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Saint-Venant's principle and its connections to topology optimization

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Abstract A version of Saint-Venant's principle is stated and proven for a scalar elliptic equation in a domain of arbitrary shape, loaded only in a small ball. Some links are pointed out to the bubble method in topology optimization: when a small hole is introduced in a given shape, the difference between the perturbed solution and the unperturbed one satisfies the hypotheses of Saint-Venant's principle. An important tool is the Poincaré-Wirtinger inequality for functions defined on a sphere; results from spectral geometry are used to determine the constant therein.

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1. Introduction

Saint-Venant's principle is a well-known result in solid mechanics. It states that, when one applies forces on a small region of a body, the local distribution of the forces has little effect on the deformation of the body far from that region. What matters is the resultant force and the resultant momentum. Initially, this principle has been stated and widely used for the study of elastic beams: when forces are applied at one end of the beam, the relevant quantities for the overall behaviour of the beam are the resultant force and the resultant momentum, not the local distribution of the forces. Later, it was realized that this principle can be stated for general three-dimensional elastic bodies.

Saint-Venant's principle has often been stated and used in a loose form, without mathematical precision. It turns out that its rigorous proof is far from trivial. Knowles [1967] gave a mathematical estimate of the decay of solutions of a scalar elliptic equation in a bar-shaped domain, in two dimensions.

In Section 2 of the present paper, we prove a version of the Saint-Venant principle for a scalar elliptic equation, in a domain of arbitrary shape, in any space dimension (Theorem 2.1). The small parameter is the radius ρ of a ball where the "forces" are applied. As noted in Remark 2.5, there is no need for the considered function to be defined inside that ball; this corresponds to a body having a small hole. It should be stressed that, for a scalar equation, the equilibrium condition involves only the resultant of the applied "forces", and not the momentum (which is not defined). We obtain a power-law decay of the energy, which contradicts the usual expectation of an exponential decay (see Remark 2.6).

Some links to topology optimization are pointed out in Section 3. We consider an infinitesimal hole in a given body and study the perturbation induced in the solution of the scalar elliptic equation. An asymptotic development of the perturbed solution is obtained (Theorem 3.2) in terms of an auxiliary function w_ρ. This result is weak in the sense that it does not imply the formula of the topological derivative for the energy functional; however it projects a new light for more general functionals (see Remark 3.4). By using Saint-Venant's principle, we can improve slightly the asymptotic development far away from the support of the infinitesimal hole (see Remark 3.6).

In the Appendix, several auxiliary results are stated and proven. One of them is Poincaré-Wirtinger inequality for functions defined on spheres (Lemma A.5). It uses a result from spectral geometry about the second eigenvalue of the Laplace operator (Remarks A.3 and A.4). Lemma A.6 states the regularity of functions having zero Laplacian. Lemma A.7 gives a formula for the derivative (with respect to variations of the boundary of the domain) of the integral of a fixed function.

2. Saint-Venant's principle

For $r > 0$ and $x_0 \in \mathbb{R}^n$ we shall denote by $B_n(x_0, r) = \{x \in \mathbb{R}^n | |x - x_0| < r\}$ the ball centered at x_0 and of radius r, and by $S_{n-1}(x_0, r) = \{x \in \mathbb{R}^n | |x - x_0| = r\}$ the sphere. We shall also use, in integrals, the short notation $|x - x_0| = r$ for $S_{n-1}(x_0, r)$.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a domain having Lipschitz boundary, and x_0 a fixed arbitrary point in Ω. Let $a \in L^\infty(\Omega, [\alpha, \beta])$ be a function such that a is constant in a ball $B_n(x_0, R)$ of fixed radius R. Let $\rho > 0$ be a small parameter and consider a function $U_\rho \in H^1(\Omega)$ verifying*

$$div(a\nabla U_\rho) = 0 \quad in \quad \Omega_\rho = \Omega \setminus \overline{B_n(x_0, \rho)} \tag{2.1}$$

$$U_\rho = 0 \quad on \quad \Gamma_D \tag{2.2}$$

$$a\nabla U_\rho n = 0 \quad on \quad \Gamma_N \tag{2.3}$$

where $\partial\Omega = \bar\Gamma_D \cup \bar\Gamma_N$ *and* $\Gamma_D \cap \Gamma_N = \emptyset$*. Suppose* U_ρ *satisfies*

$$\int_{|x-x_0|=\rho} a\nabla U_\rho n = 0 \tag{2.4}$$

where the trace of $a\nabla U_\rho$ *is considered from* Ω_ρ *onto the sphere* $S_{n-1}(x_0,\rho)$*.*
For each $r \in [\rho, R]$*, define the energy in* $\Omega_r = \Omega \setminus \overline{B_n(x_0,r)}$

