Localized solutions of anisotropic parabolic equations

S. Antontsev
CMAF, Universidade de Lisboa, Portugal
anton@ptmat.fc.ul.pt, antontsevsn@mail.ru

S. Shmarev
Departamento de Matemáticas
Universidad de Oviedo, España
shmarev@orion.ciencias.uniovi.es

May 15, 2008

Abstract

We study the localization properties of solutions of the Dirichlet problem for the anisotropic parabolic equations

$$u_t - \sum_{i=1}^{n} D_i \left(a_i(z,u)|D_i u|^{p_i-2} D_i u\right) = f(z), \quad z = (x,t),$$

with constant exponents $p_i \in (1,\infty)$. Such equations arise from the mathematical description of diffusion processes. It is shown that if the equation combines the directions of slow diffusion for which $p_i > 2$ and the directions of fast or linear diffusion corresponding to $p_i \in (1,2)$ or $p = 2$, then the solutions may simultaneously display the properties intrinsic for the solutions of isotropic equations of fast or slow diffusion. Under the assumptions that $f \equiv 0$ for $t \geq t_f$ and $u_0 \equiv 0$, $f \equiv 0$ for $x_1 > s$ we show, on one hand, that the solution instincts in a finite time if $1 < \sum\limits_{i=1}^{n} \frac{1}{p_i} \leq 1 + \frac{n}{2}$ and, on the other hand, that the support of the same solution never reaches the plane $x_1 = s + \epsilon$, provided that $\frac{1}{n-1} \geq \frac{1}{n-1} \sum\limits_{i=2}^{n} \frac{1}{p_i} > \frac{1}{p_1}$.

1 Introduction

We study the Dirichlet problem for the degenerate anisotropic parabolic equation

$$\begin{cases} u_t - \displaystyle\sum_{i=1}^{n} D_i \left(a_i(z,u)|D_i u|^{p_i-2} D_i u \right) = f(z) \quad \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \qquad u(x,0) = u_0(x) \text{ in } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a domain with Lipschitz-continuous boundary Γ, $\Gamma_T = \Gamma \times (0,T)$, $Q_T = \Omega \times (0,T)$, $z = (x,t)$,

$$p_i \in (1,\infty), \quad i = 1,\dots,n. \tag{1.2}$$

Equation (1.1) includes as a partial case the evolutional p–Laplacian

$$u_t = \Delta_p u \equiv \operatorname{div}\left(|\nabla u|^{p-2} \nabla u \right), \quad p \in (1,\infty), \tag{1.3}$$

which appears in various physical contexts. In particular, this equation arises from the mathematical description of the diffusion processes. The ranges of the exponent $p \in (1,2)$ and $p > 2$ correspond to *fast* or *slow* diffusion. It is well-known that the solutions of the nonlinear equation (1.3) with $p \neq 2$ possess some properties not displayed by the solutions of the linear diffusion equation with $p = 2$. The solutions of the linear equation obey the strong maximum principle which prevents them from attaining the maximum and minimum values in the interior of the cylinder Q_T. As distinguished from this property, for $p \neq 2$ the solutions of the Dirichlet problem for equation (1.3) are localized either in space, or in time. More precisely, the following alternative holds: if u be a solution of the Dirichlet problem (1.1) for equation (1.3), then either

$$2 > p > 1 \text{ (fast diffusion)} \;\Rightarrow\; \exists\, T_1 : \; u \equiv 0 \text{ for all } t \geq T_1,$$

or

$$\left. \begin{array}{l} p > 2 \text{ (slow diffusion)} \\ u_0 \equiv 0 \text{ in } B_r(x_0) \subset \Omega \end{array} \right\} \;\Rightarrow\; \exists\, t_*(x_0) : \; u(x_0,t) \equiv 0 \text{ for } t \in [0, t_*(x_0)],$$

where $B_s(x_0) = \{x \in \mathbb{R}^n : |x - x_0| < s\}$. These properties complement each other: the former is called *extinction in a finite time*, the latter is usually referred to as *finite speed of propagation* of disturbances from the data. If $p > 2$ and the support of the initial function u_0 is compact in Ω, then the support of the solution is expanding with time and eventually covers the whole of Ω.

Surprisingly, the localization properties intrinsic for the solutions of *anisotropic* equations of the type (1.1) conspicuously contrast to the properties typical for the solutions of equation of isotropic diffusion. For the solutions of anisotropic equations the alternative "finite speed of propagation"/"vanishing in a finite time" is no longer valid and is replaced by conditions which are similar to those known for the solutions of the diffusion-absorption equations. Roughly speaking, in the equation of anisotropic diffusion the possibility of time–space localization depends not on the values of the exponents p_i but on certain relations between them.

The aim of this note to show that if equation (1.1) combines the directions of slow diffusion ($p_i > 2$) with the directions of fast diffusion ($p_i \in (1,2)$) or linear diffusion ($p = 2$), then the solutions may be simultaneously localized both in time and space.

To describe the results we need several definitions. Let us make the agreement to say that

(a) the solution of problem (1.1) is localized in the direction x_i if

$$\exists\, s > 0 : \ u(x,t) \equiv 0 \text{ in } Q_T \cap \{x_i > s\} \ ;$$

(b) the solution vanishes in a finite time if

$$\exists\, t_* > 0 : \ u(x,t) \equiv 0 \text{ in } \Omega \text{ for } t \geq t_*;$$

(c) the solution possesses the waiting time property on the plane $x_i = s$ if

$$\begin{aligned} &u_0(x) \equiv 0 \text{ in } \Omega \cap \{x_i > s\} \\ &\qquad \Rightarrow\ u(x,t) \equiv 0 \text{ in } \Omega \cap \{x_i > s\} \text{ for } t \in [0, t_*]. \end{aligned}$$

Let us illustrate the results by the example of the equation with two independent space variables:

$$\begin{cases} u_t = \left(|u_x|^{p-2} u_x\right)_x + \left(|u_y|^{q-2} u_y\right)_y \quad \text{in } Q_T, \\ u = 0 \text{ on } \Gamma_T, \\ u(x,0) = u_0(x) \text{ in } \Omega = (0,a) \times (0,a). \end{cases} \tag{1.4}$$

Let us assume that $u_0(x,y) \equiv 0$ in $\Omega \cap \{x > x_0\}$. Then every (weak) solution of this problem

(a) is localized in the direction x if

$$1 < q < p;$$

(b) vanishes in a finite time if

$$\frac{1}{p} + \frac{1}{q} > 1;$$

(c) has the infinite waiting time on the plane $\{x = x_0\}$ if $1 < q < p$ and

$$F(x) \equiv \int_0^a |u_0(x, y)|^2 \, dy \leq \epsilon \max \left\{0, \, (x_0 - x)^\beta \right\} \quad \text{for } x < x_0$$

with a suitably small constant $\epsilon > 0$ and $\beta \equiv \beta(p, q) > 1$;

(d) in the borderline case

$$\frac{1}{p} + \frac{1}{q} = 1$$

the solution is exponentially decreasing as $t \to \infty$:

$$\|u(\cdot, t)\|^2_{2,\Omega} \leq \mathrm{e}^{-C\,t} \|u_0\|^2_{2,\Omega}, \quad C = const > 0.$$

It is easy to see that for certain p and q conditions (a), (b) are fulfilled simultaneously. In the case $n \geq 3$ the relations between the exponents p_i that allow the simultaneous presence of these localization effects become more complicated and depend on the space dimension.

