Solutions with multiple spike patterns for an elliptic system*

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Abstract. We consider a system of the form $-\varepsilon^2\Delta u + V(x)u = g(v)$, $-\varepsilon^2\Delta v + V(x)v = f(u)$ in an open domain Ω of $\mathbb{R}^N$, with Dirichlet conditions at the boundary (if any). We suppose that f and g are power-type nonlinearities, having superlinear and subcritical growth at infinity. We prove the existence of positive solutions u_ε and v_ε which concentrate, as $\varepsilon \to 0$, at a prescribed finite number of local minimum points of $V(x)$, possibly degenerate.

1 Introduction

Let Ω be a domain of $\mathbb{R}^N$, $N \geqslant 3$, not necessarily bounded, with smooth or empty boundary. We consider an elliptic system of the form

$$-\varepsilon^2\Delta u + V(x)u = g(v), \quad -\varepsilon^2\Delta v + V(x)v = f(u) \text{ in } \Omega, \quad u = v = 0 \text{ on } \partial\Omega, \tag{1.1}$$

where $u, v > 0$ in Ω and $\varepsilon > 0$ is a small parameter. Here $V(x)$ is locally Hölder continuous and $\inf_\Omega V > 0$, while $f, g \in C^1(\mathbb{R})$ fall into the typical class of superlinear and subcritical functions, namely we will assume that the following holds:

$(fg1)$ $f(0) = g(0) = f'(0) = g'(0) = 0$;

$(fg2)$ $\displaystyle\lim_{s\to+\infty} \frac{f(s)}{s^{p-1}} = \lim_{s\to+\infty} \frac{g(s)}{s^{q-1}} = 0$, for some $p, q > 2$ with $\displaystyle\frac{1}{p} + \frac{1}{q} > \frac{N-2}{N}$;

$(fg3)$ $0 < (1+\delta')f(s)s \leqslant f'(s)s^2$ and $0 < (1+\delta')g(s)s \leqslant g'(s)s^2$, for some $\delta' > 0$.

*Partially suported by FCT (Fundação para a Ciência e Tecnologia), program POCI-ISFL-1-209 (Portugal/Feder-EU). The second author was supported by a fellowship of FCT, grant SFRH/BM/22053/2005.

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We look for positive solutions of (1.1) and therefore we let $f(s) = g(s) = 0$ for $s \leqslant 0$.

Our motivation for the study of such a problem goes back to the work of Rabinowitz [23] and Wang [28] concerning the single equation

$$-\varepsilon^2 \Delta u + V(x)u = f(u) \qquad \text{in } \mathbb{R}^N. \tag{1.2}$$

In [23] a mountain-pass type argument is used in order to find a ground state solution for $\varepsilon > 0$ sufficiently small, when V is such that $\liminf_{|x|\to\infty} V(x) > \inf_{\mathbb{R}^N} V(x) > 0$. In [28] it is proved that this mountain-pass solutions concentrate around a global minimum point of V as ε tends to 0.

It should be stressed that in these papers no nondegeneracy assumptions were made upon the minimum points of V; this is in contrast with previous works (see e.g. [2, 15, 20]) where solutions with a spike shape which concentrate around nondegenerate critical points of V were constructed. A related problem concerns the case where $V(x) \equiv 1$ in a bounded domain Ω under Neumann or Dirichlet boundary conditions, the main issue being then the location of the concentrating points of the least energy solutions, see e.g. [12, 17, 18, 19]. In [5] the function V is allowed to vanish at some points of $\mathbb{R}^N$. Problems with another type of nonlinearities were considered by many authors, see e.g. [6, 8, 9, 10] and their references.

A further step in the study of such problems was made by Del Pino and Felmer in [11], where the degenerate case in (1.2) was considered in a local setting. Namely, by assuming that $\inf_\Lambda V < \inf_{\partial\Lambda} V$ with respect to a bounded open set $\Lambda \subset \Omega$, a family u_ε exhibiting a single spike in Λ, at a point x_ε such that $V(x_\varepsilon) \to \inf_\Lambda V$, is constructed. In [13] the author's approach was extended to the construction of a family of solutions with several spikes located around any prescribed finite set of local minima of V.

There are at least three difficulties in extending the quoted results to the system (1.1). Firstly, no uniqueness results seem to be known for the "limit problem" $-\Delta u + u = g(v)$, $-\Delta v + v = f(u)$ in $\mathbb{R}^N$ and this is in some cases a crucial assumption in the single equation case (compare e.g. with [10, Assumption (f5)], [6, p. 268], [13, Assumption (f4)], [17, p. 1448]).

