**TURBULENCE MODELS, $p-$FLUID FLOWS, AND $W^{2,l}$ REGULARITY OF SOLUTIONS.**

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**Abstract**

In this article we prove some sharp regularity results for the stationary and the evolution Navier-Stokes equations with shear dependent viscosity, under the no-slip boundary condition. This is a classical turbulence model, considered by von Neumann and Richtmeyer in the 50's, and by Smagorinski in the beginning of the 60's (for $p = 3$). The model was extended to other physical situations, and deeply studied from a mathematical point of view, by Ladyzhenskaya in the second half of the 60's. Actually, the above kind of systems also models a large class of non-Newtonian real fluids. Shear thickening (or dilatant) fluids if $p > 2$ and shear thinning (or pseudo plastic) fluids if $1 < p < 2$. In the sequel we consider the case $p > 2$. We are interested in regularity results in Sobolev spaces for the second order derivatives of the velocity and for the first order derivatives of the pressure *up to the boundary*, in dimension $n = 3$. In spite of the very rich literature on this subject, sharp regularity results up to the boundary are quite new. In the sequel we improve in a quite substantial way all the known results in the literature.

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# 1 Introduction

Throughout this work $u$ and $\pi$ denote, respectively, the velocity and the pressure of a viscous incompressible fluid.We are interested on regularity results for solutions to the Navier-Stokes equations for flows with shear dependent viscosity, namely

$$(1.1)\qquad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u - \nabla \cdot T(u, \pi) = f, \\ \nabla \cdot u = 0, \end{cases}$$

where $T$ denotes the Cauchy stress tensor,

$$(1.2)\qquad T = -\pi I + \nu_T(u) \mathcal{D} u,$$

and $\frac{1}{2} \mathcal{D} u$ denotes the symmetric gradient, i.e.,

$$\mathcal{D} u = \nabla u + \nabla u^T.$$

The exact form of $\nu_T(u)$ is not essential. Our proofs apply to wider classes of $\nu_T(u)$ terms, depending essentially on convexity type properties, and behavior near zero and infinity. A typical example (considered in references [1] and [2]) is

$$\nu_T(u) = \nu_0 + \nu_1 |\mathcal{D}\, u|^{p-2}, \tag{1.3}$$

where $\nu_0$ and $\nu_1$ are strictly positive constants and $p \geq 2$. The results and proofs shown in the sequel apply to this case. However we consider here the slightly more difficult case

$$\nu_T(u) = (1 + |\mathcal{D}\, u|)^{p-2}. \tag{1.4}$$

The lack of the $-\Delta\, u$ term is overcome here by appealing to a suitable device (see also the Remark 1.5 in [2]). Actually, the presence of $-\Delta\, u$ on the right hand side of (1.4) allows additional regularity results, as is easily shown.
We assume that $p \leq 3$. However this restriction is not at all necessary, in the sense that, basically, the same argument gives similar results for $p < 4$.

In our approach, the proofs of the regularity results in the presence of the convective term $(u \cdot \nabla)\, u$, and for the evolution problem, are easily obtained as corollaries of the results proved for solutions to the stationary Stokes system

$$\begin{cases} -\nabla \cdot \Big( (1 + |\mathcal{D}\, u|)^{p-2}\, \mathcal{D}\, u \Big) + \nabla\, \pi = f, \\ \nabla \cdot u = 0. \end{cases} \tag{1.5}$$

This was already shown in some of our previous papers, see for instance [2]. Since the main lines of the proofs are the same, we will state the results and leave details to the interested reader. Hence, our main concern is the study of the stationary generalized Stokes system (1.5). Our basic result is that weak solutions to (1.5) under the no-slip boundary condition

$$u_{|\Gamma} = 0, \tag{1.6}$$

satisfy

$$u \in W^{1,\, p+4}(\Omega) \cap W^{2,\, \frac{p+4}{p+1}}(\Omega). \tag{1.7}$$

Note that $(\frac{p+4}{p+1})^* < p + 4$, (see definition (2.10)). Concerning previous results, related to (1.7), in [2] it is shown that $u \in W^{2,\, \frac{3\,(4-p)}{5-p}}(\Omega)$, and in [5] that $u \in W^{2,\, \frac{8-p}{3}}(\Omega)$.
Note that the second order "tangential derivatives", see (3.2), belong to $L^2(\Omega)$. Hence, they are more regular than the remaining (purely normal) second order derivatives, see (3.4). Hence, instead of appealing to classical Sobolev embedding theorems (as done in reference [2]), we appeal here to anisotropic Sobolev-type embedding theorems, see [22], in order to prove the estimate (3.6) (see the subsection 6.2). This fruitful idea was introduced in reference [4]. Clearly, the better estimate (3.6), when applied to the boot-strap argument already introduced in [1] (see the section 7 below) leads to the better integrability exponents shown in equation (1.7).

Higher order regularity results up to the boundary, for a class of problems containing (1.1), in regular bounded open sets $\Omega \subset \mathbb{R}^3$, under the no-slip boundary condition, are studied for the first time in reference [13]. In reference [1] and [2], sharper results are obtained in the case of a flat boundary. In reference [1] we consider the stationary problem in the half space $\mathbb{R}^n_+$ under slip and no-slip boundary conditions. In reference [2], as below, we consider a cubic domain and impose our boundary condition (1.6) only on two opposite faces. On the other faces we assume periodicity, as a device to avoid unessential technical difficulties. This choice is made so that we work in a bounded domain $\Omega$ and, at the same time, with a flat boundary. Further, in reference [3], we introduce a general method that, essentially, provides the extension of results from flat to non flat boundaries. Actually, in [3], we extend to smooth boundaries the results proved in reference [2].

