Computational bounds on polynomial differential equations

Daniel S. Graça [a,b] , Jorge Buescu [c,d] ,
Manuel L. Campagnolo [e,b] ,

[a] *DM/FCT da Universidade do Algarve, Portugal*
[b] *SQIG/Instituto de Telecomunicações, Lisboa, Portugal*
[c] *DM/FCUL, University of Lisbon, Portugal*
[d] *CMAF, Lisbon, Portugal*
[e] *DM/ISA, Technical University of Lisbon, Portugal*

Abstract

In this paper we study from a computational perspective some properties of the solutions of polynomial ordinary differential equations.

We consider elementary (in the sense of Analysis) discrete-time dynamical systems satisfying certain criteria of robustness. We show that those systems can be simulated with elementary and robust continuous-time dynamical systems which can be expanded into fully polynomial ordinary differential equations in $\mathbb{Q}[\pi]$. This sets a computational lower bound on polynomial ODEs since the former class is large enough to include the dynamics of arbitrary Turing machines.

We also apply the previous methods to show that the problem of determining whether the maximal interval of definition of an initial-value problem defined with polynomial ODEs is bounded or not is in general undecidable, even if the parameters of the system are computable and comparable and if the degree of the corresponding polynomial is at most 56.

Combined with earlier results on the computability of solutions of polynomial ODEs, one can conclude that there is from a computational point of view a close connection between these systems and Turing machines.

Email addresses: dgraca@ualg.pt (Daniel S. Graça), jbuescu@ptmat.fc.ul.pt (Jorge Buescu), mlc@math.isa.utl.pt (Manuel L. Campagnolo).

Preprint submitted to Elsevier *5 July 2008*

1 Introduction

Differential equations are a powerful tool to model a diversity of phenomena in fields ranging from basic natural sciences like physics, chemistry or biology to social sciences or economics. Among these, initial value problems (IVPs) of the form $x' = f(t, x)$, with $x(t_0) = x_0$, where f is a vector field and t is the independent variable, play a predominant role. In this paper we consider the large class of polynomial IVPs (PIVPs for short) in which f is a vector of polynomials. Many well known models, like the Lorenz equations in meteorology, the Lotka-Volterra equations for predator-prey systems, or Van der Pol's equation in electronics [1] fall into this category. In Section 2 we show that in fact all the elementary functions of Analysis are solutions of PIVPs. This is a stronger version of the well established fact that all elementary functions are differentially algebraic [2]. It is also worth noticing that the solutions of PIVPs are precisely the set of functions definable with Shannon's General Purpose Analog Computer (GPAC) [3] as proved in [4].

While the qualitative behavior of linear systems (i. e. where f is linear) and planar systems (where $f : \mathbb{R}^3 \to \mathbb{R}^2$) is completely understood [5], it is not known for all but a few cases how to predict the behavior of the solutions of general PIVPs from the expression of f, which is the reason why many fundamental questions about PIVPs (e.g. Hilbert's 16th problem) are still open.

Since most nonlinear differential equations cannot be solved exactly, one has to resort to numerical methods to obtain approximate solutions. This leads to a range of questions about computational properties of PIVPs. In particular, one can ask if PIVPs have computable approximations, if the domain of the solution is computable, or even if deciding whether the maximal interval of existence is bounded is computable. Such questions have been answered for analytic IVPs (where f is analytic) in [6]. In Section 2 we point out that the results in [6] imply that the domain of existence of PIVP functions (i.e. solutions of PIVPs) is in general recursively enumerable and that the solution is computable on its domain. This last result sets an upper bound on the computability of PIVP functions since it ensures that as long as f is polynomial and computable they can be arbitrarily approximated wherever they are defined.

To obtain computational lower bounds for PIVPs, one can show that any computable function can be approximated by some PIVP function. In [7] it was proved that under a simple (and unbounded) encoding in $\mathbb{N}^3$, the evolution of Turing machines can be simulated with PIVPs. In this paper, we extend that result and show that any computable discrete dynamical system on $\mathbb{N}^m$ which admits a robust extension (to be defined) can be simulated with a PIVP.

The iteration of maps with IVPs is not new and can be found, for instance, in [8], [9]. However, those constructions are in some sense not satisfactory since they involve functions with some degree of discreteness (e.g. functions which are not analytic or even have discontinuous derivatives) which can be used to build "exact clocks" that simulate the discrete steps of the iteration.

In Sections 3 and 4 we state and prove the main result of the paper. We show that given a map ω on $\mathbb{N}^m$, one can construct a PIVP with coefficients in $\mathbb{Q}[\pi]$ that simulates the iteration of ω as long as ω is extendable to a "robust" map Ω on $\mathbb{R}^m$, in a sense to be defined in Section 3, and Ω is composition of polynomial and PIVP functions with parameters in $\mathbb{Q}[\pi]$. The simulation is robust, which is a necessity for our construction, but is also a natural requirement for a continuous-time physical system described by an IVP. The constructions in Section 4 will also provide the necessary tools to address the issues discussed in the remainder of the paper.

Finally, in Section 5, we review and extend some undecidability results on PIVPs. Our results in [7] imply that reachability for PIVPs is undecidable, i.e., given a PIVP and some open set in phase space, there is no algorithm to decide if the solution of the PIVP crosses the open set. This contrasts with the decidability of the reachability for linear differential equations [10]. In [11] we showed that the boundedness of the domain of existence for PVIPs is undecidable as long as f is polynomial of sufficiently high degree and computable. At first sight, this result might seem trivial, since one can easily construct simple PIVPs which, upon varying one parameter, exhibit a critical value where the solution is bounded on the left of this parameter value and unbounded on the right. For instance, the PIVP $x' = \alpha(x^2 - 1)t$, $x(0) = 3$ has a maximal interval which is bounded for $\alpha > 0$ and unbounded if $\alpha \leq 0$. Since we cannot compare exactly two arbitrary computable reals [12] the boundedness problem for the PIVP above is undecidable. However, in Section 5 we show that if we consider that all input parameters are "comparable", the boundedness problem remains undecidable. We also prove the claim in [11] that those undecidability results hold for PIVPs where the degree of the polynomial is less or equal to 56.

2 The GPAC, Polynomial Differential Equations, and Computable Analysis

In this section we introduce some useful definitions and results that will later be used in this paper.

Definition 1 *Let $I \subseteq \mathbb{R}$ be a non-empty open interval and let $t_0 \in I$. We say that $g : I \to \mathbb{R}$ is a PIVP function on I if it is a component of the solution of* *the initial-value problem*

$$x' = p(t, x), \quad x(t_0) = x_0 \tag{1}$$

where p is a vector of polynomials and $t_0 \in I$. We say that g is a PIVP function with parameters in $S \subseteq \mathbb{R}$ if the coefficients of p in (1), t_0, and the components of x_0 belong to S.

Similarly we say that a function $g : I \subseteq \mathbb{R} \to \mathbb{R}^k$ is a vector PIVP function if each component of g is a PIVP function.