On the other hand, let us introduce the associated energy functional $I_\varepsilon : H \times H \to \mathbb{R}$,

$$I_\varepsilon(u, v) := \int_\Omega \{\varepsilon^2 \langle \nabla u, \nabla v\rangle + V(x)uv\} - \int_\Omega F(u) - \int_\Omega G(v),$$

where $F(s) := \int_0^s f(\xi)\,d\xi$, $G(s) := \int_0^s g(\xi)\,d\xi$, and H is the Hilbert space $H := \{u \in H_0^1(\Omega) : \int_\Omega V(x)u^2 < +\infty\}$, with the inner product $\langle u, v\rangle_H := \int_\Omega \{\langle \nabla u, \nabla v\rangle + V(x)uv\}$ (at this point we assume that I_ε is well defined, i.e. that the constants p and q in assumption (fg2) are such that $2 < p, q < 2^* := 2N/(N-2)$; see below for a discussion on this). It is known that positive solutions of (1.1) correspond to critical points of the functional I_ε. But we see that, with respect to the single equation case, the quadratic part of the energy functional has no longer a positive sign. From a technical point of view, this causes some difficulties; for example, it is not clear whether the penalization method as used in [13, p. 138] can be applied to our problem.

From a more conceptual point of view, in the case of a system we also have to face the indefinite character of the energy functional, since ground-state critical points of I_ε are no longer expected to be generated from a direct (essentially) finite dimensional argument. This difficulty was bypassed in [1, 3] by means of a dual variational formulation of the problem while in [21, 22, 24, 25] a direct approach was proposed, based on a new variational characterization of the ground-state critical levels associated to (1.1). In these papers either the case $V(x) \equiv 1$ or the "coercive" case $\liminf_{|x|\to\infty} V(x) > \inf_{\mathbb{R}^N} V(x) > 0$ are considered.

Our goal here is to establish for the system (1.1) the analog of the main result in [13] concerning a single equation. Namely, we assume that V is locally Hölder continuous and

$(V1)$ $V(x) \geqslant \alpha > 0$, for all $x \in \Omega$;

$(V2)$ there exist bounded domains Λ_i, mutually disjoint, compactly contained in Ω, $i = 1, \ldots, k$, such that

$$\inf_{\Lambda_i} V < \inf_{\partial\Lambda_i} V$$

(i.e. V admits at least k local strict minimum points, possibly degenerate).

We prove the following.

Theorem 1.1. *Under assumptions* $(V1), (V2), (fg1) - (fg3)$*, there exists* $\varepsilon_0 > 0$ *such that, for any* $0 < \varepsilon < \varepsilon_0$*, problem* (1.1) *admits classical positive solutions* $u_\varepsilon, v_\varepsilon \in C^2(\Omega) \cap C^1(\overline{\Omega}) \cap H_0^1(\Omega)$*, and:*

(i) *there exist* $x_{i,\varepsilon} \in \Lambda_i$*,* $i = 1, \ldots, k$*, local maximum points of both* u_ε *and* v_ε*;*

(ii) $u_\varepsilon(x_{i,\varepsilon}), v_\varepsilon(x_{i,\varepsilon}) \geqslant b > 0$, *and* $V(x_{i,\varepsilon}) \to \inf_{\Lambda_i} V$ *as* $\varepsilon \to 0$*;*

(iii) $u_\varepsilon(x), v_\varepsilon(x) \leqslant \gamma e^{-\frac{\beta}{\varepsilon}|x - x_{i,\varepsilon}|}$, $\forall x \in \Omega \setminus \cup_{j \neq i} \Lambda_j$;

for some positive constants b, γ, β*. Moreover, the uniqueness of the local maxima holds in the following sense:*

(iv) if either u_ε *or* v_ε *have a local maximum point at some point* $z_\varepsilon \in \Omega$, $z_\varepsilon \neq x_{i,\varepsilon}$, $\forall i = 1, \ldots, k$*, then it holds:* $\lim_{\varepsilon \to 0} u(z_\varepsilon) = 0$ *and* $\lim_{\varepsilon \to 0} v(z_\varepsilon) = 0$.

