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RESEARCH ARTICLE

A scaling analysis in the SIRI epidemiological model

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For the spatial stochastic epidemic reinfection model SIRI, where susceptibles S can become infected I, then recover and remain only partial immune against reinfection R, we determine the phase transition lines using pair approximation for the moments derived from the master equation. We introduce a scaling argument that allow us to determine analytically an explicit formula for these phase transition lines and prove rigorously the heuristic results obtained previously.

Keywords: Stochastic processes; Reinfection model; Pair approximation; Phase transition lines

AMS Subject Classification: 34A34, 92D30, 92D25,

1. Introduction

In [10], we presented the phase transition lines between no-growth and compact growth, between compact growth and annular growth and between no-growth and annular growth in pair approximation for a reinfection model, called SIRI. It describes a susceptible, infected, recovered epidemic process, SIR, with additional partial reinfection, a transition from partially immune recovered R to the infected class I, hence the name SIRI. This model is a simplified version of general multi-strain models, as e.g. described in [3, 4], where after an initial infection immunity against one strain only gives partial immunity against a genetically close mutant strain. For a recent investigation of reinfection models in the biological context see for example [9]. In the physics literature models with partial immunization have also found wide interest [1, 5] due to their critical behaviour connecting directed percolation and dynamic percolation.

We compare the especially tricky phase transition line between no-growth and annular growth with simulations [10]. In pair approximation, these critical points have different values for the SIS system [7] and the SIR system [6], as opposed to the mean field values which are the same for SIS and SIR. Here in the present article,

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we prove the analytical formula of the transition line presented in [10] between no-growth and annular growth. The proof of this analytical formula of the transition line has a difficulty far beyond the calculation of the other transition lines. For the analytic calculations, we have to introduce a scaling argument that, in particular, allows us to overcome the difficulty arising from an indetermination occurring in the proof. This scaling argument consists in computing the ratio $\langle R\rangle^*/\langle I\rangle^*$ of the average of the recovered individuals $\langle R\rangle^*$ against the infected individuals $\langle I\rangle^*$, observing that several terms along the computation vanish.

In section 2 we describe the spatial epidemic model for reinfection, the SIRI model, in its stochastic description. We present the dynamic equations for the expectation values of total number of infected, recovered and for the pairs $\langle SI\rangle^*$, $\langle RI\rangle^*$ and $\langle SR\rangle^*$ closed via the pair approximation as given in [10]. In section 3 we prove the analytical formula of the transition line between no-growth and annular growth using a scaling argument.

2. The SIRI epidemic model

To describe reinfection in a simple epidemic model, we investigate an extension on classical SIS or SIR models extending to the SIRI model [10]. We consider the following transitions between host classes for N individuals being either susceptible S, infected I by a disease or recovered R

$$\begin{aligned} S+I &\xrightarrow{\beta} I+I \\ I &\xrightarrow{\gamma} R \\ R+I &\xrightarrow{\tilde{\beta}} I+I \\ R &\xrightarrow{\alpha} S \end{aligned}$$

resulting in the master equation [11] for variables S_i, I_i and $R_i \in \{0,1\}$, $i = 1, 2, ..., N$, for N individuals, with constraint $S_i + I_i + R_i = 1$.

The first infection $S + I \xrightarrow{\beta} I + I$ occurs with infection rate β, whereas after recovery with rate γ the respective host becomes resistant up to a possible reinfection $R+I \xrightarrow{\tilde{\beta}} I+I$ with reinfection rate $\tilde{\beta}$. Hence, the recovered are only partially immunized. For further analysis of possible stationary states we include a transition from recovered to susceptibles α, which might be simply due to demographic effects (or very slow waning immunity for some diseases). We will later consider the limit of vanishing or very small α. In case of demography that would be in the order of inverse 70 years, whereas for the basic epidemic processes like first infection β we would expect inverse a few weeks. We consider that the N individuals live on a regular lattice, where each corner has the same number Q of edges.