Example 2 *The following are examples of PIVP functions with parameters in $\mathbb{Z}$: the exponential function e^x, the trigonometric functions* cos, sin *[7], the inverse function $x \mapsto 1/x$ (solution of $y' = -y^2$; on $(0, +\infty)$ it can be obtained by setting the initial condition $y(1) = 1$).*

The PIVP functions are also closed under the following operations (as far as we know, these properties have only been reported in the literature for the broader case of differentially algebraic functions):

(1) Field operations $+, -, \times, /$. For instance, if $f, g : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, are PIVP functions, then so is $f + g$ in I. In fact, if f, g are the first components of the solutions of the (vector) PIVPs

$$\begin{cases} x' = p(t, x) \\ x(t_0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} y' = q(t, y) \\ y(t_0) = y_0 \end{cases}$$

respectively then, since $f'(t) + g'(t) = p_1(t, x) + q_1(t, x)$, where $p_1(t, x)$ and $q_1(t, x)$ are the first components of $p(t, x)$ and $q(t, x)$ respectively, $f + g$ is the last component of the solution of the PIVP

$$\begin{cases} x' = p(t, x) \\ y' = q(t, y) \\ z' = p_1(t, x) + q_1(t, y) \end{cases} \qquad \begin{cases} x(t_0) = x_0 \\ y(t_0) = y_0 \\ z(t_0) = x_{0,1} + y_{0,1} \end{cases}$$

where $x_{0,1}$ and $y_{0,1}$ are the first components of vectors x_0 and y_0, respectively. Similar proofs apply for the operations $-, \times, /$. It should be noted that the quotient f/g is a PIVP function in intervals which do not contain zeros of g, and that the PIVP which generates f/g is well-defined in such intervals. For instance $\tan(= \frac{\sin}{\cos})$ is a PIVP function on $(-\pi/2, \pi/2)$.

(2) Composition. If $f : I \to \mathbb{R}$, $g : J \to \mathbb{R}$, where $I, J \subseteq \mathbb{R}$ are open intervals and $f(I) \subseteq J$, are PIVP functions, then so is $g \circ f$ on I. To see this,

suppose that f, g are the first components of the solutions of the PIVPs

$$\begin{cases} x' = p(t,x) \\ x(t_0) = x_0 \end{cases} \quad \text{and} \quad \begin{cases} y' = q(t,y) \\ y(t_1) = y_0 \end{cases} \tag{2}$$

respectively, where $t_0 \in I$ and $t_1 \in J$ (no connection is assumed between these values). Then, since $(g \circ f)'(t) = g'(f(t)).f'(t)$, we construct a system that computes $f'(t)$ (just copy the left system of (2) and note that $f'(t) = p_1(t,x)$), and another that computes $g'(f(t))$ (now pick the right system of (2); the first component will give $g'(t)$, so we have to substitute the variable t by $f(t) = x_1$ so that this component yields $g'(f(t))$), obtaining the following PIVP, where $g \circ f$ is the component z_1:

$$\begin{cases} x' = p(t,x) \\ z_1' = q_1(x_1, z) p_1(t,x) \\ \quad\vdots \\ z_n' = q_n(x_1, z) p_1(t,x) \end{cases} \qquad \begin{cases} x(t_0) = x_0 \\ z(t_0) = f(x_0). \end{cases}$$

(3) Differentiation. If $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, is a PIVP function, then so is $f' : I \to \mathbb{R}$. To see this, suppose that f is the first component of the solution of the PIVP

$$\begin{cases} x' = p(t,x) \\ x(t_0) = x_0. \end{cases}$$

Then

$$f'(t) = x_1''(t) = \frac{d}{dt} p_1(t,x) = \frac{\partial p_1}{\partial t} + \sum_{i=1}^{n} \frac{\partial p_1}{\partial x_i} x_i' = \frac{\partial p_1}{\partial t} + \sum_{i=1}^{n} \frac{\partial p_1}{\partial x_i} p_i(t,x)$$

which implies that f' is the last component of the solution of the PIVP

$$\begin{cases} x' = p(t,x) \\ z' = \frac{\partial p_1}{\partial t} + \sum_{i=1}^{n} \frac{\partial p_1}{\partial x_i} p_i(t,x) \end{cases} \qquad \begin{cases} x(t_0) = x_0 \\ z(t_0) = f'(t_0). \end{cases}$$

(4) Compositional inverses. If $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, is a bijective PIVP function, then so is f^{-1}. This case will be shown in the end of this section. In particular, this result implies that log, arcsin, arccos, and arctan are also PIVP functions.

From the preceding examples, we conclude that the following corollary, where closed-form stands for the class of *elementary functions* in Analysis which, informally, correspond to the functions obtained from the rational functions, sin, cos, exp through finitely many compositions and inversions.

Corollary 3 *All closed-form functions are PIVP functions.*

When proving that some function is PIVP, we will find it most convenient to make use of ODEs not only defined with polynomials, but also with other PIVP functions. For this purpose, we have to resort to the next theorem, which can be viewed as a strengthening of the elimination theorem of Rubel and Singer for differentially algebraic functions [13] to the case of PIVPs. Its proof is given in [7] for $S = \mathbb{R}$ but applies to any subfield of $\mathbb{R}$ (a different proof is given implicitly in [14]).

Theorem 4 *Let S be a subfield of $\mathbb{R}$. Consider the IVP*

$$x' = f(t,x), \quad x(t_0) = x_0 \tag{3}$$

where $f : D \subseteq \mathbb{R}^{n+1} \to \mathbb{R}^n$, D is the domain of f, and each component of f is a composition of polynomials with coefficients in S and PIVP functions with parameters in S and $(t_0, x_0) \in D \cap S^{n+1}$. Then there exists $m \geq n$, a polynomial $p : \mathbb{R}^{m+1} \to \mathbb{R}^m$ with coefficients in S and $y_0 \in S^m$ such that the solution of (3) is given by the first n components of $y = (y_1, ..., y_m)$, where y is the solution of the PIVP

$$y' = p(t,y), \quad y(t_0) = y_0.$$

Let us now prove that the inverse function f^{-1} of a bijective PIVP function $f : I \to \mathbb{R}$, where $I \subseteq \mathbb{R}$ is an open interval, is also a PIVP function. We know that $(f^{-1})'(x) = 1/f'(f^{-1}(x))$. Then, between two consecutive (inverse images of) zeros a, b of f', with $a < b$, f^{-1} will be the solution of the IVP

$$y' = \frac{1}{f'(y)}, \quad y(f(d)) = d, \tag{4}$$

where $d \in I$ and $f(d) \in (a,b)$. Since f is a PIVP function, so is f'. Moreover $x \mapsto 1/x$ is also a PIVP function, and since PIVP functions are closed under composition, so is $x \mapsto 1/f'(x)$. Then Equation (4) and Theorem 4 ensure that $f^{-1} : (a,b) \to \mathbb{R}$ is a PIVP function.

