# The Joule-Thomson effect on the thermoelectric conductors

LUISA CONSIGLIERI
Faculty of Sciences, University of Lisbon

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**Abstract**

The transmission of an electric current in a conductor is a process in which some electrical energy is converted into heat (thermal energy). We deal with a nonlinear boundary value elliptic problem which describes the electrical heating of a solid conductor and the Joule-Thomson effect is taken into account. The existence of a weak solution is proved under both space and temperature dependence of the electrical and thermal conductivities. When the coefficients are only dependent on their temperature argument, some regularity results are stated.

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## 1 Introduction

In the electrical industry, the heat-conducting problems have been studied to minimize temperature rises inside electrical machines in order to prolong insulating life [12]. The thermistor model has been used by several authors (for instance, [6, 7, 10] and the references therein) in order to describe the heat produced by an electrical current passing through a conductor device.

In 1854, Lord Kelvin (William Thomson) derived the thermoelectric equations by applying reversible thermodynamics to the reversible parts of the thermoelectric effect whole taking no account the irreversible [1, 11]. When a temperature gradient is set up in an electrical conductor, not only does heat flow, but an electric field is also created. As the current flow takes the charge carriers through regions of different temperature, they have to absorb or reject heat in order to maintain thermal equilibrium with their surroundings. Nowadays it is known that in thermoelectricity the reversible and irreversible effects are impossible to be separated. The irreversible processes are the Joule heating due to electrical resistivity of the conductors and the thermal conduction. The reversible effect associated with the thermal energy transported by the moving charges is known as the Thomson effect: the heat is generated or absorbed in the conductor in proportion to the product of the current density and the temperature gradient.

In the works [5, 14] the Joule-Thomson effect only appears as a heat source in the heat conduction in the interior of the domain, with prescribed Dirichlet boundary conditions. The authors neglected the boundary effects and the dependence on the potential of the current density which invokes another structure in the formulation of the problem.

In the present work, the temperature dependent coefficients vary also with the spatial position because of the change in temperature throughout the material that corresponds to the real conductors [4]. Although the coefficient function in the principal part of the elliptic operators is continuous, its dependence on the temperature solution to an auxiliary boundary value problem determines the study of existence of solutions to the elliptic theory with discontinuous coefficients.

In the two-dimensional domain case, the summability class $(2+\delta)$ of the gradient of the solution implies its Hölder continuity. Also it is known that solutions to linear elliptic problems posed in the space $W^{1,p}$ enjoy interior $W^{1,\infty}$ regularity in $C^1$ domains. To prove more regularity it seems unavoidable to have a domain of class $C^2$. We provide few regularity results having in mind that the majority of industrial devices involves domains of class $C^{0,1}$.

Next section, we state the problem as a system of nonlinear elliptic partial differential equations. In the following sections 3, 4 and 5, we solve each elliptic boundary value problem. The Galerkin method used here is concerned such that the numerical analysis can be carry on. Section 6 deals with the existence result of classical solutions.

## 2 Statement of the problem

Let $\Omega$ be a bounded open subset of $\mathbb{R}^n$ $(n \geq 2)$ with Lipschitz boundary. For the thermoelectric conductor materials, the electric field $\mathbf{E}$ does not depend linearly on the current density $\mathbf{j}$, $\mathbf{j} = \sigma\mathbf{E}$ (Ohm law), where the conductivity $\sigma$ is a constant characteristic of the medium considered. Indeed the constitutive law is

$$\mathbf{E} = \frac{1}{\sigma(\theta)}\mathbf{j} + \alpha(\theta)\nabla\theta, \qquad \mathbf{E} = -\nabla\phi, \tag{1}$$

where $\phi$ is the potential and the current density satisfies the conservation of electric charge

$$\left\{ \begin{array}{l} \nabla\cdot\mathbf{j} = 0 \text{ in } \Omega \\ -\mathbf{j}\cdot\mathbf{n} = h \text{ on } \partial\Omega. \end{array} \right. \tag{2}$$

