A functional characterisation of the analytical hierarchy

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Abstract. We study an inductive class of vector functions over the reals, defined from a set of basic functions by composition, solving of first order differential equations and the taking of infinite limits. We show that we obtain exactly the class of vector functions for which $\bar{z} = f(\bar{x})$ is a predicate in the analytical hierarchy. We then prove an analogue of Post's theorem for the analytical hierarchy.

1 Introduction

In 1996 Cris Moore published a seminal paper, *Recursive theory on the reals and continuous-time computation* [Moore, 1996], where he defines an inductive class of vector valued functions over $\mathbb{R}$, aiming to provide a framework to study continuous-time phenomena from a computational perspective. This class was defined as the closure of some basic functions for the operators of composition, solving of first-order differential equations and minimalisation.

We draw deeply on previous results in the field of real recursive functions: foundational results can be found in [Mycka and Costa, 2004], where minimalisation was replaced by infinite limits in order to solve technical problems. Loff et al. [2007] show a crucial theorem for this paper: that differential recursion can be replaced by iteration, under the presence of infinite limits.

The analytical hierarchy is a hierarchy of predicates of second-order arithmetic, and is studied in a variety of contexts. It was originally devised by Lusin (1925) for the then-incipient field of descriptive set theory and discovered independently by Kleene (1955) in the study of recursion on higher types. The name 'analytical' is used because *second-order arithmetic allows for the formalisation of elementary analysis, by coding of real numbers as functions of integers* [Odifreddi, 1989, p. 376].

While seeming to come from separate directions, we show that the correspondence between real recursive functions and the analytical hierarchy is absolute.

This result, although important for real recursive function theory, is not unexpected. The class of real recursive functions is suspiciously similar to the analytically representable functions of Lebesgue (1905), and real recursive function theory is mostly a theory of definability. Also, work done by Moschovakis [1969] on abstract search computability shows that we can find analogues of Post's theorem, and infinite limits can be seen as operators of search computability over $\mathbb{R}$. In any case, we provide a simpler, quantifier-free, functional characterisation of the analytical hierarchy, which may uncover new approaches to classical problems.

We begin by reviewing the inductive definitions of recursive functionals and real recursive functions. In Section 3 we define a syntactical rank to stratify real recursive functions into a hierarchy and review previous results. We then establish a correspondence between this hierarchy and the analytical hierarchy in Section 4, and in the final section we use real recursive functions to show an analogue of Post's theorem for the analytical hierarchy.

2 Recursive classes

We aim to find a correspondence between a class of predicates of second-order arithmetic and a certain class of functions over real numbers. We encode real numbers using total functions of natural numbers with any standard continuous surjection from Baire space to $\mathbb{R}$ (This surjection exists, since $\mathbb{R}$ is a Polish space [cf. Moschovakis, 1980]. We refer to Ko [1991] for a discussion of such mappings). We denote variables ranging over $\mathbb{R}$ and variables ranging over total functions of natural numbers in the same way, with the letters w, x, y, z, and treat them indifferently throughout the paper. Variables ranging over $\mathbb{N}$ will be denoted with a, b, c.

We begin by defining two inductive classes of functions, using the succinct *function algebra* notation found in [Clote, 1999]: A function algebra

$$\mathcal{F} = [f_1, f_2, \ldots; \mathcal{O}_1, \mathcal{O}_2, \ldots]$$

is the smallest class of functions containing $f_1, f_2, \ldots$ and closed for the operators $\mathcal{O}_1, \mathcal{O}_2$, *etc.*

In order to define the classical recursive functionals we will use the following basic functionals:

1. The zero functional $\mathcal{Z}$, such that $\mathcal{Z}(; a) = 0$;
2. The successor functional, $\mathcal{S}$, given by $\mathcal{S}(; a) = a + 1$;
3. The projection functionals, where each $\mathcal{U}_i^{m,n}$ obeys

$$\mathcal{U}_i^{m,n}(x_1, \ldots, x_m; a_1, \ldots, a_n) = a_i;$$

4. The oracle functionals, $\mathcal{O}_k^{m,n}$, such that

$$\mathcal{O}_k^{m,n}(x_1, \ldots, x_m; a_1, \ldots, a_n, b) = x_k(b).$$

We write $\mathcal{C}$, $\mathcal{R}$ and μ to stand for the composition, recursion and minimalisation operators. $\mathcal{V}$ is the aggregation operator: If $F_1, \ldots, F_k$ are k functionals, then

$$\mathcal{V}[F_1, \ldots, F_k](\bar{x}; \bar{a}) = (F_1(\bar{x}; \bar{a}), \ldots, F_k(\bar{x}; \bar{a}))$$