$$E(r, U_\rho) = \int_{\Omega_r} a|\nabla U_\rho|^2. \tag{2.5}$$

Then the following estimate holds

$$E(R, U_\rho) \leq E(\rho, U_\rho)R^{-c}\rho^c\,,$$

where $c = \dfrac{2}{n-1}$*.*

Remark 2.2. The above result can be interpreted as a Saint-Venant principle. It states that, when the applied "forces" have small support (condition (2.1)) and the resultant force is zero (condition (2.4)) then, provided $E(\rho, U_\rho)$ is bounded, the energy $E(R, U_\rho)$ is small (of order ρ^c) outside a fixed ball $B_n(x_0, R)$. This implies that $\|\nabla U_\rho\|^2_{L^2(\Omega_R)}$ is of order ρ^c. In the case when Γ_D is not negligible in $\partial\Omega$, Poincaré inequality implies that $\|U_\rho\|^2_{L^2(\Omega_R)}$ is also small (of order ρ^c).

Remark 2.3. The informations given in the statement of Theorem 2.1 about the function U_ρ are not sufficient to guarantee that $E(\rho, U_\rho)$ is bounded; this assumption must be checked separately for each particular case.

Remark 2.4. The equilibrium conditon (2.4) can be expressed in any of the following equivalent forms:

$$\int_{B(x_0,\rho)} div(a\nabla U_\rho) = 0 \tag{2.6}$$

$$\int_\Omega div(a\nabla U_\rho) = 0 \tag{2.7}$$

$$\int_{|x-x_0|=r} a\nabla U_\rho n = 0, \quad r \in [\rho, R] \tag{2.8}$$

$$\int_{\partial\Omega} a\nabla U_\rho n = 0 \tag{2.9}$$

$$\begin{aligned} \int_\Omega a\nabla U_\rho \nabla\varphi = 0\,, \quad \forall\, \varphi \in C^\infty(\Omega) \text{ such that } \varphi \equiv 1 \text{ in } B_n(x_0,\rho) \\ \text{and } \varphi \equiv 0 \text{ on } \partial\Omega \end{aligned} \tag{2.10}$$

Condition (2.9) is obtained as follows : (2.1) holds in Ω_r for each $r \in [\rho, R]$; therefore $\int_{\partial\Omega_r} a\nabla U_\rho n = 0$ wherefrom $\int_{|x-x_0|=r} a\nabla U_\rho n = 0$ for each $r \in [\rho, R]$ and $\int_{\partial\Omega} a\nabla U_\rho n = 0$. This equilibrium condition is essential and is in the very spirit of Saint-Venant's principle. It can be shown that if the "resultant force" is not zero, that is, if we drop out condition (2.4), Theorem 2.1 is false.

Proof of Theorem 2.1: The proof presented below is an adapted version of the calculations performed in section 3 of [Knowles, 1967]. Without loss of generality, we shall assume that $x_0 = 0$.

One checks for the differentiability of E in its first argument, r. For values of r where a (or ∇U_ρ) has discontinuities, E may be not differentiable. But in the ball $B_n(0, R)$, the coefficient function a is constant and in this case Lemma A.6 (see the Appendix) guarantees that U_ρ is of class C^∞. The derivative of E with respect to r is given by

$$\frac{\partial E}{\partial r}(r, U_\rho) = \frac{\partial}{\partial r}\int_{\Omega_r} a|\nabla U_\rho|^2 = -\int_{|x|=r} a|\nabla U_\rho|^2 = -\int_{|x|=r} \gamma|\nabla U_\rho|^2\,, \tag{2.11}$$

where $\gamma \in [\alpha, \beta]$ is the constant value of a in the ball $B_n(0, R)$. For a proof of the derivation formula above, see Lemma A.7 in the Appendix.

By applying integration by parts in the definition of $E(r, U_\rho)$ (formula (2.5)), one obtains

$$E(r, U_\rho) = \int_{\Omega_r} a|\nabla U_\rho|^2 = \int_{|x|=r} a\nabla U_\rho n U_\rho = -\int_{|x|=r} a\frac{\partial U_\rho}{\partial r} U_\rho,$$

where we have used the hypotheses (2.1), (2.2) and (2.3). Recall that n is the outward normal to Ω_r and $\frac{\partial U_\rho}{\partial r}$ is a short notation for $\nabla U_\rho \cdot \frac{x}{|x|}$.

