Hybrid finite computation

Luís Mendes Gomes[1] and José Félix Costa[2]

[1] Departament of Mathematics, University of Azores
lmg@notes.uac.pt
[2] Departament of Mathemtics, I.S.T., Technical University of Lisbon
Centro de Matemática e Aplicações Fundamentais, Complexo Interdisciplinar, University of Lisbon
fgc@math.ist.utl.pt

Abstract. Taking the most simple kind of finite state automaton, typically used in the digital stage, whose states are continuous instead of discrete, we show that such automata can only recognize periodic infinite patterns. In our case such patterns are generated by real recursive functions, a new trend in analog stage, which are an extension to reals of Kleene' s recursive functions. And, thus, we show that automata can only recognize periodic real recursive functions, which we also show that are naturally approximated by Fourier series. With these results in hand, we are bringing together not only the concept of periodicity into the real recursive function theory, and consequently the Fourier series, but also the automata, with continuous states, and their computational limits, in a mathematical characterization of hybrid finite computation.

1 Introduction

In a broad sense, hybrid computation includes all computing techniques combining some of the features of digital computations with some of the features of analog computations. Recall that digital computation has been dominated by the unified work of Turing since mid 30s, while analog computation has not yet experienced that unification. Consequently, there is also a lack of consensus about the most appropriated formal characterization for hybrid computation. But it is well known that hybrid computation occurs at the crossroad of several scientific directions: it is based on several ideas coming from computer science and mathematics and gathered in hybrid systems (see e.g. [2]); it is at the intersection of numerical analysis and computer algebra [4].

In the theory of analog computation, that has its roots on Claude Shannon's General Purpose Analog Computer (GPAC) [11], each state of a machine is continuous rather than discrete. Much of the research that has been made, recently, in this kind of computation is included in a wider program of exploring alternative approaches to classical computation. Among these approaches we have neural networks and quantum computation, and an idealization of numerical algorithms, where real numbers are entities in themselves rather than (finite) strings of digits. Actually, one of the most interesting and elegant approaches to analog computation was introduced by Cris Moore, in his seminal paper [7], which is analogous to Kleene's recursive function theory [5].

Real recursion theory, introduced in [7], has been considered as a model of analog computation, and it has been also used to obtain analog characterizations of classical computational complexity classes [9]. One of the operators that was borrowed from classical recursion theory, the analog minimalization, is far from physical realizability and does not fit well the analytic realm of analog computation. But, as it was emphasized in [8], a most natural operator captured from Analysis, the operator of taking a limit, can be used properly to enhance the real recursion theory, providing not only good solutions to puzzling problems raised by the original model but also providing the opportunity to bring together classical computation and real and complex Analysis.

Automata theory is, usually, faced as the study of sets of strings or ω-strings over a finite alphabet accepted by finite state machines. Recently, some work has been done to lift concepts of automata theory from discrete to continuous time [10]. Instead of signals defined over a discrete sequences of time instants, it is considered signals defined over non-negative reals. An interesting subclass of such signals is the set of piecewise continuous functions, because the well-known relationship with Fourier analysis (see e.g. [15]).

In this paper, we take as our starting point the real recursive function theory of [7], but following the more recent work of [8], and, then, we define what are periodic real recursive functions and their relationship with Fourier series. Finally, we show that finite state automata, where their states are continuous instead of discrete, considered a version of automata over continuous-time found in [10], can only recognize such periodic real recursive functions.

2 Real recursive functions

In [7], it was defined a set of (vector-valued) functions over $\mathbb{R}^n$, called $\mathbb{R}$-recursive functions, following the inductive approach taken for the construction of recursive functions over $\mathbb{N}$ that we can found in e.g. [5]. The discrete recursive operator is replaced by a differential recursion operator, which is the analog counterpart of classical recurrence, and, thus, the set of $\mathbb{R}$-recursive functions generates a model of continuous time computation. More recently, in [8], the μ-operator defined over the $\mathbb{R}$-recursive functions, which is also the analog counterpart of the classical minimalization operator, was replaced by an infinite limit operator. This operator is captured from Analysis and, as we can see in [8], it can be used properly to enhance the theory of recursion over the reals, providing good solutions to puzzling problems raised by the original model in [7].

