A splitting theorem for Kähler Submanifolds of Space-forms

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Abstract

Isometric immersions from Kähler manifolds with parallel pluri-mea curvature (ppmc) generalize in a natural way the constant mean curvature (cmc) sufaces. The (2,0) part of the complexified second fundamental form is a holomorphic quadratic differential (Q) which plays a central role in the study of cmc sufaces. Likewise, for ppmc immersions, Q is also a holomorphic vector bundle quadratic differencial significant in the study of the geometry of the immersion. It is well known that those immersions with Q vanishing are extrinsically symmetric ([10] and [11]). In this work we study ppmc immersions with big nullity index of Q.

1 Introduction and statement of results

Let M^m be a Kähler manifold with complex dimension m and φ be an isometric immersion of M^m into a space form. We denote by α the second fundamental form of φ. The complexified α splits in a natural way, according to types, giving rise to

$$\alpha = \alpha^{(1,1)} + \alpha^{(2,0)} + \alpha^{(0,2)}$$

Isometric immersions with $\alpha^{(1,1)} = 0$ are called pluriminimal immersions. Holomorphic immersions between Kähler manifolds are examples of pluriminimal immersions. Pluriminimal immersions and have been extensively studied (see, by instance, [3] [4], [5], [6]). When M is a Riemann surface we have $\alpha^{(1,1)} =< .,. > H$, where $H = trace\ \alpha$ is the mean curvature of the immersion. In this case the pluriminimal immersions are precisely the minimal ones. In general, the immersion is pluriminimal if and only if its restriction to each holomorphic curve of M is a minimal immersion. The operator $\alpha^{(1,1)}$ is naturally called the plurimean curvature of the immersion. When the ambient space is R^n, it is well known that $\alpha^{(1,1)} = 0$ if and only if $H = 0$ ([4]), so that the class of pluriminimal immersions and the class of minimal immersions coincide.

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We are mainly interested in isometric immersions from Kähler manifolds which have $\alpha^{(1,1)}$ parallel, called isometric immersions with *parallel pluriminimal curvature* operator (*ppmc* isometric immersions). They constitute a natural generalization to higher dimensions of the isometric immersions from Riemann surfaces with parallel mean curvature. In fact *ppmc* isometric immersions into space-forms display some special features of the parallel mean curvature surfaces, namely the existence of a $1-$parameter deformation through a smooth family of *ppmc* isometric immersions which, up to a parallel isomorphism, have the same normal bundle ([2]). Just as in the case of immersions with parallel mean curvature, *ppmc* isometric immersions can also be characterized by the pluriharmonicity of its Gauss map ([7]).

Studying immersed surfaces in R^3, H. Hopf discovered in 1955 that for surfaces with constant mean curvature (*cmc* surfaces) the complexification of the traceless part of α is a holomorphic quadratic differential Q. This holomorphic quadratic differential has been an important ingredient in the investigation of geometric properties of *cmc* surfaces ([13], [14]). The operator Q is nothing but the $(2,0)$ part of the complex bilinear extension of α. A straightforward computation shows that, for *ppmc* isometric immersions, Q is again a vector-bundle valued holomorphic quadratic differential. Isometric immersions with $\alpha^{(2,0)} = \alpha^{(0,2)} = 0$ are called $(2,0)-$ *geodesic* immersions. Curiously that is a strong condition. Indeed, it can be deduced from Codazzi equation that $(2,0)-$ *geodesic* immersions into space forms have parallel second fundamental form. Ferus ([11], [10]), Takeuchi ([16]) and Strübing ([15]) classified the $(2,0)-$ *geodesic* immersed immersions into space forms. It turns out that they are extrinsically symmetric.

In ([3]) Dajczer and Rodrigues proved the following result:

Theorem 1 *Let M be a Kähler manifold with complex dimension m and $\varphi : M \to R^n$ be a pluriminimal immersion such that, for every $x \in M$, the index of relative nullity of α at x is greater or equal than $2m-4$. Then one of the following conditions hold:*

i *f is completly complex ruled,*

ii *$M^m = R^{2m-2k} \times M^k$ and $\varphi = id \times \varphi_2$, where $1 \leq k \leq 2$ and M^k is a Kähler manifold with complex dimension k.*

Theorem 2 *Let M be a Kähler manifold with complex dimension m and $\varphi : M \to R^n$ be a pluriminimal immersion. If M has non zero holomorphic sectional curvature and everywhere the index of relative nullity is greater or equal than $2k$, then $M^m = M^{m-r} \times M^r$ $(1 \leq r \leq k)$ and $\varphi = \varphi_1 \times \varphi_2 : M^{m-r} \times M^r \to R^{n_1} \times R^{n_2}$.*

Notice that that for pluriminimal immersions, the index of relative nullity of α and the index of relative nullity of $\alpha^{(2,0)}$ coincide.