Denoting by $\mathcal{S}$ the total entropy of the system we deduce

$$\frac{d}{dt}\mathcal{S} = -\int_\Omega \frac{\nabla\cdot\mathbf{q}}{\theta}dx = -\int_\Omega \frac{1}{\theta}[\nabla\cdot(\mathbf{q}-\phi\mathbf{j}) + \mathbf{j}\cdot\nabla\phi]dx$$

and using (1) we have

$$\frac{d}{dt}\mathcal{S} = \int_\Omega -\frac{\nabla\theta}{\theta^2}\cdot(\mathbf{q}-\phi\mathbf{j}) + \frac{1}{\theta}\left[\frac{|\mathbf{j}|^2}{\sigma(\theta)} + \alpha(\theta)\nabla\theta\cdot\mathbf{j}\right]dx \geq 0$$

assuming that

$$\begin{aligned} \mathbf{q}-\phi\mathbf{j} = -k(\theta)\nabla\theta + \alpha(\theta)\theta\mathbf{j} \quad &\text{in } \Omega \\ (\mathbf{q}-\phi\mathbf{j})\cdot\mathbf{n} = 0 \quad &\text{on } \partial\Omega. \end{aligned}$$

Here we consider different cases for the coefficients $\sigma$ and $k$. First, they are assumed known functions dependent also on the space variable. Second, they are known functions only of their argument.

The problem defined under the Joule and Thomson effects is

$$-\nabla\cdot\mathbf{q} = \nabla\cdot(k(\cdot,\theta)\nabla\theta) + \frac{|\mathbf{j}|^2}{\sigma(\cdot,\theta)} - \alpha'(\theta)\theta\nabla\theta\cdot\mathbf{j}.$$

Then the thermoelectric problem reads

$$\begin{aligned}
&(3) \qquad \text{in } \Omega: \quad && \nabla\cdot\mathbf{j} = 0 \\
&(4) && -\nabla\cdot(k(\cdot,\theta)\nabla\theta) = \frac{|\mathbf{j}|^2}{\sigma(\cdot,\theta)} - \alpha'(\theta)\theta\nabla\theta\cdot\mathbf{j} + g \\
&(5) && -\nabla\cdot(\sigma(\cdot,\theta)\nabla\phi) = \nabla\cdot(\sigma(\cdot,\theta)\alpha(\theta)\nabla\theta) \\
&(6) \qquad \text{on } \partial\Omega: && -\mathbf{j}\cdot\mathbf{n} = h \\
&(7) && k(\cdot,\theta)\nabla\theta\cdot\mathbf{n} = -\alpha(\theta)\theta h \\
&(8) \qquad \text{on } \Gamma: && \sigma(\cdot,\theta)(\nabla\phi + \alpha(\theta)\nabla\theta)\cdot\mathbf{n} = h \\
&(9) \qquad \text{on } \Gamma_0 := \partial\Omega\setminus\bar{\Gamma}: && \phi = \phi_0,
\end{aligned}$$

assuming that a known surface potential $\phi_0$ at almost every point of a part $\Gamma_0$ of the boundary $\partial\Omega$ is given. Notice that at the steady-state the heat produced by electrical dissipation $|\mathbf{j}|^2/\sigma(\theta)$ must leave through the surface.

The system (3)-(7) can be studied as follows. Find $\mathbf{j}$ satisfying (3) and (6), next we solve the elliptic boundary value problem constituted by (4) and (7) and finally find $\phi$ satisfying (5) and (8)-(9).

The data assumptions are

- $k, \sigma : \Omega\times\mathbb{R}\to\mathbb{R}$ Caratheodory such that

$$\begin{aligned}
&(10) \qquad \exists k_\# > 0: \quad k_\# \leq k(\cdot,t) \leq k^\#, \quad \forall t\in\mathbb{R}, \text{ a.e. in } \Omega; \\
&(11) \qquad \exists \sigma_\# > 0: \quad \sigma_\# \leq \sigma(\cdot,t) \leq \sigma^\#, \quad \forall t\in\mathbb{R}, \text{ a.e. in } \Omega;
\end{aligned}$$

- $\alpha\in C^1(\mathbb{R})$ such that

$$(12) \qquad \exists \alpha_\# > 0: \quad \alpha(t)\geq\alpha_\#, \quad \forall t\in\mathbb{R};$$

$$(13) \qquad \exists \alpha^\# > 0: \quad \alpha'(t) \leq \begin{cases} \alpha^\#, & \text{if } |t|\leq 1 \\ \alpha^\#/t^2, & \text{if } |t|>1\,; \end{cases}$$

- $h \in W^{m-1/p,p}(\partial\Omega)$, for $m \in \mathbb{N}$ and $p > 1$, such that

$$\exists h_\# > 0 : \quad h(s) \geq h_\#, \quad \text{a.e. } s \in \partial\Omega. \tag{14}$$

- $g \in L^r(\Omega)$, with

$$r = \begin{cases} 2n/(n+2) & \text{if } n \geq 3 \\ 1+\delta & \text{if } n = 2 \end{cases} \tag{15}$$

for all $\delta > 0$. Notice that $\alpha$ can be increasing and/or decreasing.