Definition 1. *The class of* restricted partial recursive functionals, REC($\mathbb{N}$)*, is a class of vector functions whose variables range over total functions of natural numbers and over natural numbers, given by the function algebra*

$$\mathrm{REC}(\mathbb{N}) = [\mathcal{Z}, \mathcal{S}, \mathcal{U}_i^{m,n}, \mathcal{O}_k^{m,n}; \mathcal{C}, \mathcal{R}, \mu, \mathcal{V}].$$

We then say that a predicate $P(\bar{x}, \bar{a})$ over functions and natural numbers is recursive if there is a function $\chi_P \in \mathrm{REC}(\mathbb{N})$ such that $\chi_P(\bar{x}; \bar{a}) = 1$ if $P(\bar{x}, \bar{a})$ holds and $\chi_P(\bar{x}; \bar{a}) = 0$ otherwise.

Now we move on to an inductive class of partial vector functions over real variables which was originally presented in [Moore, 1996] and reformulated by Mycka and Costa [2004]. In this class of partial vector functions recursion is replaced by differential recursion and minimalisation is replaced by infinite limits. The basic functions will be $\bar{1}^n, 1^n$ and 0^n, such that $\bar{1}^n(x_1, \ldots, x_n) = -1$, $1^n(x_1, \ldots, x_n) = 1$ and $0^n(x_1, \ldots, x_n) = 0$, and the projections U_i^n. Given an n-ary function f and an $(n+1+k)$-ary function g, both with k components, the function obtained by differential recursion, $\mathbf{R}[f, g]$, is the $(n+1)$-ary solution with k components of the initial value problem

$$\mathbf{R}[f, g](\bar{x}, 0) = f(\bar{x}),$$
$$\partial_y \mathbf{R}[f, g](\bar{x}, y) = g(\bar{x}, y, \mathbf{R}[f, g](\bar{x}, y)).$$

The application of $\mathbf{R}$ is restricted to the case when for every $\bar{x}$ one has that $\mathbf{R}[f, g](\bar{x}, \cdot)$ is unique, continuous and defined on the largest open interval containing 0 in which such a unique continuous solution exists almost everywhere. We also demand that $g(\bar{x}, y, \mathbf{R}[f, g](\bar{x}, y))$ is defined for every y in this interval.[1] Given an $(n+1)$-ary function f with k components, the function obtained by an infinite limit, $\mathbf{l}[f]$, is the n-ary function given by

$$\mathbf{l}[f](\bar{x}) = \lim_{y \to \infty} f(\bar{x}, y).$$

Identically we define the operators of infinite supremum limit and infinite infimum limit: $\mathbf{ls}[f](\bar{x}) = \limsup_{y \to \infty} f(\bar{x}, y), \mathbf{li}[f](\bar{x}) = \liminf_{y \to \infty} f(\bar{x}, y)$.[2] Finally, $\mathbf{v}$ is an aggregation operator similar to $\mathcal{V}$ but defined for vector functions over the reals.

[1] See Pouso [2001], Hassan and Rzymowski [1997] regarding existence and uniqueness of *almost-everywhere solutions* to these differential equations. The restrictions we impose imply that the operator $\mathbf{R}$ is a partial operator.

[2] Should the function f be partial and undefined for arbitrarily large values of y, then the infinite limits are also undefined.

Definition 2. *The* class of real recursive vector functions, $\mathrm{REC}(\mathbb{R})$*, is given by the function algebra*

$$\mathrm{REC}(\mathbb{R}) = [1^n, \bar{1}^n, 0^n, \mathrm{U}_i^n; \mathbf{C}, \mathbf{R}, \mathbf{l}, \mathbf{ls}, \mathbf{li}, \mathbf{v}].$$

An operator similar to classical recursion is the iteration operator $\mathbf{I}$, such that for an n-ary function f with n components[3]

$$\mathbf{I}[f](\bar{x}, y) \equiv f^{[y]}(\bar{x}) = \overbrace{f \circ f \circ \ldots \circ f}^{\lfloor |y| \rfloor \text{ times}}(\bar{x}).$$

It was shown by Loff et al. [2007, Proposition 4.3] that taking the binary sum, product and division as basic functions one can replace differential recursion with iteration:

