# Nodal solutions for perturbed symmetric elliptic equations via Morse index estimates

Miguel Ramos* and Wenming Zou†

**Abstract.** We prove the existence of an unbounded sequence of nodal solutions for an elliptic equation of the form $-\Delta u = g(x,u) + f(x,u)$, $u \in H_0^1(\Omega)$, where both $f(x,s)$ and $g(x,s)$ are subcritical and superlinear in the second variable; $g(x,s)$ is the leading term as $|s| \to \infty$ and $g(x,-s) = -g(x,s)$. The proof is based on a new min-max argument for sign-changing solutions.

## 1 Introduction

Let $\Omega$ be a bounded smooth domain of $\mathbb{R}^N$, $N \geqslant 3$. As a model problem, we consider the following equation

$$-\Delta u = a(x)|u|^{p-2}u + f(x,u), \qquad u \in H_0^1(\Omega),$$

where $a(x) \in L^\infty(\Omega)$, $\operatorname{essinf}_\Omega a > 0$, $2 < p < 2N/(N-2)$ and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $|f(x,s)s| \leqslant C(1 + |s|^\sigma)$, $0 \leqslant \sigma < p$. A long standing conjecture states that, in view of the oddness of the dominating nonlinear term $g(x,s) = a(x)|s|^{p-2}s$, such a problem should have infinitely many solutions, at least for small values of $\sigma$. A partial result on the subject was obtained in [2] (see also [23, Theorem 1] for the case $\sigma = 1$), where the authors show that the problem admits an unbounded sequence of solutions provided $2p/N(p-2) > p/(p-\sigma)$, that is $p < (2N - 2\sigma)/(N-2)$. We mention the seminal papers [1, 17] and we refer the reader to [7, 8, 21] for an account on the subject as well as recent developments.

On the other hand, the results in [13] suggest that the problem should in fact admit infinitely many *sign-changing* solutions. This was proved to be the case in at least two

*Research partially supported by FCT (Fundação para a Ciência e Tecnologia), program POCI-ISFL-1-209 (Portugal/Feder-EU).

†Research supported by NSFC (10571096), SRF-ROCS-SEM and the program of the Ministry of Education in China for New Century Excellent Talents in Universities of China.

solutions: (i) in the symmetric case, see [3, 6, 15] and also [5, Theorem 1.2] for a result with $f = 0$ and $p < 2N/(N-2)$ (among many other results on sign-changing solutions); (ii) in the non-symmetric case, see [21, Theorem 3.1] for a result with $f(x,s) = \mathrm{o}(s)$ as $s \to 0$ and $f(x,s)s \geqslant 0\ \forall s$, with the further restriction that $2p/N(p-2)+1 > p/(p-\sigma)$ (in these papers, more general odd nonlinearities $g(x,s)$ are considered, see Section 2 below for the details).

In the present paper we obtain the same conclusion as in [21], under the less stringent assumption that $2p/N(p-2) > p/(p-\sigma)$ (see Theorem 3.1 below for a precise statement). We stress that the method used in [21] cannot be extended in order to cover this range of parameters; in fact, it is known that crossing the threshold $2p/N(p-2)+1 > p/(p-\sigma)$ is not a mere technical matter. We hope that the method presented here will lead to a better understanding of the structure of the solutions' set of perturbed symmetric elliptic problems.

As a preliminary step in the proof of our main result, in Section 2 we re-obtain the existence result for the symmetric case as in [3, 5, 6, 15], except that we provide an extra information on the Morse index of the sign-changing solutions; in this context, Proposition 2.6 below will play the key role (we refer the reader to [4] and its references for related results on the Morse indices of sign-changing solutions). The proof of Proposition 2.6 combines minimax principles on invariant sets as in [10, 11] with estimates on the Morse index which go back to the work in [2, 14, 22, 23], together with a new device which will be explained later on (cf. Lemma 2.4 and the discussion which precedes it).

Section 3 is devoted to the proof of our main result. It is based on Rabinowitz's perturbation argument [17, 18] together with a recent result of Castro and Clapp [8]. Since, however, the latter result does not seem to apply directly to the energy functional associated to our problem, we rely on a further perturbation argument (on level subsets of the energy functional), which is discussed at the end of Section 2. We mention the work in [16] where a somehow similar idea is used, though in a rather different context.

**Acknowledgment.** W. Zou thanks the CMAF/University of Lisbon for the hospitality during a stay in the fall 2007, where part of this work was done.

## 2 The symmetric case

This section is devoted to the elliptic problem

$$-\Delta u = g(x,u), \qquad u \in H_0^1(\Omega),$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which is odd symmetric ($g(x,-s) = -g(x,s)\ \forall s, \text{a.e.}\, x$) and satisfies

(H1) $g(x,s)/s \to 0$ as $s \to 0$, uniformly in $x$;

(H2) $0 \leqslant g(x,s)s \leqslant C(|s|^p + 1)$, $C > 0$, $2 < p < 2^* := 2N/(N-2)$;

(H3) $g(x,s)s \geqslant \mu G(x,s) - C$, $C > 0$, $\mu > 2$, where $G(x,s) := \int_0^s g(x,\xi)\, d\xi$.

As we mentioned in the Introduction, it is known that under the above assumptions the problem admits infinitely many (pairs of) sign-changing solutions. We provide a further information on these solutions. In Theorem 2.1 we assume that $g$ is $C^1$, but not in the subsequent Theorem 2.7.

**Theorem 2.1.** *Under assumptions* (H1)–(H3)*, with $g(x,\cdot)$ odd and $C^1$, for every $k \in \mathbb{N}$, $k \geqslant 2$, the problem admits a sign-changing solution $u_k$ such that $m(u_k) \leqslant k \leqslant m^*(u_k)$.*

In the above statement, $m(u)$ stands for the Morse index associated to a given solution of the problem, that is a critical point of the energy functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u), \qquad u \in H_0^1(\Omega).$$

Namely, by definition $m(u)$ is the supremum of the dimensions of the subspaces $Y$ of $H_0^1(\Omega)$ such that $I''(u)(\varphi,\varphi) < 0\ \forall \varphi \in Y$, $\varphi \neq 0$; the augmented Morse index $m^*(u)$ is defined as $m^*(u) = m(u) + \dim \mathrm{Ker} I''(u)$.