Hencefurther, we assume $\phi_0 \in C(\Gamma_0)$ by $\phi_0 \equiv 0$. However, it is known that the existence of weak solutions relative to $W^{1,p}_{\text{loc}}(\bar{\Omega}) = W^{1,p}(\Omega)$ ($\Omega$ bounded), it is sufficient the hypothesis $\phi_0 \in W^{1,1}_{\text{loc}}(\bar{\Omega} \setminus \Gamma_0)$ [8, pp.580-581].

# 3 Existence result to the electric problem

This section is devoted to the proof of the following standard existence result for reader's convenience.

**Theorem 3.1** *For $h \in W^{m-1/p,p}(\partial\Omega)$, there exists a unique solenoidal vector field $\mathbf{j} \in \mathbf{W}^{m,p}(\Omega)$ satisfying (6).*

Proof. From the trace theorem, the surjective trace operator

$$\frac{\partial}{\partial n} \in \mathcal{L}(W^{m+1,p}(\Omega); W^{m-1/p,p}(\partial\Omega))$$

has a continuous linear right inverse $\mathcal{R} : W^{m-1/p,p}(\partial\Omega) \to W^{m+1,p}(\Omega)$ such that

$$\frac{\partial}{\partial n}(\mathcal{R}h) = h, \quad \forall h \in W^{m-1/p,p}(\partial\Omega).$$

Then if we take $\mathbf{u} = \nabla(\mathcal{R}h) \in \mathbf{W}^{m,p}(\Omega)$ the isomorphism operator div yields the existence of $\mathbf{w} \in \mathbf{W}^{m,p}_0(\Omega)$ such that

$$\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{u} \quad \text{in } \Omega.$$

Thus, defining $\mathbf{j} = \mathbf{w} - \mathbf{u} \in \mathbf{W}^{m,p}(\Omega)$, it is uniquely defined and it satisfies (3) and (6).

Remark 3.1 *If $mp > n$, we have*

$$W^{m,p}(\Omega) \hookrightarrow C(\bar{\Omega}).$$

*The condition $mp < n$ is equivalent to $(m-1/p)p < n-1$, and we have the embeddings*

$$W^{m,p}(\Omega) \hookrightarrow L^{pn/(n-mp)}(\Omega);$$
$$W^{m-1/p,p}(\partial\Omega) \hookrightarrow L^{p(n-1)/(n-mp)}(\partial\Omega).$$

## 4 Existence results to the thermic problem

We start by proving some properties to the operator related with the weak formulation of the thermic boundary value problem.

**Lemma 4.1** *Assuming (13), we have*

$$\alpha'(t)t^2 \leq \alpha^{\#} \quad , \quad \forall t \in \mathbb{R}; \tag{16}$$

$$\exists \beta > 0 : \qquad |\alpha(t)| \leq \beta \quad , \quad \forall t \in \mathbb{R}. \tag{17}$$

PROOF. The condition (16) trivially holds due to (13). For $t > t_1 > 0$ we get

$$\alpha(t) - \alpha(t_1) \leq -\alpha^{\#}\left(\frac{1}{t} - \frac{1}{t_1}\right).$$

Consequently, $|\alpha(t)| \leq \alpha(t_1) + \alpha^{\#}/t_1 + \alpha^{\#}$, for $|t| \geq 1$. Then, observing that a continuous function in the interval $[-1, 1]$ is bounded, Lemma 4.1 is finished.

**Lemma 4.2** *Assume that (10)-(14) are fulfilled and $mp \geq n$. Let $A : H^1(\Omega) \to [H^1(\Omega)]'$ defined as*

$$\langle A\theta, \eta\rangle = \int_\Omega k(\cdot,\theta)\nabla\theta\cdot\nabla\eta dx + \int_\Omega \alpha'(\theta)\theta\nabla\theta\cdot\mathbf{j}\eta dx + \int_{\partial\Omega} \alpha(\theta)\theta h\eta ds - \int_\Omega \frac{|\mathbf{j}|^2}{\sigma(\cdot,\theta)}\eta dx.$$

*Then*

1. *A is coercive:* $\frac{\langle A\theta,\theta\rangle}{\|\theta\|} \to +\infty$ *as* $\|\theta\|_{H^1(\Omega)} \to +\infty$;
2. *A is demicontinuous:* $\theta_m \to \theta$ *in* $H^1(\Omega)$ *(as* $m \to +\infty$*)* $\Rightarrow A\theta_m \rightharpoonup A\theta$ *in* $[H^1(\Omega)]'$.

