

# GENERALIZED SHEAR THINNING MODELS, AND REGULARITY

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## Abstract

In the following we consider a class of non-linear systems that covers, as very particular cases, some well known Navier-Stokes systems with shear thinning viscosity. Here  $p = p(x) \leq 2$ , may depend on  $x$  in a quite arbitrary way. We only assume point-wise  $p(x)$ -growth conditions (there are no convexity-type assumptions). We assume a non-homogeneous Dirichlet boundary condition and show that weak solutions to our class of systems have integrable gradient up to the boundary, with any finite exponent. This result extends, up to the boundary and to a much wide class of problems, some interior regularity results known in the literature. Proofs are elementary.

AMS subject classification **35Q30, 35K35, 76D03, 35K55.**

## 1 Main result

In the following we prove  $W^{1,q}(\Omega)$ -regularity up to the boundary, for any finite power  $q$ , for solutions of the system (1.4) under very weak assumptions on the non-linear term  $g(x, \nabla u)$ . Our proof, based on a bootstrap argument and Stokes-elliptic regularization (see [4]), is elementary. Nevertheless, the Theorem 1.1 below extend to a larger class of operators, and up to the boundary, some of the well known results in the literature.

In the sequel  $\Omega$  is a bounded, connected, open set in  $\mathbb{R}^3$ , locally situated on one side of its boundary  $\Gamma$ , a manifold of class  $C^2$ .

Below we consider solutions to the following class of stationary Navier-Stokes equations for flows with shear (more generally, gradient) dependent viscosity

$$(1.1) \quad \begin{cases} -\nabla \cdot T(u, \pi) + (u \cdot \nabla) u = f, \\ \nabla \cdot u = 0, \end{cases}$$

under suitable boundary conditions.  $T$  denotes the Cauchy stress tensor

$$(1.2) \quad T = -\pi I + \nu_0 \mathcal{D}u + g(x, \nabla u)$$

and  $\mathcal{D}u$  is the symmetric gradient, i.e.,

$$\mathcal{D}u = \frac{1}{2} (\nabla u + \nabla u^T).$$

Here  $\nu_0$  is a strictly positive constant and  $g$  is a tensor with components  $g_{ij}$ ,  $i, j = 1, 2, 3$ . Note that

$$g = g(x, \mathcal{D}u)$$

is a very particular case of the above one. In fact,  $g(x, \nabla u)$  may depend, in a totally unrelated way, on each of the single derivatives  $\frac{\partial u_i}{\partial x_j}$ .

We set

$$|t|^2 = \sum t_{k,l}^2,$$

where  $t = t_{kl}$  is a tensor. In the following we assume that  $g(x, t)$  satisfies the classical Caratheodory conditions together with

$$(1.3) \quad |g(x, t)| \leq c(1 + |t|)^{p-1}$$

for some  $p \in (1, 2]$ . We point out that there are not convexity assumptions here. As a very particular case,  $|g(x, t)|$  may behave like  $(1 + |t|)^{p(x)-1}$ , provided that  $p(x) \leq p$ , almost everywhere in  $\Omega$ . For instance,

$$g(x, \nabla u) = h(x, |\mathcal{D}u|) \mathcal{D}u,$$

where the real valued function  $h$  satisfies

$$|h(x, t)| \leq c(1 + |t|)^{p-2},$$

falls within the above picture. This particular case is related to the theory of electrorheological fluids. See [9].

Without loss of generality we assume that  $\nu_0 = 1$ . From (1.1) we get

$$(1.4) \quad \begin{cases} -\Delta u + \nabla \cdot g(x, \nabla u) + (u \cdot \nabla) u + \nabla \pi = f, \\ \nabla \cdot u = 0. \end{cases}$$

In order to fix ideas we assume here the non-homogeneous Dirichlet boundary condition

$$(1.5) \quad u|_{\Gamma} = a(x).$$

However, many other boundary conditions fall within the above scheme. Actually, it is sufficient that an estimate like (2.10) holds for the usual Stokes linear system (2.9) with the Dirichlet boundary condition replaced by the desired boundary condition.

