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Study of the cost functional for free material design problems

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Summary We study properties of the cost functional arising in free material optimization problems, with special emphasis on semicontinuity and its relation to convexity.

Key words homogenization theory, structural optimization, free material design

1. Introduction

The goal of the present work is to study functionals of the form

$$\Phi(A) = \int_\Omega \phi(A(x))dx \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ and A is a matrix whose components belong to $L^\infty(\Omega)$. Our study is motivated by structural optimization problems, namely by the so-called "free material design" (see Section 2 for details).

We study properties of the functional Φ which arise naturally from mechanical considerations, or ensure the well-posedeness of the optimization problem. Here is a summary of these properties

– smoothness of ϕ
– isotropy of ϕ (invariance to rotations)
– monotonicity of ϕ (and subsequently of Φ)
– lower semi-continuity of Φ (related, but not equivalent, to the convexity of ϕ)
– subadditivity of ϕ (and subsequently of Φ)

The most important and delicate point is the one concerning the semi-continuity of Φ. We point out that lower semi-continuity of Φ with respect to the weak-$*$ topology of L^∞ is not the correct hypothesis from the mechanical viewpoint (although it ensures that a solution exists for the optimization problem). We introduce a different property (lower semi-continuity with respect to the H-topology) and discuss the relations between the two, as well as the link with the convexity of ϕ.

2. Setting of the problem

Let Ω be a bounded domain in $\mathbb{R}^n$. Let α and β be two real constants such that $0 < \alpha < \beta$. Denote by $M_s^{\alpha,\beta}$ the set of symmetric $n \times n$ matrices A such that $\alpha I \le A \le \beta I$ and let $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ be the set of matrix functions $A : \Omega \to M_s^{\alpha,\beta}$ whose components are measurable functions (and consequently belong to $L^\infty(\Omega)$).

Consider some objective functional $\mathcal{J} : \mathcal{M}_s^{\alpha,\beta}(\Omega) \to \mathbb{R}$, to be minimized under a constraint on the cost functional Φ:

$$\min_{A\in\mathcal{A}} \mathcal{J}(A) \tag{2}$$

where, for a fixed, given $c \in \mathbb{R}$,

$$\mathcal{A} = \{A \in \mathcal{M}_s^{\alpha,\beta}(\Omega) : \Phi(A) = c\} \tag{3}$$

Recall that Φ is given by (1), where $\phi : M_s^{\alpha,\beta} \to \mathbb{R}$ is some scalar function. One should distinguish between the objective functional $\mathcal{J}$ (the one to be minimized), which measures the performance of the structure according to some criteria, and the cost functional Φ (giving the constraint) which measures the price of the structure (the fabrication cost). Typically, $\mathcal{J}$ could be the compliance of the structure under some loads, but any other objective functional could be considered (e.g. the distance to some target displacement). In all cases, $\mathcal{J}$ depends on A through the solution $u(A)$ of some elliptic problem in Ω; A represents the material coefficients for that elliptic problem. We choose not to state any particular form of the elliptic problem because, although $\mathcal{J}$ itself should satisfy certain properties in order for the optimization problem to be well-posed, our study is focused on the properties of the cost functional Φ.

Note that, although we use terminology from the elasticity theory, the coefficient matrix $A \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$ model other physical phenomena (like heat conduction or electrostatics). The reason is that the meaningful and interesting case for applications is in the elasticity framework, but we prefer to study functions ϕ defined on the set of $n \times n$ matrices (the elasticity tensor is a much more complicated object).

3. Basic properties

Along our study, we shall assume that the function ϕ is smooth enough. Continuity is a natural requirement, but more smoothness will be assumed when necessary.

Note that, although the function ϕ should be defined on the set $M_s^{\alpha,\beta}$, one may find it easier to consider that ϕ is defined on the whole set M_s^+ of symmetric positive definite matrices in order to obtain statements and properties independent of the parameters α and β.