The following result, extracted from [4], [14] shows that the General Purpose Analog Computer (GPAC), a model introduced by Shannon in 1941 [3], and later refined in [15, pp. 13-14], [4, p. 647], [14], is equivalent to PIVP functions. This result applies formally to the refined version of the GPAC presented in [4, p. 647], [14].

Proposition 5 *A function is generated by a GPAC iff it is a PIVP function.*

Therefore, all results stated in this paper for PIVP functions are also valid for the GPAC generable functions.

We now recall basic notions from computable analysis. See [16] for an up-to-date monograph on computable analysis from the computability point of view, [12] for a presentation from a complexity point of view, or [17] for a general introduction to the subject.

Definition 6 *A sequence* $\{r_n\}$ *of rational numbers is called a* ρ*-name of a real number* x *if there exist three functions* a, b, c *from* $\mathbb{N}$ *to* $\mathbb{N}$*, such that for all* $n \in \mathbb{N}$*,* $r_n = \frac{b(n)}{c(n)+1}(-1)^{a(n)}$ *and*

$$|r_n - x| \leq \frac{1}{2^n}. \tag{5}$$

In the conditions of the above definition, we say that the ρ-name $\{r_n\}$ is given as an oracle to an oracle Turing machine, if the oracle to be used is (a, b, c). The notion of the ρ-name can be extended to $\mathbb{R}^l$: a sequence $\{(r_{1n}, r_{2n}, \ldots, r_{ln})\}_{n\in\mathbb{N}}$ of rational vectors is called a ρ-name of $x = (x_1, x_2, \ldots, x_l) \in \mathbb{R}^l$ if $\{r_{jn}\}_{n\in\mathbb{N}}$ is a ρ-name of x_j, $1 \leq j \leq l$.

Definition 7 *A real number* x *is called computable if* a*,* b*, and* c *in (5) are computable (recursive) functions.*

Definition 8 *A function* $f : D \subseteq \mathbb{R}^m \to \mathbb{R}^k$ *is computable if there is an oracle Turing machine such that for any input* $n \in \mathbb{N}$ *(accuracy) and any* ρ*-name of* $x \in E$ *given as an oracle, the machine will output a rational vector* r *satisfying* $\|r - f(x)\|_\infty \leq 2^{-n}$*, where* $\|(y_1, \ldots, y_l)\|_\infty = \max_{1\leq i\leq l} |y_i|$ *for all* $(y_1, \ldots, y_l) \in \mathbb{R}^l$*.*

In particular, every rational number must be computable and it is not difficult to show that polynomials having computable coefficients are computable functions. The following is a corollary of Theorem 3.1 of [18].

Theorem 9 *Let* $f : \mathbb{R} \to \mathbb{R}^m$ *be a vector PIVP function with computable parameters defined on an interval* (α, β)*. Then* f *is computable in* (α, β)*.*

3 Robust Simulations of Discrete Dynamical Systems

One of the purposes of the present paper is to show that a large class of discrete systems can be simulated with vector PIVP functions. Let $\mathcal{D}$ be a discrete dynamical system (both space and time are discrete). We can associate each discrete part of the state space to an integer, so that the evolution of the system is modeled by the iteration of a map $\omega : \mathbb{N}^m \to \mathbb{N}^m$. In general, if f is a function, we denote its kth iterate by $f^{[k]}$, i.e. $f^{[0]}(x) = x$ and $f^{[k+1]} = f \circ f^{[k]}$ for all $k \in \mathbb{N}$. We now present some definitions.

Definition 10 *The map $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ is a (real)* robust extension *of the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$ if there exist δ_{in}, δ_{ev}, $\delta_{out} \in (0, 1/2)$ such that for all $x_0 \in \mathbb{R}^m$, $n_0 \in \mathbb{N}^m$, $\overline{\Omega} : \mathbb{R}^m \to \mathbb{R}^m$ one has*

(1) $\Omega(n) = \omega(n)$ *and*
(2) $\|n_0 - x_0\|_\infty \le \delta_{in}$ *and* $\left\|\Omega - \overline{\Omega}\right\|_\infty \le \delta_{ev}$ *implies* $\left\|\omega(n_0) - \overline{\Omega}(x_0)\right\|_\infty \le \delta_{out}$.

The following lemma follows easily from this definition by induction (we can "contract" δ_{out} to δ_{in} using the function σ presented in Lemma 19). For simplicity, we will usually refer to robust extensions of a map as the property described by this lemma instead of Definition 10.

Lemma 11 *If $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ is a robust extension of the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$, then there exist δ_{in}, δ_{ev}, $\delta_{out} \in (0, 1/2)$ such that for all $x_0 \in \mathbb{R}^m$, $n_0 \in \mathbb{N}^m$, $\overline{\Omega} : \mathbb{R}^m \to \mathbb{R}^m$ one has*

(1) $\Omega(n) = \omega(n)$ *and*
(2) $\|n_0 - x_0\|_\infty \le \delta_{in}$ *and* $\left\|\Omega - \overline{\Omega}\right\|_\infty \le \delta_{ev}$ *implies* $\left\|\omega^{[k]}(n_0) - \overline{\Omega}^{[k]}(x_0)\right\|_\infty \le \delta_{out}$ *for all $k \in \mathbb{N}$.*

In the continuous-time setting dynamical systems are described by ODEs instead of iteration of maps. Moreover, since time is continuous, we also allow robustness in the time instant where we read the output. Again, we could consider robustness for one time unit steps, and then generalize to give iterates for all $k \in \mathbb{N}$ as we did for robust extension. Here, for simplicity, we omit this two step procedure and present instead the following definition.

Definition 12 *Let $\phi : \mathbb{R} \to \mathbb{R}$ be the unique solution of the initial value problem*

$$x' = f(t, x), \quad x(0) = n_0.$$

We say that ϕ is a robust suspension *of the map $\omega : \mathbb{N}^m \to \mathbb{N}^m$ if there exist δ_{in}, δ_{ev}, δ_{out}, $\delta_{time} \in (0, 1/2)$, such that for all $x_0 \in \mathbb{R}^m$, $n_0 \in \mathbb{N}^m$, $k \in \mathbb{N}$, and $\overline{f} : \mathbb{R}^{m+1} \to \mathbb{R}^m$ one has that*

$$\|n_0 - x_0\|_\infty \le \delta_{in} \text{ and } \left\|\overline{f} - f\right\|_\infty \le \delta_{ev}$$

implies that the solution $\overline{\phi}$ of the initial-value problem

$$x' = \overline{f}(t, x), \quad x(0) = x_0$$

satisfies

$$\left\|\omega^{[k]}(n_0) - \overline{\phi}(t)\right\|_\infty \le \delta_{out}$$

for all $t \in \mathbb{R}_0^+$ *such that* $|t-k| \leq \delta_{time}$.

These two definitions say that whenever we have a robust extension/suspension of a map, we can perturb the system by some amount, and still obtain a result close to the desired iterate $\omega^{[k]}(n_0)$.