The proof of Theorem 2.1 will be given after we state a few lemmas. In the following we denote $P := \{u \in H_0^1(\Omega) : u \geqslant 0\}$, $-P := \{u \in H_0^1(\Omega) : u \leqslant 0\}$, $\mathcal{P} := P \cup (-P)$ and, for any $\delta > 0$, $\mathcal{P}_\delta := \{u \in H_0^1(\Omega) : d(u,\mathcal{P}) \leqslant \delta\}$. We will use the norm in $H_0^1(\Omega)$, $||u|| := (\int_\Omega |\nabla u|^2)^{1/2}$, as well as $\langle u, v\rangle := \int_\Omega \langle \nabla u, \nabla v\rangle$, $||u||_p := (\int_\Omega |u|^p)^{1/p}$.

We first recall a result from [11, Lemma 6.1 & Theorem 6.1]; for future reference and since the proof is short, we give a sketch of it (see also [5, Lemma 3.1] for a proof).

**Lemma 2.2.** *There exists $\mu > 0$ such that any solution $\sigma(t,u)$ of*

$$\frac{d}{dt}\sigma(t,u) = -\chi(\sigma(t,u))\frac{\nabla I(\sigma(t,u))}{||\nabla I(\sigma(t,u))||}, \qquad \sigma(0,u) = u,$$

*satisfies*

$$u \in \mathcal{P}_\mu \Rightarrow \sigma(t,u) \in \mathcal{P}_\mu \quad \forall t \geqslant 0.$$

*Here $\chi : H_0^1(\Omega) \to [0,1]$ is any smooth even function such that $\sigma$ is well defined in $\mathbb{R} \times H_0^1(\Omega)$.*

Proof. (sketch) We can write $\nabla I = Id - K$ where $K$ is the compact operator in $H_0^1(\Omega)$ given by $v = Ku$ iff $-\Delta v = g(x,u)$, $v \in H_0^1(\Omega)$. It follows from (H1)-(H2) that $P$ is $K$-invariant, namely that for a sufficiently small $\delta > 0$, $K(P_\delta) \subset P_{\delta/2}$. Indeed, if $d(u,P) \leqslant \delta$ and $v := Ku$ then, for any $\varepsilon > 0$, $d(v,P)||v^-|| \leqslant ||v^-||^2 = -\int_\Omega g(x,u)v^- \leqslant \int_\Omega g(x,u^-)v^- \leqslant \varepsilon||u^-||_2||v^-||_2 + C_\varepsilon||u^-||_p^{p-1}||v^-||_p \leqslant C\varepsilon\delta||v^-|| + CC_\varepsilon\delta^{p-1}||v^-||$, and the claim follows. As a consequence of that, since $\sigma(t) := \sigma(t,u) = u + t\dot{\sigma}(0) + \mathrm{o}(t)$ we see that, by denoting $\lambda := t\chi(u)/|\nabla I(u)|$,

$$\begin{aligned} d(\sigma(t),P) &= d(\lambda Ku + (1-\lambda)u + \mathrm{o}(t), P) \\ &\leqslant \lambda d(Ku,P) + (1-\lambda)d(u,P) + ||\mathrm{o}(t)|| \\ &\leqslant \delta - \lambda\delta/2 + ||\mathrm{o}(t)|| = \delta - Ct\delta + ||\mathrm{o}(t)||, \end{aligned}$$

hence $d(\sigma(t),P) < \delta$ if $t > 0$ is close to 0, and the conclusion of Lemma 2.2 follows. □

**Remark.** The $K$-invariance property yields in particular that $||\nabla I(u)|| \geqslant \delta/2$ for every $u \in \partial\mathcal{P}_\delta$, so that $I$ has no critical points lying in $\mathcal{P}_\delta \setminus \mathcal{P}$, for every small $\delta > 0$.

For any $k \in \mathbb{N}$, $k \geqslant 2$, we denote by $E_k$ the space $E_k = \mathrm{span}\{\varphi_1, \dots, \varphi_k\}$ of dimension $k$, where $\varphi_i$ is an eigenfunction corresponding to the $i$-th eigenvalue of $(-\Delta, H_0^1(\Omega))$. In order to define minimax levels for the functional $I$ which will ultimately provide sign-changing solutions to our problem, we need to find a closed set $S \subset H_0^1(\Omega)$ which intersects $\gamma(B_R(0) \cap E_k)$ far away from the cones $P$ and $-P$, for any large $R > 0$ and any continuous and odd map $\gamma$ which leaves invariant the boundary of $B_R(0) \cap E_k$. The natural choice would be to take for $S$ the unit sphere in the orthogonal space $E_{k-1}^\perp$; however, $d(S,\mathcal{P}) = 0$ for such an $S$. On the other hand, finite dimensional reductions do not seem compatible with flow invariance in the restricted cones $\mathcal{P} \cap E_m$, $m \in \mathbb{N}$.

We therefore introduce the closed set

$$S_k := \{u \in E_{k-1}^{\perp} : \frac{||u||\,||u||_p}{||u|| + ||u||_p} = 1\}.$$

**Lemma 2.3.** *There exist positive constants $C_1$, $C_2$, $C_3$ independent of $k$ such that*

$$1 \leqslant ||u||_p \leqslant C_1, \qquad \forall u \in S_k,$$

*and*

$$\inf_{S_k} I \geqslant C_2 \lambda_k^{N(2^*-p)/p2^*} - C_3 \to +\infty, \qquad \text{as } k \to \infty,$$

*where $(\lambda_k)_{k\in\mathbb{N}}$ denotes the sequence of eigenvalues of the operator $(-\Delta, H_0^1(\Omega))$.*

Proof. By writing the above identity in the equivalent form

$$\frac{1}{||u||} + \frac{1}{||u||_p} = 1,$$

our first conclusion follows from the continuous imbedding $H_0^1(\Omega) \subset L^p(\Omega)$. By combining this with the Gagliardo–Nirenberg inequality we obtain the full conclusion of Lemma 2.3. □

It follows from the previous lemma that we can fix a constant $c_0 > 0$ (independent of $k$) such that

$$\inf_{S_k} I > -c_0, \qquad \forall k.$$

For a given positive constant $R_k$, we denote

$$Q_k := B_{R_k}(0) \cap E_k, \qquad \partial Q_k := \partial B_{R_k}(0) \cap E_k;$$

in fact, in the sequel we fix $R_k$ so large that

$$\sup_{\partial Q_k} I < -c_0 \qquad \text{and} \qquad \inf\{\frac{||u||\,||u||_p}{||u|| + ||u||_p} : u \in \partial Q_k\} > 1.$$

