the u' axis close to c could not be positive for an arbitrarily large interval $[0,T]$. So we must have $u'_{a,T}(T)\to 0$ and therefore $C_T\to 0$. Since $C_T=-a\,u_{a,T}(0)^2+\frac{2}{p+1}u_{a,T}(0)^{p+1}$, we must have $u_{a,T}(0)\to\zeta_0=u_a(0)$ and $u_{a,T}(0)>u_a(0)$. □

Proposition 2.2. *The critical value $J_{a,T}(u_{a,T})$ tends to $J_a{}^*$ as T tends to $+\infty$.*

Proof. We have

$$J_{a,T}(u_{a,T}) = \int_0^T \left(u'_{a,T}(x)^2 + a\,u_{a,T}(x)^2 - \frac{2}{p+1} u_{a,T}{}^{p+1} \right) dx =$$

$$= \int_0^{u_{a,T}(0)} \left(2\sqrt{u^2\left(a - \frac{2}{p+1}u^{p-1}\right) + C_T} - \frac{C_T}{\sqrt{u^2\left(a - \frac{2}{p+1}u^{p-1}\right) + C_T}} \right) du.$$

For simplicity let $f(u) = u^2\left(a - \frac{2}{p+1}u^{p-1}\right)$. We have

$$\int_0^{u_{a,T}(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du = \int_0^{u_a(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du + \int_{u_a(0)}^{u_{a,T}(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du.$$

Since $f(u) \geq 0$ for $u \in [0, u_a(0)]$, the first integral is smaller than $u_a(0)\sqrt{C_T}$. The second integral has a singularity at $u = u_{a,T}(0)$, and considering the Taylor expansion of $f(u)$ at $u = u_{a,T}(0)$ we easily check that there exists a constant $k > 0$ such that

$$\int_{u_a(0)}^{u_{a,T}(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du \leq \int_{u_a(0)}^{u_{a,T}(0)} \frac{k\,c_T}{\sqrt{u_{a,T}(0)-u}}\,du.$$

It is now easy to conclude that $\int_0^{u_{a,T}(0)} \frac{C_T}{\sqrt{f(u)+C_T}}\,du \to 0$ and since $u_{a,T}(0) \to u_a(0)$ and $c_T \to 0$, the result follows. □

Consider now the differential equation (4), with $g(u)$ satisfying (H_1) and (H_2). In this case, multiplying the equation by u' and integrating gives us $u'^2 = a u^2 - 2G(u) + C$, $C \in \mathbb{R}$, and taking $C = 0$, we get an homoclinic u_a. Note that we can assume that $u'_a(0) = 0$, and, consequently, $u_a(0)$ is a zero of the function $a u^2 - 2G(u)$. The classic work of Berestycki and Lions [1] allows us to conclude it must be the first positive zero ζ_0 of that function.

The associated Euler-Lagrange functional is

$$J_a(u) = \int_0^{+\infty} \left(u'(x)^2 + a\,u(x)^2 - 2G(u)\right) dx,$$

and it is easily seen that

$$J_a(u_a) = 2\int_0^{u_a(0)} \sqrt{a u^2 - 2G(u)}\,du \equiv J_a{}^*.$$

The boundary value problem

$$\begin{cases} u''(x) = a\,u(x) - g\left(u(x)\right) \\ u'(0) = 0, \quad u(T) = 0 \end{cases} \tag{7}$$

has a positive solution $u_{a,T}$ and

$$J_{a,T}(u_{a,T}) = \int_0^{u_{a,T}(0)} 2\sqrt{(a u^2 - 2G(u)) - \left(a\, u_{a,T}(0)^2 - 2G(u_{a,T}(0))\right)}\, du +$$
$$+ \int_0^{u_{a,T}(0)} \frac{a\, u_{a,T}(0)^2 - 2G(u_{a,T}(0))}{\sqrt{(a u^2 - 2G(u)) - \left(a\, u_{a,T}(0)^2 - 2G(u_{a,T}(0))\right)}}\, du.$$

A careful analysis of the phase plane implies that $\lim_{T\to+\infty} u_{a,T}(0) = u_a(0)$, and after some computation, knowing that $a\,\zeta_0 - g(\zeta_0) < 0$, we can conclude again that $J_{a,T}(u_{a,T}) \to J_a{}^*$.

Remark 2.3. Concerning the problem with $g(u)$ bounded, we no longer have condition (H_1) satisfied, and therefore, mountain pass theorem cannot be applied in the same way. This will not be a problem since with g bounded, the Euler-Lagrange funcional is coercive and therefore, Palais-Smale condition is satisfied. If we ask $g(u)$ to satisfy (H_2) and if there exists $\zeta_0 > 0$ such that $a\,\zeta_0^2 - 2G(\zeta_0) = 0$, then the mountain pass geometry is preserved and the conclusions obtained above are still valid.

3. PROOF OF THE MAIN RESULT

In this section we will prove the existence of a solution for the non-autonomous problem in $\mathbb{R}^+$, with $u'(0) = 0$ and $u(+\infty) = 0$.

Let $a(x)$ be a continuous function defined in $\mathbb{R}^+$, satisfying (A_1) and (A_2). Note that we could as well have taken $a(x)$ to be a piecewise continuous function.

Remark 3.1. For $g(u) = u^3$ condition (A_2) is the inequality $A < 2^{2/3}a$.

The arguments given in the previous section concerning the existence of positive mountain-pass solutions for the autonomous case are also valid for the non-autonomous case.

Consider the boundary value problem

$$\begin{cases} u'' = a(x)\, u - u^p \\ u'(0) = 0, \quad u(T) = 0. \end{cases} \tag{8}$$

Setting

$$J_T(u) = \int_0^T \left(u'^2 + a(x)\, u^2 - \frac{2}{p+1} u_+^{p+1} \right) dx,$$

as in the constant case, this functional has a mountain-pass geometry relative to the local minimum $u = 0$ in $H_T^* \equiv \left\{H^1[0,T]\colon\, u(T) = 0\right\}$, and consequently (8) has a positive nontrivial solution u_T. We want to find a solution for the infinite domain problem as the limit of a sequence of solutions u_T, with $T \to \infty$. Let c_T be the mountain-pass critical value of J_T, that is, $c_T = J_T(u_T)$. Defining $\Gamma_T = \left\{\gamma(\tau)\colon\, [0,1] \to H_T^*\colon\, \gamma(0) = 0, \gamma(1) = u_\lambda^+\right\}$, we know that

$$c_T = \inf_{\gamma\in\Gamma_T} \max_{\tau\in[0,1]} J_T\left(\gamma(\tau)\right).$$

Since $\Gamma_{T_1} \subseteq \Gamma_{T_2}$ for $T_1 < T_2$, we have $c_T \leq c_1$ for $T \geq 1$. Also by comparison we have the following

Lemma 3.2. *The critical values c_T are such that $c_T \leq J_{A,T}(u_{A,T})$.*

Multiplying the differential equation by u and integrating, we get

$$-\int_0^T u_T'^2\, dx = \int_0^T \left(a(x)u_T^2 - u_T^{p+1}\right) dx,$$

and consequently, we have

$$J_T(u_T) = \frac{p-1}{p+1}\int_0^T u_T^{p+1}\, dx = \frac{p-1}{p+1}\int_0^T \left({u_T}'^2 + a(x)\, {u_T}^2\right) dx. \tag{9}$$

Extending u_T to $[0,+\infty[$ by $u_T(x) = 0$ for $x \geq T$, it follows that:

Proposition 3.3. *We have uniform estimates for the $L^{p+1}(0,+\infty)$ and $H^1(0,+\infty)$ norms of the solutions u_T (for $T \geq 1$).*

Proof. Since $J_T(u_T) \leq c_1$ for all $T > 1$, (9) allows us to conclude the result. □

Corollary 3.4. *There exists $k > 0$ such that, for all $T > 1$,*

$$|u_T(x)|,\ \left|{u_T}'(x)\right|,\ \left|{u_T}''(x)\right| \leq k \quad \forall x \in [0,T].$$

As a consequence, using the diagonal argument, we can pick up a sequence of values $T \to +\infty$ such that $u_T \to u$ C^1-uniformly in compact intervals and ${u_T}' \rightharpoonup u'$ weakly in $L^2(0,+\infty)$.

Proposition 3.5. ${u_T}'(T) \to 0$ *as* $T \to +\infty$.

Proof. If ${u_T}'(T) \not\to 0$, then, since ${u_T}'$ is bounded, there exists a sequence of T's tending to $+\infty$ such that ${u_T}'(T) \to d$ for some constant $d < 0$. Consider the initial value problem

$$\begin{cases} u'' = a\,u - u^p \\ u'(0) = d, \quad u(0) = 0. \end{cases} \tag{10}$$

Let $-2c$ be the largest negative zero of the solution $\bar{u}$ of (10). We have $\bar{u}'(-c) = 0$, $\bar{u}(-c) > 0$, $\bar{u}'(-3c) = 0$ and $\bar{u}(-3c) = -\bar{u}(-c)$. Defining $v_T(x) = u_T(x+T)$, we must have $v_T(x) \to \bar{u}(x)$ uniformly in compact intervals and consequently, for T large enough, we would have $u_T(T-3c) < 0$, which is a contradiction since u_T does nor vanish before T. Consequently, we must have ${u_T}'(T) \to 0$.

□

Corollary 3.6. *Setting l_T as the largest maximizer of u_T, we have $T - l_T \to +\infty$.*

In the following, let $J_T(u)\big|_{[m,n]} = \int_m^n \left(u'^2 + a(x)\, u^2 - \frac{2}{p+1} u_+^{p+1}\right) dx$.

Lemma 3.7. *Given an arbitrary positive constant ε, there exists x_ε such that for all $T > 0$ and all $x > x_\varepsilon$ we have $u_T(x) \leq \varepsilon$.*

Proof. Let $x_{\varepsilon,T} = \inf\{x : \forall t \geq x,\ u_T(t) \leq \varepsilon\}$.
Claim. Given $\varepsilon > 0$, $J_T(u_T)\big|_{[x_{\varepsilon,T},T]} \leq \eta(\varepsilon)$, *where* $\lim_{\varepsilon\to 0}\eta(\varepsilon) = 0$.
Proof of Claim. Multiplying the differential equation by u_T and integrating in $[x_{\varepsilon,T}, T]$, we have

$$J_T(u_T)\big|_{[x_{\varepsilon,T},T]} = -u_T(x_{\varepsilon,T})u_T{}'(x_{\varepsilon,T}) + \int_{x_{\varepsilon,T}}^{T} \frac{p-1}{p+1} u_T^{p+1}\,dx \leq$$
$$\leq \varepsilon\, u_T{}'(x_{\varepsilon,T}) + \varepsilon^{p-1}\frac{p-1}{p+1}\int_{x_{\varepsilon,T}}^{T} u_T^2\,dx.$$

Since $p-1>0$, the conclusion follows easily using Proposition 3.3 and Corollary 3.4.

Now suppose towards a contradiction that $x_{\varepsilon,T} \to \infty$ as $T \to \infty$. Let $[y_T, x_{\varepsilon,T}]$ be the maximal interval containing $x_{\varepsilon,T}$ such that $u_T(x)\big|_{[y_T,x_{\varepsilon,T}]} \geq \varepsilon$. If $x_{\varepsilon,T} - y_T \to +\infty$ along a sequence of T's, then, because of (9), $J_T(u_T) \to \infty$, which is a contradiction. Let l_T be the largest maximizer of u_T. If $x_{\varepsilon,T} - y_T$ does not exceed a certain constant independent of T, then we must have $l_T \to +\infty$ since the distance between l_T and $x_{\varepsilon,T}$ cannot become arbitrarily large.

Defining $v_T(x) = u_T(x+l_T)$, then, along a subsequence, we have $v_T \to v$ C^1-uniformly in compact intervals, where $v''(x) = a\,v(x) - v(x)^p$, $v'(0) = 0$ and $v > 0$ in $[0,+\infty[$ by Corollary 3.6, that is, $v = u_a$.

Given an arbitrary constant $\varepsilon > 0$, there exists a constant $c = c(\varepsilon)$ such that

$$|u_a(\pm c)| < \varepsilon,\ \left|u_a'(\pm c)\right| < \varepsilon, \tag{11}$$

and

$$J_a(u_a)\big|_{[-c,c]} > 2J_a{}^* - \varepsilon. \tag{12}$$

By Corollary 3.6, we know that for T large enough, $v_T(x)$ is well defined in $[-c,c]$ and $v_T(x)$ converges uniformly in $C^1[-c,c]$ to $u_a(x)$. Since $l_T \to \infty$, we may assume that $|a(x)-a| < \frac{\varepsilon}{2K}$ for $x > l_T - c$, where K is such that $\|u_T\|_{L^2} \leq K$. Consequently, for T large enough we have

$$\left|J_T(u_T)\big|_{[l_T-c,l_T+c]} - J_a(u_a)\big|_{[-c,c]}\right| \leq$$
$$\leq \int_{-c}^{c}\Big(\left|v_T'(x)^2 - u_a'(x)^2\right| + |a(x+l_T)-a|\,v_T(x)^2 +$$
$$+ a\left|v_T(x)^2 - u_a(x)^2\right| + \frac{2}{p+1}\left|v_T(x)^{p+1} - u_a(x)^{p+1}\right|\Big)\,dx \leq \varepsilon. \tag{13}$$

Using a slight adaptation of the *Claim,* (12) *and* (13) we have

$$\left|J_T(u_T)\big|_{[l_T-c,T]} - 2J_a{}^*\right| \leq \left|J_T(u_T)\big|_{[l_T-c,l_T+c]} - J_a(u_a)\big|_{[-c,c]}\right| +$$
$$+ J_T(u_T)\big|_{[l_T+c,T]} + 2J_a{}^* - J_a(u_a)\big|_{[-c,c[} \leq \eta\,(\varepsilon) + 2\varepsilon. \tag{14}$$

On the other hand, we have

$$J_T(u_T)\big|_{[0,l_T-c]} = \int_0^{l_T-c} \left(u_T'^2 + a(x)u_T^2 - u_T^{p+1}\right) dx + \int_0^{l_T-c} \frac{p-1}{p+1} u_T^{p+1}\, dx =$$
$$= u_T'(l_T-c)u_T(l_T-c) + \int_0^{l_T-c} \frac{p-1}{p+1} u_T^{p+1}\, dx > 0. \tag{15}$$

By Proposition 2.2, for T large enough we have $\left|J_A{}^* - J_{A,T}(u_{A,T})\right| < \varepsilon$. Taking in consideration Lemma 3.2, (14) and (15) we would have

$$J_A{}^* = J_{A,T}(u_{A,T}) + J_A{}^* - J_{A,T}(u_{A,T}) \geq J_T(u_T) - \varepsilon \geq 2J_a{}^* - 3\varepsilon - \eta(\varepsilon),$$

which, since ε was chosen arbitrarily, contradicts the assumption (A_2). □

In order to show that the limit is not the trivial solution, we need the following

Proposition 3.8. *There exists a constant $c > 0$ such that $u_T(0) > c$ for all $T > 1$.*

Proof. Suppose towards a contradiction that there exists a sequence of $T's$ tending to infinity such that $u_T(0) \to 0$. Then it is obvious that l_T tends to $+\infty$ with T and we obtain the same contradiction as in the proof of Lemma 3.7. □

We are now able to to prove the main result:

Proof of Theorem 1.1. Using Proposition 3.8 and Lemma 3.7 we have $u_T(x) \to u(x)$ C^1-uniformly in compact intervals, with $u(x)$ a positive solution of (3). □

Remark 3.9. In the case where $g(u)$ is not a power, but satisfies the hypotheses (H_1) and (H_2), we have

$$J_T(u_T) = \int_0^T (g(u)u - 2G(u))\, dx.$$

Multiplying the differential equation by u and integrating, we get

$$-\int_0^T u_T'^2\, dx = \int_0^T \left(a(x)u_T^2 - u_T\, g(u_T)\right) dx,$$

and consequently, we have

$$J_T(u_T) \geq \left(1 - \frac{2}{q}\right) \int_0^T \left(u_T'^2 + a(x)\, u_T^2\right) dx = \left(1 - \frac{2}{q}\right) \int_0^T g(u_T)u_T\, dx,$$

so that the same arguments used in the case $g(u) = u^p$ will provide us similar conclusions.

4. APPENDIX

Here we present an alternative approach, using the shooting method, for the simpler case of nondecreasing functions $a(x)$ and, without loss of generality, we consider $g(u) = u^3$, in which the explicit computations are easier to follow.

So we consider the problem

$$\begin{cases} u'' = a(x)u - u^3 = u\left(a(x) - u^2\right) \\ u'(0) = 0, \quad u(+\infty) = 0. \end{cases} \tag{16}$$

Lemma 4.1. *Let $a(x)$ be a positive and nondecreasing function defined in $[0,+\infty[$. If $u(x)$ is a solution of (16), the energy function $E(x) \equiv \frac{u'^2}{2} + \frac{u^4}{4} - \frac{a(x)u^2}{2}$ is decreasing in $\mathbb{R}^+$.*

Proof. Let $x_1 < x_2 \in \mathbb{R}^+$.Using the Stieltjes integral, we have

$$E(x_2) - E(x_1) = \int_{x_1}^{x_2} dE = \int_{x_1}^{x_2} \left(\frac{u'^2}{2} + \frac{u^4}{4}\right)' - \left[a(x)\frac{u^2}{2}\right]_{x_1}^{x_2} =$$

$$= \int_{x_1}^{x_2} a(x)u\,u'\,dx - \left[a(x)\frac{u^2}{2}\right]_{x_1}^{x_2} = -\int_{x_1}^{x_2} \frac{u^2}{2}\,da(x) \leq 0.$$

□

Positive solutions solutions of $u''(x) = a(x)u(x) - u(x)^3 = u(x)\left(a(x) - u(x)^2\right)$ are concave if $u(x) > \sqrt{a(x)}$ and convex if $u(x) < \sqrt{a(x)}$, therefore the graph of the solution u_L of

$$\begin{cases} u''(x) = a(x)u(x) - u(x)^3 \\ u(0) = L, \quad u'(0) = 0 \end{cases} \tag{17}$$

where $L > \sqrt{a(0)}$ crosses the graph of $\sqrt{a(x)}$ at $x = c_L$ for some $c_L > 0$, and we may suppose that c_L is the minimum value with this property.

Proposition 4.2. *As L tends to $+\infty$, c_L tends to 0.*

Proof. Let us first prove the result for $a(x)$ bounded. Let d_L be the minimum value such that $u_L(d_L) = \frac{L}{2}$. Suppose towards a contradiction that $d_L \not\to 0$. This means that there exists sequence $L_n \to +\infty$ such that $d_{L_n} > k$ for some constant $k > 0$. Let $p = \frac{\pi}{2k} > \frac{\pi}{2d_{L_n}}$. Since $a(x)$ is bounded, for n large enough we have $a(x) - u_{L_n}^2(x) \leq -p^2$ for $x \in [0, d_{L_n}]$, so the unique solution v of the initial value problem

$$\begin{cases} v''(x) = -p^2 v \\ v(0) = L, \quad v'(0) = 0 \end{cases} \tag{18}$$

is such that $v(x) \geq u_{L_n}(x)$ in the interval $[0, d_{L_n}]$. But $v(x) = \cos(px)$ vanishes at $x = \frac{\pi}{2p} = k < d_{L_n}$, which contradicts $v(x) \geq u_{L_n}(x)$. Consequently we have $d_L \to 0$ and a simple geometric argument implies that $c_L \leq 2d_L$, so we conclude that $c_L \to 0$.

In case $a(x)$ is unbounded, consider the bounded auxiliar function

$$\bar{a}(x) = \begin{cases} a(x), & x \leq 1 \\ a(1), & x > 1. \end{cases}$$

Applying the result obtained for bounded functions, we have that for $L > L_0$ large enough we have $c_L < 1$ and since the result only depends on the values of x smaller than c_L, the result holds for the unbounded function $a(x)$. □

Corollary 4.3. *As L tends to $+\infty$, $u_L{}'(c_L) \to -\infty$.*

Proposition 4.4. *For $L > \sqrt{a(0)}$ large enough, the solution of* (17) *has at least one zero.*

Proof. For simplicity, let us denote c_L by c. Given $L^* > \sqrt{a(0)}$ large, let c^* be the first value such that the graph of the solution of (17) with $L = L^*$ crosses the graph of $\sqrt{a(x)}$. Taking a sufficiently large $L > L^*$, the corresponding solution u_L of (17) satisfies $u_L(c) = \sqrt{a(c)}$, for some $c < c^*$. Suppose towards a contradiction that u_L does not vanish in $[0, c^*]$. Then, there exists $\hat{c} \in [c, c^*]$ such that $u_L{}'(\hat{c}) = -\frac{\sqrt{a(c)}}{c^* - c}$, which is the slope of the line connecting $(c, \sqrt{a(c)})$ and $(c^*, 0)$, and we have

$$-\frac{\sqrt{a(c)}}{c^* - c} - u'(c) = \int_c^{\hat{c}} u''(x)\, dx \leq c^* \sqrt{a(c^*)}^3.$$

Taking in consideration last corollary, we have a contradiction. □

Proposition 4.5. *Consider the initial value problem* (17) *with $L > \sqrt{a(0)}$. If its solution u_L is positive and does not have a local minimum, then $u_L(+\infty) = 0$.*

Proof. It is obvious that the graph of u_L crosses the graph of $\sqrt{a(x)}$ with negative derivative and since the derivative does not vanish again and u_L is positive, we must have $u_L'(+\infty) = 0$ and therefore $u_L(+\infty) = k > 0$. If $k \neq 0$ then

$$u_L''(+\infty) = u_L(+\infty)\left(a(+\infty) - u_L^2(+\infty)\right) > 0$$

and therefore there would exist $c \in \mathbb{R}$ such that $u_L'(c) = 0$, which is a contradiction. □

Proposition 4.6. *If $0 < L < \sqrt{2a(0)}$ then the solution u_L of* (17) *is positive in $\mathbb{R}^+$ and attains a positive minimum m for some $x_m \geq 0$.*

Proof. Since $E(0) = \frac{L^2}{2}\left(\frac{L^2}{2} - a(0)\right) < 0$ we have $E(x) < 0$ for every $x > 0$. If there exists $x_0 > 0$ such that $u_0(x_0) = 0$, then $E(x_0) = \frac{u'(x_0)^2}{2} \geq 0$, which is a contradiction.

If u_L does not attain a positive minimum, then $u_L(+\infty) = 0$ and $u_L'(+\infty) = 0$, and therefore $E(+\infty) = 0$, which is again a contradiction. □

Proposition 4.7. *If the solution u_L of* (17) *attains a positive minimum m for some $x_m \geq 0$, then u_L is positive for $x > x_m$.*

Proof. We can conclude as in the proposition above, since $E(x_m) = \frac{m^2}{2}\left(\frac{m^2}{2} - a(x_m)\right) < 0$. □

Theorem 4.8. *Let $a(x)$ be a positive, nondecreasing function. Then the problem* (16) *has at least one positive solution.*

Proof. We use a connectedness argument appearing in the papers of J. McLeod [7] and H. Berestycki, P. Lions and L. Peletier [2]. Consider the following subsets of $\mathbb{R}^+$

$$A = \left\{L > \sqrt{a(0)} \colon u_L > 0 \text{ and } u_L \text{ has a positive minimum}\right\},$$

$$B = \left\{L > \sqrt{a(0)} \colon u_L(x_0) = 0 \text{ for some } x_0 > 0\right\}.$$

Both sets are nonempty, obviously disjoint, and, by the continuous dependence of the parameters, open in $\mathbb{R}$. Let $u_0 = \inf B$. Since u_0 does not belong to A or B, we must conclude that the solution of problem (17) with $L = u_0$ is positive and tends to 0 at ∞. □

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