Nodal non-radially symmetric solutions to Emden-Fowler equations

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Abstract

We prove the existence of an unbounded sequence of sign-changing and non-radially symmetric solutions to the following Emden-Fowler equation:

$$\begin{cases} -\Delta u = |u|^{p-1}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ u(gx) = u(x), & x \in \Omega, \quad g \in G, \end{cases}$$

where Ω is a unit ball or an annulus of $\mathbb{R}^N (N \geq 3)$, $1 < p < (N+2)/(N-2)$ and G is a closed subgroup of the orthogonal group $O(N)$ of degree N but G is not transitive.

AMS Subject Classification (2000): 35J20, 35J25, 35B99.

*Research partially supported by FCT, program POCI-ISFL-1-209 (Portugal/Feder-EU).
†Supported by NSFC (10571096), SRF-ROCS-SEM and the program of the Ministry of Education in China for NCET in Universities of China.

1 Introduction

Consider the sign-changing and non-radially symmetric solutions to the following Emden-Fowler equation:

$$-\Delta u = |u|^{p-1}u, \qquad x \in \Omega, \tag{1.1}$$

$$u = 0, \qquad x \in \partial\Omega, \tag{1.2}$$

$$u(gx) = u(x), \qquad x \in \Omega, \quad g \in G, \tag{1.3}$$

where Ω is a unit ball $\Omega := \{x \in \mathbb{R}^N : |x| < 1\}$ or an annulus $\Omega := \{x \in \mathbb{R}^N : a < |x| < b\}$, $0 < a < b, N \geq 3, 1 < p < (N+2)/(N-2)$ and G is a closed subgroup of the orthogonal group $O(N)$ of degree N. Here gx is the product of the column vector x and the matrix g. Since each $g \in G$ is an orthogonal matrix, the point gx is in Ω when $x \in \Omega$. Following [11], we call a solution of (1.1)-(1.3) a G-invariant solution. It is known that (1.1)-(1.2) has infinitely many sign-changing radially symmetric solutions when $1 < p < (N+2)/(N-2)$ (cf. [7, 8, 10, 16]) and each one of them has finitely many zero points. The existence of sign-changing solutions of (1.1)-(1.2) with conclusions on nodal domains is considered in [1] but without the information on the non-radially symmetry. A radially symmetric solution becomes necessarily a G-invariant solution for any subgroup G of $O(N)$. The converse problem was considered in [6] where the author proved that there exist solutions which are G-invariant and not radially symmetric if G is not transitive on $S^{N-1} := \{x \in \mathbb{R}^N : |x| = 1\}$. In the sequel, we say that G is transitive if for any two points $x, y \in S^{N-1}$ there exists a $g \in G$ such that $y = gx$. Note that G is a transformation group on the unit sphere S^{N-1} since G is an isometric linear transformation group in $\mathbb{R}^N$. In this note, we go further and show that (1.1)-(1.3) has infinitely many sign-changing and non-radially symmetric G-invariant solutions if G is not transitive. Precisely, we have the following main results.

Theorem 1.1. *Assume that G is not transitive on S^{N-1}, then there exists a sequence $\{w_k\}$ of solutions of (1.1)-(1.3) such that each w_k is G-invariant, sign-changing and non-radially symmetric. Moreover, $\|w_k\| \to \infty$ as $k \to \infty$, where $\|\cdot\|$ denotes the norm of $H_0^1(\Omega)$.*

By Cartan's theorem (see e.g. [5, Theorem 2.3]), a closed subgroup G of $O(N)$ is also a Lie subgroup. We let $\dim G$ denote the dimension of G as a Lie group or a manifold.

Corollary 1.1. *If $\dim G \leq N-2$, then the conclusion of Theorem 1.1 holds.*

Corollary 1.2. *If G is a finite subgroup of $O(N)$, then the conclusion of Theorem 1.1 is true.*

A typical example is $G = \{Id, -Id\}$, where Id is the unit matrix. Then by Corollary 1.2, (1.1)-(1.3) has infinitely many sign-changing non-radially symmetric and even solutions. Another example is

$$G = \left\{ \begin{pmatrix} e & 0 \\ 0 & w \end{pmatrix} : e \in O(m), \ \ w \in O(N-m) \right\}, \quad 1 \leq m < N.$$

Then by Theorem 1.1, (1.1)-(1.3) has a sequence of solutions $\{u_k\}$ such that each u_k is sign-changing and $u_k(x) = u_k(|x'|, |x''|)$ for all $x' \in \mathbb{R}^m, x'' \in \mathbb{R}^{N-m}$ with $x = (x', x'') \in \Omega$, but $u_k(x) \neq u_k(|x|)$. Examples of transitive groups can be found in [4, 6, 9, 12].