We shall also assume that the cost function is invariant to rotations. This condition is difficult to characterize in the elasticity framework. But for the scalar problem that we have chosen to study, it writes simply as

$$\phi(Q^t A Q) = \phi(A)\,,\ \forall\, A \in M_s^{\alpha,\beta}\,,\ \forall Q \text{ orthogonal matrix}$$

Equivalently, ϕ should depend only on the eigenvalues of the matrix A, taken in decreasing order (see [1, sec. 2]).

Another natural hypothesis is that "stronger structures are more expensive than weaker ones", that is, if $A, B \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$ are two coefficient matrices such that $A(x) \leq B(x)$, a.e. $x \in \Omega$, then A should be cheaper than B: $\Phi(A) \leq \Phi(B)$. This is equivalent to the monotonicity of ϕ:

$$\forall\, A, B \in M_s^{\alpha,\beta}\,,\ A \leq B \Longrightarrow \phi(A) \leq \phi(B) \tag{4}$$

Note that this assumption (ϕ non-decreasing) is quite natural if one wants to obtain strong structures (for instance, when minimizing the compliance). It is far less obvious for cases like mechanism design. Being aware of that, we shall nevertheless impose this property on ϕ in the sequel.

By subadditivity of ϕ we mean

$$\phi(A + B) \leq \phi(A) + \phi(B)\,,\ \forall\, A, B \in M_s^+$$

There are reasons (of mechanical nature) for taking this property into consideration. However, its mathematical implications are not yet clear; this is why we do not focus on subadditivity in this paper.

4. Semi-continuity of Φ

It is well-known that, for an optimization problem to be well-posed, one usually requires the lower semi-continuity of both the objective functional and the cost functional. One can see this by reformulating the constrained minimization problem (2-3) as an unconstrained minimization problem

$$\min_{A \in \mathcal{M}_s^{\alpha,\beta}(\Omega)} \{\mathcal{J}(A) + \Lambda \Phi(A)\} \tag{5}$$

with the aid of the Lagrange multiplier $\Lambda \in \mathbb{R}$. Assuming that $\Lambda > 0$, we see that lower semi-continuity of both $\mathcal{J}$ and Φ ensures the well-posedeness of the optimization problem.

The question of the topology arises: one should check the lower semi-continuity of Φ with respect to some topology on $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ such that $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ be compact. An apparantly natural choice is the weak-$*$ topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$, as a subset of $L^\infty(\Omega; \mathbb{R}^{n\times n})$. This is also convenient because it is easy to characterize integrands ϕ which turn the functional Φ lower semi-continuous:

Theorem 1 [2, pp. 142–143] The functional Φ defined by (1) is lower-semicontinuous with respect to the weak-$*$ topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ if and only if the integrand ϕ is a convex function (defined on the set $M_s^{\alpha,\beta}$ of $n \times n$ matrices).

However, a closer look at the mechanical problem reveals that the weak-$*$ topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ is not an appropriate model: when the material coefficients oscillate (the case of microstructures), the limit behaviour of the material is correctly described not by the weak-$*$ limit of the coefficient matrices, but by the H-limit of the same matrices. See the last section of the present paper for details on H-convergence. Recall that $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ is a compact set when endowed with the H-topology.

Unfortunately, there is no simple characterization of the lower semi-continuity with respect to the H-topology. Under the monotonicity assumption, one can prove easily one implication:

Theorem 2 Suppose ϕ is a continuous non-decreasing function, in the sense of (4), on $M_s^{\alpha,\beta}$. If the functional Φ defined by (1) is lower-semicontinuous with respect to the weak-$*$ topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$, then Φ is also lower-semicontinuous with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$.

Proof: Consider a sequence A_ε of matrices in $\mathcal{M}_s^{\alpha,\beta}(\Omega)$, H-converging to some $A \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$. Taking into account that $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ is compact with respect to the weak-$*$ topology, consider a subsequence ε' of ε such that $A_{\varepsilon'}$ converges weakly-$*$ to some $A_+ \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$ fulfilling also the condition $\liminf \phi(A_\varepsilon) = \lim \phi(A_{\varepsilon'})$. It is known that $A \le A_+$ (see Theorem 7), so $\phi(A) \le \phi(A_+) \le \liminf \phi(A_{\varepsilon'})$ and the assertion follows. $\square$

Theorems 1 and 2 imply the following

Corollary 3 If ϕ is a continuous convex function on $M_s^{\alpha,\beta}$, non-decreasing in the sense of (4), then the functional Φ defined by (1) is lower-semicontinuous with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$.

