Perturbation from symmetry for indefinite semilinear elliptic equations

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Abstract. We prove the existence of an unbounded sequence of solutions for an elliptic equation of the form $-\Delta u = \lambda u + a(x)g(u) + f(x)$, $u \in H_0^1(\Omega)$, where $\lambda \in \mathbb{R}$, $g(\cdot)$ is subcritical and superlinear at infinity, and $a(x)$ changes sign in Ω; moreover, $g(-s) = -g(s)$ $\forall s$. The proof uses Rabinowitz's perturbation method applied to a suitably truncated problem; subsequent energy and Morse index estimates allow us to recover the original problem. We consider the case of $\Omega \subset \mathbb{R}^N$ bounded as well as $\Omega = \mathbb{R}^N$, $N \geqslant 3$.

1 Introduction

In this paper we study the problem of finding an infinite number of solutions to semilinear elliptic equations of the form:

$$-\Delta u = \lambda u + a(x)g(u) + f(x), \qquad u \in H_0^1(\Omega),$$

where $a : \overline{\Omega} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are C^1 functions, with the nonlinearity $g(s)$ essentially behaving like a superlinear and subcritical power at infinity. We further assume that g is odd symmetric ($g(-s) = -g(s)$ $\forall s$), $\lambda \in \mathbb{R}$ and $f(x) \in L^2(\Omega)$. In case $f(x) \equiv 0$, the corresponding energy functional is given by

$$I_e(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega a(x)G(u) - \frac{\lambda}{2}\int_\Omega u^2, \qquad u \in H_0^1(\Omega)$$

(here $G(x,s) := \int_0^s g(x,\xi)\,d\xi$) which is an even functional. Therefore an application of Ljusternik-Schnirelmann theory provides the existence of a sequence of critical points (i.e.

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weak solutions of PDE) whose energy levels approach infinity. If $f(x) \not\equiv 0$ the associated functional is a perturbation of the symmetric I_e, and a natural question is the persistence of infinitely many solutions.

This problem was first considered independently by A. Bahri, H. Berestycki and M. Struwe ([3, 16]) in their pioneering papers in 1980, with subsequent improvements by A. Bahri, P.L. Lions and K. Tanaka ([4, 17]). P. Rabinowitz ([10]) stated a variational principle for dealing with such "Perturbation from Symmetry" problems in 1982.These authors essentially assumed that $a(x) = 1$ and $g(s)s \sim |s|^p$ at $|s| = \infty$ and proved the existence of infinitely many solutions for $2 < p < (2N-2)/(N-2)$. Since then a number of authors have considered these type of problems under various conditions on the nonlinearity and the function $a(x)$ (see e.g. [6] for recent developments). It remains an open problem to decide whether we can allow $2 < p < 2^* := 2N/(N-2)$.

In this paper we assume that $a(x)$ changes sign in Ω (so that the nonlinear term is indefinite in sign) and the perturbation term $f(x)$ is an *arbitrary* smooth $L^2(\Omega)$ function. To our knowledge, all the works on indefinite case (cf. [2, 18, 19]) consider a perturbation term of the form $a(x)f(s)$ so that the perturbation is "comparable" to the the main term $a(x)g(s)$ and (or) assume that $a(x)$ has a so called "thick" zero set: $\overline{\{x : a(x) > 0\}} \cap \overline{\{x : a(x) < 0\}} = \emptyset$. The only exception that we are aware of is the paper of Y. Li ([9]) where the author considers a general $a(x) \in C^1(\overline{\Omega})$, and arbitrary $f(x) \in L^2(\Omega)$ but the nonlinearity is assumed to be homogeneous (i.e. $g(s)s = |s|^p$); the techniques involved do not seem to be applicable to nonhomogeneous nonlinearities, even for such simple cases as $g(s)s = A|s|^\mu + B|s|^p$ with $\mu \neq p$. The novelty of our approach is that it allows us to consider general nonlinearities as well as functions $a(x)$ without a thick zero set, for problems set either in a bounded domain or the whole space.

In fact given a smooth bounded domain $\Omega \subset \mathbb{R}^N$, $N \geqslant 3$, and $f \in C^{0,\alpha}(\overline{\Omega})$ for some $0 < \alpha < 1$, we assume g is odd symmetric ($g(-s) = -g(s)\ \forall s$), $\lambda \in \mathbb{R}$ and

(H1) $0 < (\mu - 1)g(s)s \leqslant g'(s)s^2 \leqslant (p-1)g(s)s \quad \forall |s| \geqslant R$, with $2 < \mu \leqslant p < 2^*$;

(H2) $\nabla a(x) \neq 0\ \forall x \in \overline{\Omega}$ such that $a(x) = 0$, and $\langle \nabla a(x), \nu(x) \rangle \leqslant 0\ \forall x \in \partial\Omega$ such that $a(x) = 0$. Here $\nu(x)$ denotes the unit outward normal of Ω at the point $x \in \partial\Omega$.

We point out that it follows from (H1) that

$$A_1|s|^\mu - B_1 \leqslant g(s)s \leqslant A_2|s|^p + B_2, \qquad \forall s.$$

The energy functional associated to our problem is given by

$$I(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega a(x)G(u) - \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega f(x)u, \qquad u \in H_0^1(\Omega).$$

Our result in the case of a bounded domain is:

Theorem 1.1. *Under assumptions* (H1)–(H2) *with g odd, if moreover*

$$\frac{2p}{N(p-2)} > \frac{\mu}{\mu-1},$$

then, for every $\lambda \in \mathbb{R}$ and $f \in C^{0,\alpha}(\overline{\Omega})$ there exists a sequence $(u_n) \subset H_0^1(\Omega)$ such that

$$I'(u_n) = 0 \quad \textit{and} \quad I(u_n) \to +\infty \ \textit{ as } n \to \infty.$$

In particular, the above problem admits infinitely many solutions.

Remark. We will not use the full hypothesis (H1), but rather the condition that, for some $2 < \mu \leqslant p < 2^* := 2N/(N-2)$ and every $|s| \geqslant R$,

$$g'(s)s^2 \geqslant (\mu - 1)g(s)s > 0 \quad \text{and} \quad pG(s) \geqslant g(s)s,$$

and moreover, for some $C > 0$, $0 < r < 2^*$ and every $s \in \mathbb{R}$,

$$|g'(s)s^2| \leqslant C(1 + |s|^r).$$

The latter condition ensures that the functional I is C^2 in $H_0^1(\Omega)$.

Since we do not assume that $g(s)$ is homogeneous and the function $a(x)$ does not have a thick zero set, the lack of compactness is one of the main difficulties. Our approach uses some of the ideas in [12, 14], in particular we replace the original problem by a sequence of suitably modified ones. Then we apply the perturbation method of Rabinowitz and recover the original problem by means of careful *apriori* estimates on the solutions. Showing that these estimates are indeed uniform with respect to the modifications is our main task.