PROOF. Let us show that $A$ is well defined for $mp > n$. For arbitrary $\theta \in H^1(\Omega)$, $A\theta$ is linear. Applying the assumptions (10)-(11), (13)-(14) and Lemma 4.1, we obtain $A\theta \in (H^1(\Omega))'$ considering that the embeddings $H^1(\Omega) \hookrightarrow L^4(\Omega)$ and $H^1(\Omega) \hookrightarrow L^4(\partial\Omega)$ hold for $n = 2, 3$.

1. Since

$$\nabla\left[\int_0^\theta \alpha'(\varsigma)\varsigma^2 d\varsigma\right] = \alpha'(\theta)\theta^2\nabla\theta,$$

applying the formula of integration by parts and using (2) it results

$$\int_\Omega \alpha'(\theta)\theta^2\nabla\theta\cdot\mathbf{j}dx = -\int_{\partial\Omega} h\int_0^\theta \alpha'(\varsigma)\varsigma^2 d\varsigma ds.$$

Using Hölder inequality and (16) we obtain

$$\int_{\partial\Omega} h\int_0^\theta \alpha'(\varsigma)\varsigma^2 d\varsigma ds \leq \|h\|_{2,\partial\Omega}\alpha^{\#}\|\theta\|_{2,\partial\Omega}.$$

Then, using the assumptions (10), (12) and (14) we get

$$\begin{aligned}
&\langle A\theta,\theta\rangle = \\
&= \int_\Omega k(\cdot,\theta)|\nabla\theta|^2dx + \int_\Omega \alpha'(\theta)\theta^2\nabla\theta\cdot\mathbf{j}dx + \int_{\partial\Omega}\alpha(\theta)\theta^2hds - \int_\Omega \frac{|\mathbf{j}|^2}{\sigma(\cdot,\theta)}\theta dx \\
&\geq k_\#\|\nabla\theta\|^2_{2,\Omega} - \alpha^\#\|h\|_{2,\partial\Omega}\|\theta\|_{2,\partial\Omega} + \alpha_\# h_\#\|\theta\|^2_{2,\partial\Omega} - \frac{\|\mathbf{j}\|^2_{4,\Omega}}{\sigma_\#}\|\theta\|_{2,\Omega}.
\end{aligned}$$

Thus, the operator $A$ is coercive.

2. Since $\theta_m \to \theta$ in $H^1(\Omega)$, from the continuity and boundedness properties of the Niemytski operators $k$ and $\sigma$ and the functionals $\alpha$ and $\alpha'$, we have

$$\begin{aligned}
k(\cdot,\theta_m)\nabla\theta_m \to k(\cdot,\theta)\nabla\theta &\quad \text{in} \quad \mathbf{L}^2(\Omega);\\
\sigma(\cdot,\theta_m) \to \sigma(\cdot,\theta) &\quad \text{in} \quad L^\infty(\Omega);\\
\alpha'(\theta_m)\nabla\theta_m \to \alpha'(\theta)\nabla\theta &\quad \text{in} \quad \mathbf{L}^2(\Omega);\\
\theta_m\mathbf{j} \to \theta\mathbf{j} &\quad \text{in} \quad \mathbf{L}^{2^*}(\Omega);\\
\alpha(\theta_m)\theta_m h \to \alpha(\theta)\theta h &\quad \text{in} \quad L^{2^*(n-1)/n}(\partial\Omega),
\end{aligned}$$

according to Lemma 4.1. Indeed, for all $\eta \in H^1(\Omega)$ the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ implies that $\theta_m\eta \to \theta\eta$ in $L^2(\Omega)$, for $n = 2, 3$, and the embedding $H^1(\Omega) \hookrightarrow L^{2^*(n-1)/n}(\partial\Omega)$ implies that $\theta_m\eta \to \theta\eta$ in $L^1(\partial\Omega)$. By definition of weak topology, the convergence of $A\theta_m$ to $A\theta$ arises.

If $mp = n$, $\mathbf{j} \in \mathbf{L}^q(\Omega)$ and $h \in L^q(\partial\Omega)$ for all $q > 1$. Thanks to Remark 4.1, we can proceed as for $mp > n$ to conclude that $A$ is a well defined operator satisfying 1 and 2.