Definition 1. *[8] The class* $REC(\mathbb{R})$ *of real recursive vector functions* [3] *is generated from the real recursive scalars* $0,\ 1,\ -1$*, and the real recursive projections* $I_n^i(x_1, \ldots, x_n) = x_i,\ 1 \leq i \leq n,\ n > 0$*, by the following operators:*

[3] Hereafter, for short, we will say only real recursive functions.

Composition: if f is a real recursive vector function with n k-ary components and g is a real recursive vector function with k m-ary components, then the vector function with n m-ary components, $1 \le i \le n$,

$$\lambda x_1 \dots \lambda x_m.\ f_i(g_1(x_1,\dots,x_m),\dots,g_k(x_1,\dots,x_m))$$

is real recursive.
Differential recursion: if f is a real recursive vector function with n k-ary components and g is a real recursive vector function with n $(k+n+1)$-ary components, then the vector function h of n $(k+1)$-ary components which is the solution of the Cauchy problem, $1 \le i \le n$,

$$h_i(x_1,\dots,x_k,0) = f_i(x_1,\dots,x_k),$$

$$\partial_y h_i(x_1,\dots,x_k,y) = g_i(x_1,\dots,x_k,y,h_1(x_1,\dots,x_k,y),\dots,h_n(x_1,\dots,x_k,y))$$

is real recursive whenever h is of the class C^1 on the largest interval containing 0 in which a unique solution exists.
Infinite limits: if f is a real recursive vector function with n $(k+1)$-ary components, then the vector functions h, h^{inf}, h^{sup} with n k-ary components,

$$h_i(x_1,\dots,x_k) = \lim_{y\to\infty}\ f_i(x_1,\dots,x_k,y),$$

$$h_i^{inf}(x_1,\dots,x_k) = \liminf_{y\to\infty}\ f_i(x_1,\dots,x_k,y),$$

$$h_i^{sup}(x_1,\dots,x_k) = \limsup_{y\to\infty}\ f_i(x_1,\dots,x_k,y),$$

$1 \le i \le n$, are real recursive.
Assembling and designating components: (a) arbitrary real recursive vector functions can be defined by assembling scalar real recursive function components into a vector function; (b) if f is a real recursive vector function, then each of its components is a real recursive scalar function.
Real recursive numbers: arbitrary real recursive scalar functions of arity 0 are called real recursive numbers. □

Constant functions 0_n, 1_n, -1_n which are n-ary can be derived from unary constant functions by means of projections. For example, $1_n(x_1,\dots,x_n) = 1$ can be defined as $1_1(I_n^1(x_1,\dots,x_n)) = 1$, and constant functions of arity one can be derived by differential recursion as follows: $0(0) = 0$, $\partial_y 0(y) = I_2^2(y, 0(y))$; $u(0) = c, \partial_y u(y) = 0(I_2^1(y,u(y)))$, where $c = 1, -1$.

The functions $+$, $\times$, $-$, sin, cos and $\lambda x.\ \frac{1}{x}$ are real recursive functions. Let us define $+(x,0) = I_1^1(x) = x$, $\partial_y + (x,y) = 1_3(x,y,+(x,y))$. Analogously we can get $\times(x,0) = 0_1(x)$, $\partial_y \times (x,y) = I_3^1(x,y,\times(x,y))$, hence we have by a composition $-(x,y) = +(x,\times(-1,y))$. Furthermore, the vector $(\sin(x),\cos(x))$ and its components can be defined by such differential recursion:

$$\begin{pmatrix}\sin\\ \cos\end{pmatrix}(0) = \begin{pmatrix}0\\ 1\end{pmatrix},\ \partial_y \begin{pmatrix}\sin\\ \cos\end{pmatrix}(y) = \begin{pmatrix}I_3^3\\ -I_3^2\end{pmatrix}(y,\sin y,\cos y).$$

Now for $\lambda x.\ \frac{1}{x}$, we define $h(x) = \frac{1}{x+1}$ (h is defined in the interval $(-1, \infty)$) in the following way: $h(0) = 1$, $\partial_x h(x) = \times(-1, \times(h(x), h(x)))$, and then we compose h with $\lambda x.\ x - 1$. The division is simply a composition of $\times$ and $\lambda x.\ \frac{1}{x}$ (with the domain equal to $(0, \infty)$, but we can extend the division to the negative numbers via a definition by cases).