**Remark 1.1.** In order to emphasize the very new ideas, we consider here the flat boundary case. However, all the regularity results stated here hold in the presence of smooth boundaries. This can be shown, without particular difficulties, by readers already acquainted with the approach introduced in [3]. The significant changes must be made only in a very small part of the proof in [3], concerning a couple of estimates proved in the context of the "flat" system of coordinates $y$ used in the above reference. Roughly speaking, in this last reference, we extend to the $y$ variables some basic estimates proved in [2] in the simpler context of the original $x$ variables. In order to extend to non-flat boundaries the results proved below (instead of that in [2]), it is sufficient to appeal to the equivalent basic estimates proved in the following. For instance, the fundamental estimate (10.19) in reference [3] corresponds to the estimate (3.8) in reference [2], and to the estimate (3.6) below. Roughly speaking, in the original proof of the estimate (10.19), given in [3], we extend some manipulations made in the proof of the estimate (3.8) in reference [2], from the $x$ to the $y$ variables. In the context of the new results claimed above, we extend, in a quite similar way, the corresponding manipulations made in proving the estimate (3.6) below instead of that concerning the estimate (3.8) in [2]. This calculations will be shown in detail by us, or by some collaborators, in a forthcoming paper.

The system of equations (1.1), for $p = 3$, was introduced by J.S. Smagorinsky, see [20], as a turbulence model. Actually, the case $2 \leq p \leq 3$ (specially $p = 3$) has been applied in the last forty years to model turbulence phenomena in fluid flows, a main problem in theoretical, applied and numerical Fluid Mechanics. For arbitrary $p$ the system was introduced and studied by O.A. Ladyzenskaya, already as a turbulence model, in references [7], [8], [9] and [10]. J.-L. Lions considered similar models, in which $\mathcal{D}\, u$ is replaced by $\nabla u$. See [11] and [12], Chap.2, n.5. It is worth noting that (1.2) satisfies the Stokes Principle, see [21]. A clear and rigorous discussion on this subject is given by J. Serrin in reference [19], page 231, where the above physical principle is stated in a postulational form.
Nonlinear shear dependent viscosity also models properties of certain materials. The cases $p > 2$ and $p < 2$ captures shear thickening and shear thinning phenomena, respectively. See, for instance, [6], [14], [15], and [18].

It is worth noting that the presentation of some arguments and proofs below could be reduced by appealing to some of our previous papers. However, the reading of the paper would became unclear and unpleasant to really interested readers.

## 2 Preliminaries.

In the sequel $\Omega$ denotes the 3-dimensional cube $\Omega = (]0,1[)^3$ . The symbol $\|\cdot\|_p$ denotes the canonical norm in $L^p(\Omega)$, and $\|\cdot\| = \|\cdot\|_2$ .
$W^{k,q}(\Omega)$ denotes the usual Sobolev spaces. We use the same notation for functional spaces and norms for both scalar and vector fields.

The boundary condition (1.6) will be imposed only on

$$\Gamma = \Gamma_- \cup \Gamma_+ .$$

where

$$\Gamma_- = \{x : |x_1|, |x_2| < 1 , x_3 = 0\} , \quad \Gamma_+ = \{x : |x_1|, |x_2| < 1 , x_3 = 1\} .$$

The problem is assumed to be periodic, with period equal to 1, both in the $x_1$ and the $x_2$ directions. Obviously, the "significant" boundary is $\Gamma$. We set

$$x' = (x_1, x_2) .$$

By $x'$-periodic we mean periodic of period 1 both in $x_1$ and $x_2$. A similar convention is assumed for expressions like $x'$-periodicity and so on.

We set

(2.1) $$V_p = \big\{v \in W^{1,p}(\Omega) : (\nabla \cdot v)_{|\Omega} = 0 ;\ v_{|\Gamma} = 0 ; v \text{ is } x' - \text{periodic}\big\} .$$

It is well known that there is a positive constant $c$ such that the estimate

(2.2) $$\|\nabla v\|_p + \|v\|_p \leq c\,\|\mathcal{D}v\|_p$$

holds, for each $v \in V_p$. Hence the two above quantities are equivalent norms in $V_p$. See, for instance, [17], Proposition 1.1.

Assume that $f \in (V_p)'$ , the strong dual of $V^p$ . We say that $u$ is a *weak solution* to problem (1.5), (1.6) if $u \in V_p$ satisfies (weakform)

(2.3) $$\frac{1}{2}\int_\Omega (1 + |\mathcal{D}\,u|)^{p-2}\,\mathcal{D}u \cdot \mathcal{D}v\,dx = \int_\Omega f \cdot v\,dx$$

for all $v \in V_p$. It is well known that existence and uniqueness of the weak solution follow by appealing to the method described in the Chap.2, Sect.2 of [12].

By replacing $v$ by $u$ in equation (2.3) one gets

(2.4) $$\|\mathcal{D}\,u\|^2 + \|\mathcal{D}\,u\|_p^p \leq c\,| < f, u > | ,$$

where the symbols $< \cdot, \cdot >$ denote a duality pairing.

From (2.4) and (2.2) there readily follows the basic estimates

(2.5) $$\|\nabla u\| \leq c\,\|f\| \quad \text{and} \quad \|\nabla u\|_p \leq c\,\|f\|_{p'}^{\frac{1}{p-1}} .$$

Note that the second estimate is stronger than the first one.
Well known devices show the existence of a distribution $\pi$ (determined up to a constant) such that

$$\nabla\,\pi = -\,\nabla\cdot\big[(1+\,|\mathcal{D}u|)^{p-2}\,\mathcal{D}u\,\big] + \,f\,. \tag{2.6}$$

Hence the first equation (1.5) holds in the distributions sense. Actually, by appealing to (2.6) and (2.5), one may show that $\pi \in L^{p'}(\Omega)$ and that

$$\|\pi\|_{L^{p'}_{\#}} \leq c\,(\,\|\mathcal{D}\,u\|_{p'} + \,\|\mathcal{D}\,u\|_{p}^{p-1}\,) \leq c(\,\|f\| + \|f\|_{p'})\,,$$

where, in general, $L^{\alpha}_{\#} = L^{\alpha}/\mathbb{R}$.