Remark 4.1 *From the embedding $H^1(\Omega) \hookrightarrow L^{2^*}(\Omega)$ for $2^* = 2n/(n-2)$ if $n \geq 3$ and for all $2^* > 1$ if $n = 2$, the compact embedding $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ holds for $q < 2^*$. For $\eta \in L^{2^*}(\Omega)$, $\nabla\theta \in L^2(\Omega)$ and $\theta \in L^q(\Omega)$, the term $\theta\nabla\theta\cdot\mathbf{j}\eta \in L^1(\Omega)$ if*

$$\mathbf{j} \in \begin{cases} \mathbf{L}^{2q/(q-4)}(\Omega), & \text{if } n = 2 \\ \mathbf{L}^{3q/(q-3)}(\Omega), & \text{if } n = 3. \end{cases} \tag{18}$$

*From the embeddings $H^1(\Omega) \hookrightarrow L^{2^*(n-1)/n}(\partial\Omega)$ and $H^1(\Omega) \hookrightarrow\hookrightarrow L^r(\partial\Omega)$ for $r < 2^*(n-1)/n$, for $\eta \in L^{2^*(n-1)/n}(\partial\Omega)$ and $\theta \in L^r(\partial\Omega)$ the term $h\theta\eta \in L^1(\partial\Omega)$ if*

$$h \in \begin{cases} L^{r/(r-2)}(\partial\Omega) \quad , & \text{if } n = 2 \\ L^{4r/(3r-4)}(\partial\Omega) \quad , & \text{if } n = 3. \end{cases} \tag{19}$$

Next we prove the following existence result.

**Theorem 4.1** *Assume that the conditions of Lemma 4.2 are fulfilled. For $g \in L^r(\Omega)$, with $r$ given at (15) there exists $\theta \in H^1(\Omega)$ such that*

$$\begin{aligned}
&\int_\Omega k(\cdot,\theta)\nabla\theta\cdot\nabla\eta dx + \int_\Omega \alpha'(\theta)\theta\nabla\theta\cdot\mathbf{j}\eta dx + \int_{\partial\Omega}\alpha(\theta)\theta h\eta ds = \\
&\qquad = \int_\Omega \left(\frac{|\mathbf{j}|^2}{\sigma(\cdot,\theta)} + g\right)\eta dx, \quad \forall\eta \in H^1(\Omega).
\end{aligned} \tag{20}$$

PROOF. Let $\{w_i\}_{i\in\mathbb{N}}$ be a basis in $H^1(\Omega)$, the Galerkin solution $\theta_N \in V_N = \langle w_1, \cdots, w_N\rangle$ satisfies the problem

$$\langle A\theta_N - g, w_i\rangle = 0 \quad , \quad i = 1, \cdots, N. \tag{21}$$

Since $A$ is coercive and demicontinuous (cf. Lemma 4.2) the Brouwer fixed point theorem guarantees the existence of a Galerkin solution which satisfies the estimate [15, pp.558]

$$\exists R > 0 : \quad \|\theta_N\|_V \leq R, \quad \forall N \in \mathbb{N}.$$

We can extract a weakly convergent subsequence, still denoted by $\{\theta_N\}$, i.e.

$$\exists \theta \in H^1(\Omega) : \quad \theta_N \rightharpoonup_{N\to+\infty} \theta \quad \text{ in } H^1(\Omega).$$

From the compact embeddings $H^1(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega)$ for $q < 2n/(n-2)$ and $H^1(\Omega) \hookrightarrow\hookrightarrow L^r(\partial\Omega)$ for $r < 2(n-1)/(n-2)$, we obtain for subsequences $\theta_N \to \theta$ a.e. in $\Omega$ and on $\partial\Omega$.

From the continuity and boundedness properties of the Niemytski operators $k$ and $\sigma$ and the functionals $\alpha$ and $\alpha'$, we have

$$\begin{aligned} k(\cdot, \theta_N) \to k(\cdot, \theta) \text{ and } \sigma(\cdot, \theta_N) \to \sigma(\cdot, \theta) \quad & \text{in} \quad L^\infty(\Omega); \\ \alpha'(\theta_N) \to \alpha'(\theta) \quad & \text{in} \quad L^\infty(\Omega); \\ \alpha(\theta_N) \to \alpha(\theta) \quad & \text{in} \quad L^\infty(\partial\Omega). \end{aligned}$$

Passing to the limit in the Galerkin equation (21) as $N \to +\infty$, it follows

$$\langle A\theta - g, \eta\rangle = 0, \quad \forall \eta \in \cup_{N\in\mathbb{N}} V_N$$

and by density the weak formulation (20) holds for all $\eta \in H^1(\Omega)$.

