

Qualitative properties of the ice-thickness in a 3D model

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Abstract: In this work we consider a 3D isothermal mathematical model for ice sheet flows over a horizontal bedrock. The model is derived from the mechanics and dynamics of ice sheets and experimental results carried out in Glaciology. The final formulation of the model gives rise to a degenerate quasi-linear elliptic-parabolic equation for the ice-thickness function. Under appropriated initial and Dirichlet boundary conditions, we discuss the existence and uniqueness of weak solutions for this problem. Then, we prove that the speed of propagations of local disturbances from the initial ice-thickness is finite. We prove also that the solutions of this problem have the waiting-time local behavior.

Key-Words: ice sheet dynamics, existence, uniqueness, finite speed of propagations, waiting time.

1 Introduction

Mathematical models of ice sheet flows deal with the evolution equation for the ice-thickness function H :

$$\frac{\partial H}{\partial t} + \mathbf{u}_b \cdot \nabla H = \operatorname{div} \left(\int_b^h \mathcal{A}(\theta) (h-z)^{n+1} dz |\nabla h|^{n-1} \nabla h \right) + a. \quad (1)$$

In (1), $H = h - b$, where h and b are, respectively, the top surface and the bedrock of the ice sheet, \mathbf{u}_b is the sliding velocity, $\mathcal{A}(\theta) = 2(\rho g)^n A(\theta)$, where ρ is the constant density of the ice sheet, n is called Glen's exponent and $A(\theta)$ is a temperature-depending function (θ is the absolute temperature), and a is the accumulation/ablation rate function (see Sections 3 and 4).

In this work, we shall only consider isothermal motions, which causes, in (1), that \mathcal{A} does not depend on θ . This can be a consequence of approximately zero changes of temperature in the ice sheet, or more generally if, in the Arrhenius relationship (see (12) below), $|-Q/(k\theta)| \ll 1$ and $A_0 = 1$. The exothermic model shall be considered by the authors in a future work. Another simplification of the model, results from an usual assumption in ice sheet modeling, the bedrock b is a horizontal surface, *i.e.* $b = \text{constant}$. Under these assumptions, and after an integration procedure (see Fowler [11]), (1) becomes

$$\frac{\partial H}{\partial t} + \mathbf{u}_b \cdot \nabla H = \operatorname{div} \left(\frac{H^{n+2}}{n+2} |\nabla H|^{n-1} \nabla H \right) + a. \quad (2)$$

On the other hand, it should be pointed out that, as any measure, the ice-thickness must be non-negative. A different mathematical model was considered by the authors in [3, 5]. There, it was used the arguing of Fowler [12] to justify the replacement of the sliding velocity \mathbf{u}_b by $-\nabla H$.

When formulating mathematical models for the study of ice sheets, usually it is necessary to take into account that the flow domain is not prescribed and is itself part of the solution (see Calvo *et al.* [7] and Rodrigues and Santos [21]). However, in this work, we are mainly interested with the local properties of the ice-thickness function. Therefore we may assume the ice sheet based domain is known. We assume the ice sheet occupies a sufficiently large area where can possibly vanish the ice-thickness in some relatively small subareas. On the boundary of this large area, we assume the ice-thickness vanishes.

Most of the mathematical works in Theoretical Glaciology deal only with 2D models (see *e.g.* Calvo *et al.* [7], Díaz *et al.* [9] and the references cited therein). However, in this article, we shall consider the 3D model of (1), which causes to consider two spatial coordinates in the ice sheet based domain. Let us then consider the cylinder

$$Q_T := \Omega \times (0, T) \subset \mathbb{R}^2 \times \mathbb{R}^+,$$

whose boundary is defined by $\Gamma_T := \partial\Omega \times (0, T)$ and Ω is assumed to be a large enough open bounded domain with a sufficiently smooth boundary $\partial\Omega$.

The (strong) formulation of our problem can be stated in the following terms. Given an accumulation/ablation rate function $a = a(x, y, t)$ and a sliding velocity $\mathbf{u}_b = \mathbf{u}_b(x, y)$ defined in Q_T , and an initial ice-thickness $H_0 = H_0(x, y) \geq 0$, bounded and compactly supported in Ω , to find a sufficiently smooth function $H = H(x, y, t)$ defined in Q_T such that (2) is fulfilled in Q_T ,

$$H = H_0 \quad \text{in } \Omega \quad \text{for } t = 0, \quad (3)$$

$$H = 0 \quad \text{on } \Gamma_T. \quad (4)$$

The mathematical (strong) solutions of (2)-(4) must be physically admissible, *i.e.* they have to be non-negative compactly supported solutions. A sketch of this work containing some of the ideas and formulated in a different manner was published in the conference proceedings [4]. Calvo *et al.* [7] have consider a similar problem. They have considered a 2D model and there the ice sheet domain is itself part of the solution.

2 Mathematical framework

Notation. The notation used throughout this text is largely standard in Mathematical Fluid Mechanics - see, *e.g.*, Antontsev *et al.* [2]. We distinguish vectors from scalars by using boldface letters. For functions and function spaces we will use this distinction as well. Symbols C and K will denote generic positive constants, whose value will not be specified. It can change from one inequality to another and particularly when we use different subscripts. The dependence of C and K on other parameters will always be clear from the exposition and the same subscripted symbols used in different sections account for different constant values. In this article, the notation Ω stands always for a bounded 2D domain, *i.e.*, a connected open bounded subset of \mathbb{R}^2 , whose compact boundary is denoted by $\partial\Omega$.

Function spaces. Let $1 \leq p \leq \infty$. We shall use the classical Lebesgue spaces $L^p(\Omega)$, whose norm is denoted by $\|\cdot\|_{L^p(\Omega)}$. For any nonnegative integer k , $W^{k,p}(\Omega)$ denotes the Sobolev space and its norm is denoted by $\|\cdot\|_{W^{k,p}(\Omega)}$. By $W_0^{k,p}(\Omega)$ we denote the closure of $C_0^\infty(\Omega)$ in $W^{k,p}(\Omega)$, where $C_0^\infty(\Omega)$ denotes the set of all continuously differentiable functions with compact support in Ω . For $1 \leq p < \infty$ and any nonnegative integer k , the dual spaces of $L^p(\Omega)$ and $W_0^{k,p}(\Omega)$ are denoted, respectively, by $L^{p'}(\Omega)$ and $W^{-k,p'}(\Omega)$, where $1/p + 1/p' = 1$. Given $T > 0$ and a Banach space X , $L^p(0, T; X)$ and $W^{k,p}(0, T; X)$ denote the usual Lebesgue and Sobolev spaces used in evolutive problems, with norms denoted by $\|\cdot\|_{L^p(0,T;X)}$ and $\|\cdot\|_{W^{k,p}(0,T;X)}$. The corresponding spaces of vector-valued functions are denoted by boldface letters. All these spaces are Banach spaces and the Hilbert framework corresponds to $p = 2$. Finally, $C^1(0, T; C^1(\Omega))$ denotes the space of functions $\mathbf{u} : [0, T] \rightarrow C^1(\Omega)$ whose first derivatives are continuous. For a detailed exposition of these spaces and its properties, we address the reader, for instance, to Antontsev *et al.* [2] and especially to the monograph by Adams cited therein.