We shall use $\mathbb{Q}[\pi]$, the standard algebraic ring extension of $\mathbb{Q}$ by adjoining the transcendent π, and which is the smallest ring containing $\mathbb{Q} \cup \{\pi\}$:

$$\mathbb{Q}[\pi] := \{a_n\pi^n + \ldots + a_1\pi + a_0 \in \mathbb{R} | a_0, \ldots, a_n \in \mathbb{Q}\}.$$

The following are the main results of this section, to be proved in Section 4. The next theorem shows that if the map is a composition of polynomials and PIVP functions (with parameters in $\mathbb{Q}[\pi]$), then one can constructively obtain a robust suspension of the map which is itself a PIVP function (with parameter in $\mathbb{Q}[\pi]$).

Theorem 13 *If the map* $\omega : \mathbb{N}^m \to \mathbb{N}^m$ *admits a robust extension* $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ *whose components are compositions of polynomials and PIVP functions with parameters in* $\mathbb{Q}[\pi]$, *then* ω *admits a robust suspension* ϕ *which is a vector PIVP function with parameters in* $\mathbb{Q}[\pi]$.

The next proposition follows from the proof of Theorem 12 from [7]. There the transition of a Turing machine is coded as a map over the integers in the following manner: we code the state as an integer and, using a representation of numbers in some adequate base, we code the right part of the tape as a second integer, and the left part as a third integer. We denote that encoding by η (see [7, p. 332] for more details).

Proposition 14 *Under the encoding* η, *the transition function* $\omega : \mathbb{N}^3 \to \mathbb{N}^3$ *of a Turing machine admits a robust extension* $\Omega : \mathbb{R}^3 \to \mathbb{R}^3$. *Moreover* Ω *can be chosen to be a composition of polynomials with coefficients in* $\mathbb{Q}[\pi]$ *and PIVP functions with parameters in* $\mathbb{Q}[\pi]$ *(in particular* sin, cos *and* arctan*).*

Actually in [7] we required algebraic numbers as coefficients for the polynomials. But non-rational coefficients are only needed to perform a trigonometric interpolation, and may be well approximated by rationals for the purpose at hand. This approximation will introduce some extra error to the computation of the map, but this is a minor hinderance since the map is robust. From Theorem 13 and Proposition 14, we obtain the following result.

Corollary 15 *With the above encoding, the transition function* ω *of a given Turing machine admits a robust suspension* ϕ. *Moreover* ϕ *is a vector PIVP function with parameters in* $\mathbb{Q}[\pi]$.

4 Proof of Theorem 13

This proof is based on Branicky's construction [8], and many steps are similar to those presented in [7]. So, before presenting the proof of Theorem 13, we will briefly sketch this technique, that constructively shows how a map from integers to integers can be iterated with smooth ODEs. By a smooth ODE we mean an ODE

$$y' = f(t, y) \tag{6}$$

where f is of class C^k, for some $1 \leq k \leq \infty$ (but not necessarily analytic). Instead of using the original approach of Branicky, we will use the one by Campagnolo, Costa, and Moore in [9], [19], [20].

Suppose that $\omega : \mathbb{Z}^m \to \mathbb{Z}^m$ is a map. For better readability, we break down the procedure into two constructions.

Construction 16 *Consider a point $b \in \mathbb{R}$ (the* target*), some $\gamma > 0$ (the* targeting error*), and time instants t_0 (*departure time*) and t_1 (*arrival time*), with $t_1 > t_0$. Then obtain an IVP (the* targeting equation*) defined with an ODE (6), where $f : \mathbb{R}^2 \to \mathbb{R}$, such that the solution y satisfies*

$$|y(t_1) - b| < \gamma \tag{7}$$

independent of the initial condition $y(t_0) \in \mathbb{R}$.

As pointed out in [7, p. 345] this can be done by an ODE

$$y' = c(b - y)^3 \phi(t), \tag{8}$$

where $\phi : \mathbb{R} \to \mathbb{R}_0^+$ is some function satisfying $\int_{t_0}^{t_1} \phi(t)dt > 0$ and $c > 0$ is any constant which is bigger than a constant c_0 depending on γ and ϕ. Note that the only requirement for the construction to hold is that c is large enough. We refer the reader to [7, p. 345] for details.

Construction 17 *Iterate the map $\omega : \mathbb{Z}^m \to \mathbb{Z}^m$ with a smooth ODE (6).*

Let $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ be an arbitrary smooth extension of ω to $\mathbb{R}$ (not necessarily robust). The iteration of ω may be performed [21, Proposition 3.4.2] by the initial-value problem

$$\begin{cases} z_1' = c_1(\Omega(r(z_2)) - z_1)^3 \theta_j(\sin 2\pi t) \\ z_2' = c_2(r(z_1) - z_2)^3 \theta_j(-\sin 2\pi t) \end{cases} \qquad \begin{cases} z_1(0) = x_0 \\ z_2(0) = x_0, \end{cases} \tag{9}$$

where $z_1(t), z_2(t) \in \mathbb{R}^m$, $\theta_j(x) = 0$ if $x \leq 0$ and $\theta_j(x) = x^j$ if $x > 0$, and $r(x)$ is a function that is a solution of an ODE and that satisfies $r(x) = i$ whenever $x \in [i - 1/4, i + 1/4]$ for all $i \in \mathbb{Z}$ (see the proof of Proposition 3.4.2 in [21] for the explicit definition of $r(x)$). Note that c_1 and c_2 depend on j and that all coefficients in (9) are in $\mathbb{Q}[\pi]$ [21]. In the remainder of this section we will show how to replace the non-analytic terms in (9) by PIVP functions with parameters in $\mathbb{Q}[\pi]$. As a result, by Theorem 4, it follows that the iteration can be performed with vector PIVP functions with parameters in $\mathbb{Q}[\pi]$.

However, if our purpose is to prove Theorem 13, we have some problems with the previous constructions:

(1) We have used the nonanalytic functions $\theta_j(x)$ and $r(x)$ which are obviously not PIVP functions. We will remove these functions using the fact that ω admits a robust extension. Therefore we have to study what happens when perturbations are allowed in (9) to prove Theorem 13.
(2) We would like to "read" the value of the iterated function not in time intervals of the form $[k, k+1/2]$ for $k \in \mathbb{N}$ as before, but rather in time intervals of the form $[k-1/4, k+1/4]$ so that we can use $\delta_{time} = 1/4$ for Theorem 13. This may be easily achieved by using a translation that adds 1/4 units of time. Because this construction is simple, in what follows, we will continue to stick to time intervals of the form $[k, k+1/2]$ in order to not overcomplicate our constructions.

In order to solve the previous problems, we need to recall the following two functions, σ and l_2, which were introduced and studied in [7].