Proposition 1. $\mathrm{REC}(\mathbb{R})$ *is also given by the function algebra*

$$\mathrm{REC}(\mathbb{R}) = [1^n, \bar{1}^n, 0^n, \mathrm{U}_i^n, +, \times, /; \mathbf{C}, \mathbf{I}, \mathbf{l}, \mathbf{ls}, \mathbf{li}, \mathbf{v}].$$

3 The η-hierarchy

We call *description* to the syntactical expression testifying that a certain function f is in a function algebra. Consider, for instance, the exponential. This function can be given by a simple differential recursion scheme:

$$\exp(0) = 1 \qquad \partial_y \exp(y) = \exp(y)$$

A description for the exponential function is, then, $\langle \mathbf{R}[1^0, \mathrm{U}_2^2] \rangle$. We enclose descriptions between angles to stress the difference between the function denoted by $\mathbf{R}[1^0, \mathrm{U}_2^2]$ and its expression.

It is then possible to establish syntactical measures of complexity of functions by considering their descriptions. In classical recursion theory we can, for instance, count nested primitive recursions and show that the resulting hierarchy does not collapse.

Here we establish a hierarchy of real recursive functions based on the number of nested limits needed to describe a function. In the first level we have functions which do not need limits to be defined and then at each level we allow for an additional infinite limit to be applied to functions in the previous level.

Definition 3. *The* η-hierarchy *is an* $\mathbb{N}$*-indexed family of real recursive vector functions, and the nth level of the* η*-hierarchy,* H_n*, is given by*

$$\mathrm{H}_0 = [0^n, \bar{1}^n, 1^n, \mathrm{U}_i^n; \mathbf{C}, \mathbf{R}, \mathbf{v}]$$

$$\mathrm{H}_{n+1} = [\{f, \mathbf{l}[f], \mathbf{ls}[f], \mathbf{li}[f] \mid f \in \mathrm{H}_n\}; \mathbf{C}, \mathbf{R}, \mathbf{v}]^4$$

[3] Notice that the function f is iterated only a non-negative integer number of times. By convention, $f^{[0]}(\bar{x}) = \bar{x}$.

[4] To build the $(n+1)$th level, we take the functions in the previous level and their limits, and close the resulting set under the remaining operators. The hierarchy thus becomes organized by the *rank* of the infinite limit operators.

Proposition 2. *(I) [Mycka and Costa, 2004, Loff et al., 2007] The functions* $\lambda x.\frac{1}{x}$, $\log$ *and* $\exp$ *of arity* 1 *and the functions* $+$, $\times$, $/$ *and* $\lambda xy.x^y$ *of arity* 2 *are in* H_0.

(II) [Mycka and Costa, 2004, Loff et al., 2007] Kronecker's δ *and Heaviside's* Θ*, given by*

$$\delta(x) = \begin{cases} 1 & \textit{if } x = 0 \\ 0 & \textit{otherwise} \end{cases} \textit{ and } \Theta(x) = \begin{cases} 1 & \textit{if } x \geqslant 0 \\ 0 & \textit{otherwise} \end{cases}$$

are in H_1.

(III) [Mycka and Costa, 2004, Loff et al., 2007] If a function f *is in* H_i*, then* $\mathbf{I}[f]$ *is in* $\mathrm{H}_{\max(i,1)}$.

(IV) [Mycka, 2003] There are real recursive tuple coding functions in H_3*, i.e., for every* n *and* $1 \leqslant i \leqslant n$*, we have* $\gamma_n, \gamma_{n,i} \in \mathrm{H}_3$ *and such that*

$$\gamma_n(\gamma_{n,1}(x), \ldots, \gamma_{n,n}(x)) = x, \gamma_{n,i}(\gamma_n(x_1, \ldots, x_n)) = x_i \textit{ and } \gamma_{n,i}(0) = 0.$$

One can conclude from (I,II) above that the functions given by

$$|x| = (2\theta(x) - 1)x \text{ and } \mathrm{lt}(x, y) = \theta(y - x) - \delta(y - x) = \begin{cases} 1 & \text{if } x < y, \\ 0 & \text{otherwise,} \end{cases}$$

are also in H_1 and from (IV) that $\gamma_m^{-1}(x) = (\gamma_{m,1}(x), \ldots, \gamma_{m,m}(x))$ is in H_3.