In general, there is no simple characterization of the lower semi-continuity of Φ with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$. However, in the particular case when ϕ depends only on the trace of the coefficient matrix,

$$\phi(A) = \varphi(\mathrm{tr}(A)) \,, \ \forall\, A \in M_s^{\alpha,\beta} \tag{6}$$

and assuming also the monotonicity property, one can prove that lower semi-continuity of Φ with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ is equivalent to the convexity of φ.

Theorem 4 Let $\varphi : [n\alpha, n\beta] \to \mathbb{R}$ be a continuous non-decreasing function. Define

$$\phi : M_s^{\alpha,\beta} \to \mathbb{R} \,, \quad \phi(A) = \varphi(\mathrm{tr}(A))$$

and

$$\Phi : \mathcal{M}_s^{\alpha,\beta}(\Omega) \to \mathbb{R} \,, \quad \Phi(A) = \int_\Omega \phi(A(x))dx = \int_\Omega \varphi(\mathrm{tr}(A(x)))dx.$$

Then, Φ is lower semi-continuous with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ if and only if φ is convex.

Proof for $n = 2$ ($n > 2$ is similar):

The sufficiency follows immediately from Corollary 3, since $A \mapsto \mathrm{tr}(A)$ is (linear and) non-decreasing in the sense of (4).

The necessity is proven by building a specific sequence of laminates. Consider γ, x and y arbitrary numbers in $[\alpha, \beta]$. Define

$$A_\varepsilon = \chi_\varepsilon \begin{bmatrix} \gamma & 0 \\ 0 & x \end{bmatrix} + (1 - \chi_\varepsilon) \begin{bmatrix} \gamma & 0 \\ 0 & y \end{bmatrix},$$

where $\chi_\varepsilon \overset{\star}{\rightharpoonup} \theta$ in $L^\infty(\Omega)$ (θ being a constant value in $[0, 1]$). According to Theorem 9, this sequence of laminates H-converges to

$$\begin{bmatrix} \gamma & 0 \\ 0 & \theta x + (1 - \theta) y \end{bmatrix},$$

and the lower semi-continuity condition reads

$$\varphi(\gamma + \theta x + (1 - \theta) y) \leq \theta \varphi(\gamma + x) + (1 - \theta) \varphi(\gamma + y).$$

Thus, φ is convex in the interval $[\gamma + \alpha, \gamma + \beta]$. Since γ is arbitrary in $[\alpha, \beta]$, we obtain the desired convexity of φ in $[2\alpha, 2\beta]$. □

Note that, if we drop the hypothesis (6), we have no simple characterization of the lower semi-continuity of Φ with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$.

Convexity is also necessary for φ when ϕ depends solely on the determinant of the coefficient matrix, $\phi(A) = \varphi(\det(A))$: one takes precisely the laminates above and a similar reasoning leads now to the convexity of φ in the interval $[\alpha^2, \beta^2]$. However, it is easy to prove that functions depending only on the determinant of A are not admissible:

Theorem 5 Let $\varphi : [\alpha^n, \beta^n] \to \mathbb{R}$ be a differentiable convex function. Define

$$\phi : M_s^{\alpha,\beta} \to \mathbb{R}, \quad \phi(A) = \varphi(\det(A))$$

and

$$\Phi : \mathcal{M}_s^{\alpha,\beta}(\Omega) \to \mathbb{R}, \quad \Phi(A) = \int_\Omega \phi(A(x)) dx = \int_\Omega \varphi(\det(A(x))) dx.$$

If Φ is lower semi-continuous with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$, then φ is constant.

