Nodal non-radially symmetric solutions to Emden-Fowler equations

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Abstract

We prove the existence of an unbounded sequence of sign-changing and non-radially symmetric solutions to the following Emden-Fowler equation:

\[
\begin{aligned}
-\Delta u &= |u|^{p-1}u, \quad x \in \Omega, \\
u &= 0, \quad x \in \partial \Omega, \\
u(gx) &= u(x), \quad x \in \Omega, \quad g \in G,
\end{aligned}
\]

where \( \Omega \) is a unit ball or an annulus of \( \mathbb{R}^N \) \( (N \geq 3) \), \( 1 < p < (N + 2)/(N - 2) \) and \( G \) is a closed subgroup of the orthogonal group \( O(N) \) of degree \( N \) but \( G \) is not transitive.


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1 Introduction

Consider the sign-changing and non-radially symmetric solutions to the following Emden-Fowler equation:

\[-\Delta u = |u|^{p-1}u, \quad x \in \Omega, \quad (1.1)\]
\[u = 0, \quad x \in \partial \Omega, \quad (1.2)\]
\[u(gx) = u(x), \quad x \in \Omega, \quad g \in G, \quad (1.3)\]

where \(\Omega\) is a unit ball \(\Omega := \{x \in \mathbb{R}^N : |x| < 1\}\) or an annulus \(\Omega := \{x \in \mathbb{R}^N : a < |x| < b, N \geq 3\}\) \(1 < p < (N+2)/(N-2)\) and \(G\) is a closed subgroup of the orthogonal group \(O(N)\) of degree \(N\). Here \(gx\) is the product of the column vector \(x\) and the matrix \(g\). Since each \(g \in G\) is an orthogonal matrix, the point \(gx\) is in \(\Omega\) when \(x \in \Omega\). Following [11], we call a solution of (1.1)-(1.2)-(1.3) a \(G\)-invariant solution. It is known that (1.1)-(1.2) has infinitely many sign-changing radially symmetric solutions when \(1 < p < (N+2)/(N-2)\) (cf. [7, 8, 10, 16]) and each one of them has finitely many zero points. The existence of sign-changing solutions of (1.1)-(1.2) with conclusions on nodal domains is considered in [1] but without the information on the non-radially symmetry. A radially symmetric solution becomes necessarily a \(G\)-invariant solution for any subgroup \(G\) of \(O(N)\). The converse problem was considered in [6] where the author proved that there exist solutions which are \(G\)-invariant and not radially symmetric if \(G\) is not transitive on \(S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}\). In the sequel, we say that \(G\) is transitive if for any two points \(x, y \in S^{N-1}\) there exists a \(g \in G\) such that \(y = gx\). Note that \(G\) is a transformation group on the unit sphere \(S^{N-1}\) since \(G\) is an isometric linear transformation group in \(\mathbb{R}^N\). In this note, we go further and show that (1.1)-(1.3) has infinitely many sign-changing and non-radially symmetric \(G\)-invariant solutions if \(G\) is not transitive. Precisely, we have the following main results.

**Theorem 1.1.** Assume that \(G\) is not transitive on \(S^{N-1}\), then there exists a sequence \(\{u_k\}\) of solutions of (1.1)-(1.3) such that each \(u_k\) is \(G\)-invariant, sign-changing and non-radially symmetric. Moreover, \(\|u_k\| \to \infty\) as \(k \to \infty\), where \(\|\cdot\|\) denotes the norm of \(H^1_0(\Omega)\).

By Cartan’s theorem (see e.g. [5, Theorem 2.3]), a closed subgroup \(G\) of \(O(N)\) is also a Lie subgroup. We let \(\dim G\) denote the dimension of \(G\) as a Lie group or a manifold.