Definition 4. *The* **sup** *and* **inf** *operators are given by*

$$\mathbf{sup}[f](\bar{x}) = \sup_{y \in \mathbb{R}} f(\bar{x}, y) \textit{ and } \mathbf{inf}[f](\bar{x}) = \inf_{y \in \mathbb{R}} f(\bar{x}, y).$$

More exactly, $\mathbf{sup}[f](\bar{x})$ (or $\mathbf{inf}[f](\bar{x})$) is the value z such that for all $y \in \mathbb{R}$ some $w \geqslant 0$ verifies $z - w = f(\bar{x}, y)$ (resp. $z + w = f(\bar{x}, y)$). This means that $\mathbf{sup}[f](\bar{x})$ is undefined if $f(\bar{x}, y)$ is undefined for some y. In order to show that $\mathrm{REC}(\mathbb{R})$ is closed for **sup** and **inf**, we use a function similar to the *remainder* function for the natural numbers in order to create a periodic function, and take the supremum or infimum limit of that function.

Definition 5. $(y \bmod z)$ *is the number in* $[0, |z|)$ *— or* $(-|z|, 0]$ *if* $y < 0$ *— such that* $y = n|z| + (y \bmod z)$ *for some* $n \in \mathbb{Z}$.

Proposition 3. mod *is in* H_1.

Proof. Set $(y \bmod z) = \mathrm{U}_1^2(\mathbf{I}[h](y, \mathrm{sg}(y)|z|, \frac{y}{z} + 1))$, where

$$h(y, z) = \begin{cases} (y, z) & \text{if } |y| < |z|, \\ (y - z, z) & \text{otherwise,} \end{cases} = (y - z \times (1 - \mathrm{lt}(|y|, |z|)), z),$$

and $\mathrm{sg}(y) = 2\Theta(y) - 1 - \delta(y)$. □

Proposition 4. *If $f \in \mathrm{H}_n$, then* $\mathbf{sup}[f], \mathbf{inf}[f] \in \mathrm{H}_{n+2}$.

Proof. Given an $(n+1)$-ary function $f \in \mathrm{H}_n$ we define the new periodic function $F \in \mathrm{H}_{\max(n,1)}$, given by $F(\bar{x}, y, z) = f(\bar{x}, (y \bmod z))$, and then we set

$$f^s_{\pm}(\bar{x}) = \lim_{z \to \infty} \limsup_{y \to \infty} F(\bar{x}, \pm y, z).$$

Now we have $\mathbf{sup}[f](\bar{x}) = \max(f^s_+(\bar{x}), f^s_-(\bar{x}))$, where $\max(x, y) = \mathrm{lt}(x, y) \times y + (1 - \mathrm{lt}(x, y)) \times x$. We proceed in the same way for $\mathbf{inf}[f]$. $\square$

4 The analytical hierarchy

We present the analytical hierarchy of predicates, and relate it with the η-hierarchy.

Definition 6. *The* analytical hierarchy of predicates *consists of three* $\mathbb{N}$*-indexed families of predicates over natural numbers and functions of natural numbers:*

1. Σ^1_0 *is the class of predicates that can be given using number quantifiers over a recursive predicate, and* $\Pi^1_0 = \Sigma^1_0$.
2. Σ^1_{n+1} *is the class of predicates given by* $\exists y\ \phi(\bar{x}, y, \bar{a})$, *with* ϕ *in* Π^1_n.
3. Π^1_{n+1} *is the class of predicates given by* $\forall y\ \phi(\bar{x}, y, \bar{a})$, *with* ϕ *in* Σ^1_n.
4. $\Delta^1_n = \Sigma^1_n \cap \Pi^1_n$.

We write Σ^1_ω to stand for $\cup_{n \in \mathbb{N}} \Sigma^1_n$, and in the same way for Π^1_ω and Δ^1_ω. We will use abundantly the following result.

Proposition 5. *(a)* Σ^1_{n+1} *is closed for existential quantification over functions.*
(b) Π^1_{n+1} *is closed for universal quantification over functions.*
(c) Π^1_{n+1} *and* Σ^1_{n+1} *are closed for existential and universal quantification over natural numbers.*
(d) If $P \in \Sigma^1_n$ *then some* $P^\star$ *also in* Σ^1_n *is such that* $\forall a\ P \iff \forall x\ P^\star$.
(e) If $P \in \Pi^1_n$ *then some* $P^\star$ *also in* Π^1_n *is such that* $\exists a\ P \iff \exists x\ P^\star$.

Definition 7. *We say that a function* $f : \mathbb{R}^m \to \mathbb{R}^n$ *is in* Σ^1_k *if the* $(n+m)$*-ary predicate of expression* $\bar{z} = f(\bar{x})$ *is in* Σ^1_k. *Similarly for* Π^1_k *and* Δ^1_k.