REMARK 4.2 *The existence of temperature is still valid for $mp < n$ if the conditions (18)-(19) are fulfilled. For $n = 3$, (18) and (19) read $3q/(q-2) > 6$ for $q < 6$ and $4r/(3r-4) > 2$ for $r < 4$, respectively. Thus, according to Remark 3.1, the existence of $\theta \in H^1(\Omega)$ to the problem (20) is still valid for $mp < 3$ if $p > 6/(1+2m)$. Choosing, for instance, $h \in W^{1/2,2}(\partial\Omega)$ Theorem 3.1 guarantees the existence of $\mathbf{j} \in \mathbf{W}^{1,2}(\Omega) \hookrightarrow \mathbf{L}^6(\Omega)$, but the term $\theta_N \nabla\theta_N \mathbf{j}\eta$ does not pass to the limit in the Galerkin procedure.*

**Theorem 4.2** *Under the conditions of Theorem 4.1, each solution $\theta \in H^1(\Omega)$ verifies that $\theta \in W^{1,2+\varepsilon}(\Omega)$, for some $0 < \varepsilon < 1$.*

PROOF. Let $\theta \in H^1(\Omega)$ be any solution to the problem (20), then $\theta\nabla\theta \in \mathbf{L}^q(\Omega)$ for all $1 \leq q < 2 = n$ or $q = n/(n-1)$ if $n \geq 3$. The Sobolev imbedding theorem yields $W^{1,q'n/(n+q')}(\Omega) \hookrightarrow L^{q'}(\Omega)$, thus we have $\theta\nabla\theta \in (W^{1,qn/(qn-n+q)}(\Omega))' \hookrightarrow (W^{1,r'}(\Omega))'$ for $qn/(qn-n+q) < r' < 2$ (choosing $q > 2n/(n+2)$) if $n < 4$. Using the $W^{1,q}$ regularity for solutions to second order elliptic differential equations with bounded measurable coefficients [9], we obtain $\theta \in W^{1,2+\varepsilon}(\Omega)$, for some $\varepsilon > 0$ such that $2 + \varepsilon \leq r$.

## 5 Existence results to the potential problem

The following weak formulation (22) is found as usual by integrating by parts the problem (5) and (8) for all admissible test function. Although the structure of the mixed boundary value problem does not have a standard form, the following existence and regularity results can be stated as in the elliptic theory with bounded discontinuous coefficients [13].

**Proposition 5.1** *Let the assumptions (10)-(15) be fulfilled. For* $p(n+2m) \geq 2n$ *and* $\theta \in H^1(\Omega)$*, there exists a unique solution*

$$\phi \in V := \{v \in H^1(\Omega) : \ v = 0 \ on \ \Gamma_0\}$$

*to the problem*

$$(22) \quad \int_\Omega \sigma(\theta)\nabla\phi \cdot \nabla\eta dx = -\int_\Omega \sigma(\theta)\alpha(\theta)\nabla\theta \cdot \nabla\eta dx + \int_\Gamma h\eta dx, \quad \forall \eta \in V.$$

*Moreover, if* $\theta$ *is a solution in accordance to Theorem 4.1 and* $n = 2$ *then* $\phi \in C^{0,\lambda}(\bar{\Omega})$ *for some* $0 < \lambda < 1$.

Proof. The existence of a unique solution $\phi \in V$ is a consequence of the Lax-Milgram Lemma or the Galerkin method because the left hand side of (22) defines a coercive continuous bilinear form in $V$ and remarking that Remark 3.1 can be applied since $p(n-1)/(n-mp) \geq 2(n-1)/n$ is equivalent to $p(n+2m) \geq 2n$.

Thanks to the regularity theory [13], the solution $\phi \in H^1(\Omega)$ of the mixed boundary value problem (22) is Hölder continuous up to the boundary, $\phi \in C^{0,\lambda}(\bar{\Omega})$ for some $0 < \lambda < 1$, provided that $\sigma(\theta)\alpha(\theta)\nabla\theta \in \mathbf{L}^p(\Omega)$ with $p > n$, and $h \in L^q(\Gamma)$ with $q > n - 1$. Indeed, Theorem 4.2 guarantees the existence of $p = 2 + \varepsilon > n = 2$ which concludes the proof.

**Theorem 5.1** *Under the conditions of Theorem 4.2, the solution* $\phi \in H^1(\Omega)$ *given at Proposition 5.1 verifies that* $\phi \in W^{1,2+\delta}(\Omega)$*, for some* $0 < \delta \leq \varepsilon < 1$.