It is to be specially noted that property (a) of directional localization is stronger than the property of finite speed of propagation which is typical for the solutions of (1.3) with $p > 2$. For the solutions of equation of isotropic diffusion directional localization is impossible if the equation does not contain the terms that model strong absorption or convection. We refer to [1, Ch. 2-3] for more information on the localization properties of solutions of such equations.

2 Previous work. Preliminaries.

The questions of existence, uniqueness and boundedness of solutions to anisotropic parabolic equations of the type (1.1) have been studied by many authors under various conditions on the data and with different methods - see, e.g., [5, 7, 8, 9, 10, 12, 13] and the further references therein. Anisotropic parabolic equations nonlinear with respect to the solution were studied in [14, 15]. In the present paper we are concerned with the so–called energy solutions [5] of problem (1.1).

The properties of time and space localization of solutions of anisotropic equations were studied thus far in papers [4, 6] but only in the situations when the diffusion was either fast, or slow in every space direction. The possibility of directional localization of solutions to anisotropic parabolic equations is prompted by the fact that anisotropic elliptic equations admit energy solutions that possess the same property - see [2, 3]. Examples of equations whose solutions display the property of directional localization were presented in [1, Ch. 3].

Let us introduce the Banach spaces

$$
\begin{aligned}
&\mathbf{V}(\Omega) = \left\{ u |\, u \in \bigcap_{i=1}^{n} L^{p_i}(\Omega)),\ |D_i u|^{p_i} \in L^1(\Omega),\ u = 0 \text{ on } \Gamma \right\},\\
&\mathbf{W}(Q_T) = \left\{ u |\, u \in L^{\infty}(0, T;\, L^2(\Omega)),\ |D_i u|^{p_i} \in L^1(Q_T),\ u = 0 \text{ on } \Gamma_T \right\},\\
&\|u\|_{\mathbf{W}(Q_T)} = \|u\|_{L^{\infty}(0,T;\, L^2(\Omega))} + \sum_{i=1}^{n} \|D_i u\|_{p_i, Q_T}
\end{aligned}
$$

and denote by $\mathbf{W}'(Q_T)$ the dual of $\mathbf{W}(Q_T)$ with respect to the inner product in $L^2(Q_T)$.

Definition 2.1 *A function $u \in \mathbf{W}(Q_T)$ is called weak (energy) solution of problem* (1.1) *if $u_t \in \mathbf{W}'(Q_T)$ and for every $t_1, t_2 \in [0, T]$ and every test–function $\phi \in \mathbf{W}(Q_T)$ such that $\phi_t \in \mathbf{W}'(Q_T)$*

$$
\begin{aligned}
&\int_{\Omega} u\phi \, dx \Big|_{t=t_1}^{t=t_2} \\
&\quad - \int_{Q_{t_2} \setminus Q_{t_1}} \left(u\, \phi_t - \sum_{i=1}^{n} a_i |D_i u|^{p_i - 2} D_i u\, D_i \phi - f\, \phi \right) dxdt = 0.
\end{aligned}
\tag{2.1}
$$

In the simplest case when a_i are positive constants, solvability of problem (1.1) follows from [11, Ch. 2]. We will assume that the coefficients a_i are functions satisfying the conditions

$$\left\{\begin{array}{l} a_i(z,s) \text{ are Carathéodory functions:} \\ \qquad a_i(z,s) \text{ are continuous in } s \in \mathbb{R}\ \forall\, z \in Q_T, \\ \qquad a_i(z,s) \text{ are measurable in } z \in Q_T\ \forall\, s \in \mathbb{R}, \\ a_0 \le a_i(z,s) \le A_0, \qquad a_0, A_0 = const > 0. \end{array}\right. \tag{2.2}$$

Theorem 2.1 ([5]) *Let conditions* (1.2) *and* (2.2) *be fulfilled. For every* $u_0 \in L^2(\Omega)$ *and* $f \in L^2(Q_T)$ *problem* (1.1) *has a unique energy solution* $u \in \mathbf{W}(Q_T)$ *such that* $u_t \in \mathbf{W}'(Q_T)$*, and this solution satisfies the estimate*

$$\|u\|_{\mathbf{W}(Q_T)} \le C\,\left\{\|u_0\|_{2,\Omega} + \|f\|_{2,Q_T}\right\}, \qquad C \equiv C(n, p_i, |\Omega|).$$

In what follows, we rely on the embedding theorem for the functions from Sobolev's anisotropic spaces $\mathbf{V}(\Omega)$.

Theorem 2.2 ([16]) *Let* $\Omega = \{x \in \mathbb{R}^n :\ x_i \in (0,a)\}$*. For every* $v(x) \in \mathbf{V}(\Omega)$

$$\|v\|_{r,\Omega} \le C(a,n)\left(\prod_{i=1}^{n} \|D_i v\|_{p_i,\Omega}\right)^{\frac{1}{n}},$$

with

$$r = \begin{cases} p^* = \dfrac{n\,q}{n-q} & \text{if } q < n, \\ \text{any number from } [1,\infty) & \text{if } q \ge n, \end{cases} \qquad \frac{1}{q} = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{p_i}.$$

The conclusions about localization of solutions to problem (1.1) follow from the study of the properties of the "energy functions" associated with the solution under study. These functions satisfy nonlinear ordinary differential inequalities which depend on the nonlinear structure of the equation.