We also fix any number

$$M_k > \sup_{E_k} I.$$

**Lemma 2.4.** *There exists $\mu_k > 0$ such that*

$$u \in S_k, \quad I(u) \leqslant M_k \;\Rightarrow\; d(u, \mathcal{P}) \geqslant 2\mu_k.$$

Proof. By assuming the contrary we find a sequence $(u_n) \subset S_k$ such that $I(u_n) \leqslant M_k$ and $d(u_n, \mathcal{P}) \to 0$. As observed above, the sequence $(||u_n||_p)$ is bounded and so, since $(I(u_n))$ is bounded from above, also $(||u_n||)$ is bounded. Since $E_{k-1}^{\perp} \cap \mathcal{P} = \{0\}$, this implies that, up to a subsequence, $u_n \rightharpoonup 0$ weakly in $H_0^1(\Omega)$. Since the imbedding $H_0^1(\Omega) \subset L^p(\Omega)$ is compact, this contradicts the fact that $||u_n||_p \geqslant 1 \ \forall n$. □

Let $\mu_k$ be given by Lemma 2.4 and let us set

$$\mathcal{U}_k := \{u \in H_0^1(\Omega) : d(u, \mathcal{P}) \geqslant \mu_k\}.$$

By possibly taking a smaller $\mu_k$, we can assume that the conclusion of Lemma 2.2 applies to $\mu_k$. We denote

$$\Gamma_k := \{\gamma : Q_k \to H_0^1(\Omega) \text{ continuous and odd}, \ \gamma|_{\partial Q_k} = Id, \ \sup_{\gamma(Q_k)} I < M_k\},$$

and

$$c_k := \inf_{\gamma \in \Gamma_k} \sup_{\gamma(Q_k) \cap \mathcal{U}_k} I.$$

We prove that $c_k$ is a critical value for $I$ which corresponds to a sign-changing solution of our original problem.

**Proposition 2.5.** *For any $k \in \mathbb{N}$, $k \geqslant 2$, there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = c_k$, $I'(u_k) = 0$ and $m(u_k) \leqslant k$. Moreover,*

$$\inf_{S_k} I \leqslant c_k < M_k.$$

Proof. It is clear that $c_k \leqslant \sup_{Q_k} I \leqslant \sup_{E_k} I < M_k$. On the other hand, given $\gamma \in \Gamma_k$, the set $\{u \in Q_k : \frac{||\gamma(u)|| \, ||\gamma(u)||_p}{||\gamma(u)|| + ||\gamma(u)||_p} < 1\}$ is a bounded, symmetric neighborhood of the origin in $E_k$ and so, according to the Borsuk-Ulam theorem, its boundary contains a point $u$ such that $\gamma(u) \in E_{k-1}^{\perp}$. Our choice of the constant $R_k$ implies that $u \notin \partial Q_k$ and so $\gamma(u) \in S_k$. Since moreover $I(\gamma(u)) \leqslant \sup_{\gamma(Q_k)} I \leqslant M_k$, we deduce from Lemma 2.4 that $\gamma(u) \in S_k \cap \mathcal{U}_k$. This shows that $c_k \geqslant \inf_{S_k} I$.

The conclusion that $c_k$ is indeed a critical value corresponding to a critical point in $\mathcal{U}_k$ is standard and so we will keep the proof short. In view of a contradiction, suppose there exists $\varepsilon > 0$ such that

$$||\nabla I(u)|| \geqslant 2\varepsilon, \qquad \forall u : |I(u) - c_k| \leqslant 2\varepsilon, \ d(u, \mathcal{P}) \geqslant \mu_k.$$

Then we can fix the two closed, symmetric disjoint sets $A := \{u : \nabla I(u) = 0\} \cup \{u : |I(u) - c_k| \geqslant 2\varepsilon\} \cup \{u : d(u, \mathcal{P}) \leqslant \mu_k/2\}$, $B := \{u : |I(u) - c_k| \leqslant \varepsilon,\ d(u, \mathcal{P}) \geqslant \mu_k\}$, together with a smooth, even cut-off function $\chi : H_0^1(\Omega) \to [0,1]$ such that $\chi = 0$ in $A$ and $\chi = 1$ in $B$. According to Lemma 2.2, this gives rise to a flow $\sigma : \mathbb{R} \times H_0^1(\Omega) \to H_0^1(\Omega)$ for which $\mathcal{P}_{\mu_k}$ is positively invariant. We denote $\sigma_1(u) := \sigma(1, u)$. Let us take any $\gamma \in \Gamma_k$ such that $\sup_{\gamma(Q_k)\cap\mathcal{U}_k} I \leqslant c_k + \varepsilon$. Clearly, $\sigma_1 \circ \gamma \in \Gamma_k$ and it turns out that $\sup_{(\sigma_1\circ\gamma)(Q_k)\cap\mathcal{U}_k} I \leqslant c_k - \varepsilon$. Indeed, if $(\sigma_1 \circ \gamma)(u) \in \mathcal{U}_k$ for some $u \in Q_k$ then, by the invariance property of the flow we must have that $\gamma(u) \in \mathcal{U}_k$, and in fact $\sigma(t, \gamma(u)) \in \mathcal{U}_k$ $\forall t \in [0,1]$. So, in case $I((\sigma_1 \circ \gamma)(u)) > c_k - \varepsilon$ we will have that $\chi(\sigma(t, \gamma(u))) = 1$ $\forall t \in [0,1]$, leading to the contradiction: $c_k - \varepsilon < I((\sigma_1 \circ \gamma)(u)) \leqslant I(u) - \int_0^1 2\varepsilon\, dt \leqslant c_k + \varepsilon - 2\varepsilon = c_k - \varepsilon$. In conclusion, we must have that $I((\sigma_1 \circ \gamma)(u)) \leqslant c_k - \varepsilon$, and this shows that $\sup_{(\sigma_1\circ\gamma)(Q_k)\cap\mathcal{U}_k} I \leqslant c_k - \varepsilon$. Of course, this contradicts the definition of $c_k$ and therefore, for some sequence $\varepsilon_n \to 0$ we can find a Palais-Smale sequence $(u_n) \subset H_0^1(\Omega)$ such that

$$I(u_n) \to c_k, \qquad I'(u_n) \to 0 \qquad \text{and} \qquad d(u_n, \mathcal{P}) \geqslant \mu_k.$$

It is trivial to check that $I$ satisfies the Palais-Smale condition, namely that, up to a subsequence, $u_n \to u$ in $H_0^1(\Omega)$, for some $u \in H_0^1(\Omega)$. In particular, $I(u) = c_k$, $I'(u) = 0$ and $d(u, \mathcal{P}) \geqslant \mu_k > 0$, as claimed.