Auxiliary results. Throughout this text we will make

reference, at least once, to the following inequalities (see Antontsev *et al.* [2]):

(1) Algebraic inequalities - whenever it make sense, for every $\alpha, \beta, A, B \geq 0$,

$$A^\alpha B^\beta \leq (A+B)^{\alpha+\beta}, \quad (A+B)^\alpha \leq 2^\alpha (A^\alpha + B^\alpha); \quad (5)$$

(2) Algebraic equality - whenever it make sense, for every $A, B, C, \alpha, \beta, \xi \in \mathbb{R}$ and for every $\gamma \in [0, 1]$,

$$A^\alpha B^\beta + CB^\xi = A^\alpha B^{\gamma\beta} B^{(1-\gamma)\beta} + CB^{\alpha+\gamma\beta} B^{\xi-(\alpha+\gamma\beta)}; \quad (6)$$

(3) Young's inequality - for every $a, b \geq 0, \varepsilon > 0$ and $1 < p, q < \infty$ such that $1/p + 1/q = 1$,

$$ab \leq \varepsilon a^p + C(\varepsilon)b^q;$$

(4) Hölder's inequality - for every $u \in L^p(\Omega), v \in L^q(\Omega)$, with $1 \leq p, q \leq \infty$ such that $1/p + 1/q = 1$,

$$\int_{\Omega} u v \, d\mathbf{x} \leq \|u\|_{L^p(\Omega)} \|v\|_{L^q(\Omega)}.$$

For the main properties we shall establish in this article, play an important role the following result.

Lemma 1 Assume that $B_\rho \subset \mathbb{R}^2$ is an open ball with radius ρ and $0 \leq \sigma \leq q - 1 < \infty$. Then there exists a constant C depending on σ, q and B_ρ such that, for any $u \in W^{1,q}(B_\rho)$, we have

$$\|u\|_{L^q(\partial B_\rho)} \leq C \left(\|\nabla u\|_{L^q(B_\rho)} + \rho^{-\delta} \|u\|_{L^{\sigma+1}(B_\rho)} \right)^\theta \times \|u\|_{L^{\sigma+1}(B_\rho)}^{1-\theta}, \quad (7)$$

$$\theta = 1 - \frac{(q-1)(\sigma+1)}{2q + (q-2)(\sigma+1)}, \quad \delta = 1 + 2\frac{q - (\sigma+1)}{q(\sigma+1)}. \quad (8)$$

Lemma 1 is derived from the well-known trace-interpolation inequality (see Díaz and Veron [10]).

3 Mechanics of ice sheets

The common Fluid Mechanics model adopted for cold ice is a non-Newtonian, viscous, heat-conducting, incompressible fluid. But, it should be pointed out that, strictly speaking, it is not possible to assume ice to be incompressible and yet still presume density variations under phase changes. It is, however, justified to ignore density variations since associated changes in bulk density are very small. On the other hand, it is worth to know that ice sheets are assumed to be isotropic materials, but they can develop an induced anisotropy when stressed over sufficiently long time scales.

Governing equations. The model adopted for ice sheet flows results from the principles of conservation of mass and momentum:

$$\operatorname{div} \mathbf{u} = 0; \quad \mathbf{0} = \rho \mathbf{g} + \operatorname{div} \mathbf{T}. \quad (9)$$

Notice that in (9)₂ we have neglected the inertial terms because we are in the presence of very slow flows. The notation used in (9) is well known: \mathbf{u} is the velocity field, p is the pressure, ρ is the constant density, \mathbf{g} is the gravitational force and \mathbf{T} is the Cauchy stress tensor:

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S}; \quad (10)$$

\mathbf{I} is the unit tensor and \mathbf{S} is the deviatoric part of \mathbf{T} . Notice that from (9)₁, $\text{tr}(\mathbf{S}) = 0$.

Glen's law. Extra stress tensor \mathbf{S} and strain rate tensor \mathbf{D} are related by a rheological flow law. According to the common usage in Glaciology to write stretching as a function of stress, this law states that the strain rate \mathbf{D} , at a given strain, is proportional to the stress \mathbf{S} raised to the power n :

$$\mathbf{D} = A(\theta)\text{sgn}(\mathbf{S})|\mathbf{S}|^n, \quad \text{sgn}(\mathbf{S}) = |\mathbf{S}|^{-1}\mathbf{S}. \quad (11)$$

This law was suggested by J.W. Glen and, for this reason, is called Glen's law in Glaciology. The basic postulate is that ice is an incompressible nonlinear viscous fluid. Here n is a positive constant and the function A may depend on the temperature and usually is postulated an Arrhenius-type relationship

$$A(\theta) = A_0 \exp(-k\theta^{-1}Q), \quad (12)$$

where Q is the so-called activation energy, k the Boltzman constant, θ the absolute temperature and A_0 a constant.

4 Dynamics of ice sheets

A thorough analysis of ice sheet dynamics is made in many monographs, for instance, Hutter [14] and Paterson [19]. However, many authors deal only with 2D mathematical models, see *e.g.* Fowler [11]. Present-day 3D mathematical models including full thermo-mechanical coupling are those developed by Huybrechts [15], Greve [13] and Patyn [20], to name a few. The mathematical model approach is based on the continuum mechanics equations (9)-(11). We consider a Cartesian coordinate system (x, y, z) with the z -axis vertically pointing upward and being $z = 0$ at the mean sea level.

Field Equations. Denoting the velocity components in the correspondingly directions as (u, v, w) , (9)₁ can be rewritten as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \quad (13)$$

Once the gravitational force is only important in the vertical direction, *i.e.* considering $\mathbf{g} = (0, 0, -g)$, (9)₂ becomes

$$\frac{\partial T_{\zeta x}}{\partial x} + \frac{\partial T_{\zeta y}}{\partial y} + \frac{\partial T_{\zeta z}}{\partial z} = \delta_{\zeta z} \rho g, \quad \zeta = x, y, z, \quad (14)$$

where $\delta_{\zeta z}$ is the Kronecker delta and $T_{\zeta j}$ means stress in the ζ -plane ($\zeta = \text{constant}$) along j -direction.