We denote by $D^2\,u$ the set of all the second derivatives of $u$ and by $D^2_*\,u$ the second order derivatives $\partial^2 u_j/\,\partial x_i\,\partial x_k$ with the exclusion of the normal derivatives $\partial^2 u_j/\,\partial x_3^2$, for $j = 1,\,2$. Further,

$$|D^2_*\,u|^2 := \left|\frac{\partial^2\,u_3}{\partial\,x_3^2}\right|^2 + \sum_{\substack{i,j,k=1\\(i,k)\neq(3,3)}}^{3} \left|\frac{\partial^2\,u_j}{\partial x_i\,\partial x_k}\right|^2\,. \tag{2.7}$$

Similarly, $\nabla^*$ denotes first order partial derivatives, except for $\partial\;/\partial\,x_3$.

Some integrability exponents play a crucial role in our proofs and are, for the reader's convenience, introduced here.

In the sequel $p$ denotes an exponent that lies in the interval

$$2 \leq\,p\,\leq 3 \tag{2.8}$$

and $q$ an exponent that lies in the interval

$$p \leq\,q \leq\,6\,.$$

We denote by $p'$ the dual exponent

$$p' = \frac{p}{p-\,1}\,. \tag{2.9}$$

In general, for $1 < \,a < \,3$ we define the Sobolev embedding exponent $a^*$ by the equation

$$\frac{1}{a^*} = \frac{1}{a} - \frac{1}{3}\,. \tag{2.10}$$

Moreover we define $r = \,r(q)$ by

$$\frac{1}{r(q)} = \frac{p-2}{2\,q} + \frac{1}{2}\,, \tag{2.11}$$

$\mathcal{Q} = \,\mathcal{Q}(q)$ by

$$\frac{1}{\mathcal{Q}(q)} = \frac{5\,(p-\,2)}{6\,(p-\,1)\,q} + \frac{1}{6(p-\,1)}\,, \tag{2.12}$$

and $\overline{q} = \overline{q}(q)$ by

$$\frac{1}{\overline{q}(q)} = \frac{p-2}{\mathcal{Q}(q)} + \frac{1}{2}\,. \tag{2.13}$$

Note that

$$\mathcal{Q}(q) > q\,, \tag{2.14}$$

since $q < p + 4$.
We set

$$\widetilde{q}(q) = \min\{\overline{q}(q), r(q)\}\,. \tag{2.15}$$

Note that $r(q) \geq \overline{q}(r)$ is equivalent to $q \geq 7 - 2p$.

We denote by $c$ a generic positive constant that may change from equation to equation. The positive constants $c$ do not depend on the parameters $p$ and $q$, in the usual sense (i.e., they are bounded from above for $p$ and $q$ varying in the ranges considered here).

# 3 The stationary Stokes problem. Main results.

**Theorem 3.1.** *Assume that*

$$f \in L^2(\Omega) \tag{3.1}$$

*and let $u$, $\pi$ be the weak solution to problem* (1.5) *under the boundary condition* (1.6) *plus $x'$-periodicity (problem* (2.3)*).*

*Then the derivatives $D_*^2\, u$ belong to $L^2(\Omega)$, moreover*

$$\|D_*^2\, u\|^2 + \sum_{k=1}^{2} \left\| (1 + |\mathcal{D}\, u|)^{\frac{p-2}{2}}\, \mathcal{D}\, \frac{\partial u}{\partial x_k} \right\|^2 \leq c\, \|f\|^2\,. \tag{3.2}$$

Concerning the regularity of the remaining derivatives, we start from the following conditional result.

**Theorem 3.2.** *Let $f$, $u$ and $\pi$ be as in Theorem 3.1 and assume, in addition, that*

$$\mathcal{D}\, u \in L^q(\Omega) \tag{3.3}$$

*for some $p \leq q \leq 6$. Then, in addition to* (3.2)*, one has*

$$\|\nabla^*\, \pi\|_{r(q)} + \|D^2\, u\|_{r(q)} + \|\, (1 + |\mathcal{D}\, u|)^{p-2}\, \nabla^*\, \mathcal{D}\, u\|_{r(q)} \leq \mathcal{K}_q\,, \tag{3.4}$$

*where $r = r(q)$ is given by* (2.11) *and $\mathcal{K}_q$ satisfies the estimate*

$$\mathcal{K}_q \leq c\, \|f\| + c\, \|\mathcal{D}\, u\,\|_q^{\frac{p-2}{2}}\, \|f\|\,. \tag{3.5}$$

*Furthermore,*

$$\|u\|_{1,\mathcal{Q}(q)} \leq A_q =: c_0\, \|f\|^{\frac{1}{p-1}} + c_0\, \|\nabla\, u\|_q^{\frac{5\,(p-2)}{6\,(p-1)}}\, \|f\|^{\frac{1}{p-1}} + c\, \|\nabla\, u\|_p\,, \tag{3.6}$$

*where $\mathcal{Q}(q)$ is given by* (2.12).

Since (3.3) holds for $q = p$ and $r(p) = p'$, one has the following result.

**Proposition 3.1.** *Let $f$, $u$ and $\pi$ be as in Theorem 3.1. Then*

$$\|\nabla^* \pi\|_{p'} + \|D^2 u\|_{p'} + \|(1 + |\mathcal{D} u|)^{p-2} \nabla^* \mathcal{D} u\|_{p'} \leq \tag{3.7}$$

$$c\,\|f\| + c\,\|\,1 + |\mathcal{D} u|\,\|_p^{\frac{p-2}{2}}\,\|f\|$$

*and*

$$\|u\|_{1,\mathcal{Q}(p)} \leq c_0\,\|f\|^{\frac{1}{p-1}} + c_0\,\|\nabla u\|_p^{\frac{5(p-2)}{6(p-1)}}\,\|f\|^{\frac{1}{p-1}} + c\,\|\nabla u\|_p\,, \tag{3.8}$$

*where* $\mathcal{Q}(p) = \frac{3p-5}{3p(p-1)}$ *is larger than* $p$.

Concerning the regularity of the derivative $\frac{\partial \pi}{\partial x_3}$ one has the following result.