From now on M^m will denote a connected complete Kähler manifold with complex dimension m, S^n_c $(c > 0)$ the n-dimensional euclidean sphere with sectional curvature c and H_c^n the n-dimensional hyperbolic space with constant seccional curvature c ($c < 0$).

For *ppmc* immersions we generalize theorems 1 and 2, proving:

Theorem 3 *Let $\varphi : M^m \to R^n$ be a ppmc immersion. If the index of nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2m-4$, then one of the following conditions hold:*

1 *$M^m = M^1 \times M^{m-1}$ and $\varphi = \varphi_1 \times \varphi_2 : M^1 \times M^{m-1} \to R^{n_1} \times R^{n_2}$, where $\varphi_1 : M^{m-1} \to R^{n_1}$ is extrinsically symmetric.*

2 *$M^m = M^2 \times M^{m-2}$ and $\varphi = \varphi_1 \times \varphi_2 : M^2 \times M^{m-2} \to R^{n_1} \times R^{n_2}$, where $\varphi_1 : M^{m-2} \to R^{n_1}$ is extrinsically symmetric.*

Corollary 4 *Let $\varphi : M^m \to S^n$ be a ppmc immersion such that the index of nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2m-4$. Then $M^m = M^{m-k} \times M^k$ $(1 \leq k \leq 2)$ and $\varphi = \varphi_1 \times \varphi_2 : M^{m-k} \times M^k \to S_a^{n_1} \times S_b^{n_2}$ $(a^{-1} + b^{-1} = 1)$, where $\varphi_1 : M^{m-k} \to S_a^{n_1}$ is extrinsically symmetric.*

Corollary 5 *Let $\varphi : M^m \to H^n$ be a ppmc immersion such that the index of nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2m-4$. Then $M^m = M^{m-k} \times M^k$ $(1 \leq k \leq 2)$ and either $\varphi = \varphi_1 \times \varphi_2 : M^{m-k} \times M^k \to S_a^{n_1} \times H_b^{n_2}$ $(a^{-1} + b^{-1} = -1)$, where $\varphi_1 : M^{m-k} \to S_a^{n_1}$ is extrinsically symmetric, or $\varphi = \varphi_1 \times \varphi_2 : M^{m-k} \times M^k \to H_a^{n_1} \times S_b^{n_2}$, where $\varphi_1 : M^{m-k} \to H_a^{n_1}$ is extrinsically symmetric.*

With respect to theorem 3 and corollaries 4 and 5, the case where the nullity index of $\alpha^{(2,0)}$ is gerater or equal than $2m-2$ has been treated in [8].

Theorem 6 *Let $\varphi : M^m \to R^n$ be a ppmc immersion. If the index of nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2k$ $(1 \leq k \leq m-1)$, then $M^m = M^r \times M^{m-r}$ $(k \leq r \leq m-1)$ and $\varphi = \varphi_1 \times \varphi_2 : M^r \times M^{m-r} \to R^{n_1} \times R^{n_2}$, where $\varphi_1 : M^r \to R^{n_1}$ is extrinsically symmetric.*

Corollary 7 *Let $\varphi : M^m \to S^n$ be a ppmc immersion such that the index of nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2k$ $(1 \leq k \leq m-1)$. Then $M^m = M^r \times M^{m-r}$ $(k \leq r \leq m-1)$ and $\varphi = \varphi_1 \times \varphi_2 : M^r \times M^{m-r} \to S_a^{n_1} \times S_b^{n_2}$ $(a^{-1} + b^{-1} = 1)$, , where $\varphi_1 : M^r \to S_a^{n_1}$ is extrinsically symmetric.*

Corollary 8 *Let $\varphi : M^m \to H^n$ be a ppmc immersion such that the index of nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2k$ $(1 \leq k \leq m-1)$. Then $M^m = M^r \times M^{m-r}$ $(k \leq r \leq m-1)$ and either $\varphi = \varphi_1 \times \varphi_2 : M^r \times M^{m-r} \to S_a^{n_1} \times H_b^{n_2}$ $(a^{-1} + b^{-1} = -1)$, where $\varphi_1 : M^r \to S_a^{n_1}$ is extrinsically symmetric, or $\varphi = \varphi_1 \times \varphi_2 : M^r \times M^{m-r} \to H_a^{n_1} \times S_b^{n_2}$, where $\varphi_1 : M^r \to H_a^{n_1}$ is extrinsically symmetric.*