Finally as the interested reader will no doubt realize, our method is also applicable to the case where the perturbation term $f(x)$ is replaced by a more general $f(x, u)$ if we accordingly strengthen the hypotheses. In fact, one can consider

$$-\Delta u = a(x)g(u) + f(x, u), \qquad u \in H_0^1(\Omega),$$

set in a bounded smooth domain Ω, with a and g satisfying (H1), (H2) and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ a C^1 function with

(H3) $|f(x,s)s| + |\frac{\partial f(x,s)}{\partial x_i} s| + |\frac{\partial f(x,s)}{\partial s} s^2| \leqslant C(|s|^\sigma + 1)$ $\forall s, x, i$, with $C > 0$, $0 < \sigma < \mu$.

(We stress that $f(x,s)$ may include odd symmetric terms such as, for example, linear terms λs, with $\lambda \in \mathbb{R}$). Denoting the corresponding energy functional by J, with minor changes to the proofs that follow in the subsequent sections (see also the remark at the end of section 2), one can similarly prove:

Theorem 1.2. *Under assumptions* (H1)–(H3) *with* g *odd, if moreover*

$$\frac{2p}{N(p-2)} > \frac{\mu}{\mu - \sigma},$$

then there exists a sequence $(u_n) \subset H_0^1(\Omega)$ *such that*

$$J'(u_n) = 0 \quad \textit{and} \quad J(u_n) \to +\infty \quad \textit{as } n \to \infty.$$

In particular, the above problem admits infinitely many solutions.

The rest of our paper is organized as follows: In section 2 we prove some preliminary estimates and setup the variational approach. The proof of the theorem 1.1 is given in section 3. In section 4 we will show how our approach provides a similar existence result when the problem is set in the whole space (cf. theorem 4.1). Finally for the convenience of the reader we will provide a proof of lemma 2.6 in the appendix.

2 Preliminary estimates

We assume (H1)-(H2) holds and we proceed similarly to [14, Lemma 3]. For a fixed $q \in]2, \mu[$ close enough to μ, so that $\frac{2p}{N(p-2)} > \frac{q}{q-1}$, let g_j be the odd function defined by $g_j(s) = g(s)$ if $0 \leqslant s \leqslant a_j$, $g_j(s) = A_j s^{q-1} + B_j$ if $s \geqslant a_j$, where $a_j \to +\infty$ is any increasing sequence and the coefficients are chosen in such a way that g_j is C^1. The following properties of g_j hold uniformly in j: for some positive constants c_1, c_2, c_3, c_4,

(i) $g_j(s)s \geqslant c_1|s|^q - c_2$;

(ii) $G_j(s) \leqslant g_j(s)s + c_3$;

(iii) $g_j(s)s \leqslant c_4(|s|^p + 1)$.

We denote by I_j the energy functional associated to the truncated problems

$$-\Delta u = \lambda u + a^+(x)g(u) - a^-(x)g_j(u) + f(x), \qquad u \in H_0^1(\Omega),$$

namely

$$I_j(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega a^+(x)G(u) + \int_\Omega a^-(x)G_j(u) - \frac{\lambda}{2}\int_\Omega u^2 - \int_\Omega f(x)u, \qquad u \in H_0^1(\Omega).$$

We have denoted $a^+ := \max\{a, 0\}$, $a^- := \max\{-a, 0\}$. We will use the norm in $H_0^1(\Omega)$, $||u|| := (\int_\Omega |\nabla u|^2)^{1/2}$. We also denote $\Omega^+ = \{x \in \Omega : a(x) > 0\}$, $\Omega^- = \{x \in \Omega : a(x) < 0\}$; by assumption, both Ω^+ and Ω^- are non-empty.

Lemma 2.1. *There exists $C_1 > 0$ such that*

$$\int_{\partial\Omega} |\nabla u|^2 \leqslant C_1 \left(\int_\Omega |\nabla u|^2 + \int_{\Omega^+} G(u) + \int_{\Omega^-} G_j(u) + 1 \right), \qquad \forall j, u : I_j'(u) = 0.$$

Proof. Similarly to [5, Lemma 4.2], this follows from a Pohožaev-Rellich's type identity

$$\begin{aligned} & \operatorname{div} \left(\langle \nabla u, V \rangle \nabla u - \frac{1}{2}|\nabla u|^2 V + a^+ G(u) V - a^- G_j(u) V \right) \\ = & \sum u_i u_k \frac{\partial V_i}{\partial x_k} - \frac{1}{2}|\nabla u|^2 \operatorname{div} V + a^+ G(u) \operatorname{div} V - a^- G_j(u) \operatorname{div} V \\ + & G(u)\langle \nabla a^+, V \rangle - G_j(u) \langle \nabla a^-, V \rangle - \lambda u \langle \nabla u, V \rangle - f \langle \nabla u, V \rangle, \end{aligned}$$

where $V = (V_1, \ldots, V_N)$ is any smooth vector field. In our case, we let $V(x) = \nu(x)$, the unit outward normal of Ω (extended in a smooth way to the whole set $\overline{\Omega}$). □

Lemma 2.2. *There exists $C_2 > 0$ such that*

$$\int_{\Omega^+} G(u) + \int_{\Omega^-} G_j(u) \leqslant C_2 \left(\int_\Omega |\nabla u|^2 + \int_\Omega a^+ G(u) + \int_\Omega a^- G_j(u) + 1 \right), \ \forall j, u : I_j'(u) = 0.$$

Proof. We start by recalling that there exists $k(N) \in \mathbb{N}$ with the property that for every $r > 0$ we can cover $\mathbb{R}^N$ by a sequence of balls $(B_r(y_i))_{i \in \mathbb{N}}$ in such a way that each ball $B_{2r}(y_i)$ intersects at most $k(N)$ balls $B_{2r}(y_j)$ with $j \neq i$. Let $\omega_0 := \{x \in \overline{\Omega} : a(x) = 0\}$, $2\eta := \inf_{\omega_0} |\nabla a|^2 > 0$, and let us fix $\varepsilon > 0$ so small that $\varepsilon C_1 k(N)/2\eta < 1/2$, where C_1 is given by lemma 2.1. Since $a(x) \in C^1(\overline{\Omega})$, it admits a smooth extension to the whole space $\mathbb{R}^N$. By our assumptions on $a(x)$ and by using a compactness argument, we can find $r > 0$ so small that

$$\langle \nabla a(x), \nabla a(y) \rangle \geqslant \eta, \qquad \forall x, y \in \mathbb{R}^N : \operatorname{dist}(y, \omega_0) < r, \operatorname{dist}(x, y) < 2r$$

and

$$\langle \nabla a(y), \nu(x)\rangle \leqslant \varepsilon, \qquad \forall y \in \mathbb{R}^N, x \in \partial\Omega : \mathrm{dist}(y, \omega_0) < r, \mathrm{dist}(x, y) < 2r.$$