2.1. *The ODE's for the moments*

In [10] we presented the following ordinary differential equations (ODE's), for the first moments $\langle S\rangle$, $\langle I\rangle$ and $\langle R\rangle$ (see Eq. (6)), and for the second moments $\langle SS\rangle$, $\langle II\rangle$, $\langle RR\rangle$, $\langle SI\rangle$, $\langle SR\rangle$ and $\langle IR\rangle$ (see Eq. (7)) in pair approximation:

$$\frac{d}{dt}\langle I\rangle = \beta\,\langle SI\rangle - \gamma\langle I\rangle + \tilde{\beta}\,\langle RI\rangle \tag{1}$$

$$\frac{d}{dt}\langle R\rangle = \gamma\langle I\rangle - \alpha\langle R\rangle - \tilde{\beta}\,\langle RI\rangle \tag{2}$$

$$\begin{aligned}\frac{d}{dt}\langle SI\rangle = {} & \alpha\langle RI\rangle - (\gamma+\beta)\,\langle SI\rangle + \beta(Q-1)\,\langle SI\rangle \\ & -\beta\,\frac{Q-1}{Q}\,\frac{(2\langle SI\rangle + \langle SR\rangle)\cdot\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle} \\ & +\tilde{\beta}\,\frac{Q-1}{Q}\,\frac{\langle SR\rangle\,\langle RI\rangle}{\langle R\rangle}\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{d}{dt}\langle RI\rangle = {} & \gamma\,(Q\langle I\rangle - \langle SI\rangle) - (\alpha+2\gamma+\tilde{\beta})\,\langle RI\rangle \\ & +\beta\,\frac{Q-1}{Q}\,\frac{\langle SR\rangle\quad\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle} \\ & +\tilde{\beta}\,\frac{Q-1}{Q}\,\frac{(Q\langle R\rangle - \langle SR\rangle - 2\langle RI\rangle)\cdot\langle RI\rangle}{\langle R\rangle}\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{d}{dt}\langle SR\rangle = {} & \gamma\langle SI\rangle + \alpha\,(Q\langle R\rangle - 2\langle SR\rangle - \langle RI\rangle) \\ & -\beta\,\frac{Q-1}{Q}\,\frac{\langle SR\rangle\quad\langle SI\rangle}{N-\langle I\rangle-\langle R\rangle} \\ & -\tilde{\beta}\,\frac{Q-1}{Q}\,\frac{\langle RI\rangle\,\langle SR\rangle}{\langle R\rangle}\end{aligned} \tag{5}$$

For a detailed discussion of the pair approximation in general, see [6–8] and especially applied to SIRI model see [10]. We recall that the expectation value of the total number of infected hosts $\langle I\rangle$ at a given time t is

$$\begin{aligned}\langle I\rangle(t) &= \sum_{SIR}\left(\sum_{i=1}^{N} I_i\right)\,p(S_1,I_1,R_1,S_2,...,R_N,t) \\ &= \sum_{i=1}^{N}\sum_{SIR} I_i\,p(S_1,I_1,R_1,S_2,...,R_N,t) \\ &= \sum_{i=1}^{N}\langle I_i\rangle(t) \quad .\end{aligned} \tag{6}$$

where $\sum_{SIR}$ denotes the sum $\sum_{S_1=0}^{1}\sum_{I_1=0}^{1}\sum_{R_1=0}^{1}\sum_{S_2=0}^{1}\cdots\sum_{R_N=0}^{1}$, and $p(S_1,I_1,R_1,S_2,...,R_N,t)$ is the probability of the state $S_1,I_1,R_1,S_2,...,R_N$ occurs at time t given by the master equation [2, 11] for the SIRI model [10]. Similarly, the second moment $\langle SI\rangle$ of the expectation value of two individuals neighbours in which one is susceptible and one is infected is the pair given by

$$\langle SI\rangle(t) = \sum_{SIR}\left(\sum_{i=1}^{N}\sum_{j=1}^{N} J_{ij}\;S_iI_j\right)\,p(S_1,I_1,R_1,...,R_N,t) \quad . \tag{7}$$

The other first and second moments are defined similarly. These are dynamic variables, e.g. $\langle I\rangle(t)$, and the stationary values will be denoted by $\langle I\rangle^*$, $\langle R\rangle^*$ etc.