We assume below that the surjection from functions of natural numbers to real numbers mentioned in page 2 allows us to obtain, for a given number n, the first n digits of the coded real number.

Proposition 6. *The functions* $1^n, \bar{1}^n, 0^n, \mathrm{U}^n_i, +, \times,$ / *and floor, as well as the predicates of equality and inequality over the reals, are in* Δ^1_0.

Proof. We'll begin by showing that there is a recursive way to decide the predicate over the reals given by the expression 'x and y are not different up to the nth digit', which we write $x =_n y$. An algorithm to decide this predicate needs to solve the ambiguity of the representation of a real number by binary expansion, and we can make it work the following way: given two real numbers x, y and a natural number n, we obtain the first n digits of the two reals and verify if they are the same. If they are, then we decide that $x =_n y$. If the digits are not equal we consider the first different digit — one is 0 and the other 1 — and check if the digits after the 0 are all 1s and the digits after the 1 are all 0s.[5] If so, then we decide that $x =_n y$, and we decide that $x \neq_n y$ otherwise. The predicate of real number equality is then given by: $\forall n \;\; x =_n y$, which is in $\Pi^0_1 \subset \Delta^1_0$. For the function $+$, we define a predicate, of expression $z =_n x + y$, that decides if $z = x + y$ for the first n digits of z, x and y. This function computes the sum of the truncations of x and y to the nth fractionary digit and checks if resulting rational number coincides with z to the nth digit using the method shown above. If so, the function is valued 1, and 0 otherwise. Now we have that $z = x + y$ if and only if $\forall n \; z =_n x + y$, which is Δ^1_0. The proof is similar for the remaining operations, except that the number of required significant digits varies. □

A single real number can code any finite tuple of real numbers by alternating the digits of the real numbers in the tuple (this is, incidently, how the γ functions of Proposition 2 work). In this sense, we write $y_{n,i}$ to stand for the ith real number in the n-ary tuple coded by y (again, $y_{n,i} = \gamma_{n,i}(y)$). For an m-ary tuple $\bar{y}$, we write $\bar{y}_{n,i}$ to stand for the tuple $((y_1)_{n,i}, \ldots, (y_m)_{n,i})$. Then it is not hard to see that if some n-ary predicate P is in Δ^1_n,[6] then the $(n+1)$-ary predicate $P^\star$ given by

$$P^\star(\bar{y}, n) \iff \forall i \leqslant n P(\bar{y}_{n,i})$$

is also in Δ^1_n.

Proposition 7. *All real recursive functions belong to the analytical hierarchy, in the sense of Definition 7.*

Proof. The result is proved by induction on the structure of $\mathrm{REC}(\mathbb{R})$ presented in Proposition 1. Proposition 6 gives us the result for the atomic functions. Proposition 5 will suffice for the remaining operators. If the real recursive functions f and g are in Σ^1_n, then $\mathbf{C}[f, g]$ is in Σ^1_n, since:

$$\bar{z} = \mathbf{C}[f, g](\bar{x}) \iff \exists \bar{y} \; \bar{z} = f(\bar{y}) \wedge \bar{y} = g(\bar{x}).$$

Let f be a real recursive n-ary function with n components in Σ^1_n. Then $\mathbf{I}[f]$ is in Σ^1_n, since $\bar{z} = \mathbf{I}[f](\bar{x}, y)$ if and only if

$$\exists \bar{w} \exists n \, [n = \lfloor y \rfloor \wedge \bar{w}_{n,1} = f(\bar{x}) \wedge (\forall i \leqslant n)[\bar{w}_{n,i+1} = f(\bar{w}_{n,i})] \wedge \bar{z} = \bar{w}_{n,n}]$$

If f is a real recursive $(n+1)$-ary function in Σ^1_n, then $\mathbf{ls}[f] \in \Pi^1_{n+5} \subseteq \Sigma^1_{n+6}$, since $\bar{z} = \limsup_{y \to \infty} f(\bar{x}, y)$ if and only if

$$\forall \delta > 0 \exists \epsilon \forall \epsilon^\star \geqslant \epsilon \exists s \geqslant \epsilon^\star \forall s^\star \geqslant \epsilon^\star \left[f(\bar{x}, s) \geqslant f(\bar{x}, s^\star) \wedge |\bar{z} - f(\bar{x}, s)| < \delta \right].$$

[5] e.g. $x = 101.1\underline{1}0000$ and $y = 101.1\underline{0}1111$, where the first different digit is underlined.
[6] Remember that $\Sigma^1_{n-1} \cup \Pi^1_{n-1} \subset \Delta^1_n$, which can be proven by adding extra quantifiers to predicates in $\Sigma^1_{n-1} \cup \Pi^1_{n-1}$ [see Odifreddi, 1989, p. 381].