Lemma 18 *Let $l_2 : \mathbb{R}^2 \to \mathbb{R}$ be given by $l_2(x, y) = \frac{1}{\pi}\arctan(4y(x-1/2)) + \frac{1}{2}$. Suppose also that $a \in \{0, 1\}$. Then, for any $\overline{a}, y \in \mathbb{R}$ satisfying $|a - \overline{a}| \le 1/4$ and $y > 0$,*

$$|a - l_2(\overline{a}, y)| < \frac{1}{y}.$$

Lemma 19 *Let $\sigma(x) = x - 0.2\sin(2\pi x)$ and $\varepsilon \in [0, 1/2)$. Then there is some contracting factor $\lambda_\varepsilon \in (0, 1)$ such that for all $n \in \mathbb{Z}$, $\forall \delta \in [-\varepsilon, \varepsilon]$, $|\sigma(n+\delta) - n| < \lambda_\varepsilon \delta$.*

Studying the perturbed targeting equation. (cf. Construction 16) Because the iterating procedure relies on the basic ODE (8), we have to study the following perturbed version of (8)

$$z' = c(\overline{b}(t) - z)^3\phi(t) + E(t), \tag{10}$$

where $\left|\overline{b}(t) - b\right| \le \rho$ and $|E(t)| \le \delta$. This was done in [7], where it is shown that

$$|z(1/2) - b| < \rho + \gamma + \frac{\delta}{2}. \tag{11}$$

Removing the θ_j's from (9). We must remove the θ_j's in two places: in the function r and in the terms $\theta_j(\pm \sin 2\pi t)$. Since in (9) we are using a robust extension $\Omega : \mathbb{R}^m \to \mathbb{R}^m$ of $\omega : \mathbb{N}^m \to \mathbb{N}^m$, we no longer need the corrections performed by r. There may be a problem when Ω is a robust extension of ω with $\delta_{out} > 1/4$, but this can easily be overcome by applying the function σ l times to each component of Ω until one has that $\sigma^{[l]} \circ \Omega$ is a robust extension of ω with $\delta_{in}^{\sigma} \leq 1/4$, and use $\sigma^{[l]} \circ \Omega$ instead of Ω. So, without loss of generality, we assume that $\delta_{out} \leq 1/4$ for Ω.

On the other hand we cannot use this technique to treat the terms $\theta_j(\pm \sin 2\pi t)$. We need to substitute $\phi(t) = \theta_j(\sin 2\pi t)$ with an analytic (PIVP) function $\zeta : \mathbb{R} \to \mathbb{R}$ with the following ideal behavior:

(i) ζ is periodic with period 1;

(ii) $\zeta(t) = 0$ for $t \in [1/2, 1]$;

(iii) $\zeta(t) \geq 0$ for $t \in [0, 1/2]$ and $\int_0^{1/2} \zeta(t)dt > 0$.

Of course, conditions (ii) and (iii) are incompatible for analytic functions. Instead, we approximate ζ using a function ζ_ϵ, where $\epsilon > 0$. This function must satisfy the following conditions:

$(ii)'$ $|\zeta_\epsilon(t)| \leq \epsilon$ for $t \in [1/2, 1]$;

$(iii)'$ $\zeta_\epsilon(t) \geq 0$ for $t \in [0, 1/2]$ and $\int_0^{1/2} \zeta_\epsilon(t)dt > I > 0$, where I is independent of ϵ.

In [7] an example of a PIVP function satisfying both $(ii)'$ and $(iii)'$ is constructed (function $W_0(t, y)$ in p. 346 of that paper). Similarly, $\theta_j(-\sin 2\pi t)$ will be replaced by the PIVP function $\zeta_\epsilon(-t)$. This function is defined by means of a PIVP where all coefficients are in $\mathbb{Q}[\pi]$.

Performing Construction 17 with vector PIVP functions. We are now ready to perform a simulation of an integer map with a system similar to (9), but using only PIVP (and hence analytic) functions. Choose δ_{in}, δ_{ev}, and a targeting error $\gamma > 0$ such that

$$2\gamma + \delta_{ev}/2 \leq \delta_{in} < 1/4. \tag{12}$$

We take $\delta_{time} = 1/4$. We want to determine δ_{out} and present a system of ODEs that satisfies the conditions of Theorem 13. Consider the system of ODEs

$$\begin{cases} z_1' = c_1(\Omega \circ \sigma^{[m]}(z_2) - z_1)^3 \, \zeta_{\epsilon_1}(t), \\ z_2' = c_2(\sigma^{[n]}(z_1) - z_2)^3 \, \zeta_{\epsilon_2}(-t) \end{cases} \tag{13}$$

with initial conditions $z_1(0) = z_2(0) = \overline{x}_0$, where $c_1, c_2, m, n, \epsilon_1$, and ϵ_2 are still to be defined, and σ is the error-contracting function defined in Lemma 19.

We would like (13) to satisfy the following property: on $[0, 1/2]$,

$$|z_2'(t)| \leq \gamma. \tag{14}$$

This can be achieved by taking $\epsilon_2 = \gamma/K$, where K is a bound for $c_2(\sigma^{[n]}(z_1) - z_2)^3$ in the interval $[0, 1]$. Since $|x|^3 \leq x^4 + 1$ for all $x \in \mathbb{R}$, we can take $\epsilon_2 = \frac{\gamma}{c_2(\sigma^{[n]}(z_1)-z_2)^4} + \frac{\gamma}{c_2}$. Now notice that $z_2(0)$ has an error bounded by δ_{in}. This fact, together with (14) and the fact that z_2' might be subject to perturbations of amplitude not exceeding δ_{ev} imply that

$$|z_2(t) - x_0| \leq \delta_{in} + (\delta_{ev} + \gamma)/2 = \delta_{out} < 1/2 \quad \text{for } t \in [0, 1/2]. \tag{15}$$

Therefore, for m satisfying $\sigma^{[m]}(\delta_{out}) < \gamma$, we have $\left|\sigma^{[m]}(z_2(t)) - x_0\right| < \gamma$ for all $t \in [0, 1/2]$. Hence, from the study of the perturbed targeting equation (10), where $\phi(t) = \zeta_{\epsilon_1}(t)$ and c_1 is obtained accordingly, we have (take $\rho = \gamma$ and consider (12))

$$|z_1(1/2) - \omega(x_0)| < 2\gamma + \frac{\delta_{ev}}{2} \leq \delta_{in}. \tag{16}$$

For the interval $[1/2, 1]$ the roles of z_1 and z_2 are interchanged. Similarly to the reasoning done for z_2 on $[0, 1/2]$, take $\epsilon_1 = \frac{\gamma}{c_1(\Omega \circ \sigma^{[m]}(z_2) - z_1)^4} + \frac{\gamma}{c_1}$ so that on $[0, 1/2]$

$$|z_1'(t)| \leq \gamma.$$

From this inequality, (16), and the fact that z_2' might be subject to perturbations of amplitude not exceeding δ_{ev}, we conclude that

$$|z_1(t) - \omega(x_0)| \leq \delta_{in} + (\delta_{ev} + \gamma)/2 = \delta_{out} < 1/2 \quad \text{for } t \in [1/2, 1].$$

Therefore, for $n = m$, we have $\left|\sigma^{[n]}(z_1(t)) - \omega(x_0)\right| < \gamma$ for all $t \in [1/2, 1]$. Hence, from the study of the perturbed targeting equation (10), where $\phi(t) = \zeta_{\epsilon_2}(t)$ and c_2 is obtained accordingly, we have

$$|z_2(1) - \omega(x_0)| < 2\gamma + \frac{\delta_{ev}}{2} \leq \delta_{in}.$$

Now we can repeat the procedure for intervals $[1, 2]$, $[2, 3]$, etc. to conclude that for all $j \in \mathbb{N}$ and for all $t \in [j, j + 1/2]$,

$$\left|z_1(t) - \omega^{[j]}(x_0)\right| \leq \delta_{out}.$$

Moreover, z_1 is defined as the solution of an ODE written in terms of PIVP functions, and all coefficients of this ODE are in $\mathbb{Q}[\pi]$. Then, by Theorem 4, z_1 is a vector PIVP function with parameters in $\mathbb{Q}[\pi]$.