Dynamic Boundary Condition. At the free surface, say $z = h(x, y, t)$, the model assumes that there is no applied traction, *i.e.*

$$\mathbf{T} \cdot \mathbf{n} = \mathbf{0} \quad \text{on} \quad z = h(x, y, t), \quad (15)$$

where \mathbf{n} is the exterior unit normal to the ice sheet top surface $z = h(x, y, t)$. If we write the free surface in the implicit form $s := z - h(x, y, t) = 0$, then $\mathbf{n} = |\nabla s|^{-1}\nabla s$. It is a matter of practical evidence that everywhere in an ice sheet the slopes of the free surface $z = h(x, y, t)$ are small, except in a small neighborhood of ice domes and ice margins. Thus the normal unit vector of the free surface $z = h(x, y, t)$ is approximately vertical and (15) reduces to

$$T_{xz} = 0, \quad T_{yz} = 0, \quad T_{zz} = 0 \quad \text{on} \quad z = h(x, y, t), \quad (16)$$

Hydrostatic Approximation. Applying the hydrostatic approximation in the vertical direction, *i.e.* $p_z = -\rho g$, then (14) _{$\zeta=z$} reduces to

$$\frac{\partial T_{zz}}{\partial z} = \rho g. \quad (17)$$

This means that, in all parts of an ice sheet, the shear stresses T_{xz} and T_{yz} are small compared to the vertical normal stress T_{zz} . Therefore the variational stress in the z -plane can be neglected. On the other hand, if we neglect atmospheric pressure, an integration of (17) from the surface $h(x, y, t)$ to a height z in the ice body and using (16), gives us an expression for the vertical normal stress

$$T_{zz} = \rho g(z - h). \quad (18)$$

From (10) and (18), the pressure reads as $p = \rho g(h - z) - S_{xx} - S_{yy}$ and the horizontal normal stresses can be expressed as

$$T_{xx} = 2S_{xx} + S_{yy} - \rho g(h - z), \quad (19)$$

$$T_{yy} = S_{xx} + 2S_{yy} - \rho g(h - z). \quad (20)$$

Inserting (19) and (20) in the horizontal components (x, y) of (14) _{$\zeta=x$} and (14) _{$\zeta=y$} , we achieve to

$$\frac{\partial}{\partial x}(2S_{xx} + S_{yy}) + \frac{\partial T_{xy}}{\partial y} + \frac{\partial T_{xz}}{\partial z} = \rho g \frac{\partial h}{\partial x}, \quad (21)$$

$$\frac{\partial}{\partial y}(S_{xx} + 2S_{yy}) + \frac{\partial T_{xy}}{\partial x} + \frac{\partial T_{yz}}{\partial z} = \rho g \frac{\partial h}{\partial y}. \quad (22)$$

Shallow-Ice Approximation. The major simplification of the model ensues by considering the shallow-ice approximation. This is justified, since we assume a physical process in which important length scales in the longitudinal directions are much larger, compared to those in the transverse directions. Consistently, $x, y \gg z$, and also $u, v \gg w$, and thus the dominant stresses are the shear stresses in the horizontal plane, S_{xz} and S_{yz} , which are supported by the basal drag. Moreover, normal stresses S_{xx} , S_{yy} , S_{zz} are negligible, as well the shear stress S_{xy} in the vertical planes. In consequence,

$$T_{xx} = T_{yy} = T_{zz} = -p. \quad (23)$$

and, from (15)-(18), the pressure is close to hydrostatic, *i.e.* $p = \rho g(h - z)$. Then the horizontal components of (21)-(22) simplify to

$$\frac{\partial T_{xz}}{\partial z} = \rho g \frac{\partial h}{\partial x}, \quad \frac{\partial T_{yz}}{\partial z} = \rho g \frac{\partial h}{\partial y}. \quad (24)$$

On the free surface $z = h(x, y, t)$ we obtain, after using (16) and (23),

$$T_{xz} = 0, \quad T_{yz} = 0, \quad p = 0 \quad \text{on } z = h(x, y, t). \quad (25)$$

Then a vertical integration of (24) from $h(x, y, t)$ to a height z in the ice body, and using (25), leads us to

$$T_{xz} = -\rho g(h - z) \frac{\partial h}{\partial x}, \quad T_{yz} = -\rho g(h - z) \frac{\partial h}{\partial y}. \quad (26)$$

From (11), strain rates are related with deviatoric stresses by

$$\mathbf{D} = A(\theta) \tau^{n-1} \mathbf{S}, \quad \tau = \sqrt{II_{\mathbf{S}}}, \quad (27)$$

where $II_{\mathbf{S}}$ denotes the second invariant of \mathbf{S} . Notice that (9)₁ implies $\tau = \sqrt{\text{tr}(\mathbf{S}^2)/2}$ and from the simplifications of the shallow ice approximation, especially (26),

$$\tau = \sqrt{T_{xz}^2 + T_{yz}^2} = \rho g(h - z) |\nabla h|. \quad (28)$$

A common assumption in ice sheet modeling, and which is valid for most of the ice sheet domain, is that horizontal gradients of the vertical velocity are small compared to the vertical gradient of the horizontal velocity, *i.e.* $w_x \ll u_z$ and $w_y \ll v_z$. Using this assumption, (26) and (28), we obtain from (27)

$$\frac{\partial u}{\partial z} = -2A(\theta) [\rho g(h - z)]^n |\nabla h|^{n-1} \frac{\partial h}{\partial x}, \quad (29)$$

$$\frac{\partial v}{\partial z} = -2A(\theta) [\rho g(h - z)]^n |\nabla h|^{n-1} \frac{\partial h}{\partial y}. \quad (30)$$

Integrating (29) and (30) from the ice base, say $z = b(x, y, t)$, to an arbitrary point z in the ice sheet, we obtain

$$u = u_b - 2(\rho g)^n |\nabla h|^{n-1} \frac{\partial h}{\partial x} \int_b^z A(\theta)(h - s)^n ds, \quad (31)$$

$$v = v_b - 2(\rho g)^n |\nabla h|^{n-1} \frac{\partial h}{\partial y} \int_b^z A(\theta)(h - s)^n ds, \quad (32)$$

where $\mathbf{u}_b = (u_b, v_b)$ is the ice velocity at the ice base.

Kinematic Boundary Conditions. For this model, the possible presence of attached ice shelves will be ignored. Let \mathbf{u} and \mathbf{w} denote, respectively, the ice surface velocity and the velocity at which the free surface points move. Then $\mathbf{w} \cdot \mathbf{n}$ represents the normal speed of propagation of the free surface and

$$a_h = (\mathbf{w} - \mathbf{u}) \cdot \mathbf{n} \quad (33)$$

is the ice volume flux through the free surface, also known as the accumulation/ablation function. The sign is chosen such that a supply (accumulation) is counted as positive and a loss (ablation) as negative. Then the time derivative of s (implicit form of the free

surface) following the motion of the free surface with velocity \mathbf{w} must vanish and, by using (33), we obtain

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} + v \frac{\partial h}{\partial y} - w = a_h N_h, \quad N_h = \sqrt{h_x^2 + h_y^2 + 1}. \quad (34)$$

A similar boundary condition can be derived for the ice base. We proceed as above, considering the implicit form of the ice base $r := b(x, y, t) - z = 0$, its exterior unit normal is given by $\mathbf{n} = |\nabla r|^{-1} \nabla r$ and the ice volume flux through the ice base is given by

$$a_b = (\mathbf{w} - \mathbf{u}) \cdot \mathbf{n}. \quad (35)$$

Now \mathbf{w} is the velocity at which the ice base points move and $\mathbf{w} \cdot \mathbf{n}$ represents the normal speed of propagation of the ice base. Arguing as before and using (35), we obtain

$$\frac{\partial b}{\partial t} + u \frac{\partial b}{\partial x} + v \frac{\partial b}{\partial y} - w = a_b N_b, \quad N_b = \sqrt{b_x^2 + b_y^2 + 1}. \quad (36)$$

In both cases, free surface and ice base, their interior sides are identified with the ice and therefore the exterior sides are identified with the atmosphere and the lithosphere, respectively. Provided that accumulation/ablation functions (33) and (35) are given, equations (34) and (36) govern the evolution of the free surface and ice base, respectively.