3. Analytic expression of the phase transition line

Let

$$E = \alpha + \gamma Q + \tilde{\beta} \quad , \tag{8}$$

$$F = D + \sqrt{D^2 + 4\alpha(Q-1)E} \quad , \tag{9}$$

where

$$D = \gamma Q - \tilde{\beta}(Q-1) - \alpha(Q-2) \quad . \tag{10}$$

We observe for the transition line between no-growth and annular growth that the stationary value $\langle I\rangle^*$ tends to zero and the stationary value $\langle R\rangle^*$ also tends to zero but their ratio stays finite. Hence, we conclude the following lemma:

Lemma 3.1: *The scaling limit of $\langle R\rangle^*/\langle I\rangle^*$ when $\langle I\rangle^*$ tends to zero is given by*

$$\lim_{\langle I\rangle^*\to 0} \frac{\langle R\rangle^*}{\langle I\rangle^*} = \frac{\gamma F}{2\,\alpha E} \quad . \tag{11}$$

Proof: We consider the equilibrium manifold of ODE system given by Eq. (1) to Eq. (5). We use equations (1), (2) and (5) to compute $\langle SI\rangle^*$, $\langle RI\rangle^*$ and $\langle SR\rangle^*$ and we replace their values in equation (3) and (4) giving the following two implicit equations:

$$\begin{aligned} &Q\langle I\rangle^* - \frac{\widetilde{\beta}+2\gamma}{\widetilde{\beta}}\left(\langle I\rangle^* - \frac{\alpha}{\gamma}\langle R\rangle^*\right) + \left((Q-1)\,\langle I\rangle^* + \frac{\alpha}{\gamma}\langle R\rangle^*\right) \\ &\cdot\left(1 - 2\frac{N - \langle I\rangle^* - \langle R\rangle^* - \dfrac{\alpha}{\beta Q}\langle R\rangle^*}{\dfrac{Q}{Q-1}\left(N - \langle I\rangle^* - \langle R\rangle^*\right) + \langle R\rangle^*} - \frac{2}{\widetilde{\beta}Q}\frac{\gamma\langle I\rangle^* - \alpha\langle R\rangle^*}{\langle R\rangle^*}\right) = 0 \end{aligned} \tag{12}$$

and

$$\begin{aligned} &Q\langle I\rangle^* - \frac{\alpha\,(\beta+2\gamma)}{\beta\gamma}\langle R\rangle^* - \frac{\widetilde{\beta}+2\gamma}{\widetilde{\beta}}\left(\langle I\rangle^* - \frac{\alpha}{\gamma}\langle R\rangle^*\right) \\ &+\frac{\alpha}{\gamma}\,(Q-1)\,\langle R\rangle^*\left(1 - \frac{2\alpha\langle R\rangle^*}{\beta Q\left(N - \langle I\rangle^* - \langle R\rangle^*\right)}\right) \\ &+(Q-1)\left(\langle I\rangle^* - \frac{\alpha}{\gamma}\langle R\rangle^*\right)\left(1 - \frac{2\left(\gamma\langle I\rangle^* - \alpha\langle R\rangle^*\right)}{\widetilde{\beta}Q\langle R\rangle^*}\right) = 0 \quad . \end{aligned} \tag{13}$$

We start proving that if $\langle I\rangle^*$ tends to zero then $\langle R\rangle^*$ also converges to zero. We observe that the function in Eq. (12) is continuous at $\langle I\rangle^*=0$ and its value is

$$\frac{\widetilde{\beta}+2\gamma}{\widetilde{\beta}}\frac{\alpha}{\gamma}\langle R\rangle^* + \frac{\alpha}{\gamma}\langle R\rangle^*\left(1-2\frac{N-\langle R\rangle^*-\dfrac{\alpha}{\beta Q}\langle R\rangle^*}{\dfrac{Q}{Q-1}(N-\langle R\rangle^*)+\langle R\rangle^*}+\frac{2\alpha}{\widetilde{\beta}Q}\right)=0 \quad . \tag{14}$$