We can construct also other special real recursive functions. The Kronecker δ function, the signum function, the Heaviside Θ function (equal to 1 if $x \geq 0$, otherwise 0), and the square-wave function s are real recursive functions. So, it is sufficient to take the following definitions: if $\delta(0) = 1$ and for all $x \neq 0$ we have $\delta(x) = 0$, then let us define $\delta(x) = \liminf_{y \to \infty} (\frac{1}{1+x^2})^y$. From the function $\lambda xy.\ \frac{2}{1+e^{-xy}} - 1$, we obtain

$$sgn(x) = \liminf_{y \to \infty}\ \frac{2}{1+e^{-xy}} - 1 = \begin{cases} 1\ if\ x > 0 \\ 0\ if\ x = 0 \\ -1\ if\ x < 0 \end{cases} .$$

Let $\Theta(x) = (sgn(x) + \delta(x) + 1)/2$ and $s(x) = \Theta(\sin(\pi x))$.

In some examples we can use in constructions the predicate of equality $eq = \lambda xy.\ \delta(x - y)$. Sometimes we will use Θ to control whether points are in given intervals. Then for $x \in [a, \infty)$ we have the characteristic function $\Theta(x - a)$ and for $x \in [a, b]$ we can define $\Theta_{[a,b]}(x) = \Theta(x - a)\Theta(b - x)$.

Let us add that we can find real recursive numbers (computable reals in our framework) as values of real recursive functions of arity one for, let us say, an argument equal to 0. Of course the argument can be changed to a real recursive number t by a composition of a given real recursive function with $\lambda x.\ x + t$. In this sense e and π are computable reals: $e = exp(1)$, $\pi = 4 \arctan(1)$, where $\arctan(0) = 0$, $\partial y \arctan(x) = \frac{1}{1+x^2}$.

For differential recursion, the domain is restricted to an interval of continuity and, thus, preserving the analitycity of functions that can be generated by this differential schema. For example, using differential recursion we can not define functions such as $\lambda x.|x|$. It is excluded the possibility of operations on undefined functions: functions are strict in the meaning that for undefined arguments they are also undefined. But we can define such function by $|x| = sgn(x) \times x$.

Notice that, in Definition 1, we guarantee the existence of a solution for the differential recursion whose first derivative is continuous on the largest interval containing 0, in which it is also unique. Since [3], it has been discussing a definition for a solution for such differential recursion in order to say, precisely, what is a solution for such differential recursion schema. For example, in differential recursion, it is not imposed that functions f_i and g_i, for $1 \leq i \leq n$, are of class C^1. Several examples have been taking to motivate the adequacy of the definition of a solution to a system of differential equations (see e.g. [9]).

3 Periodic real recursive functions

The classical theory of Fourier series and integrals, as well as Laplace transforms, is of great importance for physical and technical applications, since it enable us to reason about many periodic phenomena in nature, and, then, periodic functions play the main role to model them. In what follows, we will carry only Fourier series, and a particular type of periodic functions, to real recursive function theory. All background about Fourier analysis can be found in [15], which will be our source of notation and terminology.

Definition 2. *We say that a real recursive function f is periodic with a real recursive period z if, for every $x \in \mathbb{R}$, $f(x+z) = f(x)$.* □

Proposition 1. *If f is a periodic recursive function with a real recursive period z (> 0), then the Fourier coefficients a_n and b_n are real recursive numbers.* □

Proof. For every $x \in \mathbb{R}$ and every $n \in \mathbb{N}$, $\frac{2}{z}\ f(x)\ cos(\frac{2\pi nx}{z})$ and $\frac{2}{z}\ f(x)\ sin(\frac{2\pi nx}{z})$ are real recursive expressions. For some $n \in \mathbb{N}$, let g_n be a recursive function defined as follows: $g_n(0) = 0,\ \partial_x\ g_n(x) = \frac{2}{z}\ f(x)\ cos(\frac{2\pi nx}{z})$. So, we obtain the following real recursive expression:

$$g_n(y) = \frac{2}{z} \int_0^y\ f(x)\ cos(\frac{2\pi nx}{z})\ dx$$

Therefore, $g_n(z) = a_n$ is a real recursive number. Analogously, for b_n. □

The necessary and sufficient condition for the application of fundamental theorem for Fourier series say us that the function must be periodic piecewise smooth on $\mathbb{R}$. Until now, we did not say nothing about this kind of function in real recursive function framework. So