Proof. Let $\theta \in H^1(\Omega)$ be any solution to the problem (20), then $\nabla\theta \in \mathbf{L}^{2+\varepsilon}(\Omega)$ due to Theorem 4.2. Consequently, we get $\nabla\cdot(\sigma(\theta)\alpha(\theta)\nabla\theta) \in \left(\mathbf{W}^{1,(2+\varepsilon)/(1+\varepsilon)}(\Omega)\right)'$ for $\varepsilon > 0$. Since $(2+\varepsilon)/(1+\varepsilon) < 2$, then $\phi \in W^{1,2+\delta}(\Omega)$ for some $0 < \delta \leq \varepsilon < 1$ (cf. [9]).

## 6 Existence of classical solutions

In this section, we replace the Caratheodory functions $k$ and $\sigma$ by continuous real ones and we assume that $\Omega$ is a $C^2$ domain.

Let $\theta$ be a weak solution to the Poisson equation (4) with the Robin condition (7) and consider the Kirchoff transformation

$$\Theta(x) = \int_0^{\theta(x)} k(z)dz.$$

Then this new function $\Theta$ is solution to the Neumann problem for Poisson equation

$$-\Delta\Theta = \alpha'(\theta)\theta\nabla\theta\cdot\mathbf{j} + \frac{|\mathbf{j}|^2}{\sigma(\theta)} + g \quad \text{in} \quad \Omega \tag{23}$$

$$-\frac{\partial\Theta}{\partial n} = \alpha(\theta)\theta h \quad \text{on} \quad \partial\Omega. \tag{24}$$

Note that a Neumann problem for a Poisson equation, in general, is not solvable. However the regularity theory can be applied on the considered boundary value problem, since the right hand side of the equations (23)-(24) belong to some adequate Lebesgue spaces.

**Proposition 6.1** *If we assume that the problem (20) has continuous coefficient* $k$*,* $g \in L^{n+\epsilon}(\Omega)$ *with* $\epsilon > 0$ *and* $mp > n$*, then each solution* $\theta \in H^1(\Omega)$ *verifies that* $\nabla\theta \in \mathbf{C}(\bar{\Omega})$*.*

Proof. Let $\theta \in H^1(\Omega)$ be any solution to the problem (20). The proof is divided into two parts. Note that thanks to Theorem 4.2 the proof for the first case (i) can be simplified, what does not happen for the second one (ii). By this reason we decided do not use that regularization.

(i) If $n = 2$, we have $\theta\nabla\theta \in \mathbf{L}^q(\Omega)$ and $\theta \in H^1(\Omega) \hookrightarrow W^{1,q}(\Omega) \hookrightarrow W^{1-1/q,q}(\partial\Omega)$ for all $1 < q < 2$. Then the Laplace theory guarantees that

$$k(\theta)\nabla\theta = \nabla\Theta \in \mathbf{W}^{1,q}(\Omega) \hookrightarrow \mathbf{L}^{2q/(2-q)}(\Omega).$$

Applying the bootstrap argument, we have

$$\left.\begin{array}{l} \theta\nabla\theta \in \mathbf{L}^{2q/(2-q)}(\Omega) \\ \theta \in W^{1-\frac{2-q}{2q},\frac{2q}{2-q}}(\partial\Omega) \end{array}\right\} \Rightarrow \nabla\Theta \in \mathbf{W}^{1,2q/(2-q)}(\Omega) \Rightarrow \nabla\theta \in \mathbf{C}(\bar{\Omega}).$$

(ii) If $n \geq 3$, we have $\theta\nabla\theta \in \mathbf{L}^{n/(n-1)}(\Omega)$ and $\theta \in W^{1-(n-1)/n,n/(n-1)}(\partial\Omega)$. Arguing as in the above step, it follows $\nabla\Theta \in \mathbf{W}^{1,n/(n-1)}(\Omega)$. Consequently, $\nabla\theta \in \mathbf{L}^{n/(n-2)}(\Omega)$ which means that $n/(n-2) \leq 2$ if $n \geq 4$. So we are interested in proceeding only when $n = 3$. Therefore $\theta\nabla\theta \in \mathbf{L}^q(\Omega)$ and $\theta \in W^{1-1/q,q}(\partial\Omega)$ for all $q < 3$. Now we get $\nabla\Theta \in \mathbf{W}^{1,q}(\Omega)$ and $\nabla\theta \in \mathbf{L}^{3q/(3-q)}(\Omega)$. Choosing $3/2 < q < 3$, we obtain $\theta\nabla\theta \in \mathbf{L}^r(\Omega)$ and $\theta \in W^{1-1/r,r}(\partial\Omega)$ for $r > 3$ and sucessively $\nabla\Theta \in \mathbf{W}^{1,r}(\Omega)$ and $\nabla\theta \in \mathbf{C}(\bar{\Omega})$.