**Proposition 3.2.** *Under the assumptions of Theorem 3.2 one has*

$$\left\|\frac{\partial \pi}{\partial x_3}\right\|_{\widetilde{q}} \leq c\,\left[1 + A_q^{p-2}\right]\,\|f\| + c\,\mathcal{K}_q \tag{3.9}$$

*where $\widetilde{q}$ is defined in* (2.15) *and $A_q$ is the right hand side of* (3.6). *In particular, by* (3.4),

$$\|\nabla \pi\|_{\widetilde{q}} \leq c\,\left[1 + A_q^{p-2}\right]\,\|f\| + c\,\mathcal{K}_q\,. \tag{3.10}$$

The reason that leads us to separate Proposition 3.2 from Theorem 3.2 is to emphasize that the regularity of $\frac{\partial \pi}{\partial x_3}$ is simply obtained as a final by product of the regularity of all other derivatives, in contrast with the main rule of the regularity of all the other derivatives of $u$ and $\pi$ in each steep of the bootstrap argument that leads to the proof of Theorem 3.3 below.
Since (3.3) holds for $q = p$, the following result holds.
The next is our main result.

**Theorem 3.3.** *Let $f$, $u$ and $\pi$ be as in Theorem 3.1. Then, in addition to* (3.2),

$$\|u\|_{1,p+4} \leq c\|f\|^{\frac{1}{p-1}} + c\,\|f\|^{\frac{6}{p+4}} + c\,\|\nabla u\|_p\,. \tag{3.11}$$

*Furthermore,*

$$\|\nabla^* \pi\|_l + \|D^2 u\|_l + \|(1 + |\mathcal{D}u|)^{p-2} \nabla^* \mathcal{D}u\|_l \leq \tag{3.12}$$

$$c\,(1 + \|\nabla u\|_p^{\frac{p-2}{2}})\|f\| + c\,\|f\|^{\frac{2(p-2)}{p+4}}\,,$$

*where*

$$l = \frac{p+4}{p+1}. \tag{3.13}$$

*Finally,*

$$\frac{\partial \pi}{\partial x_3} \in L^m(\Omega), \tag{3.14}$$

*where* $m = \overline{q}(p+4)$. *In particular,*

$$\nabla \pi \in L^m(\Omega)\,,$$

*and*

$$\|\nabla \pi\|_m \leq c\left[1 + A_{p+4}^{p-2}\right]\|f\| + c\,\mathcal{K}_{p+4}\,. \tag{3.15}$$

Note that, by (3.11), $\mathcal{K}_{p+4}$ and $A_{p+4}$ are bounded (in terms of $\|f\|$ and $\|\nabla u\|_p$). Further, $m = 2$ if $p = 2$.

# 4 Stationary and evolution Navier-Stokes equations. Results.

The proofs of the following extensions of the above results to solutions of the Navier-Stokes stationary and evolution equations are left to the interested reader, since they are done by straightforward modifications of the corresponding proofs shown in reference [2]. We note that, in our papers, the main novelties concern the generalized Stokes system (1.5). In fact, in our opinion, the new obstacles related to the boundary value problems already appear in this particular case. Regularity results for solutions to the stationary and evolution generalized Navier-Stokes equations are proved by us as more or less straightforward consequences of the results obtained for the generalized Stokes system. Actually, we realize that a more stringent use of the estimates proved for the system (1.5) is possible. However, we did not push in this direction.

**Theorem 4.1.** *The regularity results stated in the Theorems 3.1, 3.2 and 3.3, and in the Lemma 3.2, hold for the generalized Navier-Stokes equations*

$$\begin{cases} -\nabla\cdot\Big((1+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\Big) + (u\cdot\nabla)\,u + \nabla\pi = f, \\ \\ \nabla\cdot u = 0\,. \end{cases} \tag{4.1}$$

Consider now the evolution problem

$$\begin{cases} \frac{\partial u}{\partial t} + (u\cdot\nabla)\,u - \nabla\cdot\Big((1+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\Big) + (u\cdot\nabla)\,u\,\nabla\pi = f, \\ \\ \nabla\cdot u = 0\,, \\ \\ u(0) = u_0(x)\,. \end{cases} \tag{4.2}$$

One has the following results.

**Theorem 4.2.** *Let* $u$ *be a weak solution to problem* (4.2) *under the boundary condition* (1.6) *plus* $x'$*-periodicity, where* $u_0 \in V_p$ *and* $f \in L^2(0,T;L^2)$. *Assume that* $p \geq 2 + \frac{2}{5}$. *Then*

$$\begin{cases} u \in L^2(0,T;W^{2,p'}) \cap L^\infty(0,T;W^{1,p})\,, \\ \\ \frac{\partial u}{\partial t} \in L^2(0,T;L^2)\,. \end{cases} \tag{4.3}$$

**Theorem 4.3.** *Under the assumptions of Theorem 4.2 one has*

(4.4)
$$u \in L^{\frac{p+4}{p-2}}(0,T; W^{2,\,l}) \cap L^{\infty}(0,T; W^{1,p})\,.$$

The assumption $p \geq\ 2 + \frac{2}{5}$ is superfluous if the convective term is not present. Corresponding results for the pressure are easily obtained, as well as estimates for the norms that appear in the above theorems.

# 5 Proof of Theorem 3.1.

In this section we prove the Theorem 3.1. By assumption $u \in V_p$ satisfies (2.3) for each $v \in V_p$. For arbitrary scalar or vector fields $v$ set $v^h(x) = v(x_1 + h,\, x_2,\, x_3)$ or $v^h(x) = v(x_1,\, x_2 + h,\, x_3)$ where $h \in \mathbb{R}$. We also set

$$\Delta_h\, v = \frac{v -\, v^{-h}}{h}\,.$$

By writing (2.3) with $v$ replaced by $v^h$ and by replacing, in the integrals on the left hand side, the variable $x_k$ by $x_k - h$ , $k =\ 1,\, 2$ , one easily shows that (weak2)

(5.1)
$$\frac{1}{2}\int (1 +\, |\mathcal{D}\, u^{-h}|)^{p-2}\, \mathcal{D}\, u^{-h} \cdot\, \mathcal{D}\, v\, dx = \int f\, \cdot\, v^h\, dx\,.$$

By taking the difference between equations (2.3) and (5.1), respecting the left and right sides, and by dividing by $h$ one gets