As for the information on the Morse index, let $C$ be the (non empty) compact symmetric set $C := \{u \in \mathcal{U}_k : I(u) = c_k, I'(u) = 0\}$. Since $H_0^1(\Omega)$ is an infinite dimensional separable space, $C$ can be disconnected through some hyperplane, namely we can find $e \in H_0^1(\Omega)$ such that $\int_\Omega \langle \nabla e, \nabla u\rangle \neq 0$ $\forall u \in C$, so that $C$ can be written as the disjoint union of the two compact sets $C = C_+ \cup C_- = \{u \in C : \int_\Omega \langle \nabla e, \nabla u\rangle > 0\} \cup \{u \in C : \int_\Omega \langle \nabla e, \nabla u\rangle < 0\}$; then the usual Marino-Prodi perturbation method, based on Sard's lemma, can be performed over $C_+$ and be subsequently extended by symmetry to $C_-$. As a consequence, we do not need to rely on special arguments of perturbation theory for invariant functionals, and classical arguments such as the ones in e.g. [2, 9, 12, 14, 20, 22] immediately yield the conclusion that some point $u \in C$ must have Morse index less or equal than $k$. □

Next we will be concerned with the problem of finding sign-changing solutions having Morse index greater or equal than $k$. The sets $S_k$, $\partial Q_k$ and the constant $M_k$ were defined above and we introduce now the corresponding notion of linking.

**Definition.** *Given $A \subset H_0^1(\Omega)$, we sat that $A$ and $S_k$* link *if $A$ is a compact, symmetric, $\sup_A I < M_k$, $\partial Q_k \subset A$ and moreover $\gamma(A) \cap S_k \neq \emptyset$ for every odd and continuous map $\gamma : A \to H_0^1(\Omega)$ such that $\gamma|_{\partial Q_k} = Id$.*

Similarly to [14, 20, 22], let

$$\mathcal{L}_k := \{A \subset H_0^1(\Omega) : \; A \text{ and } S_k \text{ link}\} \quad \text{ and } \quad c_k^* := \inf_{A \in \mathcal{L}_k} \sup_{A \cap \mathcal{U}_k} I.$$

Thanks to Lemma 2.4, we have that $A \cap S_k \subset \mathcal{U}_k$ $\forall A \in \mathcal{L}_k$, and therefore

$$\inf_{S_k} I \leqslant c_k^* \leqslant c_k < M_k.$$

**Proposition 2.6.** *For any $k \in \mathbb{N}$, $k \geqslant 2$, there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = c_k^*$, $I'(u_k) = 0$ and $m^*(u_k) \geqslant k$.*

Proof. By taking into account our considerations in the proof of Proposition 2.5, this is an immediate conclusion of [20, Theorem 2.5] (see also [19, Theorem 10.8]) once we establish the following claim: *given compact sets $C \subset \widetilde{C}$ of $\mathbb{R}^{k-1}$, every continuous map $h : C \to H_0^1(\Omega) \setminus S_k$ admits a continuous extension $H : \widetilde{C} \to H_0^1(\Omega) \setminus S_k$.* This, in turn, can be proved similarly to [20, Proposition 3.1]. Indeed, let $\widetilde{h} : \widetilde{C} \to H_0^1(\Omega)$ be any continuous extension of $h$, which we write as $\widetilde{h}(x) = \alpha(x) + \beta(x)$ according to the splitting $H_0^1(\Omega) = E_{k-1} \oplus (E_{k-1})^\perp$. Let $F := \{x \in \widetilde{C} : \alpha(x) = 0 \text{ and } \theta(\beta(x)) = 1\}$, where $\theta : H_0^1(\Omega) \to \mathbb{R}^+$ is the homogeneous and continuous map given by $\theta(u) := ||u|| \, ||u||_p / (||u|| + ||u||_p)$ (and $\theta(0) := 0$). By assumption, $F$ is a compact set disjoint from $C$. We can thus choose an $\varepsilon$-neighborhood $F_\varepsilon$ of $F$ in such a way that $C \cap F_\varepsilon = \emptyset$ and $\beta(x) \neq 0$ $\forall x \in F_\varepsilon$. Since $F_\varepsilon \subset \mathbb{R}^{k-1}$ and $\dim(E_{k-1} \times \mathbb{R}) = k$, the map

$$\partial F_\varepsilon \to (E_{k-1} \times \mathbb{R}) \setminus \{(0,1)\}, \qquad x \mapsto (\alpha(x), \theta(\beta(x))$$

admits a continuous extension $F_\varepsilon \to (E_{k-1} \times \mathbb{R}) \setminus \{(0,1)\}$, say $x \mapsto (\widetilde{\alpha}(x), \rho(x))$. By possibly replacing $\rho$ by its positive part $\rho^+$, we may assume that $\rho \geqslant 0$. The desired map $H$ is then given by

$$H(x) = \begin{cases} \widetilde{\alpha}(x) + \frac{\rho(x)}{\theta(\beta(x))} \beta(x) & \text{if } x \in F_\varepsilon \\ \widetilde{h}(x) & \text{if } x \notin F_\varepsilon, \end{cases}$$

and this completes the proof of Proposition 2.6. □

Proof of Theorem 2.1 completed. Following an idea introduced in [14, 22], we may restrict further the class $\mathcal{L}_k$ by setting

$$\widetilde{\mathcal{L}}_k := \{A \in \mathcal{L}_k : A \text{ has Hausdorff dimension} \leqslant k\} \quad \text{and} \quad \widetilde{c}_k := \inf_{A \in \widetilde{\mathcal{L}}_k} \sup_{A \cap \mathcal{U}_k} I.$$

We recall that the latter property means that there exists a constant $d > 0$ such that for each integer $n \in \mathbb{N}$ we can cover $A$ by $n^k$ balls of radius $d/n$. Thus, since continuous maps in $Q_k$ can be approximated by odd and Lipschitz continuous ones, we see that

$$c_k^* \leqslant \widetilde{c}_k \leqslant c_k,$$

and it follows from [14, Theorem 2.6] (see also [19, Theorem 10.8], [20, Theorem 2.9], [22, Theorem 3]) combined with our previous arguments that there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = \widetilde{c}_k$, $I'(u_k) = 0$ and $m(u_k) \leqslant k \leqslant m^*(u_k)$. □

In view of the applications in Section 3, we state a variant of Propositions 2.5 and 2.6 in which we consider a slight perturbation of the previous minimax level $c_k$.