Proposition 2. *If $f_1, \ldots, f_k$ are real recursive scalar total functions and $a_1, \ldots, a_k$ are real recursive numbers such that $a_1 < \ldots < a_k$, then*

$$\lambda x.\ \Theta_{[a_1,a_2)}(x) \times f_1(x) + \ldots + \Theta_{[a_k,+\infty)}(x) \times f_k(x)$$

is a real recursive function. □

We call the real recursive function given just above a *piecewise real recursive function* if it is according with the mentioned conditions. Thus, we are ready to generate a bridge between Fourier series and the real recursive function theory [8] and, in broad sense, to periodic piecewise functions via their Fourier expansions.

Proposition 3. *If f is a Fourier expansion with real recursive numbers as coefficients, then f is a real recursive expression.* □

Proof. The expressions $sin(\frac{2\pi nx}{z})$ and $cos(\frac{2\pi nx}{z})$ are both real recursive, since $z \neq 0$. Then,

$$a_n sin(\frac{2\pi nx}{z}) + b_n cos(\frac{2\pi nx}{z})$$

is also a real recursive expression, because a_n and b_n, the Fourier coefficients, are real recursive numbers (see Proposition 1.), as well as, its finite sum

$$\sum_{n=1}^{\lfloor y \rfloor}\ a_n sin(\frac{2\pi nx}{z}) + b_n cos(\frac{2\pi nx}{z}).$$

Then

$$\lim_{y\to 0}\sum_{n=1}^{\lfloor y\rfloor} a_n sin(\frac{2\pi nx}{z}) + b_n cos(\frac{2\pi nx}{z})$$

is also a real recursive expression. And, finally,

$$\frac{a_0}{2} + \lim_{y\to 0}\sum_{n=1}^{\lfloor y\rfloor} a_n sin(\frac{2\pi nx}{z}) + b_n cos(\frac{2\pi nx}{z})$$

is a real recursive expression, which is the expression for the Fourier expansion. □

Definition 3. *A function f is said to be partially periodic with period z if there exists z' such that, for every $x \geq z'$, $f(x+z) = f(x)$.* □

Each partially periodic function f is indeed periodic after a point z' and, between 0 and z', we will require that there exists a finite number of discontinuities. It is obvious that if $z' = 0$, f is periodic in the sense of Definition 2. In particular, piecewise functions defined on non-negative reals which are divided in two distinct parts: the first part it is defined between 0 and some non-negative real x, where there is a finite number of points of discontinuity, and the second part it is defined on non-negative reals greater than x but it is periodic. We will see that the finiteness of points of discontinuity in the left of x and periodicity of the function in the right of x is the essential property to tackle the restriction of finite state machines in the recognition of infinite signals.

4 Automata can only recognize periodic real recursive functions

In this section, we will study the computational power of continuous automata which are able to process piecewise continuous signals. For this, as we will see during the exposition and the proof of the result itself that is irrelevant the form of the function taken in each interval of time of each piecewise continuous signal. So, we restrict our attention to piecewise linear signals which are, particularly, apropriated for a representation based on ω-words, which holds the definition of the required automata in its simplest form. Previous work had been done with the simplest class of piecewise signals: the class of piecewise constant signals (see e.g. [10]); and also with piecewise constant derivatives signals [1].

We construct a certain kind of finite state automaton, whose states are continuous instead of discrete, and show that they only recognize partially periodic piecewise real recursive functions, i.e. partially-periodic piecewise real recursive functions with period z (> 0) which have a finite number of discontinuities between 0 and z. We assume that piecewise real recursive functions are defined over $\mathbb{R}_0^+$. In particular, without loss of generality, we study partially periodic piecewise linear functions because, between two points of discontinuity, the derivative of such functions is constant and, then, it is representable by a computable number in the classical sense (e.g. integer). In this case, we take the approach based on signals and automata over continuous-time found in [10] that inspired us.

In general a signal is a total function from $\mathbb{R}_0^+$ to $\mathbb{R}$. The well known square-wave and sawtooth-wave functions are examples of such signals. In what follows, we describe piecewise linear real recursive functions as signals which, in turn, are described by ω-words over $\mathbb{Z}$.