Let (y_i) be the sequence associated to an open covering of $\mathbb{R}^N$ as mentioned above, from which we extract a finite family

$$\omega_0 \subset B_r(y_1) \cup \ldots \cup B_r(y_\ell), \qquad \mathrm{dist}(y_i, \omega_0) < r, \quad \forall i = 1, \ldots, \ell.$$

For each $i = 1, \ldots, \ell$, let $\varphi_i \in \mathcal{D}(B_{2r}(y_i))$ be such that $0 \leqslant \varphi_i \leqslant 1$ and $\varphi_i = 1$ in $B_r(y_i)$. We apply the above Pohožaev-Rellich's identity with $V(x) := \varphi_i(x)\nabla a(y_i)$, yielding that

$$\int_{\Omega^+} G(u)\varphi_i \langle \nabla a(y_i), \nabla a\rangle + \int_{\Omega^-} G_j(u)\varphi_i\langle \nabla a(y_i), \nabla a\rangle \leqslant \frac{1}{2}\int_{\partial\Omega} |\nabla u|^2 \varphi_i \langle \nabla a(y_i), \nu(y)\rangle\, dy$$
$$+ \quad C\int_{\Omega\cap \mathrm{supp}\varphi_i} (|\nabla u|^2 + a^+|G(u)| + a^-|G_j(u)| + |u|^2 + 1)$$

Since $\inf_{\mathbb{R}} G > -\infty$, summing up and using the definition of r we deduce that, for $\omega := \Omega \cap (\cup_{i=1}^{\ell} B_r(y_i))$,

$$\begin{aligned}\eta\int_{\omega\cap\Omega^+} G(u) + \eta\int_{\omega\cap\Omega^-} G_j(u) \quad &\leqslant \quad \frac{\varepsilon k(N)}{2}\int_{\partial\Omega} |\nabla u|^2 \\ &+ \quad C'\int_\Omega (|\nabla u|^2 + a^+G(u) + a^-G_j(u) + |u|^2 + 1),\end{aligned}$$

for some $C' > 0$. Since ε was chosen small, it follows from lemma 2.1 that

$$\begin{aligned}\int_{\Omega^+} G(u) + \int_{\Omega^-} G_j(u) \quad &\leqslant \quad 2\int_{\Omega^+\setminus\omega} G(u) + 2\int_{\Omega^-\setminus\omega} G_j(u) \\ &+ \quad C''\int_\Omega (|\nabla u|^2 + a^+G(u) + a^-G_j(u) + |u|^2 + 1),\end{aligned}$$

for some $C'' > 0$. The conclusion follows from the fact that, by construction, $\inf_{\overline{\Omega}\setminus\omega} |a| > 0$. □

Lemma 2.3. *Given $\delta > 0$ there exists $C_\delta > 0$ such that*

$$\int_\Omega a^+g(u)u \leqslant (1+\delta)\int_\Omega |\nabla u|^2 + \delta\int_\Omega a^+G(u) + \delta\int_\Omega a^-G_j(u) + C_\delta(\int_\Omega u^2 + 1), \ \forall j, u : I_j'(u) = 0.$$

Proof. It follows from our assumptions that $g(s)s \leqslant C_3(G(s) + 1)$, for some $C_3 > 0$. Then, thanks to lemma 2.2,

$$\int_{\{a^+\leqslant\delta/C_2C_3\}} a^+g(u)u \leqslant \frac{\delta}{C_2}\int_{\Omega^+} G(u) + C_\delta \leqslant \delta\int_\Omega |\nabla u|^2 + \delta\int_\Omega a^+G(u) + \delta\int_\Omega a^-G_j(u) + C_\delta'.$$

On the other hand, by computing $I_j'(u)(u\varphi) = 0$ where $\varphi \in \mathcal{D}(\{a > 0\})$ is such that $\varphi(x) = 1$ if $a(x) \geqslant \delta/C_2C_3$ we readily see that

$$\int_{\{a^+ \geqslant \delta/C_2C_3\}} a^+ g(u)u \leqslant (1+\delta)\int_\Omega |\nabla u|^2 + C_\delta(\int_\Omega u^2 + 1),$$

and the conclusion follows. □

Lemma 2.4. *There exist $\delta, C > 0$ such that*

$$I_j(u) \geqslant \delta \int_\Omega |\nabla u|^2 + \delta \int_\Omega |u|^q - C, \qquad \forall j, u : I_j'(u) = 0.$$

Moreover, for each fixed $j \in \mathbb{N}$, given any finite dimensional eigenspace Y of $(-\Delta, H_0^1(\Omega))$,

$$I_j(u) \to -\infty \qquad \text{as } ||u|| \to \infty,\ u \in Y.$$

Proof. Since $g(s)s \geqslant \mu G(s) - C$ for some $\mu > 2$, we can find $\delta > 0$ so small that $(\frac{1}{2} - 2\delta)\frac{1}{1+\delta} g(s)s - G(s) \geqslant 2\delta G(s) - C$ $\forall s$. It follows then from lemma 2.3 that

$$\begin{aligned} I_j(u) &\geqslant 2\delta \int_\Omega |\nabla u|^2 + \delta \int_\Omega a^+ G(u) + (1-\delta)\int_\Omega a^- G_j(u) - C_\delta(\int_\Omega u^2 + 1) \\ &\geqslant \delta \int_\Omega |\nabla u|^2 + \delta(\int_\Omega |\nabla u|^2 + \int_\Omega a^+ G(u) + \int_\Omega a^- G_j(u)) - C_\delta(\int_\Omega u^2 + 1) \end{aligned}$$

Our first conclusion follows then from lemma 2.2.

Now let $j \in \mathbb{N}$ be fixed. If Y is any finite dimensional eigenspace of $(-\Delta, H_0^1(\Omega))$ then, as shown in [14, Lemma 2], $\int_\Omega a^+|u|^\mu \geqslant \eta ||u||^\mu$ $\forall u \in Y$, for some $\eta > 0$, so that

$$I_j(u) \leqslant ||u||^2 - d_1||u||^\mu + d_2||u||^q + d_2||u||^2 + d_2, \qquad \forall u \in Y,$$

for some $d_1, d_2 > 0$, yielding that $I_j(u) \to -\infty$ as $||u|| \to \infty$, $u \in Y$. □

We next introduce a penalized functional, similar to the one described in [10, 11]. Let $\chi \in \mathcal{D}(]-2,2[)$, $0 \leqslant \chi \leqslant 1$, with $\chi = 1$ in $[-1,1]$ and consider the C^2 map $\theta_j : H_0^1(\Omega) \to \mathbb{R}$,

$$\theta_j(u) = \chi\left(\frac{\delta \int_\Omega |u|^q}{2\sqrt{I_j^2(u) + 1}}\right),$$

where $\delta > 0$ is given by lemma 2.4. We introduce the functional

$$\widetilde{I}_j(u) = I_j(u) + (1 - \theta_j(u))\int_\Omega f(x)u, \qquad u \in H_0^1(\Omega).$$

Lemma 2.5. *(i) There exists $C > 0$ such that, for every $j \in \mathbb{N}$,*

$$|\widetilde{I}_j(u) - \widetilde{I}_j(-u)| \leqslant C(|\widetilde{I}_j(u)|^{1/q} + 1), \qquad \forall u \in H_0^1(\Omega).$$

(ii) There exists $M > 0$ such that, for every $j \in \mathbb{N}$, $\widetilde{I}_j$ satisfies the Palais-Smale condition at any level $c \geqslant M$.