To be precise, let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory, odd symmetric function satisfying (H1)–(H3), and let us fix any number $2 < q < \mu$ and the corresponding functional

$$I_0(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{q}\int_\Omega |u|^q, \qquad u \in H_0^1(\Omega).$$

In the sequel we also fix a small number $\lambda_0$ in such a way that $0 < \lambda_0 < (\mu - 2)/2$. The space $E_k = \operatorname{span}\{\varphi_1, \dots, \varphi_k\}$ and its subset $Q_k = B_{R_k}(0) \cap E_k$ have been defined previously, as well as the closed set $\mathcal{U} = \{u \in H_0^1(\Omega) : d(u, \mathcal{P}) \geqslant \mu_k\}$ for some $\mu_k > 0$. For a given $M_k > 0$, we introduce the number

$$d_k := \inf\{\sup_{\gamma(Q_k) \cap \mathcal{U}_k} I : \gamma \in C(Q_k; H_0^1(\Omega)) \text{ is odd}, \gamma|_{\partial Q_k} = Id, \sup_{\gamma(Q_k)} I_1 < M_k\},$$

where

$$I_1(u) := I^+(u) + \lambda_0 I_0(u), \qquad u \in H_0^1(\Omega).$$

**Proposition 2.7.** *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory, odd symmetric function satisfying* (H1)–(H3). *With the above notations, suppose $M_k > (\sup_{E_k} I_0)^2$. Then, provided $k$ is sufficiently large, there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = d_k$ and $I'(u_k) = 0$. Moreover,*

$$d_k \geqslant c k^{2p/N(p-2)},$$

*for some $c > 0$ independent of $k$.*

PROOF. Since $q < \mu$, we have that

$$I(u) \leqslant I_0(u) + C_0, \qquad \forall u \in H_0^1(\Omega),$$

for some fixed $C_0 > 0$, and therefore $\sup_{E_k} I_1 < M_k$ if $k$ is sufficiently large, yielding in particular that $d_k$ is well-defined. It follows also as in Proposition 2.5 that $d_k \geqslant \inf_{S_k} I \to +\infty$ as $k \to \infty$. We point out that, similarly to Lemma 2.4, $\mu_k$ is a small positive number associated to the sublevel set $\{I^+ + \lambda_0 I_0 < M_k\}$. In order to prove that $d_k$ is indeed a critical value for $I$ we must be able to build a flow for which this sublevel set is invariant.

In order to do this, let

$$D_k := \{u \in H_0^1(\Omega) : \ |I(u) - d_k| \leqslant 1, \ |I_1(u) - M_k| \leqslant 1\}.$$

We claim that

$$\nabla I(u) + \lambda \nabla I_0(u) \neq 0, \qquad \forall u \in D_k, \ 0 \leqslant \lambda \leqslant \lambda_0.$$

Indeed, in case $\nabla I(u) + \lambda \nabla I_0(u) = 0$ we would find that

$$||u||^2 \geqslant \frac{\mu}{1+\lambda_0} \int_\Omega G(u) - C,$$

for some $C > 0$. Since $I(u) \leqslant d_k + 1$ and since $\lambda_0$ is small, this would lead to $||u||^2 \leqslant C(d_k + 1)$ for some $C > 0$. Since, by assumption, $d_k \leqslant C\sqrt{M_k}$, this is not compatible with the fact that $I_1(u) \geqslant M_k - 1$. This proves our claim, which we write in the equivalent form:

$$\lambda \nabla I(u) + \mu \nabla I_1(u) \neq 0, \qquad \forall u \in D_k, \ \lambda, \mu \geqslant 0, \ \lambda^2 + \mu^2 = 1.$$

This shows in particular that there exists $\theta_k \in [0, 1[$ such that

$$\inf_{D_k} \frac{\langle \nabla I, \nabla I_1 \rangle}{||\nabla I|| \, ||\nabla I_1||} > -\theta_k.$$

Indeed, if this condition is violated then we can find a sequence $(u_n) \subset D_k$ such that $v_n := \nabla I(u_n)/||\nabla I(u_n)|| + \nabla I_1(u_n)/||\nabla I_1(u_n)||$ satisfies $||v_n||^2 \to 0$; since $d_k - 1 \leqslant I(u_n)$ and $I_1(u_n) \leqslant M_k + 1$, it follows easily that $(||u_n||)_n$ is bounded and therefore, up to a subsequence, $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. In particular, also $\frac{||\nabla I(u_n)|| \, ||\nabla I_1(u_n)||}{\sqrt{||\nabla I(u_n)||^2 + ||\nabla I_1(u_n)||^2}} v_n \to 0$, that is,

$$\lambda_n \nabla I(u_n) + \mu_n \nabla I_1(u_n) \to 0, \qquad u_n \in D_k, \ \lambda_n, \mu_n \geqslant 0, \ \lambda_n^2 + \mu_n^2 = 1.$$

Multiplying this expression by $u_n$ yields that actually $u_n \to u$ strongly in $H_0^1(\Omega)$. Therefore, in this way we find $u \in D_k$ such that $\lambda \nabla I(u) + \mu \nabla I_1(u) = 0$ for some $\lambda, \mu \geqslant 0$, $\lambda^2 + \mu^2 = 1$, which we already have proved to be impossible. This establishes the existence of the number $\theta_k \in [0,1[$ mentioned above.