(5.2)
$$\frac{1}{2\,h}\int \big( (1 +\, |\mathcal{D}\, u|)^{p-2}\, \mathcal{D}\, u -\, (1 +\, |\mathcal{D}\, u^{-h}|)^{p-2}\, \mathcal{D}\, u^{-h} \big) \cdot\, \mathcal{D}\, v\, dx$$
$$= \frac{1}{h}\int f\, \cdot\, (v -\, v^h)\, dx\,.$$

By setting $v =\ \Delta_h\, u$ in equation (5.2), by appealing to the identity

(5.3)
$$\|\, \mathcal{D}\, \Delta_h\, u\|^2 =\ 2\, \|\nabla\, (\Delta_h\, u)\|^2$$

and by using the estimate

(5.4)
$$\left| \frac{1}{h}\int f \cdot (v -\, v^h)\, dx \right| \leq\, \|f\|\, \left\| \frac{v -\, v^h}{h} \right\| \leq\, \|f\|\, \|\nabla v\|\,,$$

it follows that

(5.5)
$$\frac{1}{2\,h}\int \big( (1 +\, |\mathcal{D}\, u|)^{p-2}\, \mathcal{D}\, u -\, (1 +\, |\mathcal{D}\, u^{-h}|)^{p-2}\, \mathcal{D}\, u^{-h} \big) \cdot\, (\mathcal{D}\, \Delta_h\, u)\, dx$$
$$\leq\, c\, \|f\|\, \|\mathcal{D}\, (\Delta_h\, u)\|\,.$$

Next, by a well known convex analysis estimate (set $U = \mathcal{D}\, u$ and $V = \mathcal{D}\, u^{-h}$ in equation (5.1), [2]), it follows that

$$\int \big(1 + |\mathcal{D}\, u| + |\mathcal{D}\, u^{-h}|\,)\big)^{p-2}\, |\mathcal{D}\, \Delta_h\, u|^2 \leq\; c\, \|f\|\, \|\mathcal{D}\, (\Delta_h\, u)\|\,. \tag{5.6}$$

In particular,

$$\|D_*^2\, u\|^2 + \int \big(1 + |\mathcal{D}\, u| + |\mathcal{D}\, u^{-h}|\,)\big)^{p-2}\, |\mathcal{D}\, \Delta_h\, u|^2 \leq\; c\, \|f\|^2\,. \tag{5.7}$$

Note that, as a first steep, we obtain the above equation with $\|D_*^2\, u\|^2$ replaced by $\|\mathcal{D}\, \Delta_h\, u\|^2$, hence by $\|\nabla\, \Delta_h\, u\|^2$ (apply (5.3)). Further, the uniform bound of this last quantity with respect to $h$ allows us to replace it by $\|\nabla\, \nabla_*\, u\|^2$. Finally, differentiation with respect to $x_3$ of the equation $\nabla \cdot u = 0$ allows the inclusion of the derivative $\frac{\partial^2 u_3}{\partial x_3^2}$ in the above estimate, hence to replace $\|\nabla\, \nabla_*\, u\|$ by $\|D_*^2\, u\|$.
Next, (as in [2]), by passing to the limit in (5.7), as $h \to 0$, one shows that

$$\|D_*^2\, u\|^2 + \sum_{k=1}^{2} \left\| (1 + |\mathcal{D}\, u|)^{\frac{p-2}{2}}\, \mathcal{D}\, \frac{\partial u}{\partial x_k} \right\|^2 \leq\; c\, \|f\|^2\,. \tag{5.8}$$

The proof of the estimate (3.2) is accomplished.
We note that it is not strictly necessary to appeal to (5.3). See the remark 5.1 in reference [2].

# 6 Proof of the Theorem 3.2.

## 6.1 Proof of the estimate (3.4).

We start this section by recalling the following result. Let $g(x)$ be a scalar field in $\Omega$ such that $g = \nabla \cdot w_0$ and $\nabla\, g = \nabla \cdot W$, where $w_0 \in L^\beta(\Omega)$ and $W \in L^\alpha(\Omega)$, for some $\alpha \geq \beta > 1$. Then

$$\|g\|_{L^\alpha(\Omega)} \leq c\, \left(\|w_0\|_{L^\beta(\Omega)} + \|W\|_{L^\alpha(\Omega)}\right)\,. \tag{6.1}$$

For $\beta = \alpha$ this result is proved in reference [16]. The above extension is straightforward.

It is worth noting that our constants $c$ are independent of $p$, $q$, $r$ since the constants that appear in the embedding theorems used in the sequel, as well as in (6.1), are uniformly bounded from above. This follows, since the exponents lie away from the critical values. Note that $2 \leq p \leq 3$, $p \leq q \leq 6$ and $\frac{4}{3} \leq r \leq 2$.

**Lemma 6.1.** *Assume* (3.3)*. For $k = 1, 2$, the terms $(1 + |\mathcal{D}\, u|)^{p-2}\, \mathcal{D}\, \frac{\partial u}{\partial x_k}$ and the derivatives $\frac{\partial \pi}{\partial x_k}$ satisfy the estimate* (3.4)*.*

*Proof.* The proof is a straightforward copy of the proof of the lemma 6.2 in reference [2]. We present it just for the reader's convenience. Straightforward calculations show that

(6.2)
$$\frac{\partial}{\partial x_k}\,\big((1+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\big) =$$

$$(1+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,\frac{\partial u}{\partial x_k} + (p-2)\,(1+|\mathcal{D}\,u|)^{p-3}\,|\mathcal{D}\,u|)^{-1}\left(\mathcal{D}\,u\cdot\mathcal{D}\,\frac{\partial u}{\partial x_k}\right)\mathcal{D}\,u\,.$$

Hence

$$|\,\frac{\partial}{\partial x_k}\,\big((1+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\big)\,| \le c\,(1+|\mathcal{D}\,u|)^{p-2}\left|\mathcal{D}\,\frac{\partial\,u}{\partial x_k}\right|, \tag{6.3}$$

almost everywhere in $\Omega$.
On the other hand, by differentiation of equation (1.5) with respect to $x_k$, $k=1,2$, it follows that

$$\nabla\frac{\partial\pi}{\partial x_k} = \nabla\cdot\left[-\frac{\partial}{\partial x_k}\,\big((1+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,u\big)\right] + \frac{\partial f}{\partial x_k}\,. \tag{6.4}$$