2 Perliminaries

Let J denote the almost complex structure of M^m and $\varphi : M^m \to F_c$ be an isometric immersion into a space form with seccional curvature c.. We let $C(TM)$ denote the space of smooth sections of TM. We use the notation TM and $T^{\perp}M$ for the the tangent and normal bundles of φ. The complexification of TM, denoted by $T^C M$, decomposes as

$$T^C M = T'M + T''M$$

where $T'M$ and $T''M$ are the eigenbundles of J corresponding respectively to the eigenvalues i and $-i$ of J. The orthogonal projections of $T^C M$ onto $T'M$ and $T''M$ will be represented respectively by π' and π''. Of course, for any section X of TM, we have $X = \pi'(X) + \pi''(X)$.

The complex bilinear extension of the second fundamental form α splits in a natural way according to types, giving rise to

$$\alpha = \alpha^{(1,1)} + \alpha^{(2,0)} + \alpha^{(0,2)}$$

We have

$$\alpha^{(1,1)}(X,Y) = \alpha(X', Y'') + \alpha(X'', Y')$$

where $Z' = \pi'(Z)$ and $Z'' = \pi''(Z)$ for every $Z \in C(T^C M)$. We can also write $\alpha^{(1,1)}(X,Y) = C(X,Y)$, where $C(X,Y) = \frac{1}{2}\{\alpha(X,Y) + \alpha(JX,JY)\}$

Similarly

$$\alpha^{(2,0)}(X,Y) = \alpha(X', Y') = \frac{1}{2}Q(X,Y) - i\frac{1}{2}Q(X,JY)$$

where $Q(X,Y) = \frac{1}{2}\{\alpha(X,Y) - \alpha(JX,JY)\}$

We will use the same symbol ∇ to represent, either the Levi-Civita connection of TM, or the induced connections on $\varphi^{-1}TN$ and $T^*M \otimes \varphi^{-1}TN$. The symbol $\overset{\perp}{\nabla}$ will be used to represent either the induced connection on $\overset{\perp}{T}M$ or on $T^*M \otimes \overset{\perp}{T}M$.

Let $\Delta_x = \{X \in T_xM : Q(X,Y) = 0 \quad \forall\ Y \in T_xM\}$ and $\Delta_x^{\perp}$ be its orthogonal complement in T_xM.

Δ_x and $\Delta_x^{\perp}$ are J_x invariant.since $Q(X,JY) = Q(JX,Y)$, for any $X, Y \in T_xM$.

Proposition 9 *[8] On an open set where the dimension of Δ is constant, Δ is a smooth integrable distribution whose leaves are totally geodesic in M.*

We let U denote the open subset of M where $\dim \Delta$ is minimal and let $r = \dim \Delta_x$, for $x \in U$. From now on Δ will be consiedered defined on U.

For the study of this nullity foliation it is useful to consider the tensor $C_T : \Delta^{\perp} \to \Delta^{\perp}$ defined by

$$C_T(X) = -(\nabla_X T)^{\Delta^{\perp}},$$

where $T \in \Delta$ and $(\ \ )^{\Delta^{\perp}}$ denotes the orthogonal projection onto $\Delta^{\perp}$.

Proposition 10 *[8] The follwoing conditions hold*

1. *C_T commutes with J, for all $T \in \Delta$.*
2. *$Q(C_T(Y), Z) = Q(Y, C_T(Z))$, for $T \in \Delta$ and $Y, Z \in \Delta^\perp$.*

When $\xi \in \overset{\perp}{T_x}M$ let A_ξ denote the Weingarten operator at x associated to α. We represent by $N_x(M)$ the first normal space of the immersion at x.

Lemma 11 *[8] $A_{\alpha(S,T)}Y$, $A_{\alpha(S,Y)}T \in \Delta^\perp$ whenever $S, T \in \Delta$ and $Y \in \Delta^\perp$*

Lemma 12 *Let $T, S \in \Delta$ and $Y \in \Delta^\perp$ Then:*

1. *$R^M(T,Y)S = A_{\alpha(T,Y)}S$*
2. *$R^M(T,Y)T = \frac{1}{2}A_{\alpha(T,T)}Y$*

Proof. We repeat here the proof of property 1 presented in [8].