Now, assume first that $g$ is smooth enough, so that $I \in C^2(H_0^1(\Omega);\mathbb{R})$. In this case we consider the vector field

$$V(u) := \frac{1}{2}\frac{\nabla I(u)}{||\nabla I(u)||^2} + \frac{1}{2}\frac{\theta(u)}{||\nabla I(u)||}\frac{\nabla I_1(u)}{||\nabla I_1(u)||}, \qquad u \in H_0^1(\Omega),\ \nabla I(u) \neq 0,$$

where $\theta : H_0^1(\Omega) \to [0, \theta_k]$ is a cut-off function such that $\theta(u) = \theta_k$ if $u \in D_k$ and $\theta(u) = 0$ if $u$ lies in a closed small neighborhood of the critical set of $I_1$. This is easily shown to be a pseudo-gradient vector field for $I$, namely

$$\frac{1-\theta_k}{2} := \alpha \leqslant \langle V(u), \nabla I(u)\rangle \leqslant ||V(u)||\,||\nabla I(u)|| \leqslant 1, \qquad \forall u \in H_0^1(\Omega), \nabla I(u) \neq 0,$$

and moreover

$$\langle V(u), \nabla I_1(u)\rangle > 0, \qquad \forall u \in D_k.$$

Then, by going through the proof of Lemma 2.2 and Proposition 2.5 we see immediately that there exists $u_k \in \mathcal{U}_k$ such that $I(u_k) = d_k$ and $I'(u_k) = 0$. We stress that in the argument we rely on the flow defined by $\frac{d}{dt}\sigma(t,u) = -\chi(\sigma(t,u))\frac{V(\sigma(t,u))}{||V(\sigma(t,u))||}$, $\sigma(0,u) = u$, where $\chi : H_0^1(\Omega) \to [0,1]$ is a smooth cut-off function such that $\chi = 0$ in $A$ and $\chi = 1$ in $B$, and $A := \{u : \nabla I(u) = 0\} \cup \{u : |I(u) - d_k| \geqslant 2\varepsilon\} \cup \{u : d(u, \mathcal{P}) \leqslant \mu_k/2\} \cup \{u : I_1(u) \geqslant M+2\}$, $B := \{u : |I(u) - c| \leqslant \varepsilon,\ d(u,\mathcal{P}) \geqslant \mu_k,\ I_1(u) \leqslant M+1\}$.

In the general case where $g$ is merely assumed to be a Carathéodory function, the map $K : H_0^1(\Omega) \to H_0^1(\Omega)$ mentioned in the proof of Lemma 2.2 needs not to be locally Lipschitz continuous; however, given $\eta > 0$ we can find such a map $K_\eta : H_0^1(\Omega) \to H_0^1(\Omega)$ such that $||K_\eta(u) - K(u)|| \leqslant \eta\ \forall u \in H_0^1(\Omega)$. Then, since $||\nabla I||$ is bounded both from below and from above in the set $H_0^1(\Omega) \setminus A$, provided $\nu$ is small enough we can define $V(u)$ with a similar expression as above, with $\nabla I$ (resp. $\nabla I_1$) replaced by $W_\eta = \nabla I + K - K_\eta$ (resp. $W_\eta^1 = \nabla I_1 + K - K_\eta$).

As for the final conclusion in Proposition 2.7, let

$$\overline{I}(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{p}\int_\Omega |u|^p, \qquad u \in H_0^1(\Omega),$$

so that, for some $C_0 > 0$,

$$\overline{I} \leqslant I + C_0 \quad \text{ and } \quad \{I^+ + \lambda_0 I_0 < M_k\} \subset \{\overline{I}^+ + \lambda_0 I_0 < \overline{M}_k\},$$

where $\overline{M}_k := M_k + 2C_0 + M_k/\lambda_0$. This implies that $d_k \geqslant \overline{d}_k - C_0$, where $\overline{d}_k$ is the corresponding min-max level associated to $\overline{I}$. We can thus assume that $g(x,s) = |s|^{p-2}s$. In particular, $g$ is $C^1$ and by our previous considerations we know that $d_k$ is a critical level associated to some sign-changing critical point of $I$, having Morse index less or equal than $k$. By using instead the dual classes as in Proposition 2.6, we may already assume that the reversed inequality holds. The conclusion follows then as in [2, 23]. □

# 3 Perturbations from symmetry

We next consider the problem

$$-\Delta u = g(x,u) + f(x,u), \qquad u \in H_0^1(\Omega)$$

where $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory, odd symmetric function satisfying (H1)–(H3) and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that

(H4) $f(x,s)/s \to 0$ as $s \to 0$ uniformly in $x$, and $0 \leqslant f(x,s)s \leqslant C(|s|^\sigma + 1)$ $\forall s$, for some $C > 0$, $0 < \sigma < \mu$.

In the statement below, we will denote by $J$ the energy functional

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u) - \int_\Omega F(x,u), \qquad u \in H_0^1(\Omega),$$

where $G(x,s) := \int_0^s g(x,\xi)\,d\xi$ and $F(x,s) := \int_0^s f(x,\xi)\,d\xi$.

**Theorem 3.1.** *Under assumptions* (H1)–(H4) *with $g(x,\cdot)$ odd, if moreover*

$$\frac{2p}{N(p-2)} > \frac{\mu}{\mu - \sigma},$$

*the above problem admits a sequence of sign-changing solutions $(u_{k_n})_{n\in\mathbb{N}}$ whose energy levels $J(u_{k_n})$ satisfy*

$$c_1 {k_n}^{2p/N(p-2)} \leqslant J(u_{k_n}) \leqslant c_2 {k_n}^{2\mu/N(\mu-2)}.$$

*for some $c_1, c_2 > 0$ independent of $n$.*

PROOF. We use a perturbation argument similar to the one in [17, 18]. For the sake of clarity we divide the proof in several steps.

FIRST STEP. We may assume without loss of generality that

$$|J(u) - J(-u)| \leqslant C(|J(u)|^{\sigma/\mu} + 1), \qquad \forall u \in H_0^1(\Omega).$$

Indeed, otherwise we can replace $J$ by a penalized functional $\widetilde{J}$ given by $\widetilde{J}(u) = J(u) + (1 - \theta(u)) \int_\Omega F(x,u)$, where $\theta(u) := \chi\left(\delta \int_\Omega G(x,u)/\sqrt{J_s^2(u)+1}\right)$ and $J_s(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 - \int_\Omega G(x,u)$ is the symmetric (even) part of $J$; here $\chi \in \mathcal{D}(]-2,2[)$ is a smooth cut-off function, $0 \leqslant \chi \leqslant 1$, with $\chi = 1$ in $[-1,1]$, and $\delta$ is a small positive constant. The functional $\widetilde{J}$ does satisfy $|\widetilde{J}(u) - \widetilde{J}(-u)| \leqslant C(|\widetilde{J}(u)|^{\mu/\sigma} + 1)$ and moreover $\widetilde{J}'(u)\varphi = (1 + \mathrm{o}(1))\langle u, \varphi\rangle - (1 + \mathrm{o}(1))\int_\Omega g(x,u)\varphi - \theta(u)\int_\Omega f(x,u)\varphi$, $\forall u, \varphi \in H_0^1(\Omega)$, where $\mathrm{o}(1) \to 0$ as $\widetilde{J}(u) \to +\infty$. In particular, critical points of $J$ and $\widetilde{J}$ coincide at high levels of the energy. We mention that, in order to preserve the property described in Lemma 2.2, our penalized term differs slightly from Rabinowitz's one (in [17, 18], $\theta(u) = \chi\left(\delta \int_\Omega G(x,u)/\sqrt{J^2(u)+1}\right)$).