**Corollary 1.1.** If \(\dim G \leq N - 2\), then the conclusion of Theorem 1.1 holds.

**Corollary 1.2.** If \(G\) is a finite subgroup of \(O(N)\), then the conclusion of Theorem 1.1 is true.

A typical example is \(G = \{Id, -Id\}\), where \(Id\) is the unit matrix. Then by Corollary 1.2, (1.1)-(1.3) has infinitely many sign-changing non-radially symmetric and even solutions. Another example is

\[G = \left\{ \begin{pmatrix} e & 0 \\ 0 & w \end{pmatrix} : e \in O(m), \ w \in O(N-m) \right\}, \quad 1 \leq m < N.\]
Then by Theorem 1.1, (1.1)-(1.3) has a sequence of solutions \( \{u_k\} \) such that each \( u_k \) is sign-changing and \( u_k(x) = u_k(|x'|, |x''|) \) for all \( x' \in \mathbb{R}^m, x'' \in \mathbb{R}^{N-m} \) with \( x = (x', x'') \in \Omega \), but \( u_k(x) \neq u_k(|x|) \). Examples of transitive groups can be found in [4, 6, 9, 12].

2 Proof of Theorem 1.1

Let

\[ H_0^1(\Omega, G) = \{ u \in H_0^1(\Omega) : u(gx) = u(x), x \in \Omega, g \in G \} \]

with the inner product \( \langle u, v \rangle = \int_{\Omega} \nabla u \cdot \nabla v \, dx \) and the corresponding norm \( ||u|| = \langle u, u \rangle^{1/2} \). We also denote by \( ||u||_{p+1} \) the \( L^{p+1}(\Omega) \) norm of \( u \). A solution of (1.1)-(1.3) is considered as a critical point of the functional \( I \) defined by

\[ I(u) = \frac{1}{2} ||u||^2 - \frac{1}{p+1} ||u||_{p+1}^{p+1}, \quad u \in H_0^1(\Omega, G). \]

We denote by \( \lambda_k(\Omega, G) \) the \( k \)-th eigenvalue in increasing order, which is repeated a number of times equal to its multiplicity, of the problem

\[
\begin{aligned}
-\Delta u &= \lambda u, & x \in \Omega, \\
\quad u &= 0, & x \in \partial \Omega, \\
\quad u(gx) &= u(x), & x \in \Omega, \quad g \in G.
\end{aligned}
\]

(2.1)

We denote by \( \phi_k(x) \) the eigenvalue corresponding to \( \lambda_k(\Omega, G) \) with \( ||\phi_k||_2 = 1 \).

Lemma 2.1. (cf. [6, 7, 8, 10]) The set of all radially symmetric critical points of \( I \) consists of a sequence \( \{\pm u_k\}_{k \in \mathbb{N}} \) and the zero solution. Moreover,

\[ 0 < \beta_1 < \beta_2 < \cdots < \beta_k < \cdots \to \infty, \quad \text{where} \quad \beta_k = I(\pm u_k), I'(\pm u_k) = 0 \]

and there exists an \( A_0 > 0 \) independent of \( k \) such that

\[ A_0 k^{\frac{2(p+1)}{p+1}} \leq \beta_k, \quad k \in \mathbb{N}. \]

Denote

\[ G(x) = \{ gx : g \in G \}, \quad x \in S^{N-1}, \]

then \( G(x) \) is a closed submanifold of \( S^{N-1} \), we denote by \( \text{dim} \ G(x) \) the dimension of the manifold \( G(x) \). Note that \( 0 \leq \text{dim} \ G(x) \leq N-1 \) because of \( G(x) \subset S^{N-1} \). We may define

\[ m := m(G) := \max\{ \text{dim} \ G(x) : x \in S^{N-1} \}. \]

Lemma 2.2. (cf. [6]) Assume that \( G \) is not transitive on \( S^{N-1} \), then \( 0 \leq m \leq N-2 \) and there exists a positive constant \( C_1 \) independent of \( k \) such that

\[ \lambda_k(\Omega, G) \leq C_1 k^{\frac{2}{N-2}}, \quad \forall k \in \mathbb{N}. \]
By Lemma 2.2 and a simple calculation, we readily have the following lemma (see e.g. [6]).