Using Codazzi equation and fact that φ is *ppmc* we get that

$$\nabla_{Y''}X' \in \Delta' \qquad \forall\ X \in C(\Delta),\ \forall\ Y, W \in C(TM), \tag{1}$$

since $\alpha(\nabla_{Y''}X', W') = 0$.

This implies that $R(S', Y'')T' \in \Delta'$. Then we get from Gauss equation that, for any $Z \in \Delta^\perp$,

$$< R^M(T,Y)S, Z >=< R^M(T',Y'')S'', Z' > + < R^M(T'',Y')S', Z'' >=$$

$$=< \alpha(T',S''), \alpha(Y'',Z') > + < \alpha(T'',S'), \alpha(Y',Z'') >=$$

$$=< \alpha(T',Y''), \alpha(S'',Z') > + < \alpha(T'',Y'), \alpha(S',Z'') >=< \alpha(T,Y), \alpha(S,Z) >,$$

since

$$0 =< R(S'',Y')T'', Z' >= - < \alpha(Y',T''), \alpha(Z',S'') > .$$

To prove equality 2 notice that

$$< A_{\alpha(T,T)}Y, Z >= 2 < \alpha(T',T''), \alpha(Y',Z'') > +2 < \alpha(T',T''), \alpha(Y'',Z') > .$$

We also know from Gauss equation and lemma 11 that

$$< \alpha(T',T''), \alpha(Y',Z') >=< \alpha(T',Y'), \alpha(T'',Z') >= 0.$$

Then

$$< A_{\alpha(T,T)}Y, Z >= 2 < \alpha(T',Z''), \alpha(T'',Y') > +2 < \alpha(T',Y''), \alpha(T'',Z') >=$$

$$2 < \alpha(T,Y), \alpha(T,Z) >= 2 < A_{\alpha(T,Y)}T, Z >$$

■

Proposition 13 *[8] The following equality holds:*

$$(\nabla_S C_T)(Y) = C_T C_S Y + C_{\nabla_S T} Y + R^M(S,Y)T \tag{2}$$

where $S, T \in \Delta$ and $Y \in \Delta^\perp$

3 The splitting theorem

We will prove first that $\Delta^\perp$ is integrable. Assume that, on U, $\dim \Delta = 2m - r$. The case where $r = 2$ was proved in [8].

Lemma 14 *$R(X, JY)X = -JR(X, Y)X$ for all sections X, Y of TM.*

Proof. Consider an arbitrary section Z of TM.

$$\langle R(X, JY)X, Z\rangle = \langle R(X, Z)Y, JX\rangle = -\langle R(Z, Y)X, JX\rangle - \langle R(Y, X)Z, JX\rangle =$$

$$\langle R(X, Y)X, JZ\rangle = -\langle JR(X, Y)X, Z\rangle$$

■

Lemma 15 *Let $T \in \Delta$. Then:*

1 *The eigenvalues of C_T vanish identically.*

2 *$R(T, X)T = 0$ whenever $X \in \Delta^\perp$.*

Proof.
Consider a geodesic γ definided on the real line starting at T and let T denote also its velocity field. Along γ take any vector field $Y \in \Delta^\perp$.
We know, from equation 2, that

$$(\nabla_T C_T)(Y) - C_T C_T Y = R^M(T, Y)T \tag{3}$$

for all $Y \in \Delta^\perp$. Let $R_T : \Delta^\perp \to \Delta^\perp$ denote the opertor defined by $\mathbb{R}_T(Y) = R(T, Y)T$. Since the operator $\nabla_T C_T - C_T^2$ commutes with J and, by lemma 14, R_T anti-commutes with J, we must have $\nabla_T C_T - C_T^2 = 0$ and $R_T = 0$. Hence condition 2 is proved.
let us assume now that λ is a real eigenfunction of C_T along γ. We get from equation 3 that λ is a solution of the equation $\lambda' = \lambda^2$ defined on the real line, hence λ vanishes identically, that is, C_T has no non-zero real eigenvalues.
We use now an argument of [3] to conclude that zero is the only eigenvalue of C_T. Indeed, if $a + ib$ were an eigenvalue of C_T, $a^2 + b^2$ would be an eigenvalue of C_{aT-bJT} so that $a = 0$ and $b = 0$.
We have thus proved that C_T is nilpotent.
■

Proof of theorem 3 and corollaries 4 and 5 Assume now that $r = 4$ on U. The case $r = 2$ was proved in [8].