3 The energy relation

In this section we derive the "integration-by-parts" formula which is the framework of the further study of the local energy functions associated with the solution $u(x,t) \in \mathbf{W}(Q_T)$. Let us introduce the notations

$$\begin{aligned}
&\Omega(s) = \Omega \cap \{x_1 > s\}, \quad Q(s,t) = \Omega(s) \times (0,t), \\
&\omega(s) = \Omega \cap \{x_1 = s\}, \quad x' = (x_2, \ldots, x_n), \qquad x = (x_1, x').
\end{aligned}$$

Lemma 3.1 *Let the conditions of Theorem* 2.1 *be fulfilled. For every solution* $u \in \mathbf{W}(Q_T)$ *of problem* (1.1) *and a.e.* $s \in (0,a)$ *such that* $\Omega \cap \{x_1 = s\} \neq \emptyset$

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega(s)} u^2(x,\tau)dx\Big|_{\tau=0}^{\tau=t} + \sum_{i=1}^{n}\int_{Q(s,t)} a_i(z,u)|D_i u|^{p_i}\,dxdt \\
&\quad + \int_0^t dt\int_{\omega(s)} a_1(z,u)\,u\,|D_1 u|^{p_1-2}D_1 u\,dx' = \int_{Q(s,t)} u\,f\,dxdt.
\end{aligned} \tag{3.1}$$

Proof. The energy solution can be taken for test–function in identity (2.1). Let us set

$$\phi_k(x_1,x',s) = \begin{cases} 1 & \text{for } x_1 > s + \frac{1}{k}, \\ k(x_1 - s) & \text{for } x_1 \in \left[s, s+\frac{1}{k}\right], \\ 0 & \text{for } x_1 < s, \qquad k \in \mathbb{N}, \end{cases}$$

and choose $u(x,t)\phi_k(x_1,x',s)$ for the test-function in the integral identity (2.1). The resulting identity has the form

$$\begin{aligned}
\sum_{j=1}^{4} I_j(k,s) \equiv &\sum_{i=1}^{n}\int_{Q(s,t)} a_i(z,u)|D_i u|^{p_i}\,\phi_k\,dxdt \\
&+ k\int_0^t dt\int_{\Omega(s+1/k)\setminus\Omega(s)} a_1(z,u)\,u\,|D_1 u|^{p_1-2}D_1 u\,dx \\
&+ \frac{1}{2}\int_{\Omega(s)} \phi_k u^2\,dx\Big|_{\tau=0}^{\tau=t} - \int_{Q(s,t)} u\,f\,\phi_k\,dxdt = 0.
\end{aligned} \tag{3.2}$$

By the definition of $\mathbf{W}(Q_T)$

$$|D_i u|^{p_i}\,\phi_k,\; u\phi_k \in L^1(Q_T), \qquad u^2\phi_k \in L^1(\Omega) \quad \text{for a.e. } t\in(0,T),$$

which allows us to pass to the limit as $k\to\infty$ in I_1, I_3 and I_4:

$$\begin{aligned}
\lim_{k\to\infty} I_1 &= \sum_{i=1}^{n}\int_{Q(s,t)} a_i(z,u)\,|D_i u|^{p_i}\,dxdt,\\
\lim_{k\to\infty} I_3 &= \frac{1}{2}\int_{\Omega(s)} u^2\,dx - \frac{1}{2}\int_{\Omega(s)} u_0^2\,dx,\\
\lim_{k\to\infty} I_4 &= \int_{Q(s,t)} u\,f\,dxdt.
\end{aligned}$$

By virtue of (3.2) I_2 is bounded uniformly with respect to k, provided that so are the integrals I_1, I_3 and I_4. Writing I_2 in the form

$$I_2 = k\int_s^{s+1/k}\left(\int_0^t\left(\int_{\omega(s)} a_i(z,u)\,u\,|D_1u|^{p_1-2}D_1u\,dx'\right)dt\right)dx_1$$

and applying the Lebesgue theorem we conclude that there exists

$$\lim_{k\to\infty} I_2(k,s) = \int_0^t dt\int_{\omega(s)} a_1(z,u)\,u\,|D_1u|^{p_1-2}D_1u\,dx'.$$

Equality (3.2) transforms into (3.1) as $k\to\infty$. $\square$

Remark 3.1 *Let*

$$\Omega\subset\Omega_a\equiv\{x\in\mathbb{R}^n:\,x_i\in(0,a),\,i=1,\dots,n\}$$

and let $u\in\mathbf{W}(Q_T)$ *be a solution of problem* (1.1). *Set*

$$u^*(x,t)=\begin{cases} u(x,t) & \text{in } Q_T,\\ 0 & \text{in } Q_T^{(a)}\equiv(\Omega_a\setminus\Omega)\times(0,T).\end{cases}$$

The function $u^*(x,t)$ *belongs to* $\mathbf{W}(Q_T^{(a)})$ *and formally satisfies identity* (3.1) *in the cylinder* $Q_T^{(a)}$. *If* $u^*(x,t)$ *is localized in time or space, so is the function* $u(x,t)$, *which is why in what follows we study the localization properties of the energy solutions of problem* (1.1) *formulated in the domain* $\Omega\equiv\{x\in\mathbb{R}^n:\,x_i\in(0,a)\}$.

4 Large time behavior and vanishing in a finite time

Let $u(z) \in \mathbf{W}(Q_T)$ be a solution of problem (1.1). Letting in (3.1) $s = 0$, we find that for every t, $t + \Delta\, t \in [0, T]$ the solution satisfies the identity

$$\begin{aligned} &\frac{1}{2} \int_\Omega u^2(z) \Big|_{\tau=t}^{\tau=t+\Delta\, t} dx \\ &\quad = - \int_t^{t+\Delta\, t} \int_\Omega \sum_i a_i |D_i u|^{p_i} \, dz + \int_t^{t+\Delta\, t} \int_\Omega f\, u \, dz. \end{aligned} \tag{4.1}$$

4.1 Differential inequality for the energy function

Let us introduce the energy functions

$$\begin{aligned} \Theta(t) &= \int_\Omega u^2(z)\, dx, \\ \Lambda(t) &= \int_\Omega \sum_i |D_i u(z)|^{p_i} \, dx, \quad z = (x,t) \in Q_T. \end{aligned}$$

Proposition 4.1 *Let $n = 2$ and $p_1 > 1$, $p_2 > 1$, and let $\Omega = (0, a) \times (0, a)$. For every $u \in \mathbf{W}(Q_T)$ and a.e. $t \in (0, T)$*

$$\Theta(t) \le C\, \Lambda^{\frac{1}{p_1} + \frac{1}{p_2}}(t) \tag{4.2}$$

with a constant $C \equiv C(a, p_1, p_2)$.