**Proposition 6.2** *If* $g \in C(\Omega)$ *and* $h \in C^1(\partial\Omega)$*, then each solution* $\theta \in H^1(\Omega)$ *in accordance to Proposition 6.1 verifies* $k(\theta)\nabla\theta \in \mathbf{C}^1(\bar{\Omega})$*.*

Proof. Let $\Theta \in H^1(\Omega)$ be a solution to the problem (23)-(24). Applying the regularity theory [2] and using Proposition 6.1 we conclude that $k(\theta)\nabla\theta = \nabla\Theta \in \mathbf{C}^1(\bar{\Omega})$. Proposition 6.2 is proved.

**Proposition 6.3** *Let* $\theta$ *be a solution in accordance to Proposition 6.2. If* $\sigma \in C(\mathbb{R})$ *and additionally* $k \in C^1(\mathbb{R})$*, then the unique weak solution* $\phi \in V$ *to the problem (22) is a classical solution in the sense of* $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ *such that* $\sigma(\theta)(\nabla\phi + \alpha(\theta)\nabla\theta) \in \mathbf{C}^1(\Omega)$ *and* $\sigma(\theta)\nabla\phi\cdot\mathbf{n}$ *exists on* $\Gamma$*.*

Proof. Let $\theta$ be a solution in accordance to Proposition 6.2 and $\phi \in V$ the unique weak solution to the problem (22). Defining

$$\Phi(x) = \phi(x) + \int_0^{\theta(x)} \alpha(t)dt,$$

the problem (5) and (8)-(9), with $\phi_0 = 0$, can be rewriten as

$$\begin{cases} -\nabla \cdot (\sigma(\theta)\nabla\Phi) = 0 & \text{in } \Omega \\ \sigma(\theta)\nabla\Phi \cdot \mathbf{n} = h & \text{on } \Gamma \\ \Phi = \int_0^\theta \alpha(t)dt, & \text{on } \Gamma_0. \end{cases} \tag{25}$$

Then the mixed boundary value problem (25) for the elliptic equation in divergence form with coefficient in $C(\bar{\Omega})$ admits a unique classical solution (cf. Remark 6.1). Consequently, we have $\phi \in C^1(\Omega) \cap C(\bar{\Omega})$ in the conditions of Proposition 6.3.

Remark 6.1 *We refer to [8, pp.228-229, 570-580] the meaning of classical solution to mixed boundary value problem for the elliptic differential operator of the second order $L = -\nabla \cdot (a\nabla)$ as the solution which belongs to*

$$C^2_L(\Omega) \cap C(\Omega \cup \Gamma_0) \cap C^1_{n_L}(\bar{\Omega} \setminus \Gamma_0).$$

Remark 6.2 *If the electrical conductivity function $\sigma$ is assumed of class $C^1$, $\sigma \in C^1(\mathbb{R})$, any potential solution $\phi \in C^{0,\lambda}(\bar{\Omega}) \cap W^{1,2+\delta}(\Omega)$ belongs to $W^{2,p}(\Omega)$ for $p = \min\{\frac{2}{2-\lambda}, \frac{2(2+\delta)}{4+\delta}\}$, under the hypothesis $\nabla \cdot (\sigma(\theta)\alpha(\theta)\nabla\theta) \in L^p(\Omega), \phi_0 \in W^{2-(1/p),p}(\Gamma_0)$ and $h \in W^{1-(1/p),p}(\Gamma)$[3]. Moreover, it satisfies the estimate*

$$\|\phi\|_{2,p,\Omega} \le C(\|\nabla \cdot (\sigma(\theta)\alpha(\theta)\nabla\theta)\|_{p,\Omega} + \|\phi_0\|_{2-(1/p),p,\Gamma_0} \\ + \|h\|_{1-(1/p),p,\Gamma} + \|\phi\|_{0,\lambda,\Omega} + \|\phi\|_{1,q,\Omega}.$$

*This interesting regularity result as well as other ones like the $W^{1,\infty}$ or $C^{1,\lambda}$ interior and boundary estimates exclude the application to many industrial devices, since it is essential that the domain $\Omega$ is, at least, $C^2$, $C^1$ or $C^{1,\alpha}$, respectively.*

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LUISA CONSIGLIERI
Department of Mathematics and CMAF,
Faculty of Sciences, University of Lisbon,
1749-016 Lisboa, Portugal
lcconsiglieri@fc.ul.pt