As we mentioned before, as a byproduct of our approach, in the case where $f = g$ (single equation case) we improve [13, Th. 0.1] in the sense that no assumption is imposed on the limit problem associated to the equation in (1.2). As for (1.1), our main result also improves [24, Th. 1.1], which deals with global minimum points of V only. On the other hand, the uniqueness of the local maxima is established only for these points x_ε such that either $u(x_\varepsilon)$ or $v(x_\varepsilon)$ remain bounded away from zero; in the single equation case, local maxima values are always bounded away from zero, as follows from an inspection of the equation in (1.2) evaluated at a local maximum point x_ε, but this does not seem to hold true in our situation, due to the possible interaction of the solutions of the two equations.

The rest of the paper is devoted to the proof of Theorem 1.1. In Section 2 we set a general framework suitable for our proposals, mainly Proposition 2.4. The underlying ideas are already present in [25] but we provide here a more concise approach which in particular avoids an extra technical assumption that was needed in [25] and subsequently in [21, 22, 24] (namely, $f^2(s) \leqslant 2f'(s)F(s)$ and similarly for $g(s)$). Section 3 contains the core of our argument. Indeed, contrarily to the method used in [7, 13], we do not seek for k-spike solutions by minmaximizing the energy functional over a k -dimensional compact surface, but we minimize instead I_ε over the product of k suitable Nehari manifolds. Roughly speaking, each of these manifolds localizes I_ε near $H_0^1(\Lambda_i) \times H_0^1(\Lambda_i)$ $(i = 1, \ldots, k)$, thanks to a technical condition in its definition which ensures that the manifold is weakly closed. As it will be clear later on, the main estimate in the proof of Theorem 1.1 is contained in Eq. (4.7) and it turns out that our setting is rather effective in providing it.

Once these preliminary settings are established, the proof of Theorem 1.1 follows by simple arguments, as shown in Section 4. The final Section 5 concerns the following question. Under assumption $(fg2)$, the functional I_ε may not be well defined in the space

$H \times H$, because it can happen that, say, $p < 2^* = \frac{2N}{N-2} < q$. However, as explained in Section 5, we only have to prove Theorem 1.1 in the case where $2 < p = q < 2^*$. In fact, given $n \in \mathbb{N}$ we can define the truncated functions

$$f_n(s) = \begin{cases} f(s) & \text{for } s \leqslant n \\ A_n s^{p-1} + B_n & \text{for } s > n \end{cases} \qquad g_n(s) = \begin{cases} g(s) & \text{for } s \leqslant n \\ \tilde{A}_n s^{p-1} + \tilde{B}_n & \text{for } s > n \end{cases} \tag{1.3}$$

with $A_n = f'(n)/((p-1)n^{p-2})$, $B_n = f(n) - (f'(n)n)/(p-1)$, $\tilde{A}_n = g'(n)/((p-1)n^{p-2})$, $\tilde{B}_n = g(n) - (g'(n)n)/(p-1)$; we show in Section 5 that the solutions $(u_{\varepsilon_n}, v_{\varepsilon_n})$ of the corresponding system obtained by means of Theorem 1.1 applied to the truncated problem are such that $||u_{\varepsilon_n}||_\infty, ||v_{\varepsilon_n}||_\infty \leqslant C$ for some $C > 0$ independent of n, and therefore they solve the original problem (1.1) if n is taken sufficiently large. Thanks to this remark, in Sections 2, 3 and 4 we assume that $2 < p = q < 2^*$. In particular, we may assume that the following holds:

$(fg4)$ $|f'(s)| + |g'(s)| \leqslant C(1 + |s|^{p-2})$ with $2 < p < 2N/(N-2)$.

$(fg5)$ For every $\mu > 0$ there exists $C_\mu > 0$ such that

$$|f(s)t| + |g(t)s| \leqslant \mu(s^2 + t^2) + C_\mu(f(s)s + g(t)t), \qquad s, t \in \mathbb{R}.$$

As a concluding remark we mention that the extension of our results to the case where the concentration of solutions occur at critical points of V other than the local minima (in the line of the work in [14]) remains an open problem.