Definition 4. *A piecewise linear signal s over $\mathbb{Z}$ is a four-tuple $< \alpha, \beta, \theta, \tau >$, where α, β and θ are ω-words over $\mathbb{Z}$ such that, for every $i \in \mathbb{N}$, if $\alpha_i > 0$ then $\theta_i > \beta_i$; otherwise, if $\alpha_i < 0$ then $\theta_i < \beta_i$; otherwise, $\theta_i = \beta_i$, and τ is an unbounded increasing ω-word over $\mathbb{Z}$ such that $\tau_0 = 0$ and, for every $i \in \mathbb{N}$ and every $t \in [\tau_i, \tau_{i+1})$, s_i of s defined on $[\tau_i, \tau_{i+1})$ by*

$$s_i(t) = \beta_i + \int_{\tau_i}^{t} \alpha_i dt'$$

is a real recursive function. □

In each interval $[\tau_i, \tau_{i+1})$, the derivative of the (linear) real recursive function s_i is denoted by α_i, the value of $s_i(\tau_i)$ is denoted by β_i (i.e. the initial value) and, finally, θ_i denotes the maximum or minimum value taken by s_i in $[\tau_i, \tau_{i+1})$ according to the value of α_i. And, thus, we are not providing the piecewise linear signal itself but its first derivative which it is not necessarily continuous. We denote the set of piecewise linear signals over $\mathbb{Z}$ by $\mathbf{PLIN}(\mathbb{Z})$. Alternatively, each piecewise linear signal s above can also be represented by

$$s(t) = \lim_{y \to \infty} \sum_{1 \leq i \leq \lfloor y \rfloor} \Theta_{[\tau_i, \tau_{i+1})} \times s_i(t).$$

In general, $s(t)$ is not a real recursive function.

In operational sense, for $\tau_i \leq t < \tau_{i+1}$, if $\alpha_i > 0$, $s_i(t)$ increases from β_i until θ_i, since $\theta_i > \beta_i$; otherwise, if $\alpha_i < 0$ then $s_i(t)$ decreases from β_i until θ_i, since $\theta_i < \beta_i$; otherwise, $s_i(t)$ remains constant. As this construction suggests, each of such signals is a discontinuous piecewise linear function which has an infinite number of discontinuities. But, in particular, for every $i \in \mathbb{N}$, if $\beta_{i+1} = s_i(\theta_i)$ then we obtain a continuous piecewise linear signal.

With the above representation for signals it is easy to say what is a partially-periodic piecewise linear signal based on it. A particular case of such signals are the periodic piecewise linear signals.

Definition 5. *We say that a piecewise linear signal $< \alpha, \beta, \theta, \tau >$ over $\mathbb{Z}$ is partially periodic with period p if there exists $i \in \mathbb{N}$ such that, for every $j > i$, $(\alpha_j, \beta_j, \theta_j) = (\alpha_{j+p}, \beta_{j+p}, \theta_{j+p})$.* □

As we can see above, the sufficient condition for a piecewise linear signal to be partially periodic is obtained by considering a time instant for which the triple formed by the first derivative, the initial value, and the maximum (or minimum) value of the signal, after a given period greater than 0, are equal. Notice that in the beginning of time such signals exhibits a non-periodic pattern with a finite number of discontinuities which is followed by a periodic pattern that, in our case, is a periodic piecewise linear signal. We denote by $\mathbf{PPPLIN}(\mathbb{Z})$ the set of partially periodic piecewise linear signals, whose representation are generated by computable infinite sequences α, β, θ and τ. And, thus, every signal in $\mathbf{PPPLIN}(\mathbb{Z})$ is a real recursive function.

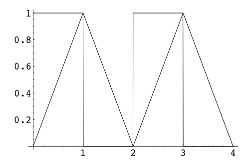

Fig. 1. Square and triangle waves

Consider the simplest form of automaton where the set of states is finite. Usually, such states are seen as abstract entities suitable to describe discrete behaviors of systems in a given level of abstraction [6]. But, here, we take an approach that has been taken by hybrid systems community (see e.g. [12]) where, in a simplest case, we have a state variable which is described by a continuous behavior over time. In this case, the automaton represents the evolution of the system through time where each transition represents the change of the regime of operation (i.e. the state of the automaton) which is described by a differential equation. So, to be coherent with signals considered above, each automaton has its state equipped with a first-order differential equation of the form $\frac{ds}{dt} = k$. Formally,