Note that the equilibrium condition (2.4) implies (see also Remark 2.4)

$$-\int_{|x|=r} a\nabla U_\rho n = \int_{|x|=r} a\frac{\partial U_\rho}{\partial r} = \int_{B_n(0,r)} div(a\nabla U_\rho) = \int_{B_n(0,\rho)} div(a\nabla U_\rho) = 0$$

thus we can write

$$E(r, U_\rho) = -\int_{|x|=r} a\frac{\partial U_\rho}{\partial r} U_\rho = -\int_{|x|=r} a\frac{\partial U_\rho}{\partial r}(U_\rho - \bar{U}_\rho)\,,$$

where $\bar{U}_\rho$ is the mean value of U_ρ on the sphere $S_{n-1}(0, r) = \partial B_n(0, r)$:

$$\bar{U}_\rho(r) = \frac{1}{|S_{n-1}(0, r)|}\int_{|x|=r} U_\rho\,.$$

We write down the following inequalities (where γ is the constant value of a in $B_n(0,R)$):

$$E(r,U_\rho) = -\int_{|x|=r} a\frac{\partial U_\rho}{\partial r}(U_\rho - \bar{U}_\rho) = -\int_{|x|=r} \gamma\sqrt{(n-1)r}\,\frac{\partial U_\rho}{\partial r}\,\frac{U_\rho - \bar{U}_\rho}{\sqrt{(n-1)r}} \leq$$

$$\leq \int_{|x|=r} \frac{\gamma}{2}\left[(n-1)r\left|\frac{\partial U_\rho}{\partial r}\right|^2 + \frac{|U_\rho - \bar{U}_\rho|^2}{(n-1)r}\right] =$$

$$= \frac{\gamma}{2}(n-1)r\int_{|x|=r}\left|\frac{\partial U_\rho}{\partial r}\right|^2 + \frac{\gamma}{2(n-1)r}\int_{|x|=r}|U_\rho - \bar{U}_\rho|^2$$

and, by applying Lemma A.5 to the function $U_\rho - \bar{U}_\rho$ (∇_τ denotes the tangential derivative):

$$E(r,U_\rho) \leq \frac{\gamma}{2}(n-1)r\int_{|x|=r}\left|\frac{\partial U_\rho}{\partial r}\right|^2 + \frac{\gamma}{2}(n-1)r\int_{|x|=r}\left|\nabla_\tau(U_\rho - \bar{U}_\rho)\right|^2 = \frac{\gamma}{2}(n-1)r\int_{|x|=r}\left|\nabla U_\rho\right|^2$$

Using formula (2.11), we obtain the differential inequality

$$E(r,U_\rho) \leq -\frac{(n-1)r}{2}\,\frac{\partial E}{\partial r}(r,U_\rho)$$

wich can be written as (recall that $E > 0$)

$$\frac{\dfrac{\partial E}{\partial r}(r,U_\rho)}{E(r,U_\rho)} \leq -\frac{2}{(n-1)r} \tag{2.12}$$

By integrating (2.12), one gets

$$E(r,U_\rho) \leq E(\rho,U_\rho)\,r^{-c}\rho^{c}$$

where $c = 2/(n-1)$.

Remark 2.5. In the statement of Theorem 2.1, there is no need for the function U_ρ to be defined inside $B_n(x_0,\rho)$. By introducing the slight change $U_\rho \in H^1(\Omega_\rho)$, that is, if U_ρ is defined only in the perforated domain $\Omega_\rho = \Omega \setminus \overline{B_n(x_0,\rho)}$, the result remains true. Of course, some of the conditions given in Remark 2.4 make no sense for $U_\rho \in H^1(\Omega_\rho)$.

Remark 2.6. Usually, when thinking about Saint-Venant's principle, one expects exponential decay of the energy (see [Knowles 1967]). This is true for bars with forces applied at their ends. This is not what we obtain for bodies of arbitrary shape. In the next section we present an example which shows that exponential decay does not occur (see Remark 3.1).

Remark 2.7. In Theorem 2.1, we assume that a is constant in $B_n(x_0,R)$. It would be nice to eliminate this hipothesis, by using in the proof the weak derivative $\dfrac{\partial E}{\partial r}$ instead of the strong derivative. This is the object of ongoing work.

3. Topology optimization

We present in this section some links between Saint-Venant's principle and topology optimization.

Structural optimization consists generically in finding the best shape of an object (elastic body) in order to fulfill some quality criteria (rigidity for instance) under certain constraints (cost). Several approaches are known: geometric shape optimization (see [Allaire 2002], [Murat Simon 1976], [Simon 1980], [Pironneau 1984], [Sokołowski Zolesio 1992]), black and white squares optimization through material interpolation, or SIMP approach (see [Bensdøe Sigmund 1999]), optimization of laminates (see [Allaire Bonnetier Francfort Jouve 1997]), optimization through homogenization (see [Allaire 2002]). Another possible approach is topology optimization, which consists in introducing new, infinitesimal holes in a given material; this is usually followed by a geometrical shape optimization step. The main issue of topology optimization is to find the optimal place in the body where to make a new infinitesimal hole. This method first appeared under the name "bubble method" (see [Eschenauer Schumacher 1994]) and then it was called "topological gradient method" (see [Garreau Guillaume Masmoudi 2001]) or "topological derivative method" (see [Sokołowski Żochowski 1999], [Nazarov Sokołowski 2003]). Note that many references focus on linear elasticity (vector elliptic equation) rather than on a scalar equation as we do in the present paper.