In the sequel we take into account any possible weak solution  $u$  to our problem for which

$$(1.6) \quad u \in W^{1,p} \cap L^2.$$

If  $a = 0$ , or if the convective term is not present in (1.4), we replace (1.6) by  $u \in W_0^{1,p}$ .

**Remark 1.1.** *We are interested in the regularity properties of any possible weak solution to our problem. Hence our assumptions do not necessarily imply the existence of a solution. However, and this is the crucial point here, many well known existence theorems fall within the assumptions made here.*

Our main result is the following.

**Theorem 1.1.** *Assume that (1.3) holds and that the data*

$$(1.7) \quad f \in L^3$$

and

$$(1.8) \quad a \in W^{1,+\infty}(\Gamma)$$

satisfy the necessary compatibility condition

$$(1.9) \quad \int_{\Gamma} a \cdot n \, d\Gamma = 0.$$

Let  $u$  be a solution of problem (1.4), (1.5) in the class (1.6). Then

$$(1.10) \quad u \in W^{1,q}(\Omega) \quad \forall q < +\infty.$$

In particular

$$(1.11) \quad u \in C^{0,\alpha}(\bar{\Omega}), \quad \forall \alpha < 1.$$

Moreover the estimate (2.7) holds with  $v$  replaced everywhere by  $u$ .

Actually the above assumptions on  $f$  and  $a$  may be replaced by the weaker assumptions

$$(1.12) \quad f \in W^{-1,q}(\Omega), \quad \forall q < +\infty$$

and

$$(1.13) \quad a \in W^{1-\frac{1}{q},q}(\Gamma), \quad \forall q < +\infty.$$

Theorem 1.1 allows the extension up to the boundary of some of the interior regularity results known in the literature (under stronger assumptions on  $g$ ). In [6], Theorem 3.2.3, it is proved that  $\mathcal{D}u \in L_{loc}^{\infty}(\Omega)$ . In Remark 3.2.5 it is pointed out that this last result implies (1.10) locally in  $\Omega$ . In Theorem 3.2.1 (1.11) is proved locally in  $\Omega$ . See also [5].

It remains open, in particular, the conjecture proposed in [6] Remark 3.2.9, namely, to prove that  $\mathcal{D}u \in L^{\infty}(\Omega)$ . Actually, by taking into account (1.10), we may even expect that  $\nabla u \in L^{\infty}(\Omega)$ .

For interior partial regularity results we also refer the reader to [2] (see, in particular, Lemma 3.14) and to [1] and references therein.

Finally we refer to [7] where the authors prove the existence of globally smooth solutions in the two dimensional case under suitable conditions.

## 2 Proof of Theorem 1.1

We define, for each non-negative integer  $n$ , the exponent

$$(2.1) \quad p_n = \frac{p}{(p-1)^n}.$$

This is a strictly increasing sequence, which tends to infinity with  $n$ . Note that  $p_0 = p$  and  $p_1 = p'$ , the dual exponent of  $p$ . We assume, moreover, the necessary compatibility condition (1.9).

By well known results, see [3], there is a linear continuous map  $R$  from  $W^{1-\frac{1}{q},q}(\Gamma)$  to  $W^{1,q}(\Omega)$  such that for each  $a \in W^{1-\frac{1}{q},q}(\Gamma)$  one has  $\nabla \cdot (Ra) = 0$  and  $\gamma(Ra) = a$ , where  $\gamma$  denotes the trace operator. We still denote by  $a$  the function  $Ra$ . If we assume that  $a$  satisfies (1.8) then we may prove directly that  $Ra \in W^{1,+\infty}(\Omega)$ .