Definition 6. *A continuous automaton $\mathcal{A}$ over $\mathbb{Z}$ is a triple (Q, δ, q_0) where*

- $Q = \{(c, b, d) \in \mathbb{Z}^3 : \ b \leq d \text{ and } c \geq 0\} \cup \{(c, b, d) \in \mathbb{Z}^3 : \ b > d \text{ and } c < 0\}$ *is a finite set (of states);*
- $\delta : Q \times \mathbb{N}^2 \to Q$ *is a function (the transition function) such that, for every* $(c, b, d), (c', b', d') \in Q$ *and* $(a, a') \in \mathbb{N}^2$ *such that* $a' > a$,

$$\delta((c, b, d), (a, a')) = (c', b', d')$$

iff $c' \leq 0$ *if* $c > 0$, *or* $c' \geq 0$ *if* $c < 0$, *or* $((c' \in \mathbb{Z} - \{0\})$ *or* $(c' = 0$ *and* $b' \neq b))$ *if* $c = 0$.
- $q_0 \in Q$ *(the initial state).* □

By the condition imposed, in the definition just above, to define the transition function, we can see that a transition take place only when the value of c changes, except in the case of $c = 0$, which can remain as 0, and in this case we must take $b' \neq b$.

We say that a piecewise linear signal $s =< \alpha, \beta, \theta, \tau >$ over $\mathbb{Z}$ is *accepted* by (or is a solution of) $\mathcal{A}$ if there exists an infinite sequence $(b_0, c_0, d_0)\ldots$ over Q such that (b_0, c_0, d_0) is the initial state, and, for every $i \in \mathbb{N}$, $b_i = \beta_i$, $c_i = \alpha_i$, $d_i = \theta_i$ and $(b_{i+1}, c_{i+1}, d_{i+1}) = \delta((b_i, c_i, d_i), (\tau_i, \tau_{i+1}))$.

As an example, consider the square wave and triangle wave signals in figure above. It is easy to see that continuous automata have, respectively,

$Q_{sq} = \{(1, 0, 1), (0, 0, 0)\}$, $q_0^{sq} = (1, 0, 1)$ and

$$\delta_{sq} = \{((1, 0, 1), (i, i + 1), (0, 0, 0)), ((0, 0, 0), (i + 1, i + 2), (1, 0, 1)) : \ i \geq 0\},$$

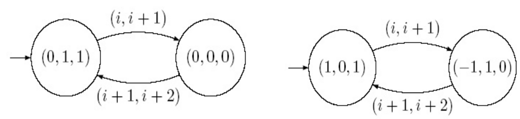

Fig. 2. Continuous automata for square and triangle waves

and $Q_{tri} = \{(0, 1, 1), (1, -1, 0)\}$, $q_0^{tri} = (0, 1, 1)$ and

$$\delta_{tri} = \{((0, 1, 1), (i, i + 1), (1, -1, 0)), ((1, -1, 0), (i + 1, i + 2), (0, 1, 1)) : \ i \geq 0\}.$$

Proposition 4. *A piecewise linear signal s is accepted by a continuous automaton $\mathcal{A}$ if and only if $s \in$ **PPPLIN**$(\mathbb{Z})$.* □

Proof. If s is a piecewise linear signal accepted by $\mathcal{A}$, then there exists an infinite sequence $(b_0, c_0, d_0)\ldots$ over Q such that (b_0, c_0, d_0) is the initial state and, for every $i \in \mathbb{N}$, $\delta((b_i, c_i, d_i), (\tau_i, \tau_{i+1})) = (b_{i+1}, c_{i+1}, d_{i+1})$. Since Q is finite, there exists $i < j$ such that $(b_i, c_i, d_i) = (b_j, c_j, d_j)$. Let $p = j - i$, Therefore, for every $n \geq i$, $(b_n, c_n, d_n) = (b_{n+p}, c_{n+p}, d_{n+p})$. Conversely, if $s =< \alpha, \beta, \theta, \tau >$ is a partially periodic piecewise linear signal, then there exist n_0 and $p > 0$ such that, for every $n \geq n_0$, $\alpha_n = \alpha_{n+p}$, $\beta_n = \beta_{n+p}$, and $\theta_n = \theta_{n+p}$. Consider the continuous automaton $\mathcal{A} = (Q, \delta, q_0)$ where $Q = \{(\beta_i, \alpha_i, \theta_i) \in \mathbb{Z}^3 : i \in \mathbb{N}\}$ and, for every $i \geq 0$, $\delta((\beta_i, \alpha_i, \theta_i), (\tau_i, \tau_{i+1}) = (\beta_{i+1}, \alpha_{i+1}, \theta_{i+1})$, and $(\beta_0, \alpha_0, \theta_0)$ is the initial state. So, $\mathcal{A}$ accepts s. Since s is partially periodic, then there exists $i \in \mathbb{N}$ such that, for every $j \geq i$, $(\beta_j, \alpha_j, \theta_j) = (\beta_{j+(j-i)}, \alpha_{j+(j-i)}, \theta_{j+(j-i)})$. Therefore, Q is finite. □