Proposition 16 $C_T \equiv 0$

Proof.
$C_T^2 = 0$, since $r = 4$.
From lemmas 12 and 15 we conclude that, for every $T \in \Delta$, $Y \in \Delta^\perp$,

$$\alpha(T, Y) = 0. \tag{4}$$

Assume that, for some T, C_T were not identically zero. Let Γ and Θ represent, respectively, the kernel and the image of C_T. Γ and Θ are parallel along γ.

Take $Y \in \Theta$ and consider $Y \in \Delta^\perp$ such that $Y = C_T(Z)$. Therefore

$$\alpha(Y, Y) = \alpha(C_T(Z), Y) = C(C_T(Z), Y) + Q(C_T(Z), Y) =$$

$$C(C_T(Z), Y) + Q(Z, C_T(Y)) = C(C_T(Z), Y)$$

Now, using equation 1, we know that $C_{T'}Z'' = C_{T''}Z' = 0$, so that

$$C(C_T(Z), Y) \quad = \quad C(C_{T'}(Z'), Y'') + C(C_{T''}(Z''), Y') =$$

$$-\alpha((\nabla_{Z'}T')^{\Delta^\perp}, Y'') - \alpha((\nabla_{Z''}T'')^{\Delta^\perp}, Y').$$

Using Codazzi equation and equation 5,

$$C(C_T(Z), Y) = -\alpha(\nabla_{Z'}T', Y'') - \alpha(\nabla_{Z''}T'', Y') \quad =$$

$$\alpha(T', \nabla_{Z'}Y'') + \alpha(T'', \nabla_{Z''}Y') = \alpha(T', (\nabla_{Z'}Y'')^\Delta) + \alpha(T'', (\nabla_{Z''}Y')^\Delta) =$$

$$= \quad \alpha(T, (\nabla_{Z'}Y'')^\Delta) + \alpha(T, (\nabla_{Z''}Y')^\Delta)$$

Notice now that, for every $S \in \Delta$, $\langle \nabla_{Z''}Y'', S' \rangle = \langle Y'', C_{S'}(Z'') \rangle = 0$, as we have seen before. Therefore,

$$\alpha(T, (\nabla_{Z'}Y'')^\Delta) + \alpha(T, (\nabla_{Z''}Y')^\Delta) = \alpha(T, (\nabla_Z Y'')^\Delta) + \alpha(T, (\nabla_Z Y')^\Delta) =$$

$$\alpha(T, (\nabla_Z Y)^\Delta)$$

We conclude then that, for any $V, W \in \Delta^\perp$,

$$\langle \alpha(Y, Y), \alpha(V, W) \rangle = \left\langle \alpha(T, (\nabla_Z Y)^\Delta), \alpha(V, W) \right\rangle =$$

$$\frac{1}{2} \left\langle \alpha(T, V), \alpha((\nabla_Z Y)^\Delta, W) \right\rangle = 0,$$

hence $\alpha(Y, Y) = 0$. We can show in the same way that, for every $Y, Z \in \Theta$, $\alpha(Y, Z) = 0$.

Now take $Y \in \Theta$ and $W \in \Gamma$. As we have seen before $Q(Y, W) = 0$. Then

$$\langle \alpha(Y, W), \alpha(Y, W) \rangle = \langle \alpha(Y', W''), \alpha(Y', W'') \rangle + \langle \alpha(Y'', W'), \alpha(Y'', W') \rangle +$$

$$2\langle \alpha(Y', W''), \alpha(Y'', W')\rangle$$

Now, from Gauss equation,

$$\langle \alpha(Y', W''), \alpha(Y'', W')\rangle = \langle \alpha(Y', Y''), \alpha(W', W'')\rangle$$

and, since $Y' = C_{T'}(Z) = C_{T'}(Z')$, we have $2\alpha(Y', Y'') = \alpha(C_{T'}(Z'), Y'') + \alpha(Y', C_{T''}(Z'')) = \alpha(T, (\nabla_Z Y)^{\Delta})$, reasoning as above. Then

$$2\langle \alpha(Y', W''), \alpha(Y'', W')\rangle = \left\langle \alpha(T, (\nabla_Z Y)^{\Delta}), \alpha(W'', W')\right\rangle = 0,$$

again by Gauss equation. Hence

$$\langle \alpha(Y, W), \alpha(Y, W)\rangle = \langle \alpha(Y', W''), \alpha(Y', W'')\rangle + \langle \alpha(Y'', W'), \alpha(Y'', W')\rangle$$