Proof for $n = 2$ ($n > 2$ is similar):

Consider $a, b, c, d \in [\alpha, \beta]$ four arbitrary points. Building the laminates

$$A_\varepsilon = \chi_\varepsilon \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + (1 - \chi_\varepsilon) \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}$$

and applying the lower semi-continuity hypothesis, we obtain the inequality

$$\varphi\left(\frac{\theta b + (1-\theta) d}{\frac{\theta}{a} + \frac{1-\theta}{c}} \right) \leq \theta\, \varphi(ab) + (1-\theta)\, \varphi(cd)$$

with equality for $\theta = 0$.

We differentiate the above inequality in $\theta = 0$:

$$c^2 \varphi'(cd) \Big[(b-d) \, \frac{1}{c} - d \left(\frac{1}{a} - \frac{1}{c} \right) \Big] \leq \varphi(ab) - \varphi(cd) \, ,$$

that is,

$$\frac{c}{a} \, \varphi'(cd) \, (ab - cd) \leq \varphi(ab) - \varphi(cd) \, .$$

Take $\lambda \in \,]\alpha, \beta[$. Choose $c = d = \lambda$, $a \in [\frac{\lambda^2}{\beta}, \lambda[\, \cap \, [\alpha, \beta]$ and define $b_\delta = \frac{\lambda^2 + \delta}{a}$, where $\delta > 0$ is a small parameter. One checks easily that, for sufficiently small δ, $b_\delta \in [\alpha, \beta]$. Taking into account that $ab_\delta = \lambda^2 + \delta$ and $cd = \lambda^2$, we rewrite the last inequality as

$$\frac{\lambda}{a} \, \varphi'(\lambda^2) \leq \frac{\varphi(\lambda^2 + \delta) - \varphi(\lambda^2)}{\delta} \, .$$

Let $\delta \searrow 0$ and recall that $a < \lambda$ in order to conclude $\varphi'(\lambda^2) \leq 0$.

Choose now $a \in \,]\lambda, \frac{\lambda^2}{\alpha}]$ and the same $c = d = \lambda$, $b_\delta = \frac{\lambda^2 + \delta}{a}$. The inequality above implies $\varphi'(\lambda^2) \geq 0$.

We have proved that $\varphi'(\lambda^2) = 0$ for all $\lambda \in [\alpha, \beta]$, hence φ is constant in $[\alpha^2, \beta^2]$. □

Theorem 5 above also shows something else: reminding that $\phi(A) = \det A$ is a polyconvex function, and thus quasiconvex and rank one convex as well, all these types of convexities are excluded as sufficient conditions. In order for Φ to be semi-continuous with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$, ϕ has to be either convex or something strictly between convex and polyconvex. It would be an interesting and non-trivial task to find an example of a function ϕ for which the functional Φ is semi-continuous with respect to the H-topology of $\mathcal{M}_s^{\alpha,\beta}(\Omega)$, but not with respect to the weak-$*$ topology.

Note also that the proofs of Theorems 4 and 5 use sequences of laminates. This means that the conditions described in these statements are necessary for lower semi-continuity in a weaker convergence than the one of the H-topology. The conditions described in Theorems 4 and 5 are necessary for the lower semi-continuity of Φ with respect to the convergence of laminates; see [3].

5. Examples

In this section we show some lower semi-continuous cost functionals, with respect to the H-topology. The third example is a highly popular choice in homogenization theory (see [4]), of which the free material design setting can be considered a direct generalization.

1. $\phi(A) = \mathrm{tr}(A)$. It is the most common choice in free material design and the lower semi-continuity of the corresponding cost functional is a direct consequence of Theorem 4.