Hence, we obtain that $\langle R\rangle^*=0$ or

$$\langle R\rangle^* = -\frac{\beta Q\left(\widetilde{\beta}+\gamma Q+\alpha\right)}{\beta\widetilde{\beta}Q(Q-2)+\widetilde{\beta}\alpha(Q-1)-\beta(\gamma Q+\alpha)} \quad . \tag{15}$$

But the value presented in Eq. (15) is not a solution of Eq. (13) for $\langle I\rangle^*=0$. Hence, the stationary state $\langle R\rangle^*$ converges to zero when $\langle I\rangle^*$ tends to zero. Let

$$\begin{aligned} N_{1,2} &= -\alpha\,\tilde{\beta}\,\gamma\,Q \quad , \\ N_{0,3} &= \tilde{\beta}\,\alpha\,(-\gamma Q+\alpha(Q-1)) \quad , \\ N_{0,2} &= \alpha\,\tilde{\beta}\,\gamma\,NQ \quad , \end{aligned} \tag{16}$$

and

$$\begin{aligned} D_{3,0} &= \gamma^2(Q-1) \quad , \\ D_{2,1} &= -\tilde{\beta}\gamma\,Q(Q-1)-2\,\alpha\gamma\,(Q-1)+2\gamma^2(Q-1) \quad , \\ D_{1,2} &= \alpha^2(Q-1)-\tilde{\beta}\gamma\,Q(Q-1)-3\,\alpha\gamma\,Q+2\,\alpha\gamma+\gamma^2 Q \quad , \\ D_{0,3} &= \alpha^2(Q-1)-\alpha\gamma\,Q \quad , \\ D_{2,0} &= -\gamma^2 N(Q-1) \quad , \\ D_{1,1} &= N\gamma(2\,\alpha\,(Q-1)+\tilde{\beta}\,Q(Q-1)-\gamma Q) \quad , \\ D_{0,2} &= N\alpha\,(\gamma Q-\alpha(Q-1)) \quad . \end{aligned} \tag{17}$$

Solving Eq. (13) in order to isolate the parameter β, we obtain that

$$\beta(\tilde{\beta}) = \frac{N_\beta(\langle I\rangle^*,\langle R\rangle^*,\tilde{\beta})}{D_\beta(\langle I\rangle^*,\langle R\rangle^*,\tilde{\beta})} \quad , \tag{18}$$

where N_β is given by

$$N_\beta(\langle I\rangle^*,\langle R\rangle^*,\tilde{\beta}) = N_{1,2}\langle I\rangle^*\langle R\rangle^{*2}+N_{0,3}\langle R\rangle^{*3}+N_{0,2}\langle R\rangle^{*2} \quad , \tag{19}$$

and D_β is given by

$$\begin{aligned} D_\beta(\langle I\rangle^*,\langle R\rangle^*,\tilde{\beta}) = {} & D_{3,0}\langle I\rangle^{*3}+D_{2,1}\langle I\rangle^{*2}\langle R\rangle^*+D_{1,2}\langle I\rangle^*\langle R\rangle^{*2} \\ & +D_{0,3}\langle R\rangle^{*3}+D_{2,0}\langle I\rangle^{*2}+D_{1,1}\langle I\rangle^*\langle R\rangle^*+D_{0,2}\langle R\rangle^{*2} \, . \end{aligned} \tag{20}$$

Substituting in Eq. (12) the expression for β given in Eq. (18), we obtain

$$\frac{N(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta})}{D(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta})} = 0 \quad , \tag{21}$$

where the denominator is given by

$$D(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta}) = \tilde{\beta} Q \langle R\rangle^* \ (\gamma Q N - \langle R\rangle^* (\gamma Q - \alpha(Q-1)) - \gamma Q \langle I\rangle^*) \\ \cdot (\langle R\rangle^* - Q(N - \langle I\rangle^*)) \quad , \tag{22}$$