2 A variational framework for superlinear systems

In this section we establish some preliminary results which are needed for the proof of Theorem 1.1. Given $f, g \in C^1(\mathbb{R}, \mathbb{R})$ and V as in the previous section (we recall that without loss of generality we also assume that (fg4) and (fg5) hold), we consider the system

$$-\Delta u + V(x)u = g(v), \qquad -\Delta v + V(x)v = f(u), \qquad u, v \in H_0^1(\Omega) \tag{2.1}$$

and the associated energy functional $I : H \times H \to \mathbb{R}$,

$$I(u, v) := \int_\Omega \{\langle \nabla u, \nabla v \rangle + V(x)uv\} - \int_\Omega F(u) - \int_\Omega G(v).$$

In the sequel, all integrations are taken over the open set Ω. For any $u, v \in H$, let $\Psi_{u,v} \in H$ be such that

$$I'((u,v) + (\Psi_{u,v}, -\Psi_{u,v}))(\phi, -\phi) = 0, \qquad \forall \phi \in H. \tag{2.2}$$

Proposition 2.1. *The map $\Theta : H \times H \to H$, $(u,v) \mapsto \Theta(u,v) = \Psi_{u,v}$ is C^1.*

Proof. The function $\Psi_{u,v}$ is the minimum point of the strictly coercive functional

$$\phi \mapsto -I((u,v) + (\phi, -\phi)) = ||\phi||^2 + \langle u - v, \phi \rangle - \langle u, v \rangle + \int F(u+\phi) + \int G(v - \phi),$$

and thus Θ is well-defined. As for its smoothness, we apply the implicit function theorem to the map $\overline{\Theta} : (H \times H) \times H^- \to H^-, \overline{\Theta}((u,v), (\Psi, -\Psi)) := PI'((u,v) + (\Psi, -\Psi))$, where P is the orthogonal projection of $H \times H$ onto $H^- := \{(\phi, -\phi),\ \phi \in H\}$. Indeed, for any fixed pair $(\mu, \nu) = (u + \Psi, v - \Psi)$, the derivative of $\overline{\Theta}$ with respect to $(\phi, -\phi)$ evaluated at (μ, ν) is given by the linear map

$$(\phi, -\phi) \mapsto T(\phi, -\phi) = PI''(\mu, \nu)(\phi, -\phi),$$

that is

$$T(\phi, -\phi)(\varphi, -\varphi) = -2\langle \phi, \varphi \rangle - \int f'(\mu)\phi\varphi - \int g'(\nu)\phi\varphi, \quad \forall \phi, \varphi.$$

Since $f'(0) = 0$ and $|f'(s)| \leqslant C|s|^{2^*}$ for $|s| \geqslant 1$ (and similarly for g), we have that $Id - T$ is a compact operator and therefore we are left to prove that T is one-to-one. Now, if $T(\phi, -\phi) = 0$ we have in particular that

$$-2||\phi||^2 = \int f'(\mu)\phi^2 + \int g'(\nu)\phi^2,$$

and so $\phi = 0$. □

Lemma 2.2. *If $(u,v) \neq (0,0)$ is such that $I'(u,v)(u,v) = 0$ and $I'(u,v)(\phi, -\phi) = 0$ for every ϕ, then*

$$\sup_{\phi \in H} I''(u,v)(u+\phi, v-\phi)(u+\phi, v-\phi) < 0.$$

Proof. This was already observed in [25, Eq. (3.2)] and follows from a straightforward computation. Indeed, our assumptions on (u,v) imply that *minus* the second derivative above equals

$$\begin{aligned} 2||\phi||^2 \quad &+ \quad \int \Big(\frac{f(u)}{u} + \frac{g(v)}{v}\Big)\phi^2 \\ &+ \quad \int \Big(f'(u) - \frac{f(u)}{u}\Big)(u+\phi)^2 + \int \Big(g'(v) - \frac{g(v)}{v}\Big)(v-\phi)^2 \end{aligned}$$

and this, as a function of ϕ, is associated to a strictly convex, positive and coercive functional (cf. (fg3)), so that its infimum is attained and is positive. $\square$

From now on, given $u, v \in H$ and $t \geqslant 0$, we denote $\Psi_t := \Psi_{tu,tv}$ according to the definition in (2.2), i.e.

$$I'(t(u,v) + (\Psi_t, -\Psi_t))(\phi, -\phi) = 0, \qquad \forall \phi \in H. \tag{2.3}$$

Lemma 2.3. *Given $u, v \in H$ such that $u \neq -v$, the map*

$$\alpha(t) := I(t(u,v) + (\Psi_t, -\Psi_t))$$

is C^2 and, for any $t \in \mathbb{R}$ and $t \neq 0$,

$$\alpha'(t) = 0 \Rightarrow \alpha''(t) < 0.$$

Moreover, $\alpha'(0) = 0$ and $\alpha''(0) > 0$.