SECOND STEP. We use notations similar to the ones in Proposition 2.7. For any large integer $k \in \mathbb{N}$ and any given $M_k > (\sup_{E_k} I_0)^2$, let

$$b_k := \inf\{\sup_{\gamma(Q_k)\cap\mathcal{U}_k} J : \gamma \in C(Q_k; H_0^1(\Omega)) \text{ is odd}, \ \gamma|_{\partial Q_k} = Id, \ \sup_{\gamma(Q_k)} J_1 < M_k\},$$

where $J_1 := J^+ + \lambda_0 I_0$. Since $J$ may not be even, $b_k$ will not be in general a critical point of $J$. However, by Proposition 2.7 (see the argument at the end of its proof) we can assert that $b_k \geqslant c_1 k^{2p/N(p-2)}$, for some $c_1 > 0$ independent of $k$. Moreover, it is known that $b_k \leqslant c_2 k^{2\mu/N(\mu-2)}$ (cf. [2, p. 1035]).

THIRD STEP. Let us fix $\gamma : Q_k \to H_0^1(\Omega)$ continuous and odd, $\gamma|_{\partial Q_k} = Id$, such that

$$\sup_{\gamma(Q_k)\cap\mathcal{U}_k} J \leqslant b_k + 1.$$

According to [8, Corollary 2.2], $\gamma$ has an odd continuous extension (which we still denote by $\gamma$), $\gamma : E_{k+1} \to H_0^1(\Omega)$ such that $\gamma(u) = u$ for every $u \in E_{k+1}$ with large norm (say, $||u|| \geqslant R_{k+1} > R_k$) and, since $\sup_{\gamma(E_k)} I_0 < M_k$, with the further property that

$$\sup_{\gamma(E_{k+1})} I_0 \leqslant \alpha M_k + \beta,$$

where $\alpha$, $\beta$ are positive constants depending only on $\Omega$ and $q$. In particular, $\sup_{\gamma(E_{k+1})} J \leqslant \alpha' M_k + \beta'$ and $\sup_{\gamma(E_{k+1})} J_1 \leqslant \alpha' M_k + \beta'$. The following number is therefore well-defined:

$$\overline{c}_k := \inf_{\gamma \in \Lambda_k} \sup_{\gamma(Q_k^+) \cap \overline{\mathcal{U}}_k} J,$$

where we have denoted $\overline{\mathcal{U}}_k := \{u : d(u, \mathcal{P}) \geqslant 2\mu_k\} \subset \mathcal{U}_k = \{u : d(u, \mathcal{P}) \geqslant \mu_k\}$, $Q_k^+ := (E_k \oplus \mathbb{R}^+ \varphi_{k+1}) \cap B_{R_{k+1}}(0)$, $\partial Q_k^+ := \big((B_{R_{k+1}}(0) \setminus B_{R_k}(0)) \cap E_k\big) \cup \big((E_k \oplus \mathbb{R}^+ \varphi_{k+1}) \cap \partial B_{R_{k+1}}(0)\big)$. By definition, the class $\Lambda_k$ consists of the continuous maps $\gamma : Q_k^+ \to H_0^1(\Omega)$ such that: (i) $\gamma|_{Q_k}$ is odd; (ii) $\gamma|_{\partial Q_k^+} = Id$; (iii) $\sup_{\gamma(Q_k) \cap \mathcal{U}_k} J \leqslant b_k + 1$; (iv) $\sup_{\gamma(Q_k^+)} J_1 < M_k^2$.

Fourth step. Suppose first that

$$b_k + 1 < \overline{c}_k.$$

In this case, it follows from the arguments in the proof of Proposition 2.7 that $\overline{c}_k$ is a critical value for $J$, namely there exists $u_k \in \mathcal{U}_k$ such that $J(u_k) = \overline{c}_k$ and $J'(u_k) = 0$. It should be noted that the flow $\sigma(t, u)$ mentioned in the proof of Proposition 2.7 is such that $\sigma(t, u) = u$ $\forall u \in \gamma(Q_k)$, $\gamma \in \Lambda_k$: indeed, this is the case for $u \in \gamma(Q_k) \cap \mathcal{U}_k$ (since $b_k + 1 < \overline{c}_k$) while if $d(u, \mathcal{P}) \leqslant \mu_k < 2\mu_k$ then also $\sigma(t, u) = u$ by construction. We stress that in the process of going from the min-max level $b_k$ to the number $\overline{c}_k$ we have increased the value of the constraint of $J_1$ (from $M_k$ to $M_k^2$) and have enlarged the invariant neighborhood of the cone $\mathcal{P}$ (by increasing $\mu_k$ by a factor of 2); this is indeed possible provided we start the process by taking a sufficiently small number $\mu_k$ and provided we perform this construction a finite number of times only. We conclude the proof of Theorem 3.1 by showing that indeed only a finite number of steps are needed in this argument. (As for the growth estimate $\overline{c}_k \leqslant c k^{2\mu/N(\mu-2)}$, it is enough to observe that rather than taking squares we can increase the value of the constraint of $J_1$ linearly, and so $\overline{c}_k \leqslant C(b_k + 1)$ for every $k$.)

Fifth step. Suppose now that

$$\overline{c}_k \leqslant b_k + 1.$$

Then we can find $\gamma \in \Lambda_k$ such that $\sup_{\gamma(Q_k^+) \cap \overline{\mathcal{U}}_k} J \leqslant b_k + 2$. We can extend $\gamma$ by symmetry to the whole space $E_{k+1}$; we still denote by $\gamma$ this extension. By taking into account the property mentioned in the first step of the proof, we see that (provided $k$ is large enough) $\sup_{\gamma(Q_{k+1}) \cap \overline{\mathcal{U}}_k} J \leqslant b_k + c\, b_k^{\sigma/\mu}$ for some $c > 0$ independent of $k$, and $\sup_{\gamma(Q_{k+1})} J_1 < M_k^3$.