**Lemma 2.3.** There exists a $B_0 > 0$ independent of $k$ such that

$$\sup_{E_k} I \leq B_0 k^{\frac{2(p+1)}{p+1}}, \quad k \in \mathbb{N},$$

where $E_k := \{\phi_1, \ldots, \phi_k\}$.

Note that $I(u) \geq 0$ if $I'(u) = 0$ and $I(u) > 0$ if $u$ is a nontrivial critical point of $I$. Let

$$N_1 := \max \left\{ c \in \mathbb{R} : c > 0 \text{ is a critical value of } I \text{ corresponding to } G\text{-invariant sign-changing and non-radially symmetric critical points} \right\}. \quad (2.2)$$

We now prove that the above set is nonempty and $N_1 = \infty$.

**Assume:** $N_1 < \infty$ (if the set on the right-hand side of (2.2) is empty, we let $N_1 = 0$).

Let $S_k := \{u \in E_{k-1}^\perp : \beta(u) := \frac{\|u\|_p + 1}{\|u\|_{p+1}} = 1\}$. Then there is a constant $C_2 > 0$ independent of $k$ such that $1 \leq \|u\|_{p+1} \leq C_2$ for all $u \in S_k$ and all $k \in \mathbb{N}$. By combining this with the Gagliardo-Nirenberg inequality we obtain the following lemma (cf. [14]).

**Lemma 2.4.** $\inf_{S_k} I \geq C_3 \lambda_k^{(1-\alpha)} - C_4$, where $C_3, C_4 > 0$ are constants independent of $k$, $\alpha = \frac{N(p-1)}{2(p+1)} \in (0, 1)$, and $(\lambda_k)_k$ stands for the sequence of eigenvalues of $(-\Delta, H^1_0(\Omega))$.

Next we fix $k_0 > 0$ such that

$$\inf_{S_k} I > N_1 \quad \text{for all } k \geq k_0. \quad (2.3)$$

Let

$$N_2 := \max\{k \in \mathbb{N} : A_0(k - k_0 + 1)^{\frac{2(p+1)}{p+1}} \leq B_0 k^{\frac{2(p+1)}{p+1}}\}.$$

Then $N_2 < \infty$. We choose $k^*$ large enough such that $k^* > k_0 + N_2$. Define

$$C^* = \sup_{E_{k^*}} I, \quad (2.4)$$

then $C^* < \infty$. Next, we just consider those $k$ with $k \geq k^*$. Since $\dim E_k < \infty$, we may find $d_k, d_k' > 0$ such that $d_k \|u\|_{p+1} \leq \|u\| \leq d_k' \|u\|_{p+1}$ for all $u \in E_k$. Furthermore, we choose $R_k > 0$ such that

$$\frac{R_k d_k}{(d_k + 1)d_k'} > 1 \quad \text{and} \quad I(u) < 0 \quad \text{for all } u \in E_k \text{ with } \|u\| \geq R_k.$$
Lemma 2.6. If $K := \{u \in H^1_0(\Omega, G) : u(x) \geq 0, x \in \Omega\}$. It is easy to check (cf. [14, Lemma 2.4]) that
\[
\text{dist}\left(\bigcup_{k \geq k_0} S_k \cap I^{C^*}, \pm P\right) := \delta > 0.
\] (2.5)