Moreover, by Hőlder's inequality and assumption (3.3), one has

$$\left\|(1+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,\frac{\partial\,u}{\partial x_k}\right\|_r \le \|1+|\mathcal{D}\,u|\,\|_q^{\frac{p-2}{2}}\left\|(1+|\mathcal{D}\,u|)^{\frac{p-2}{2}}\,\mathcal{D}\,\frac{\partial\,u}{\partial x_k}\right\|. \tag{6.5}$$

Hence, by (5.8), it follows that

$$\left\|(1+|\mathcal{D}\,u|)^{p-2}\,\mathcal{D}\,\frac{\partial\,u}{\partial x_k}\right\|_r \le c\,\|\,1+|\mathcal{D}\,u|\,\|_q^{\frac{p-2}{2}}\,\|f\|\,. \tag{6.6}$$

This proves the first statement in the Lemma. Furthermore, by using (6.1), with $g=\frac{\partial\pi}{\partial x_k}$, $\alpha = r$ and $\beta = p'$, and by (2.6), (2.5) and (6.4), it follows that

$$\left\|\frac{\partial\pi}{\partial x_k}\right\|_r \le c\,\left(\|f\| + \|f\|_{p'} + \|\,1+|\mathcal{D}\,u|\,\|_q^{\frac{p-2}{2}}\,\|f\|\right). \tag{6.7}$$

Hence,

$$\left\|\frac{\partial\pi}{\partial x_k}\right\|_r \le \mathcal{K}_q\,. \tag{6.8}$$

□

Note that from equations (6.6) and (6.8) we get the estimate (3.4) for the first and the last term on the left hand side. The missing term is the subject of the following lemma.

**Lemma 6.2.** *The derivatives* $\frac{\partial^2 u_j}{\partial x_3^2}$, $j=1,2$ *satisfy the estimate*

$$\sum_{l=1}^{2}\left\|\frac{\partial^2\,u_l}{\partial x_3^2}\right\|_{r(q)} \le \mathcal{K}_q\,. \tag{6.9}$$

*Proof.* It is worth noting that the proof is a pedestrian copy of the proof of the Lemma 6.3 in reference [2]. Nevertheless, we believe that it is pleasant to the interested reader to have it hereby.
By using (6.2), the $j.^{th}$ equation (1.5) may be written in the form

(6.10)

$$- (1 + |\mathcal{D}\, u|)^{p-2} \sum_{k=1}^{3} \frac{\partial^2 u_j}{\partial x_k^2}$$

$$-(p-2)\,(1 + |\mathcal{D}\, u|)^{p-3}\, |\mathcal{D}\, u|^{-1} \sum_{l,m,k=1}^{3} \mathcal{D}_{lm}\, \mathcal{D}_{jk} \left( \frac{\partial^2 u_l}{\partial x_m\, \partial x_k} + \frac{\partial^2 u_m}{\partial x_l\, \partial x_k} \right)$$

$$+ \frac{\partial \pi}{\partial x_j} = f_j\,,$$

where $\mathcal{D}_{ij} = (\mathcal{D}\, u)_{ij} = \frac{\partial\, u_i}{\partial x_j} + \frac{\partial\, u_j}{\partial x_i}$ and $1 \leq j \leq 3$. Let us write the first two equations (6.10), $k = 1, 2$, as follows:

$$(1 + |\mathcal{D}\, u|)^{p-2} \frac{\partial^2 u_j}{\partial x_3^2}$$

$$+\,2\,(p-2)\,(1 + |\mathcal{D}\, u|)^{p-3}\, |\mathcal{D}\, u|^{-1}\, \mathcal{D}_{j3} \sum_{l=1}^{2} \mathcal{D}_{l3} \frac{\partial^2 u_l}{\partial x_3^2} = \tag{6.11}$$

$$F_j(x) + \frac{\partial \pi}{\partial x_j} - f_j\,,$$

where the $F_j(x), j \neq 3$, are given by

(6.12)

$$F_j(x) := -\,(1 + |\mathcal{D}\, u|)^{p-2} \sum_{k=1}^{2} \frac{\partial^2 u_j}{\partial x_k^2}$$

$$-2\,(p-2)\,(1 + |\mathcal{D}\, u|)^{p-3}\, |\mathcal{D}\, u|^{-1} \left\{ \mathcal{D}_{33}\, \mathcal{D}_{j3} \frac{\partial^2 u_3}{\partial x_3^2} + \sum_{\substack{l,m,k=1 \\ (m,k)\neq(3,3)}}^{3} \mathcal{D}_{lm}\, \mathcal{D}_{jk} \frac{\partial^2 u_l}{\partial x_m\, \partial x_k} \right\}.$$

In the sequel, the equations (6.11), $j = 1, 2$, will be treated as a $2 \times 2$ linear system in the unknowns $\frac{\partial^2 u_j}{\partial x_3^2}, j \neq 3$. Note that, with an obviously simplified notation, the measurable functions $F_j$ satisfy

$$|F_j(x)| \leq c\,(\,1 + |\mathcal{D}\, u(x)|)^{p-2}\, |D_*^2\, u(x)|\,, \tag{6.13}$$

a.e. in $\Omega$.

We denote by $\widetilde{F}_j$ the right hand sides

$$\widetilde{F}_j(x) := F_j(x) + \frac{\partial \pi}{\partial x_j} - f_j \,, \tag{6.14}$$

that appear in the above $2 \times 2$ system (6.11).

Let us show that the $2 \times 2$ system (6.11) can be solved for the unknowns $\frac{\partial^2 u_j}{\partial x_3^2}$, $j = 1, 2$, for almost all $x \in \Omega$.