Ice-Thickness Equation. We integrate (13) along the vertical from the ice base $z = b(x, y, t)$ to the free surface $z = h(x, y, t)$ and we use (34) and (36) to obtain

$$\frac{\partial}{\partial x} \int_b^h u dz + \frac{\partial}{\partial y} \int_b^h v dz + \frac{\partial h}{\partial t} - N_h a_h - \frac{\partial b}{\partial t} + N_b a_b = 0. \quad (37)$$

Replacing, in (37), u and v by its expressions (31) and (32), we obtain, after an integration by parts, the evolution equation for the ice sheet thickness (see (1)), where $\mathbf{u}_b = (u_b, v_b)$ is the sliding velocity, $A(\theta) = 2(\rho g)^n A(\theta)$ and $a = a_h - a_b$ is the accumulation/ablation rate. Such as for the free surface, the slopes of the ice base $z = b(x, y, t)$ are small. Then the exterior normal vectors to $z = h(x, y, t)$ and to $z = b(x, y, t)$ are approximately vertical and this justifies why we have taken $N_h = N_b = 1$ in (1).

5 Existence and uniqueness

General formulation. In order to obtain a more general framework than (2)-(4), let us introduce the new functions $\nu = \nu(x, y, t)$ and $b = b(s)$ defined by

$$\nu := H^m = \psi(H) \implies \psi^{-1}(\nu) = \nu^{\frac{1}{m}} := b(\nu), \quad (38)$$

where $m = 2(n + 1)/n$. Notice that the new variable $\nu := H^m$ is motivated by the relation

$$\frac{H^{n+2}}{n+2} |\nabla H|^{n-1} \nabla H = \frac{m^{1-p}}{n+2} |\nabla H^m|^{p-2} \nabla H^m,$$

with $p = n + 1$. Let us assume that:

$$a \in L^\infty(\Omega); \quad \nu_0 \in L^\infty(\Omega); \quad (39)$$

$$\operatorname{div} \mathbf{u}_b = 0 \quad \text{in } Q_T; \quad \mathbf{u}_b \in \mathbf{L}^\infty(Q_T). \quad (40)$$

Notice that, according to (38), condition (39)₂ is equivalent to assume that $H_0 \in L^\infty(\Omega)$. Then the general formulation of (2)-(4) can be stated in terms of ν and b as follows. Given Ω , a constant $k = m^{1-p}/(n+2)$ and a, ν_0 and \mathbf{u}_b satisfying (39)-(40), to find a function ν defined by (38) and solution of

$$\frac{\partial b(\nu)}{\partial t} = \operatorname{div} (k|\nabla \nu|^{p-2} \nabla \nu - \mathbf{u}_b b(\nu)) + a, \quad (41)$$

$$b(\nu) = b(\nu_0) \quad \text{in } \Omega \quad \text{for } t = 0, \quad (42)$$

$$\nu = 0 \quad \text{on } \Gamma_T. \quad (43)$$

It is worth to notice that, according to (38), ν and H have the same support and both are zero on the boundary Γ_T . Moreover, if H is a solution of (2)-(4) then ν is a solution of (41)-(43) and reciprocally. The general formulation (41)-(43) is the one used to establish existence and uniqueness of solutions (see Calvo *et al.* [7]) and goes back to mathematical works on quasi-linear elliptic-parabolic differential equations (see Alt and Luckhaus [1], Otto [18], Benilan and Wittbold [6], Carrillo and Wittbold [8], Ivanov and Rodrigues [16]), being our problem a particular case.

Weak formulation We start this section by introducing the notion of solutions to the problem (41)-(43) we shall work with in the sequel. We multiply (41) by a test function ζ and integrate by parts over Q_T to obtain

$$\begin{aligned} & \int_{Q_T} \left(b(\nu) \frac{\partial \zeta}{\partial t} + a \zeta \right) dz + \int_{\Omega} b(\nu_0) \zeta_0 dx \\ & = \int_{Q_T} (k|\nabla \nu|^{p-2} \nabla \nu - b(\nu) \mathbf{u}_b) \cdot \nabla \zeta dz, \end{aligned} \quad (44)$$

where $\zeta_0 = \zeta(\cdot, 0)$ and where we have set $\mathbf{x} = (x, y)$ and $\mathbf{z} = (\mathbf{x}, t)$. Then the definition of weak solution follows as usual (see Alt and Luckhaus [1]).

Definition 2 . Let (39)-(40) be fulfilled. A function ν is a weak solution of the problem (41)-(43), if: **1**) $\nu \geq 0$ a.e. in Q_T and $\nu \in L^p(0, T; W_0^{1,p}(\Omega))$; **2**) $b(\nu) \in L^\infty(0, T; L^1(\Omega))$, $b(\nu)_t \in L^{p'}(0, T; W^{1,-p}(\Omega))$; **3**) The relation (44) holds for every $\zeta \in L^p(0, T; W_0^{1,p}(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega))$ such that $\zeta(\cdot, T) = 0$.

There are now many existence and uniqueness results which can be applied directly to the problem (41)-(43) (Alt and Luckhaus [1], Otto [18], Benilan and Wittbold [6], Ivanov and Rodrigues [16], Carrillo and Wittbold [8], to name a few). One of the first references to appear was the paper by Alt and Luckhaus [1], where is proved (Theorem 1.7) the existence of a weak solution to a general

problem which includes the case of Definition 2. There, the existence result is proved for any $u_0 = b(\nu_0)$ with $B(\nu_0) \in L^1(\Omega)$ (see (48) below) and $a \in L^{p'}(0, T; W^{-1,p'}(\Omega))$. In order to apply Alt and Luckhaus [1, Theorem 1.7] to the problem (41)-(43), let us define the following functions

$$\mathcal{B} : \mathbb{R} \rightarrow \mathbb{R}, \quad \mathcal{B}(u) = u^{\frac{1}{m}}, \quad (45)$$

$$\mathcal{A} : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2, \quad \mathcal{A}(\mathbf{v}, u) = k|\mathbf{v}|^{p-2} \mathbf{v} - u \mathbf{u}_b, \quad (46)$$

where m, k and $p = n + 1$ are constants and \mathbf{u}_b is a given vector - the sliding velocity at the ice base. One can easily see that (45) is a nondecreasing continuous function in \mathbb{R} such that $\mathcal{B}(0) = 0$ and (46) is a vector-valued continuous function in $\mathbb{R}^2 \times \mathbb{R}$ such that the growth condition

$$|\mathcal{A}(\nabla \nu, \mathcal{B}(\nu))|^{p'} \leq C_1 (1 + |\nabla \nu|^p + B(\mathcal{B}(\nu))) \quad (47)$$

hold. In (47), $C_1 = \text{const.} \geq 0$ and $B(\mathcal{B}(\nu))$ is the Legendre transform of the primitive of $\mathcal{B}(\nu)$, i.e.