Now

$$\langle \alpha(Y', W''), \alpha(Y', W'')\rangle \quad = \quad \langle \alpha(C_{T'}(Z'), W''), \alpha(Y', W'')\rangle =$$

$$-\langle \alpha(\nabla_{Z'} T', W''), \alpha(Y', W'')\rangle = \langle \alpha(T', \nabla_{Z'} W''), \alpha(Y', W'')\rangle \quad =$$

$$\left\langle \alpha(T', (\nabla_{Z'} W'')^{\Delta}), \alpha(Y', W'')\right\rangle = 0,$$

as above.

We have proved then that whenever $Y \in \Theta$, $\alpha(Y, Z) = \alpha(Y, W) = \alpha(Y, T) = 0$ for all $T \in \Delta$, $Z \in \Theta$ and $W \in \Gamma$ which entails that $Y \in \ker \alpha$, so that $Y \in \ker \alpha^{(2,0)}$, which cannot happen. Thus $\Theta = \emptyset$, and C_T vanishes identically. ∎

Δ and $\Delta^{\perp}$ are now two parallel distributions. Since Δ and $\Delta^{\perp}$ are invariant under the action of the holonomy group of M, we infer from the De Rham decomposition theorem that U is a product of two Kähler manifolds and $\varphi_{|U}$ is a product of two immersions, since $\alpha(T, Y) = 0$ whenever $T \in \Delta$, $Y \in \Delta^{\perp}$ [12]. An analiticity argument allows the conclusion that M^m is a product of two Kähler manifolds and φ is a product immersion.

The proof of corollaries 4 and 5 is analogous to that of [8] for $r = 2$.

Proof of theorem 6 The hypothesis on the holomorphic sectional curvature implies that $C_T = 0$ for all $T \in \Delta$. Indeed if $C_T \neq 0$ for some T, considering $Z \in \Delta^{\perp}$ such that $Y = C_T(Z) \neq 0$, we will show that $< R^M(Y, JY)Y, JY >= 0$, which cannot happen. In fact

$$\alpha(Y, Y) = \alpha(C_T(Z), Y) = C(C_T(Z), Y) + Q(C_T(Z), Y) = C(C_T(Z), Y),$$

since, from proposition 10-2, $Q(C_T(Z), Y) = Q(Z, C_T(Y)) = 0$.

Now using the definition of the operator C, the fact that the immersion is *ppmc* and equations 1 and 4, one can write

$$C(C_T(Z),Y) = \alpha(C_{T'}(Z'),Y'') + \alpha(C_{T''}(Z''),Y') =$$

$$\alpha(T',(\nabla_{Z'}Y'')^{\Delta}) + \alpha(T'',(\nabla_{Z''}Y')^{\Delta}) = \alpha(T,(\nabla_{Z'}Y'')^{\Delta}) + \alpha(T,(\nabla_{Z''}Y')^{\Delta})$$

Notice now that, for every $S \in \Delta$, $\left\langle (\nabla_{Z''}Y'')^{\Delta}, S\right\rangle = \langle \nabla_{Z''}Y'', S'\rangle = \langle Y'', C_{S'}(Z'')\rangle = 0$. Then

$$\alpha(Y,Y) = \alpha(T,(\nabla_Z Y'')^{\Delta}) + \alpha(T,(\nabla_Z Y')^{\Delta}) = \alpha(T,(\nabla_Z Y)^{\Delta})$$

Then, using Gauss equation,

$$< \alpha(Y,Y), \alpha(JY,JY) >=< \alpha(T,(\nabla_Y Z)^{\Delta}), \alpha(JY,JY) >=$$

$$< \alpha(T,JY), \alpha((\nabla_Y Z)^{\Delta}), JY) >= 0$$

We prove in the same way that $< \alpha(Y,JY), \alpha(Y,JY) >= 0$, from whence $< R(Y,JY)Y,JY >= 0$, which cannot happen.

Therefore C_T vanishes identically for all T in Δ and we proceed as in the proof of theorem 3 to conclude that $M = M_1 \times M_2$, where the complex dimensions of M_1 and M_2 are respectively r and $m-r$, and φ splits as a product immersion $\varphi_1 \times \varphi_2$, where φ_1 has parallel second fundamental form and φ_2 is *ppmc*.

The proof of corollaries 7 and 8 is analogous to the proof of corollaries 4 and 5.

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