We can do identically for lim inf. We also have that

$$\bar{z} = \lim_{y\to\infty} f(\bar{x}, y) \iff \forall\delta > 0 \exists\epsilon\forall\epsilon^\star \geqslant \epsilon \left[|\bar{z} - f(\bar{x}, \epsilon^\star)| < \delta\right],$$

resulting in $\mathbf{l}[f] \in \Pi^1_{n+3} \subseteq \Sigma^1_{n+4}$. If $f_1, \ldots, f_n$ are in Σ^1_n then $v[f_1, \ldots, f_n]$ is trivially also in Σ^1_n. □

It was proven by Loff et al. [2007, Proposition 4.9] that defining a function $f \in \mathrm{H}_n$ with iteration instead of differential recursion requires $n+7$ nested limits.

Corollary 1. *For every natural number n we have $\mathrm{H}_n \subseteq \Sigma^1_{6n+43}$.*

Definition 8. *The* characteristic *of a predicate P over $\mathbb{N}^m \times \mathbb{R}^n$ is the total function $\chi_P : \mathbb{N}^m \times \mathbb{R}^n \to \{0, 1\}$ such that $\chi_P(\bar{a}, \bar{x}) = 1$ if and only if $P(\bar{a}, \bar{x})$ holds. We say that such a predicate P* has a real recursive characteristic *f if f is a real recursive function such that, for every $\bar{a} \in \mathbb{N}^m, \bar{x} \in \mathbb{R}^n$, $\chi_P(\bar{a}, \bar{x}) = f(\bar{a}, \bar{x})$. We write $P \in \mathrm{H}_k$ if there is a real recursive characteristic of P in H_k.*

Proposition 8. *All predicates in the analytical hierarchy have real recursive characteristics.*

Proof. Mycka and Costa [2004, p. 855] show that all Π^1_1 predicates have real recursive characteristics in at most H_6, and so all predicates in $\Delta^1_0 \subset \Pi^1_1$ have real recursive characteristics. We now show that if P is an $(n+1)$-ary predicate with a real recursive characteristic χ_P, then there are real recursive characteristics of the predicates given by $\forall y P(\bar{x}, y)$ and $\exists y P(\bar{x}, y)$. We have shown in Proposition 4 that if a function is real recursive, then so is its supremum and infimum over the positive or negative infinite interval. So we have that $\forall y P(\bar{x}, y)$ if and only if $\mathbf{inf}[\chi_P](\bar{x}) = 1$ and that $\exists y P(\bar{x}, y)$ if and only if $\mathbf{sup}[\chi_P](\bar{x}) = 1$. This way we conclude that all analytical predicates have real recursive characteristics. □

Proposition 9. *If f is an n-ary vector function with m real components such that Γ_f, given by*

$$\Gamma_f(\bar{z}, \bar{x}) = \begin{cases} 1 & \text{if } \bar{z} = f(\bar{x}) \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

is in H_i, then f is in $\mathrm{H}_{\max(i+2,3)}$.

Proof. Remember the tuple coding functions from proposition 2. We get an $(n+1)$-ary function

$$\Gamma^\star_f(\bar{x}, z) = z \times \Gamma_f(\gamma_m^{-1}(z), \bar{x}) = \begin{cases} z & \text{if } z = \gamma_m(f(\bar{x})), \\ 0 & \text{otherwise.} \end{cases}$$

Consider

$$\Gamma^{\star\star}_f(\bar{x}) = \sup_{z\in\mathbb{R}} \Gamma^\star_f(\bar{x}) + \inf_{z\in\mathbb{R}} \Gamma^\star_f(\bar{x});$$

should $f(\bar{x})$ be defined, one has that $\Gamma_f^{\star\star}(\bar{x}) = \gamma_m(f(\bar{x}))$, and if $f(\bar{x})$ is undefined, then we have $\Gamma_f(\gamma_m^{-1}(\Gamma_f^{\star\star}(\bar{x})), \bar{x}) = 0$. So set

$$f(\bar{x}) = \frac{\gamma_m^{-1}(\Gamma_f^{\star\star}(\bar{x}))}{\Gamma_f(\gamma_m^{-1}(\Gamma_f^{\star\star}(\bar{x})), \bar{x})}.$$