5 Application – Undecidability for PIVPs with Comparable Parameters

It is well known from the basic existence-uniqueness theory of ODEs [22], [23] that if f is analytic, then the IVP

$$x' = f(t, x), \quad x(t_0) = x_0 \tag{17}$$

has a unique solution $x(t)$ defined on a maximal interval of existence $I = (\alpha, \beta) \subset \mathbb{R}$ that is analytic on I [24]. The interval is maximal in the sense that either $\alpha = -\infty$ or $x(t)$ is unbounded as $t \to \alpha^+$ with similar conditions applying to β (see Proposition 20 for details). Actually, f only needs to be continuous and locally Lipschitz in the second argument for this maximal interval to exist.

A question of interest is the following: is it possible to design an automated method that, on input (f, t_0, x_0), gives as output the maximal interval of existence for the solution of (17)? In computability theory, e.g. [25], [26], it is well known that some problems cannot be answered by the use of an algorithm (more precisely, by the use of a Turing machine). Such problems are labelled *undecidable* and many examples are known. The most prominent undecidable problem is the Halting Problem: given a universal Turing machine and some input to it, decide whether the machine eventually halts or not. To address this kind of questions for IVPs, we use the computable analysis approach [17], [12], [16], which we presented in the end of Section 2. Using that approach, it was shown in [18] that given an analytic IVP (17), defined with computable data, its corresponding maximal interval may be non-computable.

Non-computability results related to initial-value problems of differential equations are not new. For example, Pour-El and Richards [27] showed that if we relax the condition of analyticity in the IVP (17) defined with computable data, it can have non-computable solutions. In [28], [29] it is shown that there is a three-dimensional wave equation, defined with computable data, such that the unique solution is nowhere computable. However, in these examples, non-computability is not "genuine" in the sense that the problems under study are ill-posed: either the solution is not unique or it is unstable [30]. In other words, ill-posedness was at the origin of non-computability in those examples. In contrast, an analytic IVP (17) is classically well-posed and, consequently, the non-computability results do not seem to reflect computational and well-posedness deficiencies inherited by the problems.

Motivated by the non-computability result obtained in [18], this latter paper also addresses the following problem: while it is not possible to compute the maximal interval of (17) is it possible to compute some partial information about it? In particular, is it possible to decide if this maximal interval is bounded or not?

This question has interest on its own for the following reason. In many problems, we implicitly assume that t is defined for "all time". For example, if one wants to compute sinks or limit cycles associated with ODEs, this only makes sense if the solution of the ODE is defined for all times $t > t_0$. This is also implicitly assumed in problems like reachability [31], [32], [33], [34], [35], etc. For this reason, those problems only make sense when associated with ODEs for which the maximal interval is unbounded. So, it would be interesting to know which are the "maximal" classes of functions f for which the boundedness problem is decidable.

In [18], it was shown that for the general class of analytic IVPs, the boundedness problem of the maximal interval is undecidable. Here we will deepen this result: we will show that the boundedness problem is still undecidable for PIVPs of degree greater or equal than 56 with parameters in $\mathbb{Q}[\pi]$. Our result is slightly different in form from the case of the general class of analytic IVPs. Indeed, the coefficients of the polynomials are coded as finite sequences of integers and not as ρ-name satisfying (5), though from these finite sequences of integers one can easily compute ρ-names for the coefficients of the polynomials.

The boundedness problem is decidable for linear differential equations thus implying that the boundary between decidability/undecidability lies in the class of polynomials of degree n, for some $2 \leq n \leq 56$.

This result is shown using methods which differ from those employed in [18]. This result was already stated in [11], but we now present its proof.

The following result introduces the notion of maximal interval for ODEs and follows as an immediate consequence of the fundamental existence-uniqueness theory for the initial-value problem (17), where the analyticity condition is dropped for f [22], [23], [36].

Proposition 20 *Let E be an open subset of $\mathbb{R}^{n+1}$ and assume that $f : E \to \mathbb{R}^n$ is continuous on E and locally Lipschitz in the second argument (i.e. in the last n components). Then for each $(t_0, x_0) \in E$, the problem (17) has a unique solution $x(t)$ defined on a maximal interval (α, β), on which it is C^1. The maximal interval is open and has the property that, if $\beta < +\infty$ (resp. $\alpha > -\infty$), either $(t, x(t))$ approaches the boundary of E or $x(t)$ is unbounded as $t \to \beta^-$ (resp. $t \to \alpha^+$).*

Note that, as a particular case, when $E = \mathbb{R}^{n+1}$ and $\beta < \infty$, $x(t)$ is unbounded as $t \to \beta^-$. This will be the case under study in this section.

We now introduce a definition that allows us to compare real numbers of some given set, to avoid the trivial undecidability of the boundedness problem sketched in Section 1.

Definition 21 *We say that a set $D \subseteq \mathbb{R}$ is effectively comparable if D has a naming system γ, if all elements of D are γ-computable, and if given γ-names of $x, y \in D$, then $x = y$ and $x < y$ are decidable*

In the previous definition, "naming system" is either a (finite) notation or a (infinite) representation of the elements of D according to Weihrauch [16, p. 33 and p. 52]. Next we show that $\mathbb{Q}[\pi]$ is effectively comparable. Indeed, given $a_0, \ldots, a_m \in \mathbb{Q}$ (which can easily be coded as a finite sequence using a finite alphabet A), we can take the notation $f : A^* \to \mathbb{Q}[\pi]$

$$f(a_0, \ldots, a_m) = \sum_{i=0}^{m} a_i \pi^i.$$

Moreover, if $\alpha, \beta \in \mathbb{Q}[\pi]$,

$$\alpha = \sum_{i=0}^{m} a_i \pi^i \quad \text{and } \beta = \sum_{i=0}^{n} b_i \pi^i,$$

where $a_0, \ldots, a_m, b_0, \ldots, b_n \in \mathbb{Q}$. We can decide if $\alpha = \beta$ since $\alpha = \beta$ iff $a_i = b_i$ for all i and a_i and b_i are rationals. We can also compute arbitrarly close approximations of α and β. Therefore, if $\alpha \neq \beta$, we can compare these values: we just need to start computing increasing approximations of α and β until we decide whether $\alpha < \beta$ or $\alpha > \beta$. The following result is similar to Theorem 12 in [11], but here we restrict the parameters of the PIVP to an effectively comparable set. This prevents the trivial undecidability discussed in Section 1.