The elements $a_{j\,l}$ of the matrix system $A$ are given by

$$a_{j\,l} = (1 + |\mathcal{D}\,u|)^{p-2}\,\delta_{j\,l} + 2\,(p-2)\,(1 + |\mathcal{D}\,u|)^{p-3}\,|\mathcal{D}\,u|^{-1}\,\mathcal{D}_{l3}\,\mathcal{D}_{j3}\,,$$

for $j, l \neq 3$. Note that $a_{j\,l} = a_{l\,j}$. One easily shows that

$$\sum_{j,l=1}^{2} a_{j\,l}\xi_j\xi_l = (1 + |\mathcal{D}\,u|)^{p-2}\,|\xi|^2 + 2\,(p-2)\,(1 + |\mathcal{D}\,u|)^{p-3}\,|\mathcal{D}\,u|^{-1}\,\left[(\mathcal{D}\,u)\cdot\xi\right]_3^2 \,.$$

Hence the matrix $A$ is symmetric and positive definite. Moreover, the above identity shows that all the eigenvalues are larger than or equal to $(1 + |\mathcal{D}\,u|)^{p-2}$. Hence,

$$det\,A \geq \left((1 + |\mathcal{D}u|)^{p-2}\right)^2 \,.$$

Next, by setting $\xi_l = \frac{\partial^2 u_l}{\partial x_3^2}$, we get from (6.11), i.e. from

$$\sum_{l=1}^{2} a_{j\,l}\,\xi_l = \widetilde{F}_j \,, \tag{6.15}$$

that

$$\sum_{l,j=1}^{2} a_{j\,l}\,\xi_l\xi_j = \sum_{j=1}^{2} \widetilde{F}_j\,\xi_j \,. \tag{6.16}$$

Consequently $(1 + |\mathcal{D}u|)^{p-2}\,|\xi|^2 \leq |\widetilde{F}|\,|\xi|$, which shows that

$$(1 + |\mathcal{D}u|)^{p-2} \sum_{l=1}^{2} \left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \leq |\widetilde{F}| := \left(\sum_{j=1}^{2} |\widetilde{F}_j|^2\right)^{1/2} \,, \tag{6.17}$$

almost everywhere in $\Omega$. By appealing to (6.13) and (6.14) one shows that

$$\begin{aligned} &(1 + |\mathcal{D}u|)^{p-2} \sum_{l=1}^{2} \left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \\ &\leq c(\,1 + |\mathcal{D}u|)^{p-2}\,|D_*^2\,u(x)| + c\,(|\nabla^*\,\pi| + |f|) \,. \end{aligned} \tag{6.18}$$

In particular,

$$\sum_{l=1}^{2} \left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \leq c\,|D_*^2\,u(x)| + c\,(|\nabla^*\,\pi| + |f|) \,, \tag{6.19}$$

almost everywhere in $\Omega$. There readily follows, by appealing to (6.8), that (6.9) holds. The proof of the estimate (3.4) is accomplished. $\square$

## 6.2 Proof of the estimate (3.6).

The following anisotropic, Sobolev type, embedding theorem is a particular case of more general results proved by Troisi in reference [22]. It is a crucial tool for proving the Theorem 3.3.

**Proposition 6.1.** *Let* $\Omega$ *be as above, and let* $v \in W^{1,1}(\Omega)$ *. Assume that*

$$\partial_k v \in L^{p_k}\,, \text{for} \quad k = 1, 2, 3\,, \tag{6.20}$$

*where*

$$\frac{1}{\overline{p}} := \frac{1}{3} \sum_{k=1}^{3} \frac{1}{p_k} - \frac{1}{3}\,. \tag{6.21}$$

*Then* $v \in L^{\overline{p}}(\Omega)$ *and*

$$\| v \|_{\overline{p}} \leq c \prod_{k=1}^{3} \| \partial_k v \|_{p_k}^{\frac{1}{3}} + c \|v\|_p\,. \tag{6.22}$$

Obviously, we may replace $\|v\|_p$ by any other $L^s$ norm, $s \geq 1$.

An essential point in order to get the limit exponent $l$ in the proof of theorem 3.3 is that the constant $c$ on the right hand side of (6.22) does not depend on the values of the exponents $p_k$ used in the sequel. This property holds provided that $\overline{p}$ lies bounded away from $3$. This follows essentially from the equation (1.15) in the above reference (nevertheless, note that each of the values $p_k$ used in our proof lie bounded away from $3$).

We start by noting that

$$|\partial_{x_k} |\mathcal{D}\, u|^{p-1}| \leq (p-1)\, |\mathcal{D}\, u|^{p-2}\, |\partial_{x_k} \mathcal{D}\, u|\,. \tag{6.23}$$

**Lemma 6.3.** *Assume that the hypotheses in the Theorem 3.2 hold. Then*

$$\|\nabla^* |\mathcal{D}\, u\,|^{p-1}\|_r \leq \mathcal{K}_q \tag{6.24}$$

*and*

$$\|\partial_{x_3} |\mathcal{D}\, u|^{p-1}\|_s \leq c\, \|\mathcal{D}\, u\,\|_q^{p-2}\, \mathcal{K}_q\,, \tag{6.25}$$

*where* $s = s(q)$ *is given by*

$$\frac{1}{s(q)} := \frac{p-2}{q} + \frac{1}{r(q)} = \frac{3\,p + q - 6}{2\,q}\,. \tag{6.26}$$

*Proof.* The estimate (6.24) follows from (6.23), for $k = 1, 2$, together with (3.4).
On the other hand, from (6.23) and Hőlder's inequality, one gets

$$\|\partial_{x_3} |\mathcal{D}\, u|^{p-1}\|_s \leq c\, \|\mathcal{D}\, u\,\|_q^{p-2}\, \|\partial_{x_3} \mathcal{D}\, u\,\|_r\,. \tag{6.27}$$

The estimate (6.25) follows by appealing to (3.4). □

Define $\alpha = \alpha(q)$ by

$$\frac{1}{\alpha(q)} = \frac{1}{3}\left(\frac{2}{r(q)} + \frac{1}{s(q)}\right) - \frac{1}{3}\,, \tag{6.28}$$