Let $D := \{u \in H^1_0(\Omega, G) : \text{dist}(u, P) < \varepsilon_0\}$ and set $\mathcal{U} := E \setminus (-D \cup D)$, $D^* := (-D \cup D)$. Then for $\varepsilon_0$ small enough, we have that $\bigcup_{k \geq k_0} S_k \cap I^{C^*} \subset \mathcal{U}$. Moreover, $D^* \cap \mathcal{K} \subset (-P \cup P)$, where $\mathcal{K} := \{u \in H^1_0(\Omega, G) : I'(u) = 0\}$, this can be done as that in [3]. Without loss of generality, we assume that $R_k < R_{k+1} < R_{k+2} < \cdots$. For $k \in [k_0, k^*]$, we set
\[
T_k := \{h : h \in C(\Theta_k, E), h \text{ is odd}, h(u) = u \text{ on } \partial\Theta_k\},
\]
\[
\Theta_k := \{u \in E_k : ||u|| < R_k\}, \quad \partial\Theta_k := \{u \in E_k : ||u|| = R_k\}.
\]
Define
\[
Z_k := \left\{ h(\Theta_i \setminus A) : h \in T_i, i \in [k, k^*], A \in \mathcal{E}, \gamma(A) \leq i - k, \; I(h(\Theta_i \setminus A)) \leq C^* \right\}, \quad \text{ (2.6)}
\]
where $\mathcal{E}$ is the family of closed subsets $A$ of $H^1_0(\Omega, G)$ such that $0 \notin A$ and $-u \in A$ whenever $u \in A$; $\gamma(A)$ denotes the genus of $A$. Clearly, $Z_k \neq \emptyset$ since $Id \in T_k$; also, $Z_{k+1} \subset Z_k$.

Lemma 2.5. $B \cap (S_k \cap I^{C^*}) \neq \emptyset$ for any $B \in Z_k$.

Proof. We modify the proof due to [13]. It suffices to show that $B \cap S_k \neq \emptyset$. We write $B = h(\Theta_i \setminus A)$ with $h \in T_i, k^* \geq i \geq k$ and $\gamma(A) \leq i - k$. Let $W_1 := \{u \in \Theta_i : \beta(h(u)) < 1\}$. Let $W_2$ denote the component of $W_1$ containing 0. Then $W_2$ is a symmetric bounded neighborhood of 0 in $\Theta_i$ and hence $\gamma(\partial W_2) = i$. We claim that $\beta(h(\partial W_2)) = 1$. Otherwise, there is an $u \in \partial W_2$ such that $\beta(h(u)) < 1$. If $u \in \Theta_i$, then there is a neighborhood $N_u \subset \Theta_i$ of $u$ such that $\beta(h(N_u)) < 1$, it follows that $u \notin \partial W_2$. So we must have that $u \in \partial\Theta_i$ and then $||u|| = R_i$ and $1 > \beta(u) \geq \frac{R_i}{R_i + 1}$. This is a contradiction. Our claim is true. Let $W_3 := \{u \in \Theta_i : \beta(h(u)) = 1\}$. Then $\partial W_2 \subset W_3$ and $\gamma(W_3) = i$ and $\gamma(W_3 \setminus A) \geq k > k - 1$. Since $h$ is odd, we see that $\gamma(h(W_3 \setminus A)) \geq k > k - 1$. Hence $h(W_3 \setminus A) \cap E_{k-1} \neq \emptyset$. This implies that $B \cap S_k \neq \emptyset$. □

We now define $c_k = \inf_{B \in Z_k} \max_{u \in B \cap \mathcal{U}} I$, $k_0 \leq k \leq k^*$. Then by Lemma 2.5, $c_k$ is well defined. Moreover, $c_k \geq \inf S_k I > N_1$ by (2.3).

Lemma 2.6. If $c_k = c_{k+1} = \cdots = c_{k+p} = c$, then $\gamma(\mathcal{K}_c \cap \mathcal{U}) \geq p + 1$, where $\mathcal{K}_c := \{u \in H^1_0(\Omega, G) : I'(u) = c, I'(u) = 0\}$.