Proof. For any $t \in \mathbb{R}$ we denote by $\Psi_t' \in H$ the derivative of the map $t \mapsto \Psi_t$ evaluated at the point t. From (2.3) we see that

$$\begin{aligned} \alpha'(t) &= I'(t(u,v) + (\Psi_t, -\Psi_t))(u + \Psi_t', v - \Psi_t') \\ &= I'(t(u,v) + (\Psi_t, -\Psi_t))(u, v), \end{aligned}$$

and so

$$\alpha''(t) = I''(t(u,v) + (\Psi_t, -\Psi_t))(u + \Psi_t', v - \Psi_t')(u, v).$$

On the other hand, it follows also from (2.3) that

$$I''(t(u,v) + (\Psi_t, -\Psi_t))(u + \Psi_t', v - \Psi_t')(\phi, -\phi) = 0 \quad \forall t \; \forall \phi \tag{2.4}$$

and so (by letting $\phi = t^2 \Psi_t'$ in (2.4))

$$t^2 \alpha''(t) = I''(t(u,v) + (\Psi_t, -\Psi_t))(tu + t\Psi_t', tv - t\Psi_t')(tu + t\Psi_t', tv - t\Psi_t') \;\; \forall t. \tag{2.5}$$

Now, suppose $\alpha'(t_1) = 0$ for some $t_1 \neq 0$. By denoting $u_1 := t_1 u + \Psi_{t_1}$ and $v_1 := t_1 v - \Psi_{t_1}$, we have that $(u_1, v_1) \neq (0,0)$ (since $u \neq -v$),

$$I'(u_1, v_1)(u_1, v_1) = 0 \quad \text{ and } \quad I'(u_1, v_1)(\phi, -\phi) = 0 \;\; \forall \phi.$$

It follows then from Lemma 2.2 that

$$I''(u_1,v_1)(u_1+\phi,v_1-\phi)(u_1+\phi,v_1-\phi)<0 \ \ \forall\phi.$$

By letting $\phi=t_1\Psi'_{t_1}-\Psi_{t_1}$ we conclude from (2.5) that $\alpha''(t_1)<0$, as claimed.

As for the case $t_1=0$, since $I'(0,0)=0$ we have by definition that $\Psi_0=0$ and so $\alpha'(0)=0$. From (2.4) it can be checked directly that $\Psi'_0=(v-u)/2$, so that $2\alpha''(0)=\|u+v\|^2>0$, since $u\neq -v$. □

Proposition 2.4. *Let $u,v\in H$ be such that $u\neq -v$, $I'(u,v)(u,v)=0$ and $I'(u,v)(\phi,-\phi)=0$ for every ϕ, and denote*

$$\theta(t):=I'(t(u,v)+(\Psi_t,-\Psi_t))(u,v).$$

Then there exists $\delta=\delta(u,v)>0$ such that

$$\theta(t)=\delta(1-t)+\mathrm{o}(1-t) \qquad \text{as } \ t\to 1. \tag{2.6}$$

Moreover,

$$I(u,v)=\sup\{I(t(u,v)+(\phi,-\phi)):\ t\geqslant 0,\ \phi\in H\}. \tag{2.7}$$

Proof. The conclusion in (2.6) follows from Lemma 2.3, by observing that, by assumption, $\alpha'(1)=0$, so that $\theta'(1)=\alpha''(1)<0$.

As for (2.7), suppose that the supremum is attained at some $t_0\geqslant 0,\phi_0\in H$. Then we must have that $\phi_0=\Psi_{t_0}$ and $\alpha'(t_0)=0$. By Lemma 2.3, the function α has at most one positive critical point and $t_0\neq 0$, and so we must have that $t_0=1$ (and $\phi_0=\Psi_1=0$). □

For later purposes, we state a variant of (2.7) which is essentially proved in [24, Lemma 2.1] under additional assumptions on f and g.

Proposition 2.5. *Let (u_n,v_n) be a Palais-Smale sequence for the functional I, namely $0<\liminf I(u_n,v_n)\leqslant\limsup I(u_n,v_n)<+\infty$ and*

$$\mu_n:=||I'(u_n,v_n)||_{(H\times H)'}=\sup\{|I'(u_n,v_n)(\phi,\psi)|,\ \phi,\psi\in H,\ ||\phi||+||\psi||\leqslant 1\}\to 0.$$

Then

$$\sup\{I(t(u_n,v_n)+(\phi,-\phi)):\ t\geqslant 0,\ \phi\in H\}=I(u_n,v_n)+\mathrm{O}(\mu_n^2). \tag{2.8}$$