Proof. Since the set $C^1(0, T;\, C_0^1(Q_T))$ is dense in $\mathbf{W}(Q_T)$, it is sufficient to prove (4.2) for $u \in C^1(0, T;\, C_0^1(Q_T))$. For every $x = (x_1, x_2) \in \Omega$

$$\begin{aligned} u^2(x,t) &\le \max_{x_1} |u(x,t)| \cdot \max_{x_2} |u(x,t)| \\ &\le \left(\int_0^a |u_{x_2}(x,t)| \, dx_2 \right) \left(\int_0^a |u_{x_1}(x,t)| \, dx_1 \right) \\ &\le a^{2 - \frac{1}{p_1} - \frac{1}{p_2}} \left(\int_0^a |u_{x_2}|^{p_2} \, dx_2 \right)^{\frac{1}{p_2}} \left(\int_0^a |u_{x_1}|^{p_1} \, dx_1 \right)^{\frac{1}{p_1}} \\ &\equiv C(a)\, f(x_1) \cdot g(x_2). \end{aligned}$$

Integrating over Ω and applying Hölder's inequality we arrive at the required estimate:

$$\begin{aligned}\Theta(t) &\equiv \int_\Omega u^2\,dx \le C(a)\int_\Omega f(x_1)\cdot g(x_2)\,dx\\ &= C(a)\left(\int_0^a f(x_1)\,dx_1\right)\cdot\left(\int_0^a g(x_2)\,dx_2\right)\\ &\le a^{2\left(1-\frac{1}{p_1}-\frac{1}{p_2}\right)}\left(\int_\Omega |u_{x_1}|^{p_1}\,dx\right)^{\frac{1}{p_1}}\left(\int_\Omega |u_{x_2}|^{p_2}\,dx\right)^{\frac{1}{p_2}}.\end{aligned}$$

□

Proposition 4.2 *Let $n \ge 3$, $p_i > 1$, and let $\Omega \equiv \{x \in \mathbb{R}^n : x_i \in (0,a)\}$. Assume that*

$$1 < \sum_{i=1}^n \frac{1}{p_i} \le 1 + \frac{n}{2}.$$

Then for every $u(z) \in \mathbf{W}(Q_T)$ and for a.e. $t \in (0,T)$

$$\Theta^\nu(t) \le C\,\Lambda(t) \quad \textit{with} \quad \frac{1}{\nu} = \frac{2}{n}\sum_{i=1}^n \frac{1}{p_i} \in \left(\frac{2}{n},\, 1+\frac{2}{n}\right). \tag{4.3}$$

Proof. By Theorem 2.2, for a.e. $t \in (0,T)$

$$\Theta(t) \equiv \|u\|^2_{2,\Omega}(t) \le C\left(\prod_{i=1}^n \|D_i u\|_{p_i,\Omega}\right)^{\frac{2}{n}} \le C\,(\Lambda(t))^{\frac{2}{n}\sum_{i=1}^n \frac{1}{p_i}} \equiv C\,\Lambda^{\frac{1}{\nu}}(t),$$

provided that

$$2 \le \frac{2n\nu}{n-2\nu} \quad \text{and} \quad n > 2\,\nu.$$

□

Lemma 4.1 *Let the conditions of Proposition 4.1 or 4.2 be fulfilled. For every solution $u \in \mathbf{W}(Q_T)$ of problem (1.1) the function $\Theta(t)$ satisfies for a.e. $t \in (0,T)$ the ordinary differential inequality*

$$\Theta'(t) + C_1\,\Theta^{\nu}(t) \leq C_2\,\|f(\cdot,t)\|_{2,\Omega}^{\frac{2\nu}{2\nu-1}} \tag{4.4}$$

with the exponent

$$\frac{1}{\nu} = \frac{2}{n}\sum_{i=1}^{n}\frac{1}{p_i} \in \left(\frac{2}{n},\, 1+\frac{2}{n}\right)$$

and absolute constants $C_i \equiv C_i(a, p_i)$*.*

Proof. Since

$$\int_\Omega a_i(z,u)|D_i u|^{p_i}\,dx,\ \int_\Omega u\,f\,dx \in L^1(0,T),$$

then for a.e. $t \in (0,T)$ every term on the right-hand side of (4.1) has a limit as $\Delta\,t \to 0$. It follows that there exists a limit of the left-hand side as $\Delta t \to 0$, whence

$$\frac{1}{2}\Theta'(t) + \sum_{i=1}^{n}\int_\Omega a_i(z,u)|D_i u|^{p_i}\,dx = \int_\Omega f\,u\,dx$$

and

$$\Theta'(t) + a_0\,\Lambda(t) \leq 2\int_\Omega f\,u\,dx. \tag{4.5}$$

Plugging (4.3), we obtain

$$\Theta'(t) + K\,\Theta^{\nu}(t) \leq 2\int_\Omega f\,u\,dx, \quad K = const. \tag{4.6}$$

Finally, since $2\nu > 1$ we may apply Hölder's and Young's inequalities to estimate:

$$\left|\int_\Omega u\,f\,dx\right| \leq \sqrt{\Theta(t)}\ \|f(\cdot,t)\|_{2,\Omega} \leq \frac{K}{4}\Theta^{\nu}(t) + C(K,p)\,\|f(\cdot,t)\|_{2,\Omega}^{\frac{2\nu}{2\nu-1}}.$$

$\square$

4.2 Vanishing in a finite time

Theorem 4.1 *Let $u \in \mathbf{W}(Q_T)$ be a solution of problem* (1.1). *Assume that*

$$\frac{1}{\nu} = \frac{2}{n}\sum_{i=1}^{n}\frac{1}{p_i} \in (1,2).$$

(a) If $f \equiv 0$, then

$$u(x,t) \equiv 0 \text{ in } \Omega \text{ for all } t \geq t_* = \left(\frac{1}{C_1(1-\nu)}\|u_0\|^2_{2,\Omega}\right)^{\frac{1}{1-\nu}}$$

with the constant C_1 from (4.4).

(b) If $f \equiv 0$ for $t \geq t_f$, then there exists $t_* \geq t_f$ such that

$$u(x,t) \equiv 0 \quad \text{in } \Omega \text{ for all } t \geq t_*.$$

(c) If $f \equiv 0$ for $t \geq t_f$, the integral

$$K(t_f) = \int_0^{t_f} \frac{F(\tau)}{(1-\tau/t_f)^{\frac{1}{1-\nu}}}\,d\tau, \quad F(t) \equiv C_2\|f(\cdot,t)\|_{2,\Omega}^{\frac{2\nu}{2\nu-1}},$$

is convergent, and if

$$G(t_f) \equiv \|u_0\|^2_{2,\Omega} - [C_1(1-\nu)t_f]^{\frac{1}{1-\nu}} + K(t_f) \leq 0,$$

then

$$u(x,t) \equiv 0 \text{ in } \Omega \text{ for all } t \geq t_f.$$

Proof. (a) The energy function Θ satisfies the ordinary differential inequality (4.4):

$$\forall \text{ a.e. } t \in (0,T) \qquad \Theta'(t) + C_1\Theta^{\nu}(t) \leq 0, \qquad \Theta(0) = \|u_0\|^2_{2,\Omega}.$$

The straightforward integration gives

$$\Theta(t) = \left(\|u_0\|^{1-\nu}_{2,\Omega} - C_1(1-\nu)\,t\right)^{\frac{1}{1-\nu}}$$

and the conclusion follows because $\Theta(t) \geq 0$.

(b) It follows from Theorem 2.1 that

$$\Theta(t_f) \leq \Theta(0) + \|f\|^2_{2,\Omega\times(0,t_f)} \equiv \delta.$$

The conclusion follows now if we apply assertion (a) to the energy function $\Theta(t)$ on the interval (t_f, t) where $f \equiv 0$.