2 Proof of Theorem 1.1

Let

$$H_0^1(\Omega, G) = \{u \in H_0^1(\Omega) : u(gx) = u(x), x \in \Omega, g \in G\}$$

with the inner product $\langle u, v\rangle = \int_\Omega \nabla u \cdot \nabla v dx$ and the corresponding norm $\|u\| = \langle u, u\rangle^{1/2}$. We also denote by $\|u\|_{p+1}$ the $L^{p+1}(\Omega)$ norm of u. A solution of (1.1)-(1.3) is considered as a critical point of the functional I defined by

$$I(u) = \frac{1}{2}\|u\|^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}, \quad u \in H_0^1(\Omega, G).$$

We denote by $\lambda_k(\Omega, G)$ the k-th eigenvalue in increasing order, which is repeated a number of times equal to its multiplicity, of the problem

$$\begin{cases} -\Delta u = \lambda u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \\ u(gx) = u(x), & x \in \Omega, \quad g \in G. \end{cases} \tag{2.1}$$

We denote by $\phi_k(x)$ the eigenvalue corresponding to $\lambda_k(\Omega, G)$ with $\|\phi_k\|_2 = 1$.

Lemma 2.1. *(cf. [6, 7, 8, 10]) The set of all radially symmetric critical points of I consists of a sequence $\{\pm u_k\}_{k\in\mathbb{N}}$ and the zero solution. Moreover,*

$$0 < \beta_1 < \beta_2 < \cdots < \beta_k < \cdots \to \infty, \quad \text{where } \beta_k = I(\pm u_k), I'(\pm u_k) = 0$$

and there exists an $A_0 > 0$ independent of k such that

$$A_0 k^{\frac{2(p+1)}{p-1}} \leq \beta_k, \quad k \in \mathbb{N}.$$

Denote

$$G(x) = \{gx : g \in G\}, \quad x \in S^{N-1},$$

then $G(x)$ is a closed submanifold of S^{N-1}, we denote by $\dim G(x)$ the dimension of the manifold $G(x)$. Note that $0 \leq \dim G(x) \leq N-1$ because of $G(x) \subset S^{N-1}$. We may define

$$m := m(G) := \max\{\dim G(x) : x \in S^{N-1}\}.$$

Lemma 2.2. *(cf. [6]) Assume that G is not transitive on S^{N-1}, then $0 \leq m \leq N-2$ and there exists a positive constant C_1 independent of k such that*

$$\lambda_k(\Omega, G) \leq C_1 k^{\frac{2}{N-m}}, \quad \forall k \in \mathbb{N}.$$

By Lemma 2.2 and a simple calculation, we readily have the following lemma (see e.g. [6]).

Lemma 2.3. *There exists a $B_0 > 0$ independent of k such that*

$$\sup_{E_k} I \leq B_0 k^{\frac{2(p+1)}{(N-m)(p-1)}}, \quad k \in \mathbb{N},$$

where $E_k := \{\phi_1, \cdots, \phi_k\}$.

Note that $I(u) \geq 0$ if $I'(u) = 0$ and $I(u) > 0$ if u is a nontrivial critical point of I. Let

$$N_1 := \max \left\{ c \in \mathbb{R} : \begin{array}{l} c > 0 \text{ is a critical value of } I \text{ corresponding} \\ \text{to } G\text{-invariant sign-changing and} \\ \text{non-radially symmetric critical points} \end{array} \right\}. \quad (2.2)$$

We now prove that the above set is nonempty and $N_1 = \infty$.

Assume: $N_1 < \infty$ (if the set on the right-hand side of (2.2) is empty, we let $N_1 = 0$).

Let $S_k := \{u \in E_{k-1}^{\perp} : \beta(u) := \frac{\|u\| \|u\|_{p+1}}{\|u\| + \|u\|_{p+1}} = 1\}$. Then there is a constant $C_2 > 0$ independent of k such that $1 \leq \|u\|_{p+1} \leq C_2$ for all $u \in S_k$ and all $k \in \mathbb{N}$. By combining this with the Gagliardo-Nirenberg inequality we obtain the following lemma (cf. [14]).

Lemma 2.4. $\inf_{S_k} I \geq C_3 \lambda_k^{(1-\alpha)} - C_4$, *where $C_3, C_4 > 0$ are constants independent of k, $\alpha = \dfrac{N(p-1)}{2(p+1)} \in (0,1)$, and $(\lambda_k)_k$ stands for the sequence of eigenvalues of $(-\Delta, H_0^1(\Omega))$.*

Next we fix $k_0 > 0$ such that

$$\inf_{S_k} I > N_1 \quad \text{for all } k \geq k_0. \quad (2.3)$$

Let

$$N_2 := \max\{k \in \mathbb{N} : A_0(k - k_0 + 1)^{\frac{2(p+1)}{p-1}} \leq B_0 k^{\frac{2(p+1)}{(N-m)(p-1)}}\}.$$