□

Corollary 2. *For every natural n we have $\Delta_n^1 \subseteq \mathrm{H}_{2n+8}$.*

Our first main result follows from Corollaries 1 and 2:

Theorem 1. REC($\mathbb{R}$) *is the class of functions given by an analytical predicate, i.e.,*

$$\mathrm{REC}(\mathbb{R}) = \{f \mid \textit{the predicate given by } \bar{z} = f(\bar{x}) \textit{ is in } \Delta_\omega^1\}.$$

Now we prove a well-known theorem. As is now expected, results about real recursive functions imply their counterparts in the analytical hierarchy.

Proposition 10. *The analytical hierarchy does not collapse, i.e., there is no number n such that $\Delta_\omega^1 \subseteq \Delta_n^1$.*

Proof. If the analytical hierarchy collapsed to the level Δ_n^1 for some n, then one could find a universal analytical predicate, $\Psi \in \Delta_n^1$, with a real recursive characteristic $\chi_\Psi \in \mathrm{H}_{2n+6}$. By Proposition 9 one concludes that the η-hierarchy collapses to level H_{2n+8}. It was shown in [Loff et al., 2007, Theorem 6.2] that the η-hierarchy does not collapse, and so neither does the analytical hierarchy. □

5 Analogue of Post's theorem

While there is no Post theorem which relates the analytical hierarchy with natural recursive functions, we will show below an analogue of Post's theorem that relates the analytical hierarchy and real recursive functions.

We begin by defining relativised versions of REC($\mathbb{R}$) and of the η-hierarchy. We will use infimums and supremums instead of infinite limits because they are conceptually closer to quantification.

Definition 9. *Let $\mathcal{F}$ be a set of vector functions over the real numbers. The* class of real recursive vector functions relativised to *$\mathcal{F}$,* REC($\mathbb{R}, \mathcal{F}$)*, is given by*

$$\mathrm{REC}(\mathbb{R}, \mathcal{F}) = [1^n, \bar{1}^n, 0^n, \mathrm{U}_i^n, +, \times, /, \mathcal{F}; \mathbf{C}, \mathbf{I}, \mathbf{sup}, \mathbf{inf}, \mathbf{v}].$$

If P is a predicate over the reals, then class of real recursive vector functions relativised to *P is* REC($\mathbb{R}, \{\chi_P\}$)*.*

This definition is justified by the following equivalence, which states that supremums and infimums can be used to obtain infinite limits.

Proposition 11. $\mathrm{REC}(\mathbb{R},\mathcal{F}) = [1^n, \bar{1}^n, 0^n, \mathrm{U}_i^n, +, \times, /, \mathcal{F}; \mathbf{C}, \mathbf{I}, \mathbf{l}, \mathbf{li}, \mathbf{ls}, \mathbf{v}]$.

Proof. One trivially has that

$$\mathrm{REC}(\mathbb{R},\mathcal{F}) \subseteq [1^n, \bar{1}^n, 0^n, \mathrm{U}_i^n, +, \times, /, \mathcal{F}; \mathbf{C}, \mathbf{I}, \mathbf{l}, \mathbf{li}, \mathbf{ls}, \mathbf{v}],$$

by Proposition 4. We must now show that $\mathrm{REC}(\mathbb{R},\mathcal{F})$ is closed for $\mathbf{l}, \mathbf{ls}, \mathbf{li}$. If f is an $(n+1)$-ary function in $\mathrm{REC}(\mathbb{R},\mathcal{F})$, then the $(n+1)$-ary function h, given by

$$h(\bar{x}, y) = \sup_{z>y} f(\bar{x}, z) = \sup_{z\in\mathbb{R}} f(\bar{x}, y+z^2),$$

is also in $\mathrm{REC}(\mathbb{R},\mathcal{F})$. This is enough to show the closure, since

$$\limsup_{y\to\infty} f(\bar{x}, y) = \inf_{y\in\mathbb{R}} \sup_{z>y} f(\bar{x}, z)$$

and one can obtain $\mathbf{l}$ and $\mathbf{li}$ from $\mathbf{ls}$ [see Mycka, 2003]. $\square$

Now we define a relativised η-hierarchy, by counting the number of supremums and infimums needed to define a function in $\mathrm{REC}(\mathbb{R},\mathcal{F})$. We will distinguish between even and odd levels of the hierarchy. The even levels will be obtained by allowing one application of **sup** or **inf** and the odd levels will be the closure of the even levels for the remaining operators.