(iii) There exists $M > 0$ such that, for every $j \in \mathbb{N}$, if $\widetilde{I}_j(u) \geqslant M$ and $\widetilde{I}_j'(u) = 0$ then $\theta_j(u) = 1$, $\widetilde{I}_j(u) = I_j(u)$ and $\widetilde{I}_j'(u) = I_j'(u)$.

PROOF. We first prove (ii). Let $(u_n) \subset H_0^1(\Omega)$ be such that $\widetilde{I}_j(u_n) \to c$ and $\widetilde{I}_j'(u_n) \to 0$, where c is some large positive constant. Suppose first that $\theta_j(u_n) = 0$ along some subsequence. Then, since the cut-off function χ is smooth we have that $\theta_j'(u_n)\varphi = 0$ $\forall \varphi \in H_0^1(\Omega)$ and it turns out that (u_n) is a Palais-Smale sequence for the functional $I_{0,j}$, where

$$I_{0,j}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega a^+ G(u) + \int_\Omega a^- G_j(u) - \frac{\lambda}{2}\int_\Omega u^2, \qquad u \in H_0^1(\Omega).$$

Now, for a given $r \in]q, \mu[$ we can find $\varepsilon, C > 0$ such that, for every $s \in \mathbb{R}$,

$$g(s)s - rG(s) \geqslant \varepsilon|s|^\mu - C \quad \text{and} \quad rG_j(s) - g_j(s)s \geqslant \varepsilon|s|^q - C.$$

Since, for every $u \in H_0^1(\Omega)$,

$$\begin{aligned}(\frac{r}{2} - 1)\int_\Omega |\nabla u|^2 \ &+ \ \int_\Omega a^+(g(u)u - rG(u)) + \int_\Omega a^-(rG_j(u) - g_j(u)u) \\ &= \ rI_{0,j}(u) - I_{0,j}'(u)u + \lambda(\frac{r}{2} - 1)\int_\Omega u^2,\end{aligned}$$

we deduce that $||u_n||^2 + \int_\Omega |a|\,|u_n|^q \leqslant C(\int_\Omega u_n^2 + 1)$ for some $C > 0$. In particular, $\int_\Omega |a|\,|u_n|^q \leqslant C(\int_\Omega u_n^2 + 1)$. Thus, assuming by contradiction that $||u_n|| \to \infty$ along a subsequence, since $2 < q < 2^*$ we deduce that $v_n := u_n/||u_n||$ converges weakly to some function v such that $\int_\Omega |a|\,|v|^q = 0$ and so, since $a(x)$ vanishes on a set of measure zero, $v = 0$. This contradicts the fact that $||u_n||^2 \leqslant C(\int_\Omega u_n^2 + 1)$, and so $(||u_n||)$ is bounded. By usual arguments, (u_n) admits a subsequence which converges strongly in $H_0^1(\Omega)$ (here, of course, the value c of the energy level is irrelevant).

So, from now on we assume that $\theta_j(u_n) \neq 0$, which implies that $\int_\Omega |u_n|^q \leqslant C(|I_j(u_n)| + 1)$ for some $C > 0$. By Hölder's inequality, $|I_j(u_n) - \widetilde{I}_j(u_n)| \leqslant \int_\Omega |f(x)u_n| \leqslant C'(|I_j(u_n)| +$

$1)^{1/q}$, and therefore $I_j(u_n)$ is large whenever $\widetilde{I}_j(u_n)$ is. By means of a simple computation, this implies that

$$\theta_j'(u_n)u_n \int_\Omega f(x)u_n = \mathrm{o}(1)I_j(u_n) + \mathrm{o}(1)I_j'(u_n)u_n,$$

with $\mathrm{o}(1) \to 0$ as $c \to \infty$, uniformly in j, from which we see that

$$\widetilde{I}_j'(u_n)u_n - I_j'(u_n)u_n = \mathrm{o}(1)I_j(u_n) + \mathrm{o}(1)I_j'(u_n)u_n.$$

Since, by assumption, $\widetilde{I}_j'(u_n)u_n = \mathrm{o}_j(u_n)$ with $\mathrm{o}_j(u_n)/||u_n|| \to 0$, we derive that

$$I_j'(u_n)u_n = \mathrm{o}_j(u_n) + \mathrm{o}(1)I_j(u_n),$$

and therefore, for a given $r \in]q, \mu[$,

$$rI_j(u_n) + \mathrm{o}_j(u_n) = (r + \mathrm{o}(1))I_j(u_n) - I_j'(u_n)u_n.$$

Since, for every $u \in H_0^1(\Omega)$,

$$\begin{aligned}(\frac{r}{2} - 1)\int_\Omega |\nabla u|^2 \quad &+ \quad \int_\Omega a^+(g(u)u - rG(u)) + \int_\Omega a^-(rG_j(u) - g_j(u)u) \\ &= \quad rI_j(u) - I_j'(u)u + \lambda(\frac{r}{2} - 1)\int_\Omega u^2 + (r-1)\int_\Omega f(x)u,\end{aligned}$$

by replacing r with $r+\mathrm{o}(1)$, we conclude as above that, up to a subsequence, (u_n) converges in $H_0^1(\Omega)$.

The proof of property (i) is similar to the one in [11, Prop. 10.16], by observing that $|\widetilde{I}_j(u) - \widetilde{I}_j(-u)| \leqslant C(\theta_j(u) + \theta_j(-u)) \int_\Omega |f(x)u|$ and $|\widetilde{I}_j(u) - I_j(u)| \leqslant (1 - \theta_j(u)) \int_\Omega |f(x)u|$.