To fix ideas, consider a domain $\Omega \subset \mathbb{R}^n$ $(n \geq 2)$. Consider two positive constants $0 < \alpha < \beta$ and a function a in $L^\infty(\Omega, [\alpha, \beta])$. The physical state of the body is described by the solution u of the problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -div(a\nabla u) = f \end{cases} \tag{3.1}$$

The goal of topology optimization is to describe the behaviour of an objective function $J(u)$ when the material coefficient a is perturbed by introducing a microscopic inclusion of material α in a zone previously occupied by material β. Consider a point x_0 in Ω such that $a = \beta$ in a ball of center x_0 and fixed radius R. Consider a small parameter $\rho > 0$ and denote by a_ρ the function $a_\rho \in L^\infty(\Omega, [\alpha, \beta])$ defined by

$$a_\rho(x) = \begin{cases} a(x), & \text{if } x \in \Omega \setminus B_n(x_0, \rho) \\ \alpha, & \text{if } x \in B_n(x_0, \rho). \end{cases}$$

In other words, $a_\rho = a - (\beta - \alpha)\chi_{B_n(x_0,\rho)}$, where $\chi_{B_n(x_0,\rho)}$ is the characteristic function of the ball $B_n(x_0, \rho)$. For very small ρ, a_ρ represents the material coefficient of a body with a microscopic inclusion at x_0. By taking α much smaller than β, a_ρ can be seen as the material coefficient of a body with a microscopic hole at x_0. Denote by u_ρ the solution of the problem :

$$\begin{cases} u_\rho \in H_0^1(\Omega), \\ -div(a_\rho \nabla u_\rho) = f. \end{cases} \tag{3.2}$$

Although many different functionals can be considered, a typical example of functional to be minimized is the energy

$$J_\rho(u_\rho) = \int_\Omega a_\rho \nabla u_\rho \nabla u_\rho = \int_\Omega f u_\rho \, .$$

In topology optimization, the main issue is to study the variation of the objective functional $J_\rho(u_\rho) - J(u)$, where

$$J(u) = \int_\Omega a \nabla u \nabla u = \int_\Omega f u \, .$$

In several papers (see [Garreau Guillaume Masmoudi 2001], [Lewiński Sokołowski 2003], [Sokołowski Zochowski 1999]), for the framework of linear elasticity, an asymptotic development is deduced for J_ρ in the form

$$J_\rho(u_\rho) = J(u) + \lambda \rho^n + o(\rho^n) \tag{3.3}$$

where n is the space dimension. The scalar λ appearing in the formula above is called <u>topological derivative</u> of the considered functional. A formal definition of the topological derivative is:

$$\lambda = \lim_{\rho \to 0} \frac{J_\rho(u_\rho) - J(u)}{\rho^n}$$

As it turns out, this value depends on the (unperturbed) state u and on the location x_0 of the hole. Then, one chooses a location where λ is minimum in order to perform a (microscopic) hole which decreases the value of the objective functional. One is interested in comparing the variation of the objective functional with the variation of other quantities, like cost. If we define the cost to be the integral of a or of a_ρ, respectively, then the variation of the cost will be also of order of ρ^n and the comparison makes sense.

Remark 3.1. The difference $u_\rho - u$ belongs to $H^1_0(\Omega)$ and satisfies

$$div(a \nabla (u_\rho - u)) = 0 \quad \text{in } \Omega \setminus B_n(x_0, \rho)$$

Suppose that in Saint-Venant's principle we obtained an exponential decay of the energy (see Remark 2.6). This means that, for a fixed radius R, the norm of $u_\rho - u$ in $L^2(\Omega \setminus B_n(x_0, R))$ goes to zero exponentially as $\rho \to 0$ (in particular, it goes to zero faster than any power of ρ). Consider now the case when $f = 0$ in some neighbourhood of x_0 (e.g. in the ball $B_n(x_0, R)$). Recall that

$$J_\rho(u_\rho) - J(u) = \int_\Omega f(u_\rho - u) = \int_{\Omega \setminus B_n(x_0, R)} f(u_\rho - u)$$

This value goes to zero (as $\rho \to 0$) faster than any power of ρ, which implies that the topological derivative, denoted by λ in formula (3.3), must be zero. Thus, the topological derivative should be zero in those regions of Ω where $f = 0$. This contradicts mechanical common sense (there are many meaningful examples with forces applied only on small parts of the body) and also contradicts results obtained in the literature (see [Garreau Guillaume Masmoudi 2001], [Lewiński Sokołowski 2003], [Sokołowski Zochowski 1999]).