Theorem 22 *Let D be an effectively comparable set such that $\mathbb{Q}[\pi] \subseteq D$. The following problem is undecidable: "Given $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$ with polynomial components with coefficients in D (these coefficients are given by their names, as described in Definition 21), and $(t_0, x_0) \in \mathbb{Q} \times \mathbb{Q}^n$, decide whether the maximal interval of the IVP (1) is bounded or not".*

Actually, if we are given the description of a universal Turing machine, we can constructively define a set of polynomial ODEs simulating it that encodes the Halting Problem. If we use the small universal Turing machine presented in [37], having 4 states and 6 symbols, we obtain the following theorem.

Theorem 23 *Let D be an effectively comparable set such that $\mathbb{Q}[\pi] \subseteq D$. There is a vector $p : \mathbb{R}^{n+1} \to \mathbb{R}^n$, with $n \geq 1$, defined by polynomials with coefficients in D (these coefficients are given by their names, as described in Definition 21), where each component has degree less than or equal to 56, such that the following problem is undecidable: "Given $(t_0, x_0) \in \mathbb{Q} \times \mathbb{Q}^n$, decide whether the maximal interval of the IVP (1) is bounded or not".*

Proof. The idea to prove this theorem is to simulate with a set of polynomial ODEs Rogozhin's small universal Turing machine [37]. We can obtain a set of PIVPs simulating this Turing machine as described by Theorem 13, Proposition 14, and Corollary 15. Then we expand this PIVP system as a polynomial ODE using the techniques introduced in the proof of Theorem 4. Since the entire procedure is constructive and bottom-up, it is possible to determine the degrees of the polynomials appearing in the IVP. This will be done later in the proof.

The important point is that we can obtain a PIVP (1), with solution x, that satisfies for every $k \in \mathbb{N}$

$$\begin{cases} x_q(t) \leq m - \frac{11}{16} & \text{if } M \text{ has not halted at step } k \text{ and } t \leq k \\ x_q(t) \geq m - \frac{5}{16} & \text{if } M \text{ has already halted at step } k \text{ and } t \geq k \end{cases} \tag{18}$$

where the states of the Turing machine are encoded by numbers in $\{1, \ldots, m\}$ and $m = 4$ is the Halting state. Consider the IVP

$$\begin{cases} z_1' = x_q - (m - 1/2) \\ z_2 = \frac{1}{z_1} \end{cases} \iff \begin{cases} z_1' = x_q - (m - 1/2) \\ z_2' = ((m - 1/2) - x_q)z_2^2 \end{cases} \tag{19}$$

where $z_1(0) = z_2(0) = -1$. Since x_q appears as a component, we assume that this IVP is coupled with the PIVP defined by Proposition 14 and Theorem 4. It is easy to see that while M hasn't halted, $x_q - (m - 1/2) \leq -3/16$. Thus z_1 keeps decreasing and the IVP is defined in $(0, +\infty)$, i.e. the maximal interval is unbounded, if M never halts.

On the other hand, if M eventually halts, z_1 starts increasing at a rate of at least $3/16$ and will do that forever. So, at some time it will have to assume the value 0. When this happens, a singularity appears for z_2 and the maximal interval is therefore (right-)bounded. For negative values of t just replace t by $(-t)$ in the PIVP (1) and assume t to be positive. It can be shown that the behavior of the system will be similar, and we reach the same conclusions for the left bound of the maximal interval. So M halts iff the maximal interval of the PIVP (19) is bounded, i.e. boundedness is undecidable.

It remains to determine the degree of the polynomials appearing in the definition of (1) and (19). We will now sketch how this is done. In what follows we assume that x and y are variables in an IVP, whose derivatives can be written as a polynomial (possibly involving other variables of the IVP) of degrees k and n, respectively (for short, we will simply say that x and y have degree k and n). Then our task is to know what is the degree of the PIVP giving functions like $\sin x$, etc.

(1) The case of sin and cos. We have

$$\begin{cases} (\sin x)' = x' \cos x \\ (\cos x)' = -x' \sin x \end{cases} \implies \begin{cases} y_1' = x' y_2 \\ y_2' = -x' y_1 \end{cases}$$

where y_1 and y_2 substitute $\sin x$ and $\cos x$, respectively. So, if x has degree k, $\sin x$ and $\cos x$ can be replaced by variables having degree $k+1$.

(2) The case of arctan. One has

$$\begin{cases} (\arctan x)' = \frac{x'}{1+x^2} \\ \left(\frac{1}{1+x^2}\right)' = -\frac{2x'x}{(1+x^2)^2} \end{cases} \implies \begin{cases} y_1' = x' y_2 \\ y_2' = -2x' x y_2^2 \end{cases}$$

where y_1 replaces $\arctan x$. So, $\arctan x$ can be replaced by a variable of degree $k+1$, but also introduces another variable of degree $k+3$.

(3) There are other functions that we didn't describe in detail previously, and that are used in our simulation (the reader is referred to [7]). But they are built from polynomials and the functions arctan and sin. So a straightforward application of the proof of Theorem 4 and the cases 1 and 2 above are enough to understand what happens with the degree of variables which derivative is described in terms of these functions.

Carrying out all the steps mentioned above, one can see that 56 is the highest degree for a variable that appears in the polynomial expansion of the ODE simulating Rogozhin's small universal Turing machine. ■

Let us remark that, while the boundedness problem of the maximal interval for unrestricted PIVPs is in general undecidable, this is not the case for some subclasses of polynomials. For instance, the boundedness problem is decidable for the class of linear differential equations (the maximal interval is always $\mathbb{R}$ — see e.g. [36, p. 79]) or for the class of one-dimensional autonomous differential equations where f is a polynomial of any degree (the ODE is separable, yielding an integral of a rational function that can be algorithmically solved). It would be interesting to investigate maximal classes where the boundedness problem is decidable.

6 Conclusion

In this paper we provide further results that establish a bridge between the theory of ODEs and computation (see [38] for an up-to-date review). We focus on polynomial initial value problems with computable and comparable parameters.

With respect to computation, our main result is that the boundness of the maximal interval of definition is undecidable even for PIVPs with comparable parameters and degree up to 56. We can view this result as a ODE analog to the undecidability of the Halting problem for Turing machines.

With respect to polynomial ODEs, we show that they can simulate a large class of dynamical systems – including Turing machines – in the presence of noise.

Based on the previous results we argue that polynomial ODEs, which are a well known model of physical phenomena, are also a powerful, yet realistic, model of continuous time computation.