As for (iii), suppose $\widetilde{I}_j'(u) = 0$ and $\widetilde{I}_j(u) \geqslant M$ for some large $M > 0$. Assume first that $\theta_j(u) = 0$. Then u is actually a critical point of the functional $I_{0,j}$ defined above and, as in lemma 2.4,

$$\delta \int_\Omega |u|^q \leqslant C + I_{0,j}(u) = C + \widetilde{I}_j(u).$$

So, for a fixed small $\eta > 0$ such that $(1+\eta)/(1-\eta) < 2$ we have that

$$\begin{aligned}\delta \int_\Omega |u|^q \quad &\leqslant \quad C - \eta I_{0,j}(u) + (1+\eta)I_{0,j}(u) \\ &= \quad C - \eta I_{0,j}(u) + (1+\eta)\int_\Omega f(x)u + (1+\eta)I_j(u) \\ &\leqslant \quad C_\delta - \eta I_{0,j}(u) + \delta\eta \int_\Omega |u|^q + (1+\eta)I_j(u)\end{aligned}$$

and so, if $I_{0,j}(u)$ is sufficiently large,

$$\delta \int_\Omega |u|^q \leqslant \delta\eta \int_\Omega |u|^q + (1+\eta) I_j(u).$$

This implies that $\delta \int_\Omega |u|^q < 2I_j(u) \leqslant 2\sqrt{I_j^2(u)+1}$, whence $\theta_j(u) = 1$, contradicting our assumption that $\theta_j(u) = 0$.

So, from now on we assume that $\theta_j(u) \neq 0$. Then, as observed in the proof of property (ii) above, $I_j(u)$ is large whenever $\widetilde{I}_j(u)$ is, and u satisfies in $H_0^1(\Omega)$ an equation of the form

$$-\Delta u = \lambda u + a^+(x)g(u) - a^-(x)g_j(u) + \alpha|u|^{q-2}u + \beta f(x),$$

where $\alpha = \alpha(j,u)$ and $\beta = \beta(u,j)$ are real constants which remain bounded uniformly in j; in fact, $\alpha \to 0$ and $|\beta| \leqslant 2$ as $M \to \infty$. An inspection of the proof of lemma 2.4 will show that the constants δ and C in that lemma could have been chosen in such a way that

$$\widetilde{I}_j(u) \geqslant \delta \int_\Omega |u|^q - C, \qquad \forall u : \widetilde{I}_j'(u) = 0.$$

Proceeding as above (case $\theta_j(u) = 0$) we conclude that $\theta_j(u) = 1$ in a neighborhood of u and this completes the proof of lemma 2.5. □

In the sequel we denote by $m(u)$ the Morse index of a critical point u of the functional I_j. Namely, by definition $m(u)$ is the supremum of the dimensions of the subspaces Z of $H_0^1(\Omega)$ such that $I_j''(u)(\varphi,\varphi) < 0$ $\forall \varphi \in Z$, $\varphi \neq 0$. Our next lemma states that if u is a critical point of I_j then actually u solves a non-truncated equation, provided $m(u)$ is bounded independently of j.

Lemma 2.6. *For every $k \in \mathbb{N}$ there exists $N(k) > 0$ and $j(k) \in \mathbb{N}$ such that*

$$I_j'(u) = 0,\ m(u) \leqslant k+1 \Rightarrow ||u||_{H_0^1(\Omega)} + ||u||_{L^\infty(\Omega)} \leqslant N(k) < a_j,\ \forall j \geqslant j(k).$$

Proof. Under more stringent assumptions on the functions $a(x)$ and $g(s)$, this was proved in [14, Sect. 3] by means of a blow-up argument. For the reader's convenience, in the Appendix we sketch an alternate proof which combines the previous estimates with the argument in [12, Th. 1] (see also [20] for a similar argument). □

Remark. In connection with theorem 1.2, the reader can check that under assumptions (H1)–(H3) we obtain similar lemmas for the related equation. The two more substantial changes in the arguments presented above are the following: 1. In the proof of (the corresponding) lemma 2.1, the vector field should include an extra term $F(x,u)V$ and it should be observed that $|F(x,u)| + |\nabla_x F(x,u)| \leqslant C + C\min\{G(u), G_j(u)\}$, and 2. In the verification of the Palais-Smale condition (lemma 2.5 (ii)), one uses the fact that if $\theta_j(u_n) \neq 0$ then $\int_\Omega (|F(x,u_n)| + |f(x,u_n)u_n|) \leqslant C(|I_j(u_n)| + 1)$.

3 Proof of Theorem 1.1

In this section we look for critical points of the functionals $\widetilde{I}_j$. Similarly to lemma 2.4, to each $k, j \in \mathbb{N}$ we can associate a number $R_k^j > 0$ such that

$$\widetilde{I}_j(u) < 0, \qquad \forall u \in E_k : ||u|| \geqslant R_k^j.$$

Here $E := H_0^1(\Omega)$ and E_k is spanned by the first k eigenfunctions of $(-\Delta, H_0^1(\Omega))$. Accordingly, let

$$b_k^j := \inf_{\gamma \in \Gamma_k^j} \sup_{\gamma(B_{R_k^j}(0)\cap E_k)} \widetilde{I}_j,$$

where

$$\Gamma_k^j := \{\gamma : B_{R_k^j}(0) \cap E_k \to E \text{ continous and odd: } \gamma(u) = u,\ \forall u \in \partial B_{R_k^j}(0)\}.$$

Lemma 3.1. *For every $k \in \mathbb{N}$ there exists $M(k) > 0$ such that*

$$b_k^j \leqslant M(k), \qquad \forall j \in \mathbb{N}.$$

Proof. Let $\overline{I}_j$ be the functional defined by

$$\overline{I}_j(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - c_0 \int_\Omega a^+|u|^\mu + \int_\Omega a^- G_j(u) + \frac{c_1}{2}\int_\Omega u^2, \qquad u \in H_0^1(\Omega);$$

the constants $c_0, c_1 > 0$ are chosen in such a way that, for some $C_0 > 0$,

$$\widetilde{I}_j(u) \leqslant \overline{I}(u) + C_0, \quad \forall j \in \mathbb{N},\ u \in H_0^1(\Omega).$$

We denote by $\overline{b}_k^j$ the corresponding min-max levels; clearly, $b_k^j \leqslant \overline{b}_k^j + C_0$. It is then enough to prove that $\overline{b}_k^j \leqslant M(k)$. Similarly to the proof of lemma 2.5 (ii) (case $\theta_j(u_n) = 0$), $\overline{I}_j$ satisfies the Palais- Smale condition in $H_0^1(\Omega)$.