The above result show us that each continuous finite state automaton can accepted only periodic piecewise signals, no matters what function we take between consecutive discontinuities. And, thus, periodicity impose the computational power for the continuous finite state automata.

5 Conclusions and further work

Real recursion theory, introduced in [7], has been considered as a model of analog computation. As it was enhanced in [8], the operator of taking a limit captured from analysis, can be also used properly to provide the opportunity to bring together classical computation and real and complex analysis. In this paper we bring together Fourier analysis and continuous finite state automata. Then, we show that the ingredients needed to deal with Fourier series, namely the piecewise smooth periodic functions, can be embodied in the framework of real recursive functions, originally introduced in [7] and revised and expanded in [8]. It seems obvious that not all piecewise smooth periodic functions can be accepted by continuous finite state automata (see e.g. [10]). Then, we also show that a special kind of automaton over continuous time can only be able to accept periodic piecewise linear signals, for which it is possible a characterization by ω-words is provided.

Further work is envisaged to explore how our model of hybrid computation can be extended to take into account interaction in a given notion of analog network (see e.g. [14]), following the definitons and suggestions for interaction introduced in [13].

References

1. E. Asarin, O. Maler, and A. Pnueli. Reachability analysis of dynamical systems having piecewise-constant derivatives. *Theoretical Computer Science*, 138:35–65, 1995.
2. M. S. Branicky. Universal computation and other capabilities of hybrid and continuous dynamical systems. *Theoretical Computer Science*, 138:67–100, 1995.
3. M. L. Campagnolo. *Computational Complexity of Real Valued Recursive Functions and Analog Circuits*. PhD thesis, IST, Universidade Técnica de Lisboa, July 2001.
4. J. D. Dora, A. Maignan, M. Mirica-Ruse, and S. Yovine. Hybrid computation. In *Proceedings of International Symposium on Symbolic and Algebraic Computation, ISSAC'01*, pages 101–108. ACM Press, 2001.
5. S. Kleene. Arithmetical predicates and quantifiers. *Transactions American Mathematical Society*, 79:312–340, 1955.
6. M. Minsky. *Computation: Finite and Infinite Machines*. Prentice Hall, 1972.
7. C. Moore. Recursion theory on the reals and continuous-time computation. *Theoretical Computer Science*, 162:23–44, 1996.
8. J. Mycka and J. F. Costa. Real recursive functions and their hierarchy. *Journal of Complexity*, 20:835–857, 2004.
9. J. Mycka and J. F. Costa. The P$\neq$ NP conjecture in the context of real and complex analysis. *Journal of Complexity*, 22 (2):287–303, 2006.
10. A. Rabinovich. Automata over continuous time. *Theoretical Computer Science*, 300:331–363, 2003.
11. C. Shannon. Mathematical theory of the differential analyser. *Journal Mathematical Physics*, 20:337–354, 1941.
12. B. Trakhtenbrot. Automata and hybrid systems. Technical report, Uppsala University, 1998.
13. B. Trakhtenbrot. Automata and their interaction: definitional suggestions. *Fundamentals of Computation Theory*, 1684:54–89, 1999.
14. J.V. Tucker and J.I. Zucker. Computability of analog networks. *Theoretical Computer Science*, 2006. to appear.
15. A. Vretblad. *Fourier Analysis and Its Applications*, volume 223 of *Graduate Texts in Mathematics*. Springer-Verlag, 2003.