Acknowledgments. The authors wish to thank Pieter Collins, Kerry Ojakian, and Ning Zhong, and the anonymous referees for useful remarks and comments. DG and MC were partially supported by *Fundação para a Ciência e a Tecnologia* and EU FEDER POCTI/POCI via CLC, SQIG - Instituto de Telecomunicações and grant SFRH/BPD/39779/2007 (DG). Additional support to DG was also provided by the *Fundação Calouste Gulbenkian* through the *Programa Gulbenkian de Estímulo à Investigação*. JB was partially supported by CMAF through FEDER and FCT-Plurianual 2007.

References

[1] M. W. Hirsch, S. Smale, Differential Equations, Dynamical Systems, and Linear Algebra, Academic Press, 1974.

[2] J. F. Ritt, Integration in Finite Terms. Liouville's Theory of Elementary Methods., Columbia Univ. Press, 1948.

[3] C. E. Shannon, Mathematical theory of the differential analyzer, J. Math. Phys. MIT 20 (1941) 337–354.

[4] D. S. Graça, J. F. Costa, Analog computers and recursive functions over the reals, J. Complexity 19 (5) (2003) 644–664.

[5] J. H. Hubbard, B. H. West, Differential Equations: A Dynamical Systems Approach — Higher-Dimensional Systems, Springer, 1995.

[6] D. S. Graça, N. Zhong, J. Buescu, The ordinary differential equation defined by a computable function whose maximal interval of existence is non-computable, in: G. Hanrot, P. Zimmermann (Eds.), Proceedings of the 7th Conference on Real Numbers and Computers (RNC 7), LORIA/INRIA, 2006, pp. 33–40.

[7] D. S. Graça, M. L. Campagnolo, J. Buescu, Computability with polynomial differential equations, Adv. Appl. Math. 40 (3) (2008) 330–349.

[8] M. S. Branicky, Universal computation and other capabilities of hybrid and continuous dynamical systems, Theoret. Comput. Sci. 138 (1) (1995) 67–100.

[9] M. L. Campagnolo, C. Moore, J. F. Costa, Iteration, inequalities, and differentiability in analog computers, J. Complexity 16 (4) (2000) 642–660.

[10] E. Hainry, Reachability in linear dynamical systems, in: A. Beckmann, C. Dimitracopoulos, B. Löwe (Eds.), Computability in Europe 2008 (CiE 2008), Vol. 5028 of LNCS, 2008.

[11] D. S. Graça, J. Buescu, M. L. Campagnolo, Boundedness of the domain of definition is undecidable for polynomial ODEs, in: R. Dillhage, T. Grubba, A. Sorbi, K. Weihrauch, N. Zhong (Eds.), Proceedings of the 4th International Conference on Computability and Complexity in Analysis (CCA 2007), FernUniversität in Hagen, 2007, pp. 127–135.

[12] K.-I. Ko, Computational Complexity of Real Functions, Birkhäuser, 1991.

[13] L. A. Rubel, F. Singer, A differentially algebraic elimination theorem with application to analog computability in the calculus of variations, Proc. Amer. Math. Soc. 94 (4) (1985) 653–658.

[14] D. S. Graça, Some recent developments on Shannon's General Purpose Analog Computer, Math. Log. Quart. 50 (4-5) (2004) 473–485.

[15] M. B. Pour-El, Abstract computability and its relations to the general purpose analog computer, Trans. Amer. Math. Soc. 199 (1974) 1–28.

[16] K. Weihrauch, Computable Analysis: an Introduction, Springer, 2000.

[17] M. B. Pour-El, J. I. Richards, Computability in Analysis and Physics, Springer, 1989.

[18] D. Graça, N. Zhong, J. Buescu, Computability, noncomputability and undecidability of maximal intervals of IVPs, Trans. Amer. Math. Soc.To appear.

[19] M. Campagnolo, C. Moore, Upper and lower bounds on continuous-time computation, in: I. Antoniou, C. Calude, M. Dinneen (Eds.), 2nd International Conference on Unconventional Models of Computation - UMC'2K, Springer, 2001, pp. 135–153.

[20] M. L. Campagnolo, Computational complexity of real valued recursive functions and analog circuits, Ph.D. thesis, Instituto Superior Técnico/Universidade Técnica de Lisboa (2002).

[21] M. L. Campagnolo, The complexity of real recursive functions, in: C. S. Calude, M. J. Dinneen, F. Peper (Eds.), Unconventional Models of Computation (UMC'02), LNCS 2509, Springer, 2002, pp. 1–14.

[22] E. A. Coddington, N. Levinson, Theory of Ordinary Differential Equations, McGraw-Hill, 1955.

[23] S. Lefshetz, Differential Equations: Geometric Theory, 2nd Edition, Interscience, 1965.

[24] V. I. Arnold, Ordinary Differential Equations, MIT Press, 1978.

[25] M. Sipser, Introduction to the Theory of Computation, PWS Publishing Company, 1997.

[26] J. E. Hopcroft, R. Motwani, J. D. Ullman, Introduction to Automata Theory, Languages, and Computation, 2nd Edition, Addison-Wesley, 2001.

[27] M. B. Pour-El, J. I. Richards, A computable ordinary differential equation which possesses no computable solution, Ann. Math. Logic 17 (1979) 61–90.

[28] M. B. Pour-El, J. I. Richards, The wave equation with computable initial data such that its unique solution is not computable, Adv. Math. 39 (1981) 215–239.

[29] M. B. Pour-El, N. Zhong, The wave equation with computable initial data whose unique solution is nowhere computable, Math. Log. Quart. 43 (1997) 499–509.

[30] K. Weihrauch, N. Zhong, Is wave propagation computable or can wave computers beat the Turing machine?, Proc. London Math. Soc. 85 (3) (2002) 312–332.

[31] R. Alur, D. L. Dill, Automata for modeling real-time systems, in: Automata, Languages and Programming, 17th International Colloquium, LNCS 443, Springer, 1990, pp. 322–335.

[32] E. Asarin, O. Maler, A. Pnueli, Reachability analysis of dynamical systems having piecewise-constant derivatives, Theoret. Comput. Sci. 138 (1995) 35–65.

[33] T. A. Henzinger, P. W. Kopke, A. Puri, P. Varaiya, What's decidable about hybrid automata?, J. Comput. System Sci. 57 (1) (1998) 94–124.

[34] O. Bournez, Achilles and the Tortoise climbing up the hyper-arithmetical hierarchy, Theoret. Comput. Sci. 210 (1) (1999) 21–71.

[35] V. D. Blondel, J. N. Tsitsiklis, A survey of computational complexity results in systems and control, Automatica 9 (36) (2000) 1249–1274.

[36] J. K. Hale, Ordinary Differential Equations, 2nd Edition, Robert E. Krieger Pub. Co, 1980.

[37] Y. Rogozhin, Small universal Turing machines, Theoret. Comput. Sci. 168 (2) (1996) 215–240.

[38] O. Bournez, M. L. Campagnolo, New Computational Paradigms. Changing Conceptions of What is Computable, Springer-Verlag, New York, 2008, Ch. A Survey on Continuous Time Computations, pp. 383–423.