On the other hand, we can assume that $\overline{I}_j(u) < 0, \forall u \in E_k : ||u|| \geqslant R_k^j$, and moreover there exists $\alpha > 0$ such that,

$$\overline{I}_j(u) \geqslant \frac{1}{2}||u||^2 - \alpha||u||^\mu \qquad \forall j \in \mathbb{N},\, u \in H_0^1(\Omega).$$

Therefore for ρ sufficiently small, we have

$$m := \inf\{\overline{I}_j(u) :\ j \in \mathbb{N},\, ||u|| = \rho\} > 0;$$

clearly, we can also assume that $\rho < R_k^j, \forall j, k \in \mathbb{N}$. Since $\overline{I}_j$ is an even functional, the well-known symmetric version of the Mountain Pass theorem [1] yields a critical point u_k^j of $\overline{I}_j$ at energy level $\overline{b}_k^j$. Moreover, u_k^j can be chosen with the further property that its Morse index is less or equal than k (see e.g. [7, 8, 13, 15]). According to (a slight variant of) lemma 2.6, we have that $||u_k^j||_{H_0^1(\Omega)} + ||u_k^j||_{L^\infty(\Omega)} \leqslant C(k)$ for some constant $C(k)$ which does not depend on j. Since $\overline{I}'_j(u_k^j)u_k^j = 0$, also $\int_\Omega a^- g_j(u_k^j)u_k^j \leqslant C'(k)$. Thus $\int_\Omega a^- G_j(u_k^j) \leqslant C''(k)$ and so $\overline{b}_k^j = \overline{I}_j(u_k^j) \leqslant M(k)$, uniformly in j. □

Lemma 3.2. *There exist $k_0 \in \mathbb{N}$ and $c > 0$ such that*

$$b_k^j \geqslant ck^{2p/N(p-2)}, \qquad \forall k \geqslant k_0, j \in \mathbb{N}.$$

Proof. Since $|a^+(x)G(s) - a^-(x)G_j(s)| \leqslant C(|s|^p + 1)$ for some $C > 0$, this follows from an estimate given in [4, 17] which relies on the information on the Morse index of the solutions of the equation $-\Delta u = |u|^{p-2}u$, $u \in H_0^1(\Omega)$. □

Lemma 3.3. *Given $C > 0$ and large $k_0 \in \mathbb{N}$, there exists $k_1 \in \mathbb{N}$, $k_1 > k_0$, such that for $j \in \mathbb{N}$, it is not possible to have*

$$b_{k+1}^j \leqslant b_k^j + C\left((b_k^j)^{1/q} + 1\right), \qquad \forall k :\ k_0 \leqslant k \leqslant k_1.$$

Proof. Indeed, according to the argument in [3, Lemma 5.3] (see also [11, Prop. 10.46]), if for some $j \in \mathbb{N}$ the inequalities are satisfied, this would lead to the existence of $C' > 0$ (depending only on k_0, q and C) such that

$$b_k^j \leqslant C'\, b_{k_0}^j\, k^{q/(q-1)}, \qquad \forall k :\ k_0 \leqslant k \leqslant k_1.$$

Taking lemmas 3.1 and 3.2 into account, the validity of this inequality for k_1 sufficiently large (depending only on k_0, C, q and c of lemma 3.2) would imply that $2p/N(p-2) \leqslant q/(q-1)$. However, q was chosen in such a way that the reversed (strict) inequality holds, and this is a contradiction. □

Proof of theorem 1.1 completed. We need to recall the framework in [11, pp. 67–68]. Let us denote

$$U_k^j := \{u = v + te_{k+1},\ ||u|| \leqslant R_{k+1}^j, v \in E_k, t \geqslant 0\},$$

where e_{k+1} is an eigenfunction associated to the $(k+1)$-th eigenvalue of $(-\Delta, H_0^1(\Omega))$,

$$Q_k^j := (\partial B_{R_{k+1}^j}(0) \cap E_{k+1}) \cup ((B_{R_{k+1}^j}(0) \setminus B_{R_k^j}(0)) \cap E_k)$$

(so that $\sup_{Q_k^j} \widetilde{I}_j < 0$),

$$\Lambda_k^j := \{\gamma : U_k^j \to E :\ \gamma|_{Q_k^j} = Id \ \text{ and } \ \gamma(-u) = -\gamma(u)\ \forall u \in B_{R_k^j}(0) \cap E_k\},$$

and

$$c_k^j := \inf_{\gamma \in \Lambda_k^j} \max\{\widetilde{I}_j(\gamma(u)) : u \in U_k^j\} \geqslant b_k^j.$$

We recall that for each $k \in \mathbb{N}$ an integer $j(k)$ was defined in lemma 2.6. Clearly, we can assume that $j(k+1) \geqslant j(k) \geqslant k$ for every k.

Given any large positive number d, using lemma 3.2 we choose k_0 such that $b_k^j > d$, for all $k \geqslant k_0, j \in \mathbb{N}$ and take k_1 as in lemma 3.3 for this choice of k_0 and C given in lemma 2.5 (i). Next we fix $j_0 \geqslant j(k_1)$ and claim that there exists an integer $k_* \in [k_0, k_1]$ such that

$$b_* := b_{k_*}^{j_0} < c_{k_*}^{j_0} := c_*.$$

Indeed, otherwise we would have $b_k^{j_0} = c_k^{j_0}$ for every integer $k \in [k_0, k_1]$ and, as in [11, Prop. 10.46], this together with lemma 2.5 (i) would lead to a contradiction with lemma 3.3. Now, let

$$\widetilde{\Lambda} := \{\gamma \in \Lambda_{k_*}^{j_0} :\ \widetilde{I}_{j_0}(\gamma(u)) \leqslant (b_* + c_*)/2, \quad \forall u \in B_{R_{k_*}^{j_0}}(0) \cap E_{k_*}\}$$

and

$$\widetilde{c} := \inf_{\gamma \in \widetilde{\Lambda}} \max\{\widetilde{I}_{j_0}(\gamma(u)) :\ u \in U_{k_*}^{j_0}\} \geqslant c_*.$$

Then $\widetilde{c} \geqslant c_* > b_* > d$ and, by standard deformation arguments (and thanks also to lemma 2.5 (ii)) there exists $u \in H_0^1(\Omega)$ such that

$$\widetilde{I}_{j_0}(u) = \widetilde{c} \qquad \text{and} \qquad \widetilde{I}'_{j_0}(u) = 0.$$

According to lemma 2.5 (iii), it turns out that u is actually a critical point of the functional I_{j_0}. Moreover, using the definition of $\widetilde{c}$, the arguments in e.g. [7, 8, 13, 15] show that u can be chosen in such a way that its Morse index is less or equal than $k_* + 1 \leqslant k_1 + 1$. Then, by definition of j_0 (cf. lemma 2.6), u is a critical point of the original functional I, hence a solution of our original problem with $I(u) > d$. Since d was arbitrary, this completes the proof of theorem 1.1. □

4 The case of $\mathbb{R}^N$

We this section we consider the problem

$$-\Delta u + u = a(x)g(u) + f(x), \qquad u \in H^1(\mathbb{R}^N)$$

where now $a : \mathbb{R}^N \to \mathbb{R}$ is a C^1 function changing sign in $\mathbb{R}^N$, and $f \in L^2(\mathbb{R}^N) \cap L^{\mu/(\mu-1)}(\mathbb{R}^N) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ for some $0 < \alpha < 1$. The odd function $g : \mathbb{R} \to \mathbb{R}$ satisfies (H1) and we further assume

(H4) $a(0) > 0$, $\limsup_{|x|\to\infty} a(x) \in [-\infty, 0[$ and a has only non-degenerate zeros, i.e. $\nabla a(x) \neq 0$ $\forall x \in \mathbb{R}^N$ such that $a(x) = 0$;

(H5) $0 \leqslant G(s) \leqslant Cg(s)s \;\; \forall s$.