(c) Let $S(t)$ be the nonnegative solution of the problem

$$S'(t) + C_1\, S^{\nu}(t) = 0, \qquad S(0) = S_0 > 0.$$

The function $S(t)$ is given by the explicit formula

$$S(t) = \max\left\{0,\ \left(S(0)^{1-\nu} - C_1(1-\nu)\, t\right)^{\frac{1}{1-\nu}}\right\}.$$

Let us fix the initial value $S(0)$ by the condition

$$t_f = \frac{S^{1-\nu}(0)}{C_1(1-\nu)},$$

i.e., $S(t_f) = 0$. Let us consider the function $W(t) \equiv \Theta(t) - S(t)$ which satisfies on the interval $(0, t_f)$ the inequality

$$W'(t) + C_1\nu \int_0^1 \frac{d\lambda}{(\lambda\Theta(t) + (1-\lambda)S(t))^{1-\nu}}\, W(t) \leq F(t).$$

Multiplying this inequality by

$$\exp\left(C_1\nu \int_0^t \int_0^1 \frac{d\lambda d\tau}{(\lambda\Theta(\tau) + (1-\lambda)S(\tau))^{1-\nu}}\right)$$

and then integrating over the interval $(0, t_f)$, we transform it to the form

$$\begin{aligned}\Theta(t_f) &\leq \Theta(0) - S(0) \\ &\quad + \int_0^{t_f} F(\tau) \exp\left(C_1\nu \int_0^{\tau} \int_0^1 \frac{d\lambda\, dz}{(\lambda\Theta(z) + (1-\lambda)S(z))^{1-\nu}}\right) d\tau.\end{aligned}$$

Let us notice that by virtue of the equation for $S(t)$

$$\begin{aligned}&C_1\nu \int_0^{\tau} \int_0^1 \frac{d\lambda\, dz}{(\lambda\Theta(z) + (1-\lambda)S(z))^{1-\nu}} \\ &\qquad \leq C_1 \int_0^{\tau} \frac{dz}{S^{1-\nu}(z)} \int_0^1 \frac{\nu\, d\lambda}{\lambda^{1-\nu}} = \int_0^{\tau} \frac{S'(z)}{S(z)}\, dz = \ln\frac{S(\tau)}{S(0)}.\end{aligned}$$

It follows that

$$\Theta(t_f) \leq \Theta(0) - S(0)\left(1 - \int_0^{t_f} \frac{F(\tau)}{S(\tau)} d\tau\right) \equiv G(t_f) \leq 0$$

with $G(t_f)$ defined in the conditions of Theorem 4.1. Since $\Theta(t) \geq 0$ for all $t > 0$, it is necessary that $\Theta(t_f) = 0$. Considering now the differential inequality for $\Theta(t)$ on the interval (t_f, T),

$$\begin{cases} \Theta'(t) + C_1\,\Theta^{\nu}(t) \leq 0 \quad \text{for } t \geq t_f, \\ \Theta(t_f) = 0, \qquad \Theta(t) \geq 0, \end{cases}$$

we conclude that these conditions are satisfied only if $\Theta(t) \equiv 0$ for $t \geq t_f$, that is, if $\|u\|_{2,\Omega}(t) \equiv 0$. □

4.3 Large time behavior

According to Theorem 4.1, the solution vanishes at a finite moment t_*, provided that $\nu < 1$ and $\|f\|_{2,\Omega}(t) \equiv 0$ from some t_f on. Let us now study the behavior of $\|u\|_{2,\Omega}(t)$ in the cases when these conditions are not fulfilled.

Theorem 4.2 *Let*

$$M = \|u_0\|^2_{2,\Omega} + \int_0^{\infty} \|f\|^2_{2,\Omega}(t)\,dt < \infty. \tag{4.7}$$

(a) If $\nu \leq 1$, then

$$\|u\|^2_{2,\Omega}(t) \leq \mathrm{e}^{-C_3 t}\left(\|u_0\|^2_{2,\Omega} + C_2 \int_0^t \mathrm{e}^{C_3\tau} \|f(\cdot,t)\|_{2,\Omega}^{\frac{2\nu}{2\nu-1}}\,d\tau\right)$$

with the constant $C_3 = C_1 M^{\nu-1}$.

(b) If $\nu > 1$ and

$$\|f\|^2_{2,\Omega}(t) \leq f_0(1+t)^{-\frac{\nu}{\nu-1}}$$

from some t_0 on with a positive constant f_0. Then there exists $C =$ const such that

$$\|u\|^2_{2,\Omega}(t) \leq C(1+t)^{-\frac{\nu}{\nu-1}} \quad \textit{for } t \geq t_0.$$

Remark 4.1 *Gathering assertions (a) and (b), we have that for $f \equiv 0$ for $t \geq t_*$ and $\nu = 1$*

$$\|u\|_{2,\Omega}^2 \leq \mathrm{e}^{-C\,t} \|u_0\|_{2,\Omega}^2 \quad \textit{as } t \to \infty.$$

Proof. (a) Using (4.7) we write inequality (4.4) in the form

$$\Theta'(t) + C_1 M^{\nu-1} \Theta(t) \leq \Theta'(t) + C_1 M^{\nu} \left(\frac{\Theta(t)}{M} \right)^{\nu} \leq F(t)$$

with $F(t) = C_2 \|f(\cdot, t)\|_{2,\Omega}^2$, and the conclusion follows by Gronwall's inequality.

(b) Without loss of generality we may assume that $t_0 = 0$. The energy function $\Theta(t)$ satisfies the differential inequality

$$\Theta'(t) + C_1 \Theta^{\nu}(t) \leq f_0 (1+t)^{-\frac{\nu}{\nu-1}} \quad \text{for } t \geq t_0.$$

Let $Y(t)$ be a solution of the equation

$$Y'(t) + C_1 Y^{\nu}(t) = f_* (1+t)^{-\frac{\nu}{\nu-1}}$$

with the constant $f_* > 0$ to be defined. A solution of this equation is given by the explicit formula

$$Y(t) = A(1+t)^{-\frac{1}{\nu-1}}$$

with the parameter A chosen from the condition

$$G(A) \equiv -\frac{A}{\nu-1} + A^{\nu} - f_* = 0.$$

This algebraic equation always has a solution $A^* > 0$ because $G(0) = -f_* < 0$, $G(\infty) = \infty$. Moreover, the solution A^* is estimated from below

$$A^* = \left[\frac{1}{C_1} \left(\frac{A^*}{\nu-1} + f_* \right) \right]^{\frac{1}{\nu}} \geq \left[\frac{1}{C_1} f_* \right]^{\frac{1}{\nu}}.$$