$$B(\mathcal{B}(\nu)) := \int_0^\nu s d\mathcal{B}(s). \quad (48)$$

It should be noticed that B is super-linear in the sense that for any $\delta > 0$, there exists a $C(\delta) < \infty$, such that for all $u \in \mathbb{R}$, $|u| \leq \delta B(u) + C(\delta)$. From this property of B it is a easy task to prove (47). The proof of the strict monotonicity condition

$$(\mathcal{A}(\mathbf{v}, u) - \mathcal{A}(\mathbf{w}, u)) \cdot (\mathbf{v} - \mathbf{w}) \geq C_2 |\mathbf{v} - \mathbf{w}|^p, \quad (49)$$

$C_2 = \text{const.} > 0$, is more involved. In fact, after some algebraic manipulations, we can prove successively

$$\begin{aligned} & k^{-1} (\mathcal{A}(\mathbf{v}, u) - \mathcal{A}(\mathbf{w}, u)) \cdot (\mathbf{v} - \mathbf{w}) \\ & = |\mathbf{v}|^p + |\mathbf{w}|^p - (|\mathbf{v}|^{p-2} - |\mathbf{w}|^{p-2}) \cdot \mathbf{v} \cdot \mathbf{w} \\ & = (|\mathbf{v}|^{p-2} + |\mathbf{w}|^{p-2}) |\mathbf{v} - \mathbf{w}|^2 + |\mathbf{v}|^{p-1} |\mathbf{w}| + |\mathbf{v}| |\mathbf{w}|^{p-1} \\ & \geq C |\mathbf{v} - \mathbf{w}|^p, \quad C = C(p), \quad p \geq 2. \end{aligned}$$

For our purposes, it is enough to consider $p \geq 2$, because $p = n + 1$ and, as we shall see in Section 7, $n \geq 1, 9$. Anyway, an extension of the result presented in Alt and Luckhaus [1] to the case $1 < p < 2$ is given by Ivanov and Rodrigues [16]. Moreover, Alt and Luckhaus [1] have shown that the natural energy associated to a weak solution ν of the problem (41)-(43) is given by the finite sum

$$\sup_{t \in (0, T)} \int_{\Omega} B(\mathcal{B}(\nu(\cdot, t))) dx + \int_{Q_T} |\nabla \nu|^p dz < \infty, \quad (50)$$

where $B(\mathcal{B}(\nu(\cdot, t)))$ is defined in (48). Benilan and Wittbold [6] under rather general assumptions than Alt and Luckhaus [1], and using the nonlinear semigroup theory, have proved the existence of mild solutions, which under certain conditions were shown to be weak solutions. Uniqueness of weak solutions of (41)-(43) is a much more difficult task because of the

nonlinear term $b(\nu)$. The usual approach consists in to prove the L^1 -contraction principle

$$\int_{\Omega} |\mathcal{B}(\nu_1(\cdot, t) - \mathcal{B}(\nu_2(\cdot, t))| dx \leq e^{Lt} \int_{\Omega} |\mathcal{B}(\nu_1(\cdot, 0)) - \mathcal{B}(\nu_2(\cdot, 0))| dx \quad (51)$$

for any two weak solutions ν_1 and ν_2 satisfying (50) - L is the assumed Lipschitz constant of a . Under the additional continuity property

$$|\mathcal{A}(\mathbf{v}, u) - \mathcal{A}(\mathbf{v}, z)|^{p'} \leq C(1 + B(u) + B(z) + |\mathbf{v}|^p)|u - z|, \quad (52)$$

Alt and Luckhaus [1, Theorem 2.3] also have proved the uniqueness of a weak solution ν provided

$$\frac{\partial \nu}{\partial t} \in L^1(Q_T). \quad (53)$$

It is a easy task to prove that (46) satisfies (52). Later, Otto [18], by using Kruzhkov method of doubling variables both in space and time, have proved (51), and consequently the uniqueness result, for ν_i , $i = 1, 2$, satisfying (50) without assuming (53). Carrillo and Wittbold [8] have generalized the uniqueness result of Otto [18] and have proved a comparison result by using also Kruzhkov method.

6 Qualitative properties

Existence and uniqueness of a weak solution $\nu = H^m$ to the equivalent problem (41)-(43) have been established in the previous section. According to (38), those results allow us to state the existence and uniqueness of a weak solution H for (2)-(4) and such that: 1) $H \geq 0$ a.e. in Q_T and $H^m \in L^p(0, T; W_0^{1,p}(\Omega))$; 2) $H \in L^\infty(0, T; L^1(\Omega))$ and $H_t \in L^{p'}(0, T; W^{1,-p}(\Omega))$; 3) for every $\zeta \in L^{n+1}(0, T; W_0^{1,n+1}(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega))$, with $\zeta(\cdot, T) = 0$, the equivalent of (44) holds:

$$\int_{Q_T} \left(H \frac{\partial \zeta}{\partial t} + a\zeta \right) dz + \int_{\Omega} H_0 \zeta_0 dx = \int_{Q_T} (k|\nabla H^m|^{p-2} \nabla H^m - H \mathbf{u}_b) \cdot \nabla \zeta dz.$$

It should be stressed that to guarantee $H \geq 0$, we need to assume $a \geq 0$. On the other hand, from (50) and by using (48), one can easily proves that the energy associated with the problem (2)-(4) is finite

$$E(Q_T) := \sup_{t \in [0, T]} \int_{\Omega} |H(\cdot, t)|^{m+1} dx + \int_{Q_T} |\nabla H^m|^p dz < \infty, \quad (54)$$

where we already have seen that

$$p = n + 1, \quad m = \frac{2(n+1)}{n}. \quad (55)$$

In order to establish the qualitative properties, let us fix \mathbf{x}_0 in Ω , $\rho_0 \in (0, \text{dist}(\mathbf{x}_0, \partial\Omega))$ and assume that

$$H_0(\mathbf{x}) = 0 \text{ in } B_{\rho_0}(\mathbf{x}_0) = \{\mathbf{x} \in \Omega : |\mathbf{x} - \mathbf{x}_0| < \rho_0\} \subset \Omega. \quad (56)$$

In this section, we shall assume that

$$\text{div } \mathbf{u}_b = 0 \text{ in } Q_T, \quad \mathbf{u}_b \in C^1(0, T; C^1(\Omega)). \quad (57)$$