In the sequel, in order to simplify the presentation, we assume that (1.7) and (1.8) hold, even if (1.12) and (1.13) are sufficient for our purposes.

As usual, we consider as a new unknown the function

$$v = u - a.$$

This leads to the problems

$$(2.2) \quad \begin{cases} -\Delta v + \nabla \cdot g(x, \nabla a + \nabla v) + (v \cdot \nabla)v + \\ (a \cdot \nabla)v + (v \cdot \nabla)a + \nabla \pi = f + \Delta a - (a \cdot \nabla)a; \\ \nabla \cdot v = 0; \\ v|_{\Gamma} = 0. \end{cases}$$

From (1.6) it follows that  $v \in W_0^{1,p} \cap L^2$ . The spaces  $W_0^{1,q}$  are endowed here with the norm  $\|\nabla u\|_q$ .

From (2.2) we get the "energy estimate"

$$(2.3) \quad \begin{cases} \|\nabla v\|_2^2 \leq (\|f\|_{-1,2} + \|\nabla a\|_2 + \|a\|_3 \|\nabla a\|_2) \|\nabla v\|_2 + \|\nabla a\|_{\infty} \|v\|_2^2 + \\ c \|1 + |\nabla a|\|_p + c \|\nabla v\|_p^p. \end{cases}$$

Note that

$$(2.4) \quad |g(x, \nabla a + \nabla v)| \leq c(1 + \|\nabla a\|_{\infty} + |\nabla v|)^{p-1}.$$

The symbol  $c$  denotes, here and in the sequel, positive constants that may depend, at most, on  $\Omega$ . The same symbol may denote distinct constants.

Note that (2.3) shows that  $v$  and  $u$  belong to  $W^{1,2}(\Omega)$ .

We may improve the estimate (2.3), however this is not necessary for our purposes. Note, however, that under the more usual assumptions that lead to an existence theorem the term  $\|\nabla v\|_p^p$  appears in the left hand side instead of in the right hand side. Moreover, if  $a = 0$ , the term  $\|\nabla a\|_{\infty} \|v\|_2^2$  is not present. One has the following result.

**Lemma 2.1.** *Assume that (1.7) and (1.8) hold. Let  $v \in W_0^{1,p}$  be a solution of problem (1.4), (1.5). If, for some index  $n \geq 0$ ,*

$$(2.5) \quad \nabla v \in L^{p^n}$$

then

$$(2.6) \quad \nabla v \in L^{p^{n+1}}.$$

Moreover

$$(2.7) \quad \|\nabla v\|_{p_{n+1}} \leq C_{p_{n+1}} (1 + \|a\|_{1,\infty}^2 + \|\nabla v\|_2^2 + \|f\|_3 + \|\nabla v\|_{p_n}^{p-1}).$$

We denote by  $C_q$  positive constants, defined in the sequel (see (2.10)). These constants depend on the particular value of the integrability exponent  $q$ . See the Remark 2.1.

The above result holds if the assumptions (1.7) and (1.8) are replaced by  $f \in W^{-1,p_{n+1}}(\Omega)$  and  $a \in W^{1-\frac{1}{p_{n+1}},p_{n+1}}(\Gamma)$ .

*Proof.* By appealing to (2.4), straightforward calculations show that

$$\|g(\cdot, \nabla a + \nabla v)\|_{p_{n+1}}$$

is bounded by the right hand side of equation (2.8) below. Hence

$$(2.8) \quad \|\nabla \cdot g(\cdot, \nabla a + \nabla v)\|_{-1,p_{n+1}} \leq c_0 (1 + \|\nabla a\|_\infty^{p-1} + \|\nabla v\|_{p_n}^{p-1}),$$

where the constant  $c_0$  can be chosen independent of  $p_n$ .