Then we may define

$$b_{k+1} := \inf\{\sup_{\gamma(Q_{k+1})\cap\overline{\mathcal{U}}_k} J : \gamma \in C(Q_{k+1}; H_0^1(\Omega)) \text{ is odd}, \ \gamma|_{\partial Q_{k+1}} = Id, \ \sup_{\gamma(Q_{k+1})} J_1 < M_k^3\},$$

and we know that

$$b_{k+1} \leqslant b_k + c\, b_k^{\sigma/\mu}.$$

Starting from $b_{k+1}$, we iterate this process as in Step 2 above. Now, for a given $k_0$ and $k_1 > k_0$ sufficiently large, it cannot happen that $b_{k+1} \leqslant b_k + c'\, b_k^{\sigma/\mu}$ for every $k = k_0, \dots, k_1$, otherwise (cf. [1, Lemma 5.3] or [18, Proposition 10.46]) $b_k \leqslant C(k_0)k^{\mu/(\mu-\sigma)}$ for every such $k$, contradicting the facts that $2p/N(p-2) > \mu/(\mu-\sigma)$ and $b_k \geqslant ck^{2p/N(p-2)}$ for every large $k$. This shows that, given any large $k_0 \in \mathbb{N}$, the process described above must stop after a finite number of iterations (depending only on $k_0$), that is, we must have that $b_k + 1 < \overline{c}_k$ for some $k$. As we saw above, in this case we may conclude that $\overline{c}_k$ is a critical value of $J$ and the proof of Theorem 3.1 is complete. □

**Corollary 3.2.** *Under the assumptions of Theorem* 3.1*, suppose moreover that for a.e. $x$ the following partial derivatives exist, are continuous and:*

$$g(x,s)s + f(x,s)s < \frac{\partial g}{\partial s}(x,s)s^2 + \frac{\partial f}{\partial s}(x,s)s^2 \leqslant C(|s|^p + 1), \qquad \forall s.$$

*Then the sign-changing solutions $u_{k_n}$ can be chosen with the further property that $u_{k_n}$ has at most $k_n + 1$ nodal domains.*

Proof. If follows by our assumptions that $J$ is a $C^2$ functional. Then, similarly to Proposition 2.5, the critical point $u_{k_n}$ of $J$ at level $\overline{c}_{k_n}$ can be chosen in such a way that its Morse index is less or equal than $k_n + 1$. The conclusion follows. □

## References


[1] A. Bahri, H. Berestycki, A perturbation method in critical point theory and applications, *Trans. Amer. Math. Soc.* 267 (1981), 1–32.

[2] A. Bahri, P.L. Lions, Morse-index of some min-max critical points. I. Application to multiplicity results, *Comm. Pure Appl. Math.* 41 (1988), 1027–1037.

[3] T. Bartsch, Critical point theory on partially ordered Hilbert spaces, *J. Funct. Anal.* 186 (2001), 117–152.

[4] T. Bartsch, K.C. Chang, Z.Q. Wang, On the Morse indices of sign-changing solutions for nonlinear elliptic problems, *Math. Z.* 233 (2000), 655–677.

[5] T. Bartsch, Z. Liu, T. Weth, Sign changing solutions of superlinear Schrödinger equations, *Comm. Partial Differential Equations* 29 (2004), 25–42.

[6] T. Bartsch, T. Weth, A note on additional properties of sign changing solutions to superlinear elliptic equations, *Topol. Methods Nonlinear Anal.* 22 (2003), 1–14.

[7] Ph. Bolle, N. Ghoussoub, H. Tehrani, The multiplicity of solutions to non-homogeneous boundary value problems, *Manuscripta Math.* 101 (2000), 325–350.

[8] A. Castro, M. Clapp, Upper estimates for the energy of solutions of nonhomogeneous boundary value problems, *Proc. Amer. Math. Soc.* 134 (2005), 167–175.

[9] K.C. Chang, "Infinite dimensional Morse theory and multiple solution problems", Progress in Nonlinear Differential Equations and their Applications, vol. 6, Birkhäuser, Boston, Mass., 1993.

[10] M. Conti, L. Merizzi, S. Terracini, Remarks on variatianal methods and lower-upper solutions, *NoDEA* 6 (1999), 371–393.

[11] M. Conti, L. Merizzi, S. Terracini, On the existence of many solutions for a class of superlinear elliptic systems, *J. Differential Equations* 167 (2000), 357–387.

[12] N. Ghoussoub, "Duality and perturbation methods in critical point theory", Cambridge Tracts in Mathematics, vol. 17, Cambridge University Press, Cambridge, 1993.

[13] B. Gidas, J. Spruck, A prori bounds for positive solutions of a nonlinear elliptic equation, *Comm. Partial Differential Equations* 6 (1981), 883–901.

[14] A.C. Lazer, S. Solimini, Nontrivial solutions of operator equations and Morse indices of critical points of min-max type, *Nonlinear Anal. TMA* 12 (1988), 761–775.

[15] S. Li, Z.Q. Wang, Ljusternik-Schnirelmann theory in partially ordered Hilbert spaces, *Trans. Amer. Math. Soc.* 354 (2002), 3207–3227.

[16] P. Majer, S. Terracini, Periodic solutions to some problems of $n$-body type, *Arch. Rational Mech. Anal.* 124 (1993), 381–404.

[17] P. Rabinowitz, Multiple critical points of perturbed symmetric functionals, *Trans. Amer. Math. Soc.* 272 (1982), 753–770.

[18] P. Rabinowitz, "Minimax methods in critical point theory with applications to differential equations", CBMS Reg. Conf. 65, Amer. Math. Soc., Providence, R.I., 1986.

[19] M. Ramos, "Linking theorems in critical point theory" (in Portoghese), University of Lisbon, Department of Mathematics, 1993.

[20] M. Ramos, L. Sanchez, Homotopical linking and Morse index estimates in min-max theorems, *Manuscripta Math.* 87 (1995), 269–284.

[21] M. Schechter, W. Zou, Infinitely many solutions to perturbed elliptic equations, *J. Funct. Anal.* 228 (2005), 1–38.

[22] S. Solimini, Morse index estimates in min-max theorems, *Manuscripta Math.* 63 (1989), 421–453.

[23] K. Tanaka, Morse indices at critical points related to the symmetric mountain pass theorem and applications, *Comm. Partial Differential Equations* 14 (1989), 99–128.

University of Lisbon, CMAF - Faculty of Science, Av. Prof. Gama Pinto 2, 1649-003 Lisboa, Portugal

*E-mail address:* mramos@ptmat.fc.ul.pt

Department of Mathematical Sciences, Tsinghua University, Beijing 100084, China

*E-mail address:* wzou@math.tsinghua.edu.cn