2. $\phi(A) = \max_{\|\xi\|=1} A\xi \cdot \xi$. This is the spectral radius, since only positive definite matrices are dealt with. Being the pointwise supremum of the family $\{\phi_\xi\}_{\|\xi\|=1}$, $\phi_\xi(A) = A\xi \cdot \xi$, it is surely a convex function. The monotonicity, although not completely obvious,

is a simple exercise. Lower semi-continuity of the cost functional is due to Corollary 3. The isotropy is straightforward: as an ortogonal matrix Q yields an isometry, one has

$$\phi(Q^t AQ) = \max_{\|\xi\|=1}(Q^t AQ)\xi\cdot\xi = \max_{\|\xi\|=1} A(Q\xi)\cdot(Q\xi) = \max_{\|\eta\|=1} A\eta\cdot\eta = \phi(A).$$

3. We can perform optimization in the set of all possible mixtures beteween materials αI and βI. More precisely, define the set $G = \cup_{0\leq\theta\leq 1} G_\theta \subset \mathcal{M}_s^{\alpha,\beta}$ (see the next section for the definition of G_θ). For each matrix $A \in G$, one can compute explicitely the lowest value θ such that $A \in G_\theta$. This represents the "cheapest" mixture between αI and βI producing the given material tensor A. Consider then the cost function

$$\phi(A) = \inf\{\theta \in [0,1] : A \in G_\theta\}$$

We obtain a cost functional Φ defined not on the entire $\mathcal{M}_s^{\alpha,\beta}(\Omega)$ but only on those matrix functions taking values in G. It is a simple exercise to prove that the set of these functions is compact for the H-topology.

The cost function ϕ is continuous on G, non-decreasing and isotropic (as it is defined in terms of eigenvalues of the matrix).

We now prove its lower semi-continuity. Let $A_\varepsilon \xrightarrow{H} A$; the local costs $\theta_\varepsilon(x) = \phi(A_\varepsilon(x))$ can be supposed to converge weakly-$*$, say $\theta_\varepsilon \overset{\star}{\rightharpoonup} \bar\theta$, because $L^\infty(\Omega;[0,1])$ is weak-$*$ compact. From Theorem 8 it is known that each A_ε is a H-limit of some sequence $\chi_\eta^\varepsilon(\beta I) + (1-\chi_\eta^\varepsilon)(\alpha I)$, with $\chi_\eta^\varepsilon \overset{\star}{\rightharpoonup} \theta_\varepsilon$. Since $L^\infty(\Omega;[0,1]) \times \mathcal{M}_s^{\alpha,\beta}(\Omega)$ is a metrizable space, we can take a diagonal sequence $(\chi_{\eta_\varepsilon}^\varepsilon)$ such that

$$\chi_{\eta_\varepsilon}^\varepsilon \overset{\star}{\rightharpoonup} \bar\theta \quad \text{and} \quad \chi_{\eta_\varepsilon}^\varepsilon(\beta I) + (1-\chi_{\eta_\varepsilon}^\varepsilon)(\alpha I) \xrightarrow{H} A;$$

again from Theorem 8, this means precisely that $A(x) \in G_{\bar\theta(x)}$, a.e. $x \in \Omega$. But by definition of $\theta(x) = \phi(A(x))$ it is then obvious that

$$\int_\Omega \theta(x)dx \leq \int_\Omega \bar\theta(x)dx = \liminf \int_\Omega \theta_\varepsilon(x)dx,$$

thus proving the assertion.

6. Recall on H-convergence and bounds

This section is devoted to a review of definitions and results about homogenization theory. For details, see e.g. [5].

Definition 6 Consider a sequence of matrix functions $A_\varepsilon \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$. We say that A_ε H-converges to some $A \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$ if, for all $f \in H^{-1}(\Omega)$ and for all $\bar u \in H^{1/2}(\partial\Omega)$, the solution $u_\varepsilon \in H^1(\Omega)$ (which exists and is unique) of the problem

$$\begin{cases} -div(A_\varepsilon \nabla u_\varepsilon) = f & \text{in } \Omega \\ u_\varepsilon = \bar u & \text{on } \partial\Omega \end{cases}$$

converges, weakly in $H^1(\Omega)$, to the solution $u_0 \in H^1(\Omega)$ (which exists and is unique) of the problem

$$\begin{cases} -div(A\nabla u_0) = f & \text{in } \Omega \\ u_0 = \bar{u} & \text{on } \partial\Omega \end{cases}$$

We denote the H-convergence by $A_\varepsilon \xrightarrow{H} A$. Note that it derives from a metrizable topology on $\mathcal{M}_s^{\alpha,\beta}(\Omega)$. It is known that $\mathcal{M}_s^{\alpha,\beta}(\Omega)$, when endowed with the H-topology, is compact.