Definition 10. *The nth level of the* η-hierarchy relativised to *a set of real functions* $\mathcal{F}$, $\mathrm{H}_n^{\mathcal{F}}$, *is inductively defined by*

$$\mathrm{H}_0^{\mathcal{F}} = \{1^n, \bar{1}^n, 0^n, \mathrm{U}_i^n, +, \times, /\} \cup \mathcal{F}$$

$$\mathrm{H}_{2n+1}^{\mathcal{F}} = [\mathrm{H}_{2n}^{\mathcal{F}}; \mathbf{C}, \mathbf{I}, \mathbf{v}]$$

$$\mathrm{H}_{2n+2}^{\mathcal{F}} = \{f, \mathbf{sup}[f], \mathbf{inf}[f] \mid f \in \mathrm{H}_{2n+1}^{\mathcal{F}}\}$$

Since $y = \chi_Q(\bar{x}) \iff (y = 1 \wedge Q(\bar{x})) \vee (y = 0 \wedge \neg Q(\bar{x}))$, we get the following.

Proposition 12. $Q \in \Delta_n^1$ *if and only if* $\chi_Q \in \Delta_n^1$.

The new hierarchy gives the following analogue of Post's (1948) theorem:

Theorem 2. *For every* $n \geqslant 1$

(1) If $Q \in \Delta_n^1$ *and* $\chi_P \in \mathrm{H}_2^Q$ *then* $P \in \Delta_{n+1}^1$.
(2) If $P \in \Delta_{n+1}^1$ *then* $\chi_P \in \mathrm{H}_2^Q \cap \mathrm{H}_2^R$ *for some* $Q \in \Pi_n^1$ *and* $R \in \Sigma_n^1$.

Proof. To prove (1), we show that every function in H_2^Q is in Δ_{n+1}^1. By Propositions 6 and 12 we conclude that $\mathrm{H}_0^Q \subseteq \Delta_n^1 \subset \Sigma_n^1$. The proof shown for Proposition 7 is sufficient to show that functions obtained by composition, iteration or aggregation — the functions in H_1^Q — are in Σ_n^1. Now suppose a function is given by $\mathbf{inf}[f]$ for some function $f \in \Sigma_n^1$. See that

$$z = \inf_{y\in\mathbb{R}} f(\bar{x}, y) \iff \forall y[z \leqslant f(\bar{x}, y)] \wedge \forall t > z \exists u[t > f(\bar{x}, u)] \iff$$

$$\forall y \forall t \exists u \exists v \exists w [v \geqslant 0 \wedge w > 0 \wedge z + v = f(\bar{x}, y) \wedge (t > z \Rightarrow t - w = f(\bar{x}, u))]$$

gives a predicate in Π^1_{n+1}, and we can do similarly for **sup**. So $\mathrm{H}_2^Q \subseteq \Pi^1_{n+1}$. But if $\chi_P \in \Pi^1_{n+1}$ then both P and $\neg P$ are in Π^1_{n+1} and so we get $P \in \Delta^1_{n+1}$. To prove (2) take P in Δ^1_{n+1}. This means that $P(\bar{x}) \iff \exists y Q(\bar{x}, y) \iff \forall y R(\bar{x}, y)$ for some $Q \in \Pi^1_n, R \in \Sigma^1_n$. So immediately we get $\chi_P = \mathbf{sup}[\chi_Q] \in \mathrm{H}_2^Q$ and $\chi_P = \mathbf{inf}[\chi_R] \in \mathrm{H}_2^R$ □

6 Concluding remarks

We have seen that the inductive closure of some very basic functions for the operations of solving differential equations and taking infinite limits gives us exactly the same expressive power as the analytical hierarchy.

Effectively, this will trivialise the proof that some given function is real recursive. For instance, $\chi_{\mathbb{Q}}$ is real recursive simply because

$$z = \chi_{\mathbb{Q}}(x) \iff (z = 1 \wedge \exists a \exists b\ ax = b) \vee (z = 0 \wedge \neg \exists a \exists b\ ax = b)$$

gives an analytical predicate.

Alas, the analogue of Post's theorem that we obtained is not as good as one would wish: an equivalence would be better. We cannot seem to be able to settle the question *if a predicate P is in Δ^1_{n+1} then can we find a predicate Q in Δ^1_n such that $P \iff \forall Q$?* This would provide the intended result.

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