The energy functional associated to our problem is now given by

$$I(u) := \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla u|^2 + u^2) - \int_{\mathbb{R}^N} a(x)G(u) - \int_{\mathbb{R}^N} f(x)u, \qquad u \in H^1(\mathbb{R}^N).$$

We prove the following:

Theorem 4.1. *Under assumptions* (H1), (H4)–(H5) *with g odd, if moreover*

$$\frac{2p}{N(p-2)} > \frac{\mu}{\mu - 1},$$

then, for every $f \in L^2(\mathbb{R}^N) \cap \cap L^{\mu/(\mu-1)}(\mathbb{R}^N) \cap C^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ there exists a sequence $(u_n) \subset H^1(\mathbb{R}^N)$ such that

$$I'(u_n) = 0 \quad \text{and} \quad I(u_n) \to +\infty \;\; \text{as } n \to \infty.$$

In particular, the above problem admits infinitely many solutions.

Proof. We apply our previous result (with $\lambda = -1$) for a bounded domain $\Omega = B_R(0)$ for every large $R > 0$. A review of the proofs in the previous two sections indicates that all our estimates are uniform in R, this being due to (H5) and the following three facts: 1. $\lambda = -1 < 0$, 2. $f \in L^{q/(q-1)}(\mathbb{R}^N)$, since $2 < q < \mu$, and 3. Ω^+ is bounded. We will indicate how this uniformity is achieved in more detail below.

Incorporating the notation used in sections 2 and 3, we write $I_{j,R}, \widetilde{I}_{j,R}, b^j_{k,R}, c^j_{k,R}, \ldots$ to display their dependence on the domain $B_R := B_R(0)$. Then we can prove the following versions of the relevant lemmas in the previous two sections.

Lemma 4.2. *There exists $C > 0$ such that*

$$|\widetilde{I}_{j,R}(u) - \widetilde{I}_{j,R}(-u)| \leqslant C(|\widetilde{I}_{j,R}(u)|^{1/q} + 1), \qquad \forall R > 0, j \in \mathbb{N}, u \in H^1_0(B_R(0))).$$

Proof. In can be checked that the estimates in section 2 are uniform in R; we point out that if R is chosen large enough so that a^+ is supported in $B_R(0)$ then (a corresponding version of) lemma 2.1 is not needed for the remaining estimates. The uniformity in R eventually leads to the property stated in lemma 2.5 (i). □

Lemma 4.3. *For every $k \in \mathbb{N}$ there exists $N(k) > 0$ and $j(k) \in \mathbb{N}$ such that*

$$I'_{j,R}(u) = 0,\ m(u) \leqslant k + 1 \Rightarrow ||u||_{L^2} + ||u||_{H^1_0} + ||u||_{L^\infty} \leqslant N(k) < a_j,\ \forall R > 0, j \geqslant j(k).$$

Proof. The proof in the appendix is still valid since Ω^+ is bounded, $f \in L^2(\mathbb{R}^N)$ and $0 \leqslant G_j(s) \leqslant Cg(s)s \ \ \forall s, j$. In addition we will get the extra term $||u||_{L^2}$ on the left hand side of the inequality since now $\lambda = -1$. □

Lemma 4.4. *For every $k \in \mathbb{N}$ there exists $M(k) > 0$ such that*

$$b^j_{k,R} \leqslant M(k), \qquad \forall R > 0, j \in \mathbb{N}.$$

Proof. We follow the proof of lemma 3.1, keeping in mind that since Ω^+ is bounded, the constants c_0, c_1 and C_0 in the definition of $\overline{I}_{j,R}$ can be chosen independent of $R > 0$. We also use the fact that $G_j(s) \geqslant 0 \ \ \forall s, j$ □

Lemma 4.5. *There exist $k_0 \in \mathbb{N}$ and $c > 0$ such that*

$$b^j_{k,R} \geqslant ck^{2p/N(p-2)}, \qquad \forall R > 0, k \geqslant k_0, j \in \mathbb{N}.$$

Proof. Since $g(s)s \geqslant 0$, setting

$$J_R(u) := \frac{1}{2}\int_{B_R} |\nabla u|^2 - \alpha_1 \int_{B_R} a^+|u|^p, \qquad u \in H_0^1(B_R)$$

we can choose α_1 and $C > 0$ such that

$$\widetilde{I}_{j,R}(u) \geqslant J_R(u) - C, \qquad \forall R > 0, j \in \mathbb{N}, u \in H_0^1(B_R).$$

We can now compare our problem with the problem $-\Delta u = a^+(x)|u|^{p-2}u$, $u \in H_0^1(B_R(0))$ and, again using the fact that Ω^+ is bounded, conclude as in lemma 3.2. □

Lemma 4.6. *Given $C > 0$ and large $k_0 \in \mathbb{N}$, there exists $k_1 \in \mathbb{N}$, $k_1 > k_0$, such that for $R > 0, j \in \mathbb{N}$, it is not possible to have*

$$b^j_{k+1,R} \leqslant b^j_{k,R} + C\left((b^j_{k,R})^{1/q} + 1\right), \qquad \forall k:\ k_0 \leqslant k \leqslant k_1.$$

Proof. We will follow the proof of lemma 3.3 and finish up using lemmas 4.4 and 4.5. □

Now we may complete the argument as in section 3. Given any large positive number d, let k_0 be chosen large enough so that $b^j_{k,R} > d$, for all $k \geqslant k_0$, $R > 0, j \in \mathbb{N}$ (see lemma 4.5). According to lemma 4.6, there exists $k_1 \geqslant k_0$ such that for $R > 0, j \in \mathbb{N}$, we cannot have $b^j_{k,R} = c^j_{k,R}$ for every $k = k_0, \ldots, k_1$; we stress that k_1 is independent of j and R, it depends on k_0 and on the constants C and c given by lemmas 4.2 and 4.5 only. As a consequence, fixing $j_0 \geqslant j(k_1)$, we can assert that for every large R there exists $k \in \{k_0, \ldots, k_1\}$ such that $c^{j_0}_{k,R} > b^{j_0}_{k,R}$. As explained before, this yields a solution in $H_0^1(B_R(0))$ to the equation $-\Delta u + u = a(x)g(u) + f(x)$, whose energy level is greater than $b^{j_0}_{k,R}$ and whose norm in $H^1(\mathbb{R}^N)$ is bounded by some constant which depends only on k_1. By taking sequences $R_n \to \infty$ and corresponding solutions (u_n) and by taking weak limits in $H^1(\mathbb{R}^N)$, in this way we find a solution $u \in H^1(\mathbb{R}^N)$ to our problem. On the other hand, applying the Fatou's lemma (recall that $g(s)s \geqslant 0$) to the identity