If we claim that $f_* \geq \max(f_0, C_1 \Theta^{\nu}(0))$, then

$$\Theta(0) \leq Y(0) = A^* \quad \text{and} \quad f_* \geq f_0.$$

Let us introduce the function $W(t) = \Theta(t) - Y(t)$. The function $W(t)$ satisfies the linear differential inequality

$$W'(t) + D\,W(t) \leq (f_0 - f_*)(1+t)^{\frac{-\nu}{\nu-1}} \leq 0 \quad \text{for } t > 0, \qquad W(0) \leq 0,$$

with the coefficient

$$D = C_1 \frac{\Theta^\nu(t) - Y^\nu}{\Theta(t) - Y} \equiv \nu \int_0^1 (\lambda\Theta(t) + (1-\lambda)Y(t))^{\nu-1}\,d\lambda \geq 0.$$

It follows that $W(t) \leq 0$ for all $t > 0$, i.e.,

$$0 \leq \Theta(t) \leq Y(t) = A^*(1+t)^{-\frac{1}{\nu-1}}.$$

□

As was already mentioned in the Introduction, the solution may vanish in a finite time if the equation combines direction of fast diffusion with the directions of slow diffusion. Let $n = 2$, $p_2 > 1$ and $p_1 \in (1,2)$. In this case the condition sufficient for the finite time vanishing has the form

$$\frac{1}{p_1} + \frac{1}{p_2} \in (1,2) \qquad \Longleftrightarrow \qquad 1 < p_2 < \frac{p_1}{p_1 - 1} \to \begin{cases} \infty & \text{as } p_1 \to 1+, \\ 2+ & \text{as } p_1 \to 2-. \end{cases}$$

For the isotropic equation with $p_1 = p_2$ this means that $p_i \in (1,2)$.

5 Space localization

Let us introduce the functions

$$\begin{aligned} E(s,t) &= \sum_{i=1}^n \int_{Q(s,t)} |D_i u|^{p_i}\,dxdt, \\ b(s,\tau) &= \int_{\Omega(s)} |u|^2(x,\tau)\,dx, \\ \overline{b}(s,t) &= \sup_{\tau\in(0,t)} b(s,\tau) \end{aligned}$$

and notice that by virtue of the definition of the space $\mathbf{W}(Q_T)$ the function $E(s)$ is differentiable for a.e. $s \in (0,a)$ and

$$E_s = -\sum_{i=1}^{n} \int_0^t \int_{\omega(s)} |D_i u|^{p_i}\, dx' dt.$$

5.1 Differential inequality for the energy function $E(s,t)$

We want to transform the energy relation (3.1) into the differential inequality for the energy function $E(s,t)$. Let us denote

$$J(s,t) = -\int_0^t dt \int_{\omega(s)} a_1\, u\, |D_1 u|^{p_1-2} D_1 u\, dx'$$

and write (3.1) in the form

$$\begin{aligned} \frac{1}{2}\int_{\Omega(s)} u^2(x,\tau)dx\Big|_{\tau=0}^{\tau=t} + \sum_{i=1}^{n} \int_{Q(s,t)} a_i |D_i u|^{p_i}\, dxdt \\ = J(s,t) + \int_{Q(s,t)} u\, f\, dxdt. \end{aligned} \tag{5.1}$$

We assume that $\Omega = \{x \in \mathbb{R}^n :\ x_i \in (0,a)\}$ (see Remark 3.1). Fix an arbitrary $\theta \in (0,1)$ and set

$$\lambda = 1 + 2\frac{p_1 - 1}{p_1}. \tag{5.2}$$

The following representation holds:

$$\begin{aligned} |u(s,x',t)| &= |u|^{\theta}|u|^{1-\theta} = |u|^{\theta} \left(\int_s^a D_1((u^2)^{\lambda/2})dx\right)^{\frac{1-\theta}{\lambda}} \\ &= (\lambda/2)^{\frac{1-\theta}{\lambda}} |u|^{\theta} \left(\int_s^a (u^2)^{\frac{\lambda}{2}-1} D_1(u^2)dx\right)^{\frac{1-\theta}{\lambda}} \\ &\le \lambda^{\frac{1-\theta}{\lambda}}\, |u|^{\theta} \left(\int_s^a |D_1 u|^{p_1} dx\right)^{\frac{1-\theta}{p_1\lambda}} \left(\int_s^a u^2 dx\right)^{\frac{(1-\theta)(\lambda-1)}{2\lambda}}. \end{aligned} \tag{5.3}$$

Applying Hölder's inequality we find that

$$
\begin{aligned}
|J| &\le A_0 \int_0^t \int_{\omega(s)} |D_1 u|^{p_1-1}|u|\, dx' dt \\
&\le \left(\int_0^t \int_{\omega(s)} |D_1 u|^{p_1}\, dx' dt\right)^{\frac{p_1-1}{p_1}} \left(\int_0^t \int_{\omega(s)} |u|^{p_1}\, dx' dt\right)^{\frac{1}{p_1}} \\
&\equiv (-E_s(s,t))^{\frac{p_1-1}{p_1}} \cdot I^{\frac{1}{p_1}}.
\end{aligned} \tag{5.4}
$$

Applying Hölder's inequality once again, we estimate I as follows:

$$
\begin{aligned}
I &\le \int_0^t \int_{\omega(s)} |u|^{\theta p_1} \left(\int_s^a |D_1 u|^{p_1}\, dx_1\right)^{\frac{1-\theta}{\lambda}} \left(\int_s^a u^2\, dx_1\right)^{\frac{(1-\theta)(p_1-1)}{\lambda p_1}} dx' dt \\
&\le \int_0^t \Bigg[\left(\int_{\omega(s)} \left(|u|^{\theta p_1} \cdot \left(\int_s^a |D_1 u|^{p_1}\, dx_1\right)^{\frac{1-\theta}{\lambda}}\right)^{\frac{1}{\sigma}} dx'\right)^{\sigma} \\
&\qquad \times \left(\int_{\Omega(s)} u^2 dx\right)^{1-\sigma} \Bigg] dt
\end{aligned}
$$

with the exponent

$$
\sigma = 1 - \frac{(1-\theta)(p_1-1)}{\lambda p_1} \equiv 1 - \frac{(1-\theta)(\lambda-1)}{2\lambda} \in (0,1). \tag{5.5}
$$

Further,

$$
\begin{aligned}
I &\le \bar{b}^{1-\sigma}(s,t) \int_0^t \left[\left(\int_{\omega(s)} |u|^{\frac{\theta p_1}{\sigma}} \cdot \left(\int_s^a |D_1 u|^{p_1}\, dx_1\right)^{\frac{1-\theta}{\sigma\lambda}} dx'\right)^{\sigma}\right] dt \\
&\le \bar{b}^{1-\sigma}(s,t) \int_0^t \left[\left(\int_{\omega(s)} |u|^{\frac{p_1\theta}{\mu}}\, dx'\right)^{\mu} \left(\int_{\Omega(s)} |D_1 u|^{p_1}\, dx\right)^{\frac{1-\theta}{\lambda}}\right] dt
\end{aligned} \tag{5.6}
$$

with the exponent

$$
\mu = \sigma - \frac{1-\theta}{\lambda} \equiv 1 - \frac{(1-\theta)(\lambda+1)}{2\lambda} > 0.
$$