Then $N_2 < \infty$. We choose k^* large enough such that $k^* > k_0 + N_2$. Define

$$C^* = \sup_{E_{k^*}} I, \quad (2.4)$$

then $C^* < \infty$. Next, we just consider those k with $k \geq k^*$. Since $\dim E_k < \infty$, we may find $d_k, d_k' > 0$ such that $d_k \|u\|_{p+1} \leq \|u\| \leq d_k' \|u\|_{p+1}$ for all $u \in E_k$. Furthermore, we choose $R_k > 0$ such that

$$\frac{R_k d_k}{(d_k + 1) d_k'} > 1 \quad \text{and} \quad I(u) < 0 \text{ for all } u \in E_k \text{ with } \|u\| \geq R_k.$$

Let P denote the positive cone of $H_0^1(\Omega, G)$, that is $P := \{u \in H_0^1(\Omega, G) : u(x) \geq 0, x \in \Omega\}$. It is easy to check (cf. [14, Lemma 2.4]) that

$$\mathrm{dist}\Big(\big(\cup_{k\geq k_0}^{k^*} S_k\big) \cap I^{C^*}, \pm P\Big) := \delta > 0. \tag{2.5}$$

Let $D := \{u \in H_0^1(\Omega, G) : \mathrm{dist}(u, P) < \varepsilon_0\}$ and set $\mathcal{U} := E\backslash(-D \cup D), D^* := (-D \cup D)$. Then for ε_0 small enough, we have that $\big(\cup_{k\geq k_0}^{k^*} S_k\big) \cap I^{C^*} \subset \mathcal{U}$. Moreover, $D^* \cap \mathcal{K} \subset (-P \cup P)$, where $\mathcal{K} := \{u \in H_0^1(\Omega, G) : I'(u) = 0\}$, this can be done as that in [3]. Without loss of generality, we assume that $R_k < R_{k+1} < R_{k+2} < \cdots$. For $k \in [k_0, k^*]$, we set

$$T_k := \{h : h \in C(\Theta_k, E), h \text{ is odd}, h(u) = u \text{ on } \partial\Theta_k\},$$

$$\Theta_k := \{u \in E_k : \|u\| < R_k\}, \quad \partial\Theta_k := \{u \in E_k : \|u\| = R_k\}.$$

Define

$$Z_k := \left\{ h(\overline{\Theta_i\backslash A}) : \begin{array}{c} h \in T_i,\ i \in [k, k^*],\ A \in \mathcal{E}, \\ \\ \gamma(A) \leq i - k, \quad I(h(\overline{\Theta_i\backslash A})) \leq C^* \end{array} \right\}, \tag{2.6}$$

where $\mathcal{E}$ is the family of closed subsets A of $H_0^1(\Omega, G)$ such that $0 \notin A$ and $-u \in A$ whenever $u \in A$; $\gamma(A)$ denotes the genus of A. Clearly, $Z_k \neq \emptyset$ since $Id \in T_k$; also, $Z_{k+1} \subset Z_k$.

Lemma 2.5. *$B \cap (S_k \cap I^{C^*}) \neq \emptyset$ for any $B \in Z_k$.*

Proof. We modify the proof due to [13]. It suffices to show that $B \cap S_k \neq \emptyset$. We write $B = h(\overline{\Theta_i\backslash A})$ with $h \in T_i, k^* \geq i \geq k$ and $\gamma(A) \leq i - k$. Let $W_1 := \{u \in \Theta_i : \beta(h(u)) < 1\}$. Let W_2 denote the component of W_1 containing 0. Then W_2 is a symmetric bounded neighborhood of 0 in Θ_i and hence $\gamma(\partial W_2) = i$. We claim that $\beta(h(\partial W_2)) = 1$. Otherwise, there is an $u \in \partial W_2$ such that $\beta(h(u)) < 1$. If $u \in \Theta_i$, then there is a neighborhood $N_u \subset \Theta_i$ of u such that $\beta(h(N_u)) < 1$, it follows that $u \notin \partial W_2$. So we must have that $u \in \partial\Theta_i$ and then $\|u\| = R_i$ and $1 > \beta(u) \geq \frac{R_i d_i}{d_i'(d_i+1)}$. This is a contradiction. Our claim is true. Let $W_3 := \{u \in \Theta_i : \beta(h(u)) = 1\}$. Then $\partial W_2 \subset W_3$ and $\gamma(W_3) = i$ and $\gamma(\overline{W_3\backslash A}) \geq k > k - 1$. Since h is odd, we see that $\gamma(h(\overline{W_3\backslash A})) \geq k > k - 1$. Hence $h(\overline{W_3\backslash A}) \cap E_{k-1}^{\perp} \neq \emptyset$. This implies that $B \cap S_k \neq \emptyset$. □

We now define $c_k = \inf_{B\in Z_k} \max_{u\in B\cap\mathcal{U}} I, \quad k_0 \leq k \leq k^*$. Then by Lemma 2.5, c_k is well defined. Moreover, $c_k \geq \inf_{S_k} I > N_1$ by (2.3).