This means that exponential decay of the energy is not to be expected in this version of Saint-Venant's principle.

The remaining of this section is devoted to the study of the behaviour of the state function u_ρ as $\rho \to 0$. An asymptotic development of u_ρ itself could be used to obtain asymptotic developments of any functional $J_\rho(u_\rho)$. Taking into account formula (3.3), we expect the difference $u_\rho - u$ to be of order ρ^n (in some norm). However, this is not what we obtain, see Remark 3.4.

We introduce two auxiliary problems and study their solutions. First, recall that $u_\rho - u$ verifies

$$\begin{cases} u_\rho - u \in H_0^1(\Omega), \\ div(a_\rho \nabla(u_\rho - u)) = (\beta - \alpha) div(\chi_\rho \nabla u), \end{cases} \tag{3.4}$$

χ_ρ being the characteristic function of $B_n(x_0, \rho)$.

We assume that the coefficient $a(x)$ is constantly equal to β in some neighbourhood $B_n(x_0, R)$ of x_0. Thus, Lemma A.6 (see the Appendix) ensures that the solution u of problem (3.1) is smooth in the same neighbourhood : $u \in C^\infty(B_n(x_0, R))$. Thus, in $B_n(x_0, R)$ we can approximate $\nabla u(x)$ by $\nabla u(x_0)$. As, for ρ small enough, $B_n(x_0, \rho) \subset B_n(x_0, R)$, one has

$$|\nabla u(x) - \nabla u(x_0)| \le M\rho\,,\ \forall\, x \in B_n(x_0, \rho)\,,$$

where M is some upper bound for $|D^2 u|$ (the norm of the Hessian matrix) in $B_n(x_0, R)$.

Having in mind this approximation, we change problem (3.4) into the following problem, whose solution we denote by v_ρ:

$$\begin{cases} v_\rho \in H_0^1(\Omega), \\ div(a_\rho \nabla v_\rho) = (\beta - \alpha)\, div(\chi_\rho \nabla u(x_0)), \end{cases} \tag{3.5}$$

We subtract (3.5) from (3.4) :

$$\begin{cases} u_\rho - u - v_\rho \in H_0^1(\Omega), \\ div(a_\rho \nabla(u_\rho - u - v_\rho)) = (\beta - \alpha)\, div(\chi_\rho(\nabla u - \nabla u(x_0))). \end{cases} \tag{3.6}$$

One has

$$\|\chi_\rho(\nabla u - \nabla u(x_0))\|^2_{L^2(\Omega)} = \int_{B_n(x_0,\rho)} |\nabla u - \nabla u(x_0)|^2 \le M^2 \rho^2 |B_n(x_0, \rho)| = M^2 \rho^{2+n} |B_n(0,1)|\,,$$

thus

$$\|\chi_\rho(\nabla u - \nabla u(x_0))\|_{L^2(\Omega)} \le M \sqrt{|B_n(0,1)|}\, \rho^{1+n/2}$$

and by applying Lemma A.8 (see the Appendix) with problem (3.6) we obtain the following estimate

$$\|\nabla(u_\rho - u - v_\rho)\|_{L^2(\Omega)} \le M \frac{\beta - \alpha}{\alpha} \sqrt{|B_n(0,1)|}\, \rho^{1+n/2}. \tag{3.7}$$

The function v_ρ is not explicit enough (it still depends on the shape of Ω). We shall approximate it by another function w_ρ, defined in the whole $\mathbb{R}^n$. This new function w_ρ can be defined as the solution of the elliptic problem

$$\begin{cases} w_\rho & = & 0 \text{ at infinity,} \\ div(a_\rho \nabla w_\rho) & = & (\beta-\alpha)\, div(\chi_\rho \nabla u(x_0)) \text{ in } \mathbb{R}^n \end{cases}$$

which can then be solved explicitely. Instead, we prefer to give directly its analytic expression:

$$w_\rho(x) = \begin{cases} \dfrac{\beta-\alpha}{\alpha+(n-1)\beta}\, \dfrac{\rho^n}{|x-x_0|^n} \langle \nabla u(x_0), x-x_0 \rangle\,, & \text{if } |x-x_0| > \rho \\ \dfrac{\beta-\alpha}{\alpha+(n-1)\beta} \langle \nabla u(x_0), x-x_0 \rangle\,, & \text{if } |x-x_0| < \rho \end{cases} \tag{3.8}$$