$$\int_{\mathbb{R}^N} (|\nabla u_n|^2 + u_n^2) + \int_{\mathbb{R}^N} a^- g(u_n)u_n = \int_{\mathbb{R}^N} a^+ g(u_n)u_n + \int_{\mathbb{R}^N} f(x)u_n,$$

we see that $u_n \to u$ strongly in $H^1(\mathbb{R}^N)$ and also that $\int_{\mathbb{R}^N} a^- g(u_n)u_n \to \int_{\mathbb{R}^N} a^- g(u)u$. Since $0 \leqslant G(s) \leqslant Cg(s)s\ \forall s$, also $\int_{\mathbb{R}^N} a^- G(u_n) \to \int_{\mathbb{R}^N} a^- G(u)$, and so $I(u) \geqslant d$. Since d is arbitrarily large, this completes the proof of theorem 4.1. □

5 Appendix

In this section we provide a sketch of the proof of lemma 2.6. Hereafter, u is a critical point of I_j whose Morse index depends only on k. For the sake of clarity we divide the proof into several steps.

FIRST STEP ("conservation of energy"). Given $\varepsilon > 0$, we observe that

$$\int_\Omega a^+ g(u)u = \int_{\{a^+<\varepsilon\}} a^+ g(u)u + \int_{\{a^+\geqslant\varepsilon\}} a^+ g(u)u \leqslant \varepsilon \int_{\Omega^+} g(u)u + C + \int_{\{a^+\geqslant\varepsilon\}} a^+ g(u)u.$$

By combining the identity $\int_\Omega(|\nabla u|^2 + a^- g_j(u)u) = \int_\Omega(a^+ g(u)u + \lambda u^2 + fu)$ with lemma 2.2, it follows from the inequality above that we can fix $\varepsilon > 0$ so small that

$$\int_\Omega |\nabla u|^2 + \int_\Omega a^+ g(u)u \leqslant C(\int_\Omega u^2 + 1) + C\int_{\{a^+\geqslant\varepsilon\}} a^+ g(u)u,$$

for some positive constant C. We stress that the value of ε depends only on the constants relating $g(s)$ and $G(s)$, and also $g_j(s)$ and $G_j(s)$, and these are uniform with respect to j.

SECOND STEP (Pohožaev-Rellich's identity). Let ε be given by the first step above. We claim that we can find a finite number of points $x_1, \ldots, x_m \in \overline{\Omega}$ and a small constant $r > 0$ in such a way that

$$\{a^+ \geqslant \varepsilon\} \subset \cup_{i=1}^m B_r(x_i), \qquad B_{2r}(x_i) \subset \{a^+ \geqslant \varepsilon/2\},$$

and, for every smooth function $\varphi \in \mathcal{D}(B_{2r}(x_i))$,

$$\int_\Omega |\nabla u|^2\varphi^2 + \int_\Omega ag(u)u\varphi^2 \leqslant C(\int_\Omega u^2\varphi^2 + 1) + C\int_\Omega ((|\nabla u|^2 + u^2 + a|G(u)|)|\nabla\varphi^2| + u^2|\nabla\varphi|^2),$$

for some constant $C > 0$.

Indeed, using a compactness argument, it is sufficient to prove this in a neighborhood of a given point x_0 such that $a(x_0) \geqslant \varepsilon$. This, in turn, follows from the classical Pohožaev-Rellich's identity stated in lemma 2.1, with $V(x) = \varphi^2(x)(x - x_0)$ and by using the fact that $g(s)s \leqslant pG(s) + C$ with $p < 2^*$. We refer the reader to [12, Lemma 2.1] for the details. We point out that in case $x_0 \in \partial\Omega$ one must in fact use a convenient vector field V such that $||V - (Id - x_0)||_{C^1(B_{2r}(x_0))}$ is small enough.

THIRD STEP (Morse index estimates). Let $r > 0$ and x_i be given by the second step. For simplicity of notations, in the following we assume $x_i = 0$. For any numbers $r < s_0 < s_1 < 2r$, we denote

$$\mathcal{B}(s_0, s_1) := \{x : s_0 < |x| < s_1\} \subset B_{2r}(0) \setminus B_r(0).$$

Let $\psi \in \mathcal{D}(\mathbb{R}^N)$ be any function with support in a neighborhood of $\mathcal{B}(s_0, s_1)$ and $\psi = 1$ in $\mathcal{B}(s_0, s_1)$. Since the Morse index of u is bounded above, we can find s_0 and s_1 in such a way that $I_j''(u)(u\psi, u\psi) \geqslant 0$, that is,

$$\begin{aligned}\int_\Omega ag'(u)u^2\psi^2 &\leqslant \int_\Omega |\nabla(u\psi)|^2 - \lambda \int_\Omega u^2\psi^2 \\ &= \int_\Omega ag(u)u\psi^2 + \int_\Omega u^2|\nabla\psi|^2 + \int_\Omega f(x)u.\end{aligned}$$

By our assumptions, $g'(s)s^2 \geqslant (\mu-1)g(s)s - C$ for some $\mu > 2$, and so the above inequality implies that

$$\int_{\mathcal{B}(s_0,s_1)} (|\nabla u|^2 + ag(u)u) \leqslant C(\int_\Omega u^2 + 1).$$

Now, let $\varphi \in \mathcal{D}(B_{s_1}(0))$ be such that $\varphi = 1$ in $B_{s_0}(0)$. Since $\nabla\varphi = 0$ except in $\mathcal{B}(s_0, s_1)$, it follows from the previous estimate together with the conclusion in the second step of the proof that

$$\int_{B_{s_0}(0)} (|\nabla u|^2 + ag(u)u) \leqslant C'(\int_\Omega u^2 + 1).$$

This shows that

$$\int_{B_r(x_i)} (|\nabla u|^2 + ag(u)u) \leqslant C''(\int_\Omega u^2 + 1), \qquad \forall i = 1, \ldots, m.$$

By recalling that $\{a^+ \geqslant \varepsilon\} \subset \cup_{i=1}^m B_r(x_i)$ and by taking the conclusion of step 1 into account, we arrive at the estimate

$$\int_\Omega (|\nabla u|^2 + a^+ g(u)u) \leqslant C(\int_\Omega u^2 + 1),$$

for some constant $C > 0$.

Fourth step (conclusion). By combining the conclusion in the third step with lemma 2.2 we get an a priori bound

$$\int_\Omega |\nabla u|^2 \leqslant C$$

which is uniform in u and j. Using elliptic estimates, this implies that $||u||_{L^\infty(\Omega)} \leqslant C'$, as claimed. □

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