Theorem 7 Consider a sequence of matrix functions $A_\varepsilon \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$ such that $A_\varepsilon \xrightarrow{H} A$, $A_\varepsilon \xrightarrow{\star} A_+$ and $A_\varepsilon^{-1} \xrightarrow{\star} A_-^{-1}$. Then $A_-(x) \le A(x) \le A_+(x)$ a.e. $x \in \Omega$.

For each $\theta \in [0,1]$, define the numbers

$$\begin{cases} \mu_+(\theta) = \theta\beta + (1-\theta)\alpha, \\ \dfrac{1}{\mu_-(\theta)} = \dfrac{\theta}{\beta} + \dfrac{1-\theta}{\alpha}; \end{cases}$$

and define the set $K_\theta \subset \mathbb{R}^n$ by

$$\mu \in K_\theta \iff \begin{cases} \mu_-(\theta) \le \mu_i \le \mu_+(\theta),\ i = 1,\dots,n\ , \\ \displaystyle\sum_{i=1}^n \frac{1}{\mu_i - \alpha} \le \frac{1}{\mu_-(\theta) - \alpha} + \frac{n-1}{\mu_+(\theta) - \alpha}, \\ \displaystyle\sum_{i=1}^n \frac{1}{\beta - \mu_i} \le \frac{1}{\beta - \mu_-(\theta)} + \frac{n-1}{\beta - \mu_+(\theta)}. \end{cases}$$

Define also the set $G_\theta \subset \mathcal{M}_s^{\alpha,\beta}$ by $A \in G_\theta$ if and only if the n-tuple of eigenvalues $(\mu_1,\dots,\mu_n)$ of A belongs to K_θ.

Theorem 8 Assume that $A_\varepsilon \xrightarrow{H} A$ and that $A_\varepsilon(x) = a_\varepsilon(x)I$, where a_ε takes only values α and β. Assume also that, for some $\theta \in L^\infty(\Omega;[0,1])$,

$$a_\varepsilon \xrightarrow{\star} \theta\beta + (1-\theta)\alpha \quad \text{in } L^\infty(\Omega). \tag{7}$$

Then we have

$$A(x) \in G_{\theta(x)} \quad \text{a.e. } x \in \Omega. \tag{8}$$

Conversely, if $A \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$ and $\theta \in L^\infty(\Omega;[0,1])$ satisfy (8), there exists a sequence a_ε of measurable functions taking only values α and β, satisfying (7) and such that $a_\varepsilon I \xrightarrow{H} A$.

In general, there are no explicit formulae for the H-limit of a given sequence of tensors A_ε. In the case of laminated materials, however, the H-limit can be computed explicitely:

Theorem 9 [6, p. 11] If the coefficients a_{ij}^ε of a sequence of matrices $A_\varepsilon \in \mathcal{M}_s^{\alpha,\beta}(\Omega)$ are functions of the first coordinate x_1 only, then $A_\varepsilon \xrightarrow{H} A$ is equivalent to:

(a)

$$\frac{1}{a_{11}^\varepsilon} \xrightarrow{\star} \frac{1}{a_{11}};$$

(b) For $i \neq 1$,

$$\frac{a_{i1}^{\varepsilon}}{a_{11}^{\varepsilon}} \stackrel{\star}{\rightharpoonup} \frac{a_{i1}}{a_{11}};$$

(c) For $i \neq 1$ and $j \neq 1$,

$$a_{ij}^{\varepsilon} - \frac{a_{i1}^{\varepsilon} a_{1j}^{\varepsilon}}{a_{11}^{\varepsilon}} \stackrel{\star}{\rightharpoonup} a_{ij} - \frac{a_{i1} a_{1j}}{a_{11}}.$$

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