and simply check that it satisfies $div(a_\rho \nabla(w_\rho)) = (\beta-\alpha) div(\chi_\rho \nabla u(x_0))$.
The comparison between v_ρ and w_ρ is difficult because w_ρ is not zero on $\partial\Omega$. This is why we shall multiply w_ρ by a fixed function $\psi \in C^1(\Omega)$ such that $\psi \equiv 1$ in $B_n(x_0, R)$ and $\psi \equiv 0$ on $\partial\Omega$ (a cutting function).
In order to estimate the difference $v_\rho - \psi w_\rho$, we study the quantity $div(a_\rho \nabla(v_\rho - \psi w_\rho))$, using the fact that $div(a_\rho \nabla v_\rho) = div(a_\rho \nabla w_\rho)$ in Ω:

$$\begin{aligned} div(a_\rho \nabla(v_\rho - \psi w_\rho)) = div(a_\rho \nabla v_\rho - a_\rho \psi \nabla w_\rho - a_\rho w_\rho \nabla \psi) = \\ = div(a_\rho \nabla w_\rho - a_\rho \psi \nabla w_\rho - a_\rho w_\rho \nabla \psi) = div((1-\psi) a_\rho \nabla w_\rho - a_\rho w_\rho \nabla \psi)\,. \end{aligned}$$

Therefore, $v_\rho - \psi w_\rho$ is the solution of the problem

$$\begin{cases} v_\rho - \psi w_\rho \in H_0^1(\Omega) \\ div(a_\rho \nabla(v_\rho - \psi w_\rho)) = div((1-\psi) a_\rho \nabla w_\rho - a_\rho w_\rho \nabla \psi) \end{cases} \tag{3.9}$$

Both $1-\psi$ and $\nabla\psi$ vanish in $B_n(x_0, R)$, thus

$$\|(1-\psi) a_\rho \nabla w_\rho - a_\rho w_\rho \nabla \psi\|_{L^2(\Omega)} = \|(1-\psi) a_\rho \nabla w_\rho - a_\rho w_\rho \nabla \psi\|_{L^2(\Omega_R)}$$

where $\Omega_R = \Omega \setminus B_n(x_0, R)$. By looking at the definition (3.8) of w_ρ one verifies that both $w_\rho(x)$ and $\nabla w_\rho(x)$ are of order of ρ^n for $x \in \Omega_R$, that is, there is a constant $C > 0$ independent of ρ such that

$$\|(1-\psi) a_\rho \nabla w_\rho - a_\rho w_\rho \nabla \psi\|_{L^2(\Omega)} \le C \rho^n$$

and by applying again Lemma A.8 with problem (3.9) we conclude that

$$\|\nabla(v_\rho - \psi w_\rho)\|_{L^2(\Omega)} \le \frac{C}{\alpha}\, \rho^n \tag{3.10}$$

On the other hand, the difference $w_\rho - \psi w_\rho$ is also of order ρ^n:

$$\|w_\rho - \psi w_\rho\|_{H^1(\Omega)} = \|(1-\psi)w_\rho\|_{H^1(\Omega_R)} \le C\rho^n \,, \tag{3.11}$$

where C is a constant, not necessarily the same as in (3.10).
We are now in a position to state our main approximation result.

Theorem 3.2. *The perturbed solution u_ρ admits the following asymptotic development:*

$$u_\rho = u + w_\rho + O(\rho^{1+n/2})\,.$$

More precisely, the sequence of functions

$$\rho^{-1-n/2}(u_\rho - u - w_\rho)$$

is bounded in $H^1(\Omega)$, uniformly in ρ.

Proof : We combine formulas (3.7) and (3.10) in order to get

$$\|\nabla(u_\rho - u - \psi w_\rho)\|_{L^2(\Omega)} = O(\rho^{1+n/2})\,.$$

We then use the Poincaré inequality — recall that $u_\rho - u - \psi w_\rho$ belongs to $H_0^1(\Omega)$ — and obtain

$$\|u_\rho - u - \psi w_\rho\|_{H^1(\Omega)} = O(\rho^{1+n/2})\,.$$

Finally, inequality (3.11) gives the desired result. □

Remark 3.3. The function w_ρ itself is small; we conjecture it is $O(\rho^{n/2})$. It is important to check that w_ρ is not too small: for instance, if it were of order $\rho^{1+n/2}$ then the asymptotic expansion given in Theorem 3.2 would reduce to $u_\rho = u + O(\rho^{1+n/2})$. It is easy to verify that w_ρ is <u>not</u> $o(\rho^{n/2})$:

$$\|\nabla w_\rho\|^2_{L^2(\Omega)} = \int_\Omega |\nabla w_\rho|^2 \ge \int_{B_n(x_0,\rho)} |\nabla w_\rho|^2 = C\,|B_n(x_0,\rho)| = C\,\rho^n\,,$$

that is, $\|\nabla w_\rho\|_{L^2(\Omega)} \ge C\,\rho^{n/2}$ for some constant $C > 0$.