Let us apply Theorem 2.2 to the function $u(x_1, x', t)$ considered as a function of the variables $x' \in \omega(s)$. Let us claim that

$$\frac{1}{q} = \frac{1}{n-1} \sum_{i=2}^{n} \frac{1}{p_i} \le \frac{1}{n-1}. \tag{5.7}$$

(Notice that since $p_i > 1$, this claim is automatically fulfilled if $n = 2$.) For every $v \in \mathbf{V}(\omega(s))$ and every $r \in [1, \infty)$

$$\|v\|_{r,\omega(s)} \le C \left(\prod_{i=2}^{n} \|D_i v\|_{p_i,\omega(s)} \right)^{\frac{1}{n-1}} .. \tag{5.8}$$

If $p_1\theta \ge \mu$, inequality (5.8) yields the estimate

$$\int_{\omega(s)} |u|^{\frac{p_1\theta}{\mu}} \, dx' \le C \left(\sum_{i=2}^{n} \int_{\omega(s)} |D_i u|^{p_i} \, dx' \right)^{\frac{p_1\theta}{\mu}\frac{1}{q}},$$

otherwise $\frac{p_1\theta}{\mu} \in (0,1)$ and we have

$$\int_{\omega(s)} |u|^{\frac{p_1\theta}{\mu}} \, dx' \le C(a,n) \, \|u\|_{1,\omega(s)}^{\frac{p_1\theta}{\mu}} \le C' \left(\sum_{i=2}^{n} \int_{\omega(s)} |D_i u|^{p_i} \, dx' \right)^{\frac{p_1\theta}{\mu}\frac{1}{q}}.$$

Plugging these estimates to (5.6), we finally have:

$$I \le C \, \overline{b}^{1-\sigma}(s,t) \int_0^t (-E_{st})^{\frac{p_1}{q}\theta} (E_t)^{\frac{1-\theta}{\lambda}} \, dt.$$

Recall that all the previous arguments hold true for every $\theta \in (0,1)$. Let us now choose θ from the condition

$$\theta \, \frac{p_1}{q} + \frac{1-\theta}{\lambda} = 1. \tag{5.9}$$

Equation (5.9) always has a solution in the interval $(0,1)$, provided that $p_1 > q$. Indeed: let

$$\Phi(\theta) = \theta \, \frac{p_1}{q} + \frac{1-\theta}{\lambda} - 1;$$

this function is continuous in $(0,1)$,

$$\Phi(0) = \frac{1}{\lambda} - 1 < 0 \quad \text{and} \quad \Phi(1) = \frac{p_1}{q} - 1 > 0,$$

which is why $\Phi(\theta)$ has a root in the interval $(0,1)$.

It follows now by Hölder's inequality that

$$I \le C\,(\bar{b}(s,t) + E(s,t))^{1-\sigma+\frac{1-\theta}{\lambda}} \cdot (-E_s)^{\frac{p_1}{q}\theta}.$$

Gathering this estimate with (5.10) we arrive at the estimate

$$|J| \le C\,(\bar{b}(s,t) + E(s,t))^{\frac{1}{p_1}-\frac{\sigma}{p_1}+\frac{1-\theta}{p_1\lambda}}\,(-E_s(s,t))^{\frac{p_1-1}{p_1}+\frac{\theta}{q}}. \tag{5.10}$$

Lemma 5.1 *Let the exponents p_i satisfy conditions* (1.2) *and*

$$\frac{1}{n-1} \ge \frac{1}{n-1}\sum_{i=2}^{n}\frac{1}{p_i} = \frac{1}{q} > \frac{1}{p_1}. \tag{5.11}$$

Then

$$|J(s,t)| \le \frac{1}{2}(\bar{b}+E) + C_1\,(-E_s)^{\alpha} \tag{5.12}$$

with absolute constant C_1 and the exponent

$$\alpha = \frac{\frac{p_1-1}{p_1} + \frac{\theta}{q}}{\frac{p_1-1}{p_1} + \frac{\sigma}{p_1} - \frac{1-\theta}{p_1\lambda}} > 1 \tag{5.13}$$

with θ, σ, λ defined in (5.9), (5.5), (5.2).

Proof. By virtue of (5.11) conditions (5.7) and (5.9) are fulfilled, whence (5.10). The claim $\alpha > 1$ is equivalent to the inequality

$$0 < \frac{p_1\theta}{q} - \sigma + \frac{1-\theta}{\lambda} = 1 - \sigma = \frac{(1-\theta)(\lambda-1)}{2\lambda}.$$

$\square$

Plugging (5.12) into (5.1), we obtain the differential inequality for the energy function $E(s,t)$:

$$\bar{b} + E - C\,(-E_s)^{\alpha} \le K\int_{Q(s,t)} u\,f\,dxdt + K\int_{\Omega(s)} u_0^2\,dx \tag{5.14}$$

with the exponent $\alpha > 1$ from (5.13) and absolute constants C, K.

5.2 Stable localization.

The properties of directional localization of the solution follow from the properties of the functions satisfying the ordinary differential inequality (5.14).

Theorem 5.1 *Let the conditions of Theorem* 2.1 *be fulfilled and let*

$$\frac{1}{n-1} \geq \frac{1}{n-1}\sum_{i=2}^{n}\frac{1}{p_i} = \frac{1}{q} > \frac{1}{p_1}. \tag{5.15}$$

If

$$\begin{cases} u_0(x) \equiv 0 \ \ in\ \Omega(s_0) \equiv \Omega \cap \{x_1 > s_0\}, \\ f \equiv 0 \ \ in\ Q(s_0,t) \equiv \Omega(s_0) \times (0,T), \end{cases}$$

and if

$$M = \|u_0\|^2_{2,\Omega} + \|f\|^2_{Q_T}$$

is sufficiently small, then every solution of problem (1.1) *is localized in the direction x_1: there exists $s' \geq s_0$ such that for every solution of problem* (1.1)

$$u(x,t) \equiv 0 \ \ in\ Q(s',T) \equiv \Omega(s') \times [0,T].$$

The value of s' is independent of T.