PROOF. We prove a slightly more general result. Let

$$\nu_n := |I'(u_n, v_n)(u_n, v_n)| + \sup\{|I'(u_n, v_n)(\phi, -\phi)|,\ \phi \in H,\ ||\phi|| \leqslant 1\} \leqslant 2\mu_n$$

and let us show that the quantity appearing in (2.8) is $I(u_n, v_n) + \mathrm{O}(\nu_n^2)$. Some of this follows as in [24, 25] and so we only stress the differences. We denote $\Psi_t^n := \Psi_{tu_n, tv_n}$ (cf. (2.2)) while $\Psi_t'^n$ stands for the derivative of the map $t \mapsto \Psi_t^n$ evaluated at the point t. Since $I(u_n, v_n) > 0$ we must have that $u_n \neq -v_n$. Moreover, since

$$\begin{aligned}\int (f(u_n)u_n + g(v_n)v_n) &= 2\langle u_n, v_n\rangle - I'(u_n, v_n)(u_n, v_n)\\ &\geqslant 2I(u_n, v_n) - I'(u_n, v_n)(u_n, v_n),\end{aligned}$$

our assumptions imply that

$$\liminf_{n\to\infty} \int (f(u_n)u_n + g(v_n)v_n) > 0. \tag{2.9}$$

Also, from

$$I'(u_n + \Psi_1^n, v_n - \Psi_1^n)(\Psi_1^n, -\Psi_1^n) - I'(u_n, v_n)(\Psi_1^n, -\Psi_1^n) = -I'(u_n, v_n)(\Psi_1^n, -\Psi_1^n) = \mathrm{O}(\nu_n)$$

we readily see that

$$||\Psi_1^n|| = \mathrm{O}(\nu_n). \tag{2.10}$$

Now, let

$$\alpha_n(t) := I(t(u_n, v_n) + (\Psi_t^n, -\Psi_t^n)).$$

It can be shown that $\sup_{t\geqslant 0} \alpha_n(t) = \alpha_n(t_n)$ for some $t_n \geqslant 0$ and that the sequences $||u_n||$, $||v_n||$ and $|t_n|$ are bounded (cf. [24, p. 160-162]; here we make use of property (fg5)). In particular, by recalling the definition of Ψ_t^n we see that $(\Psi_t^n)_n$ is bounded as long as t remains bounded. We claim that the same conclusion holds for $(\Psi_t'^n)_n$ as well. Indeed, by differentiating (2.3) and by letting $\phi = \Psi_t'^n$ we see that

$$I''(t(u_n, v_n) + (\Psi_t^n, -\Psi_t^n))(u_n + \Psi_t'^n, v_n - \Psi_t'^n)(\Psi_t'^n, -\Psi_t'^n) = 0,$$

that is

$$\begin{aligned}2||\Psi_t'^n||^2 &= \langle \Psi_t'^n, v_n - u_n\rangle - \int f'(tu_n + \Psi_t^n)(u_n + \Psi_t'^n)\Psi_t'^n + \int g'(tv_n - \Psi_t^n)(v_n - \Psi_t'^n)\Psi_t'^n\\ &\leqslant \langle \Psi_t'^n, v_n - u_n\rangle - \int f'(tu_n + \Psi_t^n)u_n\Psi_t'^n + \int g'(tv_n - \Psi_t^n)v_n\Psi_t'^n\end{aligned}$$

and the conclusion follows. This, combined with (2.10), shows that $\Psi_t^n \to 0$ as $t \to 1$, uniformly in n. Then, as shown in [25, Eq. (3.10)], there exist some small $\delta, \eta > 0$ such that, for every large n,

$$\sup_{t\in[1-\delta,1+\delta]} \alpha_n''(t) \leqslant -\eta \int (f(u_n)u_n + g(v_n)v_n) < 0, \tag{2.11}$$

where we have used (2.9) in the last inequality.

Our conclusion follows easily from (2.11). Indeed, we see from (2.10) that

$$\alpha_n'(1) = I'(u_n + \Psi_1^n, v_n - \Psi_1^n)(u_n, v_n) = I'(u_n, v_n)(u_n, v_n) + \mathrm{O}(\nu_n) = \mathrm{O}(\nu_n) \tag{2.12}$$

and so, thanks to (2.11), $\alpha_n'(1-\delta) > 0 > \alpha_n'(1+\delta)$ for large values of n. By Lemma 2.3 we must have then that $t_n \in [1-\delta, 1+\delta]$. Going back to (2.11)-(2.12) we deduce that

$$|t_n - 1| = \mathrm{O}(\nu_n).$$

As a consequence,

$$\alpha_n(1) = \alpha_n(t_n) + \mathrm{O}((1-t_n)^2) = \alpha_n(t_n) + \mathrm{O}(\nu_n^2). \tag{2.13}$$