Proof. Note that $0 \notin \mathcal{K}_c \cap \mathcal{U} := \mathcal{K}_c^e$ and $\mathcal{K}_c \cap \mathcal{U}$ is compact. Assume that $\gamma(\mathcal{K}_c \cap \mathcal{U}) \leq p$, then there is an open neighborhood $U$ of $\mathcal{K}_c^e$ such that $\gamma(U) \leq p$. Let $V$ be an open neighborhood of $\mathcal{K}_c \cap (-P \cup P) := \mathcal{K}_c^{P^n}$. In particular, we
may assume that $V \subset D^{*}$. Then by the deformation lemma (cf. [13]), for 
$\varepsilon > 0$ small enough, we find a flow $\eta \in C([0,1] \times E, E)$ such that $\eta(1, u)$ is
odd in $u$, $\eta(1, I^{-\varepsilon} \setminus (U \cup V)) \subset I^{c-\varepsilon}$ and $\eta(1, \cdot) = Id$ on $\partial H_i$ for $i \in \{k, k^{*}\}$
(since $I < 0$ on $\partial H_i$ and $c > N_1 \geq 0$). Under our assumptions, the flow $\eta$
keeps $\pm D$ invariant, that is $\eta(1, \pm D) \subset \pm D$ (see for example [2, 3, 15]). Hence,
$\eta(1, I^{-\varepsilon} \setminus U) \subset I^{c-\varepsilon} \cup D^{*}$. Choose $B \in Z_{k+p}$ such that $\max_{\beta} I \leq c + \varepsilon$. We
write $B = h(\Theta_i, A)$ with $h \in T_i, i \in \{k + p, k^{*}\}, \gamma(A) \leq i - (k + p), I(B) \leq C^{*}$.
Note that $\gamma(U) \leq p$ and $B \setminus U = h(\Theta_i, A \cup h^{-1}(U))$ and $\gamma(A \cup h^{-1}(U)) \leq i - k$, then we have $B \setminus U \in Z_k$. Since $\eta$ is a descending flow, then we see that
$\eta(1, B \setminus U) \in Z_k$. But $\eta(1, B \setminus U) \cap U \subset \eta(1, (B \cap U) \setminus U) \cap U \subset \eta(1, I^{-\varepsilon} \setminus U) \cap U \subset (I^{c-\varepsilon} \cup D^{*}) \cap U \subset I^{c-\varepsilon}$, we get a contradiction. □

**Proof of Theorem 1.1.** Since $c_k > N_1$ for all $k \in \{k_0, k^{*}\}$, by Lemma 2.1, we
see that $\{c_{k_0}, c_{k_0 + 1}, \ldots, c_{k^{*}}\} \subset \{\beta_1, \beta_2, \ldots\}$. Note that $c_{k_0} \leq c_{k_0 + 1} \leq \cdots \leq c_{k^{*}}$.
Assume $c_k = c_{k+1}$ for some $k \in \{k_0, k^{*} - 1\}$. Then by Lemma 2.6, $\gamma(K_{\Theta_k} \cap U) \geq 2$.
But $c_k = \beta_i$ for some $i$, then $\{K_{\Theta_k} \cap U\} = \{u_i, -u_i\}$. This is a contradiction.
This implies that $\{c_k\}_{k=k_0}^{k^{*}}$ is strictly increasing. Therefore, $c_{k^{*}} = \beta_j$ with
$j \geq k^{*} - k_0 + 1$. Hence,

$$A_0(k^{*} - k_0 + 1)\frac{2^{p+1}}{p+1} \leq \beta_j = c_{k^{*}} \leq B_0(k^{*})\frac{2^{p+1}}{p+1}.$$  

This is impossible and is due to the assumption that $N_1 < \infty$. □

**Proofs of Corollaries 1.1 and 1.2.** Under the assumption of Corollary 1.1,
$G$ is not transitive since $\dim G(x) \leq N - 2$ and then $G(x) \neq S^{N-1}$. As for
Corollary 1.2, we observe that no finite group can be transitive. □

**References**


25–42.