and the numerator is given by

$$N(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta}) = C_{4,0}\langle I\rangle^{*4} + C_{3,1}\langle I\rangle^{*3}\langle R\rangle^* + C_{2,2}\langle I\rangle^{*2}\langle R\rangle^{*2} \\ + C_{1,3}\langle I\rangle^*\langle R\rangle^{*3} + C_{0,4}\langle R\rangle^{*4} + C_{3,0}\langle I\rangle^{*3} \\ + C_{2,1}\langle I\rangle^{*2}\langle R\rangle^* + C_{1,2}\langle I\rangle^*\langle R\rangle^{*2} + C_{0,3}\langle R\rangle^{*3} \\ + C\langle I\rangle^{*2} + B\langle I\rangle^*\langle R\rangle^* + A\langle R\rangle^{*2} \quad , \tag{23}$$

with

$$A = -2\alpha N^2 Q^2 (\alpha + \gamma Q + \tilde{\beta}) \quad , \tag{24}$$

$$B = 2\gamma N^2 Q^2 (\gamma Q - \tilde{\beta}(Q-1) - \alpha(Q-2)) \quad , \tag{25}$$

$$C = 2\gamma^2 N^2 Q^2 (Q-1) \quad . \tag{26}$$

The other coefficients $C_{i,j}$ of the numerator are not presented here, because we will not use them in the future computations. We are going to find the limit of the ratio $\langle R\rangle^*/\langle I\rangle^*$, when $\langle I\rangle^*$ tends to zero, such that

$$N(\langle I\rangle^*, \langle R\rangle^*; \tilde{\beta}) = 0 \tag{27}$$

is satisfied. Dividing Eq. (27) by $\langle I\rangle^{*2}$ and furthermore defining the ratio of recovered over infected $\langle V\rangle^* = \langle R\rangle^*/\langle I\rangle^*$, we obtain that

$$C_{4,0}\langle I\rangle^{*2} + C_{3,1}\langle I\rangle^*\langle R\rangle^* + C_{2,2}\langle R\rangle^{*2} + C_{1,3}\langle V\rangle^*\langle R\rangle^{*2} + C_{0,4}\langle V\rangle^{*2}\langle R\rangle^{*2} \\ + C_{3,0}\langle I\rangle^* + C_{2,1}\langle R\rangle^* + C_{1,2}\langle V\rangle^*\langle R\rangle^* + C_{0,3}\langle V\rangle^{*2}\langle R\rangle^* \\ + C + B\langle V\rangle^* + A\langle V\rangle^{*2} = 0 \quad . \tag{28}$$

When $\langle I\rangle^*$ tends to zero we already proved that $\langle R\rangle^*$ converges to zero. Hence, from Eq. (28), we obtain

$$A\langle V\rangle^{*2} + B\langle V\rangle^* + C = 0 \quad , \tag{29}$$

in the limiting case when $\langle I\rangle^*$ tends to 0. Therefore, there are two solutions $\langle V\rangle^*_{1,2}$ for $\langle V\rangle^*$ given by

$$\langle V\rangle^*_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad . \tag{30}$$

Since $C = 2\gamma^2 N^2 Q^2 (Q-1) > 0$ and $A = -2\alpha N^2 Q^2 (\alpha + \gamma Q + \tilde{\beta}) < 0$, we conclude that $-4AC > 0$ and so $B^2 - 4AC > B^2$. Hence, Eq. (29) has a unique positive solution

$$\langle V \rangle^* = \frac{-B - \sqrt{B^2 - 4\,A\,C}}{2A} \quad . \tag{31}$$

Inserting into Eq. (31) the expressions of A, B and C presented in Eqs. (24) to (26), we obtain Eq. (11). □

Now we will use the value of the ratio $\langle R \rangle^* / \langle I \rangle^*$ at criticality to obtain the analytic expression of the phase transition line.

Let

$$G(\tilde{\beta}) = \gamma \tilde{\beta} Q \cdot F^2 \quad , \tag{32}$$

and

$$\begin{aligned} H(\tilde{\beta}) = 2 \Big(2\alpha(Q-1) + \tilde{\beta} Q(Q-1) - \gamma Q \Big) \cdot E \cdot F \\ + (\gamma Q - \alpha(Q-1)) \cdot F^2 \\ -4\alpha(Q-1) \cdot E^2 \quad , \end{aligned} \tag{33}$$

where E and F are defined in Eq. (8) and Eq. (9) respectively.