On the other hand, by expanding the map $t \mapsto I(u_n + t\Psi_1^n, v_n - t\Psi_1^n)$ up to the order 2 and by using (2.10) we see that

$$\alpha_n(1) = I(u_n, v_n) + \mathrm{O}(\nu_n^2). \tag{2.14}$$

It follows from (2.13)-(2.14) that $\alpha_n(t_n) = I(u_n, v_n) + \mathrm{O}(\nu_n^2)$ and this proves Proposition 2.5. □

Remarks. 1) In the case $\Omega = \mathbb{R}^N$, for a given $\lambda > 0$ let us denote by I_λ the energy functional associated to the problem

$$-\Delta u + \lambda u = g(v), \qquad -\Delta v + \lambda v = f(u), \qquad u, v \in H^1(\mathbb{R}^N),$$

and by $c(\lambda)$ the corresponding ground-state critical level, that is $c(\lambda) = \inf\{I_\lambda(u,v) : (u,v) \neq (0,0) \text{ and } I_\lambda'(u,v) = 0\}$. Then, under the assumptions of Proposition 2.5, we also have that

$$I_\lambda(u_n, v_n) \geqslant c(\lambda) + \mathrm{o}(1), \qquad \text{as } n \to \infty. \tag{2.15}$$

Indeed, let $u, v \in H$ be such that, up to a subsequence, $u_n \rightharpoonup u$ and $v_n \rightharpoonup v$ weakly in $H^1(\mathbb{R}^N)$. Clearly, $I'_\lambda(u,v) = 0$ and by using the invariance by translation of the problem we may assume that $(u,v) \neq (0,0)$. By applying Fatou's lemma we see that

$$2I_\lambda(u,v) = \int (f(u)u - 2F(u) + g(v)v - 2G(v)) \leqslant 2 \liminf_{n\to\infty} I_\lambda(u_n, v_n)$$

and the conclusion follows.

2) Thanks to Proposition 2.5, we may apply the argument in [24, Lemma 3.1] to deduce that the map $\mathbb{R}^+ \to \mathbb{R}^+$, $\lambda \mapsto c(\lambda)$, is increasing. Also, as a simple consequence of (2.7), the ground-state critical levels are monotonic with respect to the potentials F and G.

3 Nehari manifold

In the following, for each $i = 1, \dots, k$, we fix open sets such that $\Lambda_i \Subset \Lambda'_i \Subset \widetilde{\Lambda}_i \Subset \Omega$ and cut-off functions ϕ_i such that $\phi_i = 1$ in Λ'_i and $\phi_i = 0$ in $\mathbb{R}^N \setminus \widetilde{\Lambda}_i$; we also denote $\Lambda := \cup_i \Lambda_i$, $\widetilde{\Lambda} := \cup_i \widetilde{\Lambda}_i$.

Following an idea introduced in [11, 13], we fix small numbers a_1, $a_2 > 0$ in such a way that $2f'(a_1) \leqslant \alpha := \inf_\Omega V$, $f'(s) \geqslant f'(a_1)$ $\forall s \geqslant a_1$, $2g'(a_2) \leqslant \alpha$, $g'(s) \geqslant g'(a_2)$ $\forall s \geqslant a_2$, and set $\widetilde{f}(s) := f(s)$ if $s \leqslant a_1$, $\widetilde{f}(s) := f'(a_1)s + (f(a_1) - a_1 f'(a_1))$ if $s \geqslant a_1$ and similarly for $\widetilde{g}(s)$. Then we introduce

$$f(x,s) := \chi_\Lambda(x) f(s) + (1 - \chi_\Lambda(x))\widetilde{f}(s)$$

and a similar function $g(x,s)$, and the corresponding energy functional

$$J_\varepsilon(u,v) := \int_\Omega \{\varepsilon^2 \langle \nabla u, \nabla v\rangle + V(x)uv\} - \int_\Omega F(x,u) - \int_\Omega G(x,v),$$

where $F(x,s) := \int_0^s f(x,\xi)\,d\xi$, $G(x,s) := \int_0^s g(x,\xi)\,d\xi$. Similarly to [11, 13], this truncation technique will be helpful in both bringing compactness to the problem and locating the maximum points of our solutions. The relevant properties of $f(x,s)$ and $g(x,s)$ are displayed in the next lemma, whose proof is elementary.