To proceed our study, let us consider the Lagrange variables \mathbf{X} defined as usual in Continuum Mechanics (see, e.g., Meirmanov *et al.* [17]):

$$\frac{d\mathbf{X}(\mathbf{x}, t)}{dt} = \mathbf{u}_b(\mathbf{X}, t), \quad t \in (0, T); \quad \mathbf{X}(\mathbf{x}, 0) = \mathbf{x}, \quad \mathbf{x} \in \Omega. \quad (58)$$

Under conditions expressed in (57), there exists a unique solution $\mathbf{X}(\mathbf{x}, t)$ of the problem (58), which is a homeomorphism between Ω and $\Omega^t = \{\mathbf{y} : \mathbf{z} = \mathbf{X}(\mathbf{x}, t), \mathbf{x} \in \Omega\}$ for any $t \in [0, T]$. This solution transforms the ball $B_{\rho}(\mathbf{x}_0)$ into

$$B_{\rho}^t(\mathbf{x}_0) = \{\mathbf{z} : \mathbf{y} = \mathbf{X}(\mathbf{x}, t), \text{ for some } \mathbf{x} \in B_{\rho}(\mathbf{x}_0)\}.$$

Moreover, the following formulas hold

$$\frac{d}{dt} \int_{B_{\rho}^t(\mathbf{x}_0)} \Phi dz = \int_{B_{\rho}^t(\mathbf{x}_0)} \left(\frac{\partial \Phi}{\partial t} + \mathbf{u}_b \nabla \Phi \right) dz, \quad (59)$$

$$\frac{dJ}{dt} = J \text{div } \mathbf{u}_b, \quad J = \det \left(\frac{\partial \mathbf{X}(\mathbf{x}, t)}{\partial \mathbf{x}} \right), \quad J(\mathbf{x}, 0) = 1. \quad (60)$$

In the considered case ($\text{div } \mathbf{u}_b = 0$), we have that $J(\mathbf{x}, 0) = J(\mathbf{x}, t) = 1$. In order to simplify the expressions, let us introduce the energy functions

$$E(\rho, s) := \int_0^s \int_{B_{\rho}^t(\mathbf{x}_0)} |\nabla H^m|^p dz dt \quad (61)$$

and

$$B(\rho, s) := \int_{B_{\rho}^s(\mathbf{x}_0)} H^{m+1} dz. \quad (62)$$

6.1 Finite speed of propagations

Theorem 3 *Let H be a weak solution to the problem (2)-(4) with $a = 0$. Assume \mathbf{u}_b satisfies (57) and (54) is finite. If (56) is verified, then there exists t^* , $0 < t^* < T$, such that*

$$H(\mathbf{x}, t) = 0 \quad \text{a.e. in } B_{\rho(t)}(\mathbf{x}_0) \quad \forall t \in [0, t^*],$$

where

$$\rho^\nu(t) = \rho_0^\nu - \frac{\nu}{C\tau} t^\lambda E(\rho_0, t)^\tau,$$

for some positive constants τ , λ , ν and C .

PROOF. We formally multiply (2) (with $a = 0$) by H , a weak solution of (2)-(4) and integrate by parts over $B_{\rho}^t(\mathbf{x}_0) \times (0, s)$, with $s \leq t \leq T$. To be precise, we should multiply (2) by a regularized H function, with compact support in Ω , and then pass to the limit in the obtained integral inequality (see Díaz & Veron [10, Lemma 2.1]). Using (57)₁ and the notations introduced in (61)-(62), we obtain the following energy relation

$$\frac{1}{m+1} B(\rho, s) + kE(\rho, s) \leq \frac{1}{m+1} B(\rho, 0) + kI(\rho, s), \quad (63)$$

where $k = m^{1-p}/(n+2)$,

$$I(\rho, s) := \int_0^s \int_{S_{\rho}^t(\mathbf{x}_0)} |\nabla H^m|^{p-2} \nabla H^m \cdot \mathbf{n} H^m dS,$$

$S_\rho^t(\mathbf{x}_0)$ is the boundary of $B_\rho^t(\mathbf{x}_0)$, i.e. $S_\rho^t(\mathbf{x}_0) = \partial B_\rho^t(\mathbf{x}_0)$, and \mathbf{n} is the unit exterior normal to $S_\rho^t(\mathbf{x}_0)$. Taking the supreme for $s \in [0, t]$ in (63), we achieve to

$$\sup_{0 \leq s \leq t} B(\rho, s) + C_1 E(\rho, t) \leq B(\rho, 0) + C_1 I(\rho, t), \quad (64)$$

where $C_1 = (m+1)k$. Noticing that

$$\frac{\partial E(\rho, t)}{\partial \rho} = \int_0^t \int_{S_\rho^t(\mathbf{x}_0)} |\nabla H^m|^p dS,$$

we obtain, by using Hölder's inequality,

$$|I(\rho, t)| \leq (J(\rho, t))^{\frac{1}{p}} \left(\frac{\partial E(\rho, t)}{\partial \rho} \right)^{\frac{p-1}{p}}, \quad (65)$$

$$J(\rho, t) := \int_0^t J(\rho) dt, \quad J(\rho) := \int_{S_\rho^t} H^{mp} dS.$$

Now, we shall estimate $J(\rho, t)$ in terms of $E(\rho, t)$ and $B(\rho, t)$ by using Lemma 1. We apply (7) with $q = p$ and $\sigma = m$ to obtain

$$J(\rho) \leq C^p \left(\|\nabla H^m\|_{L^p(B_\rho)} + \rho^{-\delta} \|H\|_{L^{m+1}(B_\rho)}^m \right)^{p\theta} \times \|H\|_{L^{m+1}(B_\rho)}^{mp(1-\theta)}, \quad (66)$$

where, from (8) and (55),

$$\theta = 1 - \frac{n(3n+2)}{5n^2+n-2}, \quad \delta = 1 + 2 \frac{n^2-2n-2}{(3n+2)(n+1)}. \quad (67)$$

The analysis of (67), taking into account that Glen's exponent n is positive, shows us that

$$\theta > 0 \text{ iff } n > (1 + \sqrt{17})/4 \text{ and } \delta > 0 \text{ for any } n > 0. \quad (68)$$

Integrating (66) over the interval $(0, t)$ and then using Hölder's inequality and the algebraic inequality (5)₂,

$$J(\rho, t) \leq C_2 \left(E(\rho, t) + \rho^{-\delta p} \int_0^t B(\rho, s)^{\frac{pm}{m+1}} ds \right)^\theta \times \left(\int_0^t B(\rho, s)^{\frac{pm}{m+1}} ds \right)^{1-\theta}, \quad C_2 = (2^\theta C)^p. \quad (69)$$