Theorem 3.2: *Let $\alpha > 0$. The phase transition line $\beta(\tilde{\beta}) = \beta_c(\tilde{\beta}, \alpha, \gamma, Q, N)$ between no-growth and annular growth for the spatial epidemic SIRI model in pair approximation is given by*

$$\beta(\tilde{\beta}) = \frac{G(\tilde{\beta})}{H(\tilde{\beta})} \quad , \tag{34}$$

with $0 \le \tilde{\beta} \le \gamma/(Q-1)$.

Proof: We observe that Eq. (18) can be rewritten in terms of $\langle I \rangle^*$, $\langle R \rangle^*$ and $\langle V \rangle^* = \langle R \rangle^* / \langle I \rangle^*$ as follows:

$$\beta(\tilde{\beta}) = \frac{L_1 \, \langle R \rangle^* + N_{0,2} \langle V \rangle^{*2}}{D_{3,0} \langle I \rangle^* + L_2 \langle R \rangle^* + D_{2,0} + D_{1,1} \langle V \rangle^* + D_{0,2} \langle V \rangle^{*2}} \quad , \tag{35}$$

where

$$L_1 = N_{1,2} \langle V \rangle^* + N_{0,3} \langle V \rangle^{*2} \quad , \tag{36}$$

$$L_2 = D_{2,1} + D_{1,2} \langle V \rangle^* + D_{0,3} \langle V \rangle^{*2} \quad , \tag{37}$$

and the coefficients $N_{i,j}$ and $D_{i,j}$ are presented in Eqs. (16) and (17) respectively. The phase transition curve follows form Eq. (35) by letting $\langle I \rangle^*$ tends to zero. Under this limit, Eq. (35) reduces to

$$\beta(\tilde{\beta}) = \frac{N_{0,2} \langle V \rangle^{*2}}{D_{2,0} + D_{1,1} \langle V \rangle^* + D_{0,2} \langle V \rangle^{*2}} \quad . \tag{38}$$

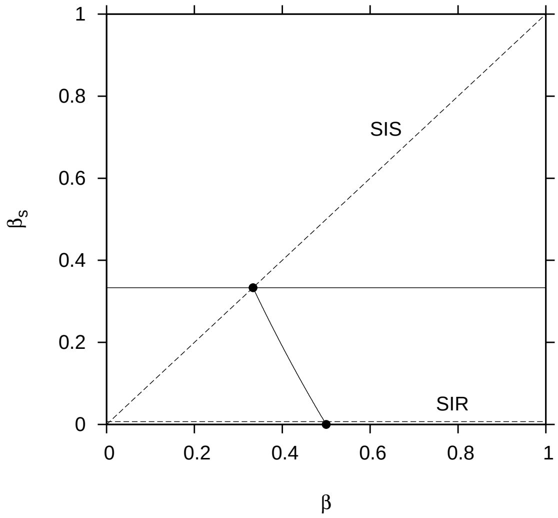

Figure 1. The phase transition line between no-growth and annular growth determined from the analytic solution in the limiting case when α tends to zero which is explicitly given in Eq. (41). The horizontal transition line of the SIRI limiting case when $\alpha = 0$ and the phase transition points of SIS and SIR limiting cases are also presented as calculated in [10]. The SIS and SIR limiting cases are given by dashed lines. (Parameters $Q = 4$ appropriate for spatial two dimensional systems and $\gamma = 1$ were used.)

Hence, the numerator of Eq. (38) is given by

$$N_{0,2}\langle V\rangle^{*2} = \alpha\,\tilde{\beta}\,\gamma\,NQ\frac{\gamma^2 F^2}{4\,\alpha^2 E^2}$$

$$= \tilde{\beta}\,\gamma^3\,NQ\frac{F^2}{4\,\alpha E^2} \quad , \tag{39}$$

and the denominator is given by

$$-\gamma^2 N(Q-1) + N\gamma(2\,\alpha\,(Q-1) + \tilde{\beta}\,Q(Q-1) - \gamma Q)\frac{\gamma\,F}{2\,\alpha E}$$
$$+N\alpha\,(\gamma Q - \alpha(Q-1))\frac{\gamma^2\,F^2}{4\,\alpha^2 E^2}$$

$$= \gamma^2 N\left(-(Q-1) + (2\,\alpha\,(Q-1) + \tilde{\beta}\,Q(Q-1) - \gamma Q)\frac{F}{2\,\alpha E}\right.$$
$$\left. +(\gamma Q - \alpha(Q-1))\frac{F^2}{4\,\alpha E^2}\right) \ . \tag{40}$$