Lemma 3.1. *The function $f(x,s)$ (and also $g(x,s)$) satisfies:*

(i) $f(x,s) = \mathrm{o}(s)$ *as* $s \to 0$, *uniformly in* $x \in \Omega$;

(ii) $|\frac{\partial f}{\partial s}(x,s)| \leqslant C(1 + |s|^{p-2})$ *with* $2 < p < 2N/(N-2)$, $\forall x \in \Omega$, $s \in \mathbb{R}$;

(iii) $(1+\delta')f(x,s)s \leqslant s^2 \dfrac{\partial f}{\partial s}(x,s)$, *with* $\delta' > 0$, $\forall x \in \Lambda$, $s \in \mathbb{R}$*;*

(iv) $0 < 2F(x,s) \leqslant f(x,s)s$, $\forall x \in \Omega \setminus \Lambda$, $s \in \mathbb{R}, s \neq 0$*;*

(v) $f(x,s) \leqslant f(s)$, $\forall x \in \Omega$, $s \in \mathbb{R}$*;*

(vi) for some (arbitrarily small) $\delta = \delta(a_1, a_2) > 0$,

$$|f(x,s)| + |g(x,s)| \leqslant \delta |s|, \quad \forall x \in \Omega \setminus \Lambda, \; s \in \mathbb{R}; \tag{3.1}$$

(vii) for every $\mu > 0$ *there exists* $C_\mu > 0$ *such that*

$$|f(x,s)t| + |g(x,t)s| \leqslant \mu\,(s^2+t^2) + C_\mu\,(f(x,s)s + g(x,t)t), \; \forall x \in \Omega, \; s,t \in \mathbb{R}. \tag{3.2}$$

We denote by N_ε the set of functions $(u,v) \in H \times H$ satisfying

$$J_\varepsilon'(u,v)(\phi,-\phi) = 0 \;\forall \phi \in H, \quad J_\varepsilon'(u,v)(u\phi_i, v\phi_i) = 0, \quad \int_{\Lambda_i} (u^2+v^2) > \varepsilon^{N+1}, \; \forall i = 1,...,k$$

and $||(u,v)||_\varepsilon^2 = ||u||_\varepsilon^2 + ||v||_\varepsilon^2 := (\varepsilon^2 \int_\Omega |\nabla u|^2 + \int_\Omega V(x)u^2) + (\varepsilon^2 \int_\Omega |\nabla v|^2 + \int_\Omega V(x)v^2)$. It can be shown that N_ε is nonempty. Indeed, let us fix points $x_i \in \Lambda_i$ such that $V(x_i) = \inf_{\Lambda_i} V$ and let us consider a fixed pair of solutions $u_i, v_i \in H^1(\mathbb{R}^N)$ of the system

$$-\Delta u_i + V(x_i)u_i = g(v_i), \qquad -\Delta v_i + V(x_i)v_i = f(u_i),$$

corresponding to the ground-state critical level (cf. the Remark following Proposition 2.5)

$$c_i = I_{V(x_i)}(u_i, v_i).$$

We let

$$u_{i,\varepsilon}(x) := \phi_i(x)u_i((x-x_i)/\varepsilon), \qquad v_{i,\varepsilon}(x) := \phi_i(x)v_i((x-x_i)/\varepsilon).$$

Our next proposition shows that N_ε is nonempty if ε is sufficiently small. Its proof is postponed to the end of the present section.

Proposition 3.2. *There exists* $\varepsilon_0 > 0$ *such that for every* $0 < \varepsilon \leqslant \varepsilon_0$ *and every* $i = 1, \ldots, k$ *there is* $\Psi_\varepsilon \in H_0^1(\Omega)$ *and points* $t_{1,\varepsilon}, \ldots, t_{k,\varepsilon} \in [0,1]$ *such that the functions*

$$\overline{u}_\varepsilon := \sum_{i=1}^k t_{i,\varepsilon} u_{i,\varepsilon} + \Psi_\varepsilon \quad \textit{and} \quad \overline{v}_\varepsilon := \sum_{i=1}^k t_{i,\varepsilon} v_{i,\varepsilon} - \Psi_\varepsilon$$

satisfy

$$J'_\varepsilon(\overline{u}_\varepsilon,\overline{v}_\varepsilon)(\phi_i\overline{u}_\varepsilon,\phi_i\overline{v}_\varepsilon