Note that

$$\frac{1}{\alpha(q)} = \frac{1}{r(q)} + \frac{p-2}{3\,q} - \frac{1}{3}\,, \tag{6.29}$$

moreover, recall (2.12),

$$\mathcal{Q}(q) = (p-1)\,\alpha(q)\,.$$

**Lemma 6.4.** *Assume that the hypotheses in the Theorem 3.2 hold. Then,*

$$\nabla u \in L^{\mathcal{Q}(q)}\,,$$

*moreover,*

$$\|\nabla u\|_{\mathcal{Q}(q)} \leq c_0\,\|f\|^{\frac{1}{p-1}} + c_0\,\|\mathcal{D}\,u\|_q^{\frac{5\,(p-2)}{6\,(p-1)}}\,\|f\|^{\frac{1}{p-1}} + c\,\|\nabla u\|_p\,. \tag{6.30}$$

*Proof.* From (6.24), (6.25) and proposition 6.1, it follows that

$$\|\,|\nabla u|^{p-1}\,\|_\alpha \leq c\,\mathcal{K}_q\,\|\mathcal{D}\,u\|_q^{\frac{p-2}{3}} + c\,\|\nabla u\|_p^{p-1}\,. \tag{6.31}$$

Our thesis follows from (6.31) and (3.5). □

# 7 Proof of Theorem 3.3.

In the sequel $\|\;\;\|_{k,s}$ denotes the norm in the Sobolev space $W^{k,s}(\Omega)$. We have already proved that if a solution $u$ belongs to $W^{1,q}$ then $u$ belongs to $W^{1,\mathcal{Q}}$, where $\mathcal{Q}(q)$ is given by (2.12). Moreover (3.6) holds.
Define the increasing sequence

$$\begin{cases} q_1 = p\,, \\ q_{n+1} = \mathcal{Q}(q_n)\,, \end{cases} \tag{7.1}$$

appeal to an induction argument. For convenience we set $b_n = \|u\|_{1,q_n}$, $a = \|f\|^{\frac{1}{p-1}}$, $\beta = \frac{5(p-2)}{6(p-1)}$ and $C = c\,\|\nabla u\|_p$. From (3.6) it follows that

$$b_{n+1} \leq c_0\,a\,(1 + b_n^\beta) + C\,.$$

Clearly

$$q_\infty = p + 4 \tag{7.2}$$

is a fixed point of the strictly increasing function $\mathcal{Q}(q)$, moreover

$$\lim_{n\to\infty} q_n = q_\infty\,. \tag{7.3}$$

Note that for $n = 1$ one has

$$\|u\|_{1,q_1} = \|u\|_{1,p} \,.$$

If we are able to show that the quantities $b_n = \|u\|_{1,q_n}$, at least for large values of $n$, are uniformly bounded by a finite number $L$ then, well know results in Functional Analysis, together with (7.3), yield

$$\|u\|_{1,q_\infty} \leq L \,. \tag{7.4}$$

Note that $0 \leq \beta < 1$. Denote by $\lambda$ the (unique) solution of the equation $\lambda = c_0\, a + c_0\, a\, \lambda^\beta + C$. It is easily seen that $a_n < 2\,\lambda$ for large values of $n$. On the other hand one easily shows that

$$\lambda \leq 3\,c_0\, a + (3\,c_0\, a)^{\frac{1}{1-\beta}} + 3\,C \,.$$

Hence,

$$\|u\|_{1,q_\infty} \leq c\|f\|^{\frac{1}{p-1}} + c\,\|f\|^{\frac{6}{p+4}} + c\,\|\nabla\, u\|_p \,,$$

which is just (3.11).
Finally, the estimates (3.12) follow by applying once more the Theorem 3.2, now with $q = q_\infty$ given by (7.2). In this case the equation (2.11) shows that $r = r(q_\infty) = l$, with $l$ given by (3.13). Hence, from (3.4), it follows that

$$\|\nabla^*\, \pi\|_l + \|D^2\, u\|_l + \|\,(1+|\mathcal{D}\, u|)^{p-2}\, \nabla^*\, \mathcal{D}\, u\|_l \leq \mathcal{K}_{q_\infty} \leq c\,\|f\| + c\,\|1+|\mathcal{D}\, u|\,\|_{q_\infty}^{\frac{p-2}{2}}\,\|f\| \,. \tag{7.5}$$

Recall the definition (3.5). Finally, by appealing to (3.11) we show that $\mathcal{K}_{q_\infty}$ is bounded by the right hand side of (3.12). So, this last estimate holds.

Regularity and estimates for $\frac{\partial\, \pi}{x_3}$ (hence, for $\nabla\, \pi$), see (3.15), follows immediately by appealing to the Lemma 3.2 with $q = q_\infty = p + 4$. Note that $m = \widetilde{q}(p+4) = \overline{q}(p+4)$ since $p + 4 > 7 - 2\,p$ (see the remark after (2.15)).

# 8 Proof of proposition 3.2

*Proof.* From equation (6.10) written for $j = 3$, we get

$$\left|\frac{\partial \pi}{\partial x_3}\right| \leq c\,(1 + |\mathcal{D}\, u(x)|)^{p-2}\, |D^2_*\, u(x)| +$$

$$c\,(p-2)\,(1 + |\mathcal{D}\, u(x)|)^{p-2} \sum_{l=1}^{2} \left|\frac{\partial^2\, u_l}{\partial x_3^2}\right| + |f_3(x)| \,, \tag{8.1}$$

almost everywhere in $\Omega$. Hence, by (6.18), one has

$$\left|\frac{\partial \pi}{\partial x_3}\right| \leq c\,(1 + |\mathcal{D}\, u|)^{p-2}\, |D^2_*\, u| + c\,(|\nabla^*\, \pi| + |f|\,) \,. \tag{8.2}$$

On the other hand, by Hőlder's inequality,

$$\|\,(1 + |\mathcal{D}u|)^{p-2}\, D^2_* u\|_{\overline{q}(q)} \leq \|\,1 + |\mathcal{D}u|\,\|^{p-2}_{\mathcal{Q}(q)}\, \|\, D^2_* u\,\| \,. \tag{8.3}$$

By (3.2) and (3.6) one gets

$$\| (1 + |\mathcal{D}u|)^{p-2} D^2_* u\|_{\overline{q}} \leq c \left[1 + A_q^{p-2}\right] \|f\| \,, \tag{8.4}$$

where $A_q$ is the right hand side of (3.6). Finally, (3.9) follows by appealing to (8.2), (8.4) and (3.4).

□

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