We use

$$\int_0^t B(\rho, s)^{\frac{pm}{m+1}} ds \leq t \sup_{0 \leq s \leq t} B(\rho, s)^{\frac{pm}{m+1}}$$

and the algebraic inequality (5)₂ in (69), which yield

$$J(\rho, t)^{\frac{1}{p}} \leq C_3 t^{(1-\theta)\frac{1}{p}} K(\rho, t)^\theta, \quad C_3 = (2C_2)^{1/p},$$

$$K(\rho, t) := E(\rho, t)^{\frac{1}{p}} \left(\sup_{0 \leq s \leq t} B(\rho, s) \right)^{\frac{m}{m+1} \frac{1-\theta}{\theta}} + \rho^{-\delta} t^{\frac{1}{p}} \left(\sup_{0 \leq s \leq t} B(\rho, s) \right)^{\frac{m}{m+1} \frac{1}{\theta}}. \quad (70)$$

Then, we use the algebraic equality (6) on the term $K(\rho, t)$, and we obtain for any $\gamma \in [0, 1]$

$$K(\rho, t) = E(\rho, t)^{\frac{1}{p}} \left(\sup_{0 \leq s \leq t} B(\rho, s) \right)^{\gamma b} \left(\sup_{0 \leq s \leq t} B(\rho, s) \right)^{(1-\gamma)b}$$

$$+ \rho^{-\delta} t^{\frac{1}{p}} \left(\sup_{0 \leq s \leq t} B(\rho, s) \right)^{\frac{1}{p} + \gamma b} \times \quad (71)$$

$$\left(\sup_{0 \leq s \leq t} B(\rho, s) \right)^{\frac{m}{m+1} \frac{1}{\theta} - \left(\frac{1}{p} + \gamma b \right)}, \quad b = \frac{m}{m+1} \frac{1-\theta}{\theta}.$$

The following obvious relations $E(\rho, t) \geq 0$, $t \leq T$, $\rho \leq \rho_0$ and (68)₂ imply $B(\rho, t) \leq B(\rho_0, t)$ and $\rho^\delta \leq \rho_0^\delta$, which together with the algebraic inequality (5)₁, allow us to estimate (71) in the form

$$K(\rho, t) \leq C_4 \rho^{-\delta} \left(E(\rho, t) + \sup_{0 \leq s \leq t} B(\rho, s) \right)^\alpha, \quad (72)$$

where

$$C_4 = \max \{1, \rho_0^\delta\} \max \{1, T^{\frac{1}{p}}\} \times \max \left\{ \left(\sup_{0 \leq t \leq T} B(\rho_0, t) \right)^\alpha, \left(\sup_{0 \leq t \leq T} B(\rho_0, t) \right)^\beta \right\}, \quad (73)$$

$$\alpha = \frac{1}{p} + \gamma \frac{m}{m+1} \frac{1-\theta}{\theta}, \quad \beta = \frac{m}{m+1} \frac{1}{\theta} - \left(\frac{1}{p} + \gamma \frac{m}{m+1} \frac{1-\theta}{\theta} \right).$$

Applying successively (72) in (70) and the resulting inequality in (65), we prove that

$$|I(\rho, t)| \leq C_5 \rho^{-\delta\theta} t^{\frac{1-\theta}{p}} \times \left(E(\rho, t) + \sup_{0 \leq s \leq t} B(\rho, s) \right)^{\alpha\theta} \left(\frac{\partial E(\rho, t)}{\partial \rho} \right)^{\frac{p-1}{p}}, \quad (74)$$

where $C_5 = C_3 C_4^\theta$ and α is given by (73)₂. Now, we shall use (74) in (64), but first we notice that, if $\rho \leq \rho_0$, we are inside the ball B_{ρ_0} and, from (56), $B(\rho, 0) = 0$. In consequence, we obtain

$$\sup_{0 \leq s \leq t} B(\rho, s) + E(\rho, t) \leq C_6 \rho^{-\delta\theta} t^{\frac{1-\theta}{p}} \times \left(\sup_{0 \leq s \leq t} B(\rho, s) + E(\rho, t) \right)^\eta \left(\frac{\partial E(\rho, t)}{\partial \rho} \right)^{\frac{p-1}{p}}, \quad (75)$$

where $C_6 = C_1 C_5 / \min \{1, C_1\}$ and

$$\eta = \alpha\theta = \frac{\theta}{p} + \gamma \frac{m}{m+1} (1-\theta). \quad (76)$$

Rising both sides of (75) to the power $p/(p-1)$, we achieve to

$$\left(\sup_{0 \leq s \leq t} B(\rho, s) + E(\rho, t) \right)^{1-\tau} \leq C_7 \rho^{1-\nu} t^\lambda \frac{\partial E(\rho, t)}{\partial \rho}, \quad (77)$$

where $C_7 = C_6^{\frac{p}{p-1}}$ and

$$\tau = \frac{1}{p-1} \left(\gamma p \frac{m}{m+1} - 1 \right) (1-\theta),$$

$$\nu = 1 + \delta\theta \frac{p}{p-1}, \quad \lambda = \frac{1-\theta}{p-1}. \quad (78)$$

Once that $B(\rho, s) \geq 0$, we obtain, from (77), the following ordinary differential inequality for the energy function E with respect to the variable ρ :

$$E(\rho, t)^{\tau-1} \frac{\partial E(\rho, t)}{\partial \rho} \geq C_8 \rho^{\nu-1} t^{-\lambda}, \quad C_8 = C_7^{-1}. \quad (79)$$

Notice that in (79) the variable t is considered as a parameter. Integrating (79) between ρ and $\rho_0 \geq \rho$, we get

$$E(\rho, t)^\tau \leq E(\rho_0, t)^\tau - C_8 \frac{\tau}{\nu} t^{-\lambda} (\rho_0^\nu - \rho^\nu).$$

In consequence, for positive constants τ, ν and λ (see (78)), $E(\rho, t) = 0$ if and only if

$$\rho^\nu \leq \rho_0^\nu(t) := \rho_0^\nu - \frac{\nu}{C_8 \tau} t^\lambda E(\rho_0, t)^\tau. \blacksquare$$

6.2 Waiting time property

Theorem 4 *Let H be a weak solution to the problem (2)-(4) with $a = 0$. Assume \mathbf{u}_b satisfies (57) and (54) is finite. Additionally to (56), assume that the following condition holds*

$$\int_{B_\rho(\mathbf{x}_0)} |H_0|^{m+1} d\mathbf{x} \leq \varepsilon (\rho - \rho_0)^\mu, \quad (80)$$

for some $\rho > \rho_0$, $\mu = \mu(n) > 0$, $\varepsilon > 0$. Then, there exist t^* , $0 < t^* < T$, and $\varepsilon^* > 0$, $0 < \varepsilon \leq \varepsilon^*$ such that $H(\mathbf{x}, t) = 0$ a.e. in $B_{\rho_0}(\mathbf{x}_0)$ for all $t \in [0, t^*]$.