Lemma 2.6. *If $c_k = c_{k+1} = \cdots = c_{k+p} = c$, then $\gamma(\mathcal{K}_c \cap \mathcal{U}) \geq p + 1$, where $\mathcal{K}_c := \{u \in H_0^1(\Omega, G) : I(u) = c, I'(u) = 0\}$.*

Proof. Note that $0 \notin \mathcal{K}_c \cap \mathcal{U} := \mathcal{K}_c^s$ and $\mathcal{K}_c \cap \mathcal{U}$ is compact. Assume that $\gamma(\mathcal{K}_c \cap \mathcal{U}) \leq p$, then there is an open neighborhood U of $\mathcal{K}_c^s$ such that $\gamma(U) \leq p$. Let V be an open neighborhood of $\mathcal{K}_c \cap (-P \cup P) := \mathcal{K}_c^{pn}$. In particular, we may assume that $V \subset D^*$. Then by the deformation lemma (cf. [13]), for $\varepsilon > 0$ small enough, we find a flow $\eta \in C([0,1] \times E, E)$ such that $\eta(1,u)$ is odd in u, $\eta(1, I^{c+\varepsilon} \backslash (U \cup V)) \subset I^{c-\varepsilon}$ and $\eta(1,\cdot) = Id$ on $\partial\Theta_i$ for $i \in [k,k^*]$ (since $I < 0$ on $\partial\Theta_i$ and $c > N_1 \geq 0$). Under our assumptions, the flow η keeps $\pm D$ invariant, that is $\eta(1,\pm D) \subset \pm D$ (see for example [2, 3, 15]). Hence, $\eta(1, I^{c+\varepsilon}\backslash U) \subset I^{c-\varepsilon} \cup D^*$. Choose $B \in Z_{k+p}$ such that $\max_{B\cap\mathcal{U}} I \leq c+\varepsilon$. We write $B = h(\overline{\Theta_i \backslash A})$ with $h \in T_i, i \in [k+p, k^*], \gamma(A) \leq i-(k+p), I(B) \leq C^*$. Note that $\gamma(U) \leq p$ and $\overline{B\backslash U} = h(\overline{\Theta_i \backslash (A \cup h^{-1}(U))})$ and $\gamma(A \cup h^{-1}(U)) \leq i-k$, then we have $\overline{B\backslash U} \in Z_k$. Since η is a descending flow, then we see that $\eta(1, \overline{B\backslash U}) \in Z_k$. But $\eta(1, \overline{B\backslash U}) \cap \mathcal{U} \subset \eta(1, \overline{(B\cap\mathcal{U})\backslash U}) \cap \mathcal{U} \subset \eta(1, I^{c+\varepsilon}\backslash U) \cap \mathcal{U} \subset (I^{c-\varepsilon} \cup D^*) \cap \mathcal{U} \subset I^{c-\varepsilon}$, we get a contradiction. □

Proof of Theorem 1.1. Since $c_k > N_1$ for all $k \in [k_0, k^*]$, by Lemma 2.1, we see that $\{c_{k_0}, c_{k_0+1}, \cdots, c_{k^*}\} \subset \{\beta_1, \beta_2, \cdots\}$. Note that $c_{k_0} \leq c_{k_0+1} \leq \cdots \leq c_{k^*}$. Assume $c_k = c_{k+1}$ for some $k \in [k_0, k^*-1]$. Then by Lemma 2.6, $\gamma(\mathcal{K}_{c_k} \cap \mathcal{U}) \geq 2$. But $c_k = \beta_i$ for some i, then $(\mathcal{K}_{c_k} \cap \mathcal{U}) = \{u_i, -u_i\}$. This is a contradiction. This implies that $\{c_k\}_{k=k_0}^{k^*}$ is strictly increasing. Therefore, $c_{k^*} = \beta_j$ with $j \geq k^* - k_0 + 1$. Hence,

$$A_0(k^* - k_0 + 1)^{\frac{2(p+1)}{p-1}} \leq \beta_j = c_{k^*} \leq B_0(k^*)^{\frac{2(p+1)}{(N-m)(p-1)}}.$$

This is impossible and is due to the assumption that $N_1 < \infty$. □

Proofs of Corollaries 1.1 and 1.2. Under the assumption of Corollary 1.1, G is not transitive since $\dim G(x) \leq N-2$ and then $G(x) \neq S^{N-1}$. As for Corollary 1.2, we observe that no finite group can be transitive. □

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