Remark 3.4. We initially expected $u_\rho - u$ to be of order ρ^n (in some norm). This is not what we obtain in Theorem 3.2: we get that $\|w_\rho\|_{H^1(\Omega)}$ and consequently $\|u_\rho - u\|_{H^1(\Omega)}$ are of order $\rho^{n/2}$ or larger (see also the previous Remark). On the other hand, this is not in direct contradiction with (3.3); it is well possible that $\displaystyle\int_\Omega f(u_\rho - u)$ be smaller (of a different order of magnitude) than the norm of $u_\rho - u$. It simply means that in order to prove (3.3) one needs to use more precise informations on u_ρ than provided by Theorem 3.2.

However, the results obtained in the present paper suggest that, for more general objective functionals, the asymptotic development may have a different form than (3.3). For instance, a term of order $\rho^{n/2}$ may appear.

Remark 3.5. The asymptotic development

$$u_\rho = u + \psi w_\rho + O(\rho^{1+n/2})$$

is also valid. It has the advantage of involving only functions in $H_0^1(\Omega)$; note that $w_\rho \notin H_0^1(\Omega)$ but $\psi w_\rho \in H_0^1(\Omega)$. It has the disadvantage of involving a fixed arbitrary function ψ alien to the problem.

Remark 3.6. For $n \geq 3$, one can apply Saint-Venant's principle and improve the estimates far away from x_0. Note that the function $u_\rho - u - v_\rho$ satisfies the hipotheses of Theorem 3.1. Thus, for fixed $R > 0$, one has

$$\begin{aligned}\int_{\Omega_R} a|\nabla(u_\rho - u - v_\rho)|^2 &\leq R^{-c}\rho^c \int_{\Omega_\rho} a|\nabla(u_\rho - u - v_\rho)|^2 \\ &\leq R^{-c}\rho^c \int_{\Omega} a|\nabla(u_\rho - u - v_\rho)|^2\end{aligned}$$

where $c = \dfrac{2}{n-1}$. Combining with (3.7), one obtains that $\nabla(u_\rho - u - v_\rho)$ is of order $\rho^{1+n/2+c/2}$ in the norm of $L^2(\Omega_R)$. Using the estimate (3.10) and noting that $1+c/2 \leq n/2$, we conclude that $u_\rho = u + w_\rho + O(\rho^{1+n/2+c/2})$ in the norm of $H^1(\Omega_R)$.

Appendix

We begin by recalling the known Poincaré-Wirtinger inequality, see [Meyers 1978].

Lemma A.1. *Let Ω be a bounded and connected domain in $\mathbb{R}^n$. Then there exists a constant $C > 0$ such that, for any $u \in H^1(\Omega)$ with $\int_\Omega u = 0$, one has*

$$\|u\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}$$

The above inequality holds even for functions defined not on open subsets of $\mathbb{R}^n$, but on compact Riemannian manifolds. See [Aubin 1982, chapter 2] for the definition and properties of Sobolev spaces on Riemannian manifolds.

Lemma A.2. *Let M be a compact and connected Riemannian manifold of dimension n. Then there exists a constant $C > 0$ such that, for any $u \in H^1(M)$ with $\int_M u = 0$, one has*

$$\|u\|_{L^2(M)} \leq C\|\nabla u\|_{L^2(M)}$$

Proof: The proof found in [Meyers 1978] can be adapted to the case of a Riemannian manifold. We present here a short proof, for completeness. Let us suppose that there is no positive constant C with the property stated in Lemma A.2. Then, for each integer k, there is a function $v_k \in H^1(M)$ such that $\int_M v_k = 0$ and $\|v_k\|_{L^2(M)} > k\|\nabla v_k\|_{L^2(M)}$. Denote by $w_k = \dfrac{v_k}{\|v_k\|_{L^2(M)}}$; thus $\|w_k\|_{L^2(M)} = 1$ and $\|\nabla w_k\|_{L^2(M)} < 1/k$. So, $\|w_k\|_{L^2(M)}$ is bounded while $\|\nabla w_k\|_{L^2(M)} \to 0$. In particular, $\|w_k\|_{H^1(M)}$ is bounded, so there is a subsequence of w_k (still denoted by w_k) which converges weakely in $H^1(M)$, let w^* be its weak limit. As $\nabla w_k \to 0$ strongly in $L^2(M)$, we conclude that w^* is constant in M. The equality $\int_{L^2(M)} w_k = 0$ passes to the limit as $\int_{L^2(M)} w^* = 0$, which means that $w^* = 0$. On the other side, $w_k \to w^*$ strongly in $L^2(M)$, therefore $\|w_k\|_{L^2(M)} = 1$ implies $\|w^*\|_{L^2(M)} = 1$. We obtain a contradiction, which proves the result. $\square$

Remark A.3. The above constant C depends of course of the considered manifold. The best constant C coincides with the second eigenvalue of the operator $-\Delta$ on the considered manifold, see for instance [Berger Gauduchon Mazet 1971, Lemme D.II.3] or [Chavel 1984, section I.5, Rayleigh's Theorem].