PROOF. We proceed as we did in the proof of Theorem 3 (see (64)) and we obtain, after using the assumption (80),

$$B(\rho, s) + K_1 E(\rho, s) \leq \varepsilon (\rho - \rho_0)^\mu + K_1 I(\rho, s), \quad (81)$$

where $K_1 = (m+1)k$ and ε is a positive constant to be defined later on. Using the same reasoning as we did to prove (74), we obtain

$$\begin{aligned} |I(\rho, t)| &\leq K_2 t^{\frac{1-\theta}{p}} \left(E(\rho, t) + \sup_{0 \leq s \leq t} B(\rho, s) \right)^\eta \\ &\times \left(\frac{\partial E(\rho, t)}{\partial \rho} \right)^{\frac{p-1}{p}}, \quad K_2 = C_3 C_4^\theta, \end{aligned} \quad (82)$$

where η is given in (76), C_3 in (70) and C_4 has the form given in (72) but with the term $\max\{1, \rho_0^\delta\}$ replaced by $\max\{1, \rho_0^{-\delta}\}$ - notice that, in this case, $\rho_0 \leq \rho$. Gathering (81) and (82), we obtain

$$\begin{aligned} K_3 \left(E(\rho, t) + \sup_{0 \leq s \leq t} B(\rho, s) \right) &\leq \varepsilon (\rho - \rho_0)^\mu + \\ K_2 t^{\frac{1-\theta}{p}} \left(E(\rho, t) + \sup_{0 \leq s \leq t} B(\rho, s) \right)^\eta &\left(\frac{\partial E(\rho, t)}{\partial \rho} \right)^{\frac{p-1}{p}}, \end{aligned} \quad (83)$$

where $K_3 = \min\{1, K_1\}$. Using Young's inequality with $\epsilon = K_3/(2K_2)$ in the second term of the right-hand side of (83) and knowing that $B(\rho, s) \geq 0$, we obtain

$$E(\rho, t) \leq \varepsilon K_4 (\rho - \rho_0)^\mu + K_5 \left(t^{\frac{1-\theta}{p-1}} \frac{\partial E(\rho, t)}{\partial \rho} \right)^{\frac{1}{\xi}}, \quad (84)$$

where $K_4 = 2K_3^{-1}$, $K_5 = 2K_2 K_3^{-1} C(\epsilon)$ and

$$\xi = \frac{p}{p-1} (1-\eta) = 1 + \left(1 - \gamma p \frac{m}{m+1} \right) \frac{1-\theta}{p-1}. \quad (85)$$

Rising both sides of (84) to the power ξ and defining

$$\mu := \frac{1}{1-\xi} = \frac{(p-1)(m+1)}{[\gamma p m - (m+1)](1-\theta)}, \quad (86)$$

we obtain the ordinary differential inequality in the variable ρ

$$E(\rho, t)^\xi \leq \varepsilon K_6 (\rho - \rho_0)^{\frac{\xi}{1-\xi}} + K_7 \frac{\partial E(\rho, t)}{\partial \rho}, \quad (87)$$

where $K_6 = (2\varepsilon K_4)^\xi$ and $K_7 = (2K_5)^\xi t^{\frac{1-\theta}{p-1}}$. The analysis of (87) shows us that its solutions are of the form $E(\rho, t) = K(\rho - \rho_0)^{1/(1-\xi)}$, where the constant K should satisfy to

$$\left(K^\xi - K_6 - \frac{K_7}{1-\xi} K \right) (\rho - \rho_0)^{\frac{\xi}{1-\xi}} \leq 0.$$

Finally, once that $\rho_0 \leq \rho$, we obtain $E(\rho, t) = 0$ if and only if $K^\xi - K_6 - K_7/(1-\xi)K \geq 0$. Writing this inequality in terms of t and ε , we obtain $E(\rho, t) = 0$ for all t such that

$$t \leq \left(\frac{K^\xi - (2\varepsilon K_4)^\xi}{(2K_5)^\xi K} (1-\xi) \right)^{\frac{p-1}{1-\theta}} := t^*,$$

provided $\mu > 0$. Finally, ε is chosen such that $\varepsilon \leq K/(2K_4) := \varepsilon^*$. \blacksquare

Remark 5 *The results of Theorem 3 and 4 are still valid for a global non-zero accumulation/ablation rate function a . Indeed, finite speed of propagations property (Theorem 3) holds, provided we assume $a = 0$ in $B_{\rho_0}(\mathbf{x}_0) \times [0, t^*]$. As for the waiting-time property (Theorem 4), it holds if we assume $a = 0$ in $B_{\rho_0}(\mathbf{x}_0) \times [0, T]$.*

7 Conclusions

Ice sheets are vast and slow-moving edifices of solid ice, which flow under their own weight by solid state creep processes such as the creep of dislocation in the crystalline lattice structure of the ice. Experimental results have showed different creep curves (graph of strain versus time) when a polycrystalline aggregate of ice is subjected to a constant stress. An elastic deformation is followed by a small period of transient or primary creep in which the strain rate decreases continuously until a minimum value. Then the secondary creep is reached and it remains for a long period as the strain rate is approximately constant. After that, the strain rate increases and tertiary creep is reached. If the test is carried on for long enough, a steady value is reached. In good approximation, we can assume that in deforming ice masses like ice sheets, secondary creep prevails for low temperatures (below -10°C), whereas tertiary creep prevails for higher temperatures. Numerous laboratory experiments have shown

that, for secondary creep, the relation between shear strain rate and shear stress is given by Glen's law (11), where n varies from about 1.9 to 4.8. However, most of evidence from either laboratory tests and deformation measurements in Antarctic and Greenland have showed that $n = 3$ is more appropriated (see Hutter [14] and Paterson [19]). We see that our results agree with these experiments. Indeed the obtained properties for the ice-thickness function hold provided the constant parameters τ , λ , ν and μ are positive. Finite speed of propagations (FSP) of local disturbances from the initial (zero) ice thickness (see Theorem 3) is obtained on the basis that τ , λ and ν are positive. The analysis of (78), taking into account (55) and (67), and once that Glen's exponent n is positive, leads us to: $\tau > 0$ iff $\gamma > (3n + 2)/[2(n + 1)^2]$ and $\theta < 1$; $\nu > 0$ for any $n > 0$; $\lambda > 0$ iff $\theta < 1$. From (67)₁, we see that $\theta < 1$ for any $n > 0$. Then γ must be chosen in (71) such that $(3n + 2)/[2(n + 1)^2] < \gamma \leq 1$, and this is possible because $0 < (3n + 2)/[2(n + 1)^2] < 1$ for any $n > 0$. In consequence the only requirement for the FSP is that $n > (1 + \sqrt{17})/4 \simeq 1.3$ (see (68)₁). On the other hand, there is a positive waiting time (WT) until which the ice-thickness remains locally equal to zero, provided the global initial ice thickness is suitably small and the parameter μ is positive (see Theorem 4). The analysis of (86) leads us to say that, such as for the FSP, the only requirement for the WT is that $n > (1 + \sqrt{17})/4 \simeq 1.3$. Finally, since the ice-thickness is written as $H = h - b$, we can analyze the properties FSP and WT in terms of the ice top surface $z = h$. As discussed in Fowler [11, 12] the slope of the surface $z = h$ is singular in advance but finite in retreat. This distinction causes the finite speed of propagations and the waiting time behaviors. Indeed, after a local retreat, the margin slope must rebuilt itself before another ice advance it is possible.

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