Proof. By virtue of (5.14) the energy function $E(s,t)$ satisfies the conditions

$$\begin{cases} C\,E^{\frac{1}{\alpha}} + E_s \leq 0 \quad \text{for } s > s_0, \\ 0 \leq E(s,t) \leq M \text{ for all } (s,t) \in (0,a) \times (0,T), \\ E_s \leq 0 \end{cases}$$

with the exponent $\alpha > 1$ defined in (5.13). The straightforward integration leads to the inequality

$$E^{1-\frac{1}{\alpha}}(s,t) = E^{1-\frac{1}{\alpha}}(s_0,t) - \frac{C\,\alpha}{\alpha-1}(s-s_0) \leq M^{1-\frac{1}{\alpha}} - \frac{C\,\alpha}{\alpha-1}(s-s_0).$$

It follows that for all $t \in [0,T]$

$$E(s',t) = 0 \quad \text{for } s' = s_0 + \frac{\alpha - 1}{C\alpha} M^{1-\frac{1}{\alpha}}.$$

Let us now claim that $s' < a$. Due to the energy estimate of Theorem 2.1 this condition is always fulfilled if M is sufficiently small. □

5.3 Infinite waiting time

Theorem 5.2 *Let the conditions of Theorem* 5.1 *be fulfilled and, in addition, let us assume that the integral*

$$K(s) \equiv \int_s^{s_0} \frac{\Phi(z)}{(z - s_0)^{\alpha/(\alpha-1)}} dz, \quad \Phi(s) = \|u_0\|^{\frac{2}{\alpha}}_{2,\Omega(s)} + \|f\|^{\frac{2}{\alpha}}_{2,Q(s,T)},$$

is convergent. Then there exists a constant $\epsilon > 0$ *such that if*

$$M = M = \|u_0\|^2_{2,\Omega} + \|f\|^2_{Q_T} < \epsilon,$$

then the solution of problem (1.1) $u \in \mathbf{W}(Q_T)$ *possesses the property of infinite waiting time on the plane* $\Omega \cap \{x_1 = s_0\}$*:*

$$u(x,t) \equiv 0 \quad \textit{in } Q_T \cap \{x_1 \geq s_0\} \textit{ for every } t \geq 0.$$

Proof. The function $\Theta(t)$ satisfies the differential inequality (5.14). Let us estimate the first term on the right–hand side of this inequality in the following way: by Hölder's and Young'sa inequalities

$$\begin{aligned} 2\int_{Q(s,t)} u\, f\, dxdt &\leq 2\int_0^t \|u(\cdot,t)\|_{2,\Omega}\|f(\cdot,t)\|_{2,\Omega}\, dt \\ &\leq 2\sqrt{\bar{b}(s,t)} \int_0^t \|f(\cdot,t)\|_{2,\Omega}\, dt \\ &\leq \frac{1}{2}\bar{b}(s,t) + C\, \|f\|^2_{2,Q(s,t)}. \end{aligned}$$

Substituting this estimate into (5.14) and simplifying, we get

$$E \leq C\,(-E_s)^{\alpha} + K_1(\|f\|^2_{2,Q(s,t)} + \|u_0\|^2_{2,\Omega}).$$

Rising the both sides to the power $1/\alpha$, we finally arrive at the inequality: for $s < s_0$

$$\begin{cases} E^{\frac{1}{\alpha}} + E_s \le F(s), \qquad F(s) \equiv K_2 \left(\|f\|^{\frac{2}{\alpha}}_{2,Q(s,t)} + \|u_0\|^{\frac{2}{\alpha}}_{2,\Omega}\right), \\ 0 \le E \le M, \qquad E_s \le 0, \end{cases}$$

with a constant K_2 independent of f, u_0 and E. The conclusion follows as in the proof of Theorem 4.1 (b) $\square$

Acknowledgements

The first author was partially supported by the research project POCI/-MAT/61576/2004, FCT/MCES (Portugal). The second author acknowledges the support of the research grant MTM2007-65088 (Spain).

References

[1] S. N. Antontsev, J. I. Díaz, and S. Shmarev, *Energy methods for free boundary problems*, Progress in Nonlinear Differential Equations and their Applications, 48, Birkhäuser Boston Inc., Boston, MA, 2002. Applications to nonlinear PDEs and fluid mechanics.

[2] S. N. Antontsev and S. I. Shmarev, *On the localization of solutions of elliptic equations with nonhomogeneous anisotropic degeneracy*, Sibirsk. Mat. Zh., 46 (2005), pp. 765–782.

[3] ——, *Elliptic equations with anisotropic nonlinearity and nonstandard growth conditions*, Elsevier, 2006. Handbook of Differential Equations. Stationary Partial Differential Equations, Elsevier, Vol. 3, Chapter 1, pp.1-100.

[4] ——, *Parabolic equations with anisotropic nonstandard growth conditions*, in Internat. Ser. Numer. Math. 154, Birkhäuser, Verlag Basel/Switzerland, 2006, pp. 33–44.

[5] ——, *Anisotropic parabolic equations with variable nonlinearity*, Preprint, CMAF, University of Lisbon, Portugal, downloadable at: http://www.ptmat.fc.ul.pt/preprints/pdf/2007, 13 (2007), pp. 1–34.

[6] ——, *Extinction of solutions of parabolic equations with variable anisotropic nonlinearities*, in Proceedings of the Steklov Institute of Mathematics, 261, Steklov Institute of Mathematics, Moscow, Russia, 2008, pp. 1–11.

[7] M. Bendahmane and K. H. Karlsen, *Renormalized entropy solutions for quasi-linear anisotropic degenerate parabolic equations*, SIAM J. Math. Anal., 36 (2004), pp. 405–422 (electronic).

[8] ——, *Nonlinear anisotropic elliptic and parabolic equations in* $\mathbb{R}^N$ *with advection and lower order terms and locally integrable data*, Potential Anal., 22 (2005), pp. 207–227.

[9] ——, *Renormalized solutions of an anisotropic reaction-diffusion-advection system with* L^1 *data*, Commun. Pure Appl. Anal., 5 (2006), pp. 733–762.

[10] F. Li, *Anisotropic parabolic equations with measure data. II*, Math. Nachr., 279 (2006), pp. 1585–1596.

[11] J.-L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires*, Dunod, 1969.

[12] M. Sango, *On a doubly degenerate quasilinear anisotropic parabolic equation*, Analysis (Munich), 23 (2003), pp. 249–260.

[13] K. N. Soltanov, *Some nonlinear equations of the nonstable filtration type and embedding theorems*, Nonlinear Anal., 65 (2006), pp. 2103–2134.

[14] B. Song, *Anisotropic diffusions with singular advections and absorptions. part 1. existence*, Applied Mathematics Letters, 14 (2001), pp. 811–816.

[15] ——, *Anisotropic diffusions with singular advections and absorptions. part 2. uniqueness*, Applied Mathematics Letters, 14 (2001), pp. 817–823.

[16] M. Troisi, *Teoremi di inclusione per spazi di Sobolev non isotropi*, Ricerche Mat., 18 (1969), pp. 3–24.