We shall need to apply the Poincaré-Wirtinger inequality for functions defined on a sphere of variable radius r (a manifold of dimension $n-1$ embedded in $\mathbb{R}^n$).

Remark A.4. For the unit sphere $S_{n-1}(0,1) = \{x \in \mathbb{R}^n : |x| = 1\}$, the best constant in Lemma A.2 is $C = n-1$, see for instance [Berger Gauduchon Mazet 1971, Proposition C.I.1] or [Chavel 1984, section II.4].

The following result specifies the way the constant C depends on the radius of the sphere.

Lemma A.5. *Let u be a function in $H^1(S_{n-1}(0,r))$, where $S_{n-1}(0,r)$ is the sphere centered at 0 and of radius r, $S_{n-1}(0,r) = \{x \in \mathbb{R}^n : |x| = r\}$. If $\int_{S_{n-1}(0,r)} u = 0$, then $\|u\|_{L^2(S_{n-1}(0,r))} \leq r\,(n-1)\,\|\nabla u\|_{L^2(S_{n-1}(0,r))}$.*

Proof : Let $u \in H^1(S_{n-1}(0,r))$. Define $w \in H^1(S_{n-1}(0,1))$ defined by $w(x) = u(rx)$, $\forall\, x \in S_{n-1}(0,1)$. We apply Lemma A.2 for $M = S_{n-1}(0,1)$, together with remark A.4:

$$\|w\|_{L^2(S_{n-1}(0,1))} \leq (n-1)\,\|\nabla w\|_{L^2(S_{n-1}(0,1))}$$

then compute

$$\|u\|^2_{L^2(S_{n-1}(0,r))} = r^{n-1}\|w\|^2_{L^2(S_{n-1}(0,1))}\,,$$

$$\|\nabla u\|^2_{L^2(S_{n-1}(0,r))} = r^{n-3}\|\nabla w\|^2_{L^2(S_{n-1}(0,1))}\,.$$

We infer that

$$\|u\|_{L^2(S_{n-1}(0,r))} \le r\,(n-1)\,\|\nabla u\|_{L^2(S_{n-1}(0,r))}$$

which concludes the proof. □

The following is a regularity result, used in section 2.

Lemma A.6. *Let $u \in H^1(\Omega)$ such that $\Delta u = 0$ in Ω. Then $u \in C^\infty(\Omega)$.*

Proof: For two dimensions, this is a known result in complex analysis (u may be viewed as the real part of a complex analytic function). For arbitrary dimensions, the assertion is a consequence of Theorem 6.6 in [Agmon 1965] and of the remark following it. □

Lemma A.7. *Let $\Omega \subset \mathbb{R}^n$ be a domain containing the origin, and consider f a continuous function defined in Ω. For each $r > 0$, define*

$$E(r) = \int_{\Omega_r} f\,, \quad \textit{where } \Omega_r = \Omega \setminus \bar{B}(0,r)$$

Then E is differentiable and

$$E'(r) = -\int_{|x|=r} f$$

Proof: This is a particular case of a more general result, see for instance [Murat Simon 1976b, Théorème 4.2 ii] or [Sokolowsky Zolezio 1992, Proposition 2.46] We include here a quick proof for completeness. One has

$$\frac{E(r+\delta r) - E(r)}{\delta r} = -\frac{1}{\delta r}\int_{r\le|x|\le r+\delta r} f\,.$$

By splitting the above integration operation into an integration on the sphere followed by a one-dimensional integration along each direction, one concludes without difficulty that

$$\lim_{\delta r \searrow 0} \frac{1}{\delta r}\int_{r\le|x|\le r+\delta r} f = \int_{|x|=r} f\,.$$

The following estimate is used in section 3.

Lemma A.8. *Let $a \in L^\infty(\Omega,[\alpha,\beta])$ and $\vec{b} \in L^2(\Omega,\mathbb{R}^n)$. Then the solution w of*

$$\begin{cases} w \in H^1_0(\Omega), \\ div(a\nabla w) = div\,\vec{b} \end{cases}$$

satisfies

$$\|\nabla w\|_{L^2(\Omega)} \leq \frac{1}{\alpha}\,\|\vec{b}\|_{L^2(\Omega)}\,.$$

Proof : This result is a consequence of the estimates :

$$\alpha\|\nabla w\|^2_{L^2(\Omega)} \leq \Big|\int_\Omega a\nabla w\nabla w\Big| = \Big|\int_\Omega -\vec{b}\nabla w\Big| \leq \|\vec{b}\|_{L^2(\Omega)}\|\nabla w\|_{L^2(\Omega)}. \;\square$$

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