Next consider the linear Stokes system

$$(2.9) \quad \begin{cases} -\Delta w + \nabla \tilde{\pi} = F, \\ \nabla \cdot w = 0, \\ w|_\Gamma = 0. \end{cases}$$

We know regularity results, see [4], show that

$$(2.10) \quad \|\nabla w\|_q \leq C_q \|F\|_{-1,q},$$

where  $C_q$  denotes a suitable positive constant. By setting

$$F = f + \Delta a - (a \cdot \nabla) a - (\nabla \cdot g(\nabla a + \nabla v) + (v \cdot \nabla) v + (a \cdot \nabla) v + (v \cdot \nabla) a)$$

it readily follows, by appealing to (2.8) and (2.10), that (2.7) holds. The constant  $c_0$  was incorporated inside  $C_{p_{n+1}}$ .

In order to prove our thesis for  $u$  it is clearly sufficient to prove the same thesis for  $v$  due to the relation  $u = a + v$  together with the regularity of  $a$ . Since  $v \in W^{1,p_0}(\Omega)$ , the thesis follows by induction, by appealing to Lemma 2.1.  $\square$

**Remark 2.1.** *It is worth noting that if the constants  $C_q$  were uniformly bounded for large values of  $q$  then the estimates (2.7) would immediately yield the uniform boundedness of the norms  $\|\nabla v\|_{p_{n+1}}$ , and hence the Lipschitz continuity of  $v$  and  $u$  up to the boundary (in this context, the assumption (1.13) would be strictly necessary). This result would hold even if  $C_q$  were to go to  $+\infty$  sufficiently slowly. However, it may be proved, see [10], that for the scalar equation  $-\Delta w = F$  under the homogeneous Dirichlet boundary condition, one has  $\|w\|_{2,q} \leq C_q \|F\|_q$ , where  $C_q = O(q)$ . At best, we expect this same behavior for the constants  $C_q$  in the case of the Stokes problem (2.9). Merely assuming this behavior, we cannot prove that  $K_n$  is bounded as  $n \rightarrow \infty$ .*

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## References

- [1] Apushkinskaya D.; Bildhauer, M.; Fuchs, M. Steady states of anisotropic generalized Newtonian fluids. *J. Math. Fluid Mech.*, **7** (2005), 261-297.
- [2] Bildhauer, M. *Convex Variational Problems*. Lecture Notes in Mathematics, **1818**. Springer-Verlag, Berlin 2003.
- [3] Bogovski, M.E. Solution of the first boundary value problem for the equation of continuity of an incompressible medium. *Soviet Math. Dokl.*, **20** (1979), 1094-1098.
- [4] Cattabriga, L. Su un problema al contorno relativo al sistema di equazioni di Stokes. *Rend. Sem. Mat. Univ. Padova*, **31** (1961), 308-340.
- [5] Fuchs, M., Regularity for a class of variational integrals motivated by nonlinear elasticity. *Asymp. Analysis*, **9** (1994), 23-38.
- [6] Fuchs, M.; Seregin G. *Variational Methods for Problems from Plasticity Theory and for Generalized Newtonian Fluids*. Lecture Notes in Mathematics, **1749**. Springer-Verlag, Berlin 2000.
- [7] Kaplický, P., Málek, J., Stará, J.,  $C^{1,\alpha}$ -solutions to a class of nonlinear fluids in two dimensions-stationary Dirichlet problem. *POMI*, **259** (1999), 89-121.
- [8] Ladyzhenskaya, O.A. On nonlinear problems of continuum mechanics. *Proc. Int. Congr. Math.(Moscow, 1966)*, 560-573. Nauka, Moscow, 1968. English transl. in Amer.Math. Soc. Transl.(2) 70, 1968.
- [9] Ružička M. Modeling, mathematical and numerical analysis of electrorheological fluids. *Applications of Mathematics* **49** (2004), 565-609.
- [10] Yudovich, V.I. Some estimates connected with integral operators and with solutions of elliptic equations. *Soviet Math. Dokl.* **2** (1961), 746-749.