Dividing Eq. (39) by Eq. (40), we obtain the explicit formula for the phase transition curve of the SIRI model, that can be written as given in Eq. (34). □

This completes the expression for the critical curve $\beta(\tilde{\beta})$ for the general α and γ case. When α tend to 0, we obtain the following expression for $\beta(\tilde{\beta})$:

Corollary 3.3: *In the limit when α tends to zero the phase transition line between no-growth and annular growth $\beta(\tilde{\beta})$ for the spatial epidemic SIRI model is* *given by*

$$\lim_{\alpha\to 0}\beta(\tilde{\beta}) = \frac{\gamma^2 Q - \gamma\tilde{\beta}(Q-1)}{\gamma Q(Q-2)+\tilde{\beta}(Q-1)} \quad . \tag{41}$$

In Fig. 1, we show the horizontal transition line corresponding in the left hand side to transition from no-growth to compact growth and in the right hand side to transition from annular growth to compact growth (see [10]) and the obliqual line is the phase transition between no-growth and annular growth as determined in corollary 3.3. The intersection of these two lines is the phase transition for the SIS model and the intersection of the obliqual line with the horizontal axis is the phase transition line for the SIR model.

4. Summary

We have computed the analytic expression of the phase transition line between no-growth and annular-growth for the spatial reinfection SIRI model using pair approximation for the moments derived from the master equation. We have introduced a scaling argument that allowed us to determine analytically an explicit formula for the phase transition line between no-growth and annular growth. This scaling argument consisted in computing the ratio $\langle R\rangle^*/\langle I\rangle^*$ of the average of the recovered individuals $\langle R\rangle^*$ against the infected individuals $\langle I\rangle^*$.

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References

[1] S.M. Dammer, and H. Hinrichsen, *Spreading with immunization in high dimensions*, J. Stat. Mech: Theor Exp. P07011 (2004), 17 pages.
[2] R.J. Glauber, *Time-dependent statistics of the Ising model*, J. Math. Phys. 4 (1963), pp. 294–307.
[3] M.G.M. Gomes, L.J., White, and G.F. Medley, *Infection, reinfection and vaccination under suboptimal protection: epidemiological perspective*, J. Theor. Biology 228 (2004), pp. 539–549.
[4] ———, *The reinfection threshold*, J. Theor. Biology 236 (2005), pp. 111–113.
[5] P. Grassberger, H. Chaté, and G. Rousseau, *Spreading in media with long-time memory*, Phys. Rev. E 55 (1997), pp. 2488–2495.
[6] J. Joo, and J.L. Lebowitz, *Pair approximation of the stochastic susceptible-infected-recovered-susceptible epidemic model on the hypercubic lattice*, Physical Review E 70 (2004), pp. 036114.
[7] S.A. Levin, and R. Durrett, *From individuals to epidemics*, Phil. Trans. Royal Soc. London B 351 (1996), pp. 1615–1621.
[8] D.A. Rand, *Correlation equations and pair approximations for spatial ecologies*, in: Advanced Ecological Theory, ed. J. McGlade,(Blackwell Science, Oxford, London, Edinburgh, Paris), 1999, 100–142.
[9] M. Safan, H. Heesterbeek, and K. Dietz, *The minimum effort required to eradicate infections in models with backward bifurcation*, J. Math. Biol. 53 (2006), pp. 703-718.
[10] N. Stollenwerk, J. Martins, and A. Pinto, *The phase transition lines in pair approximation for the basic reinfection model SIRI*, Phys. Lett. A 371 (2007), pp. 379–388.
[11] N.G. van Kampen, *Stochastic Processes in Physics and Chemistry*, North-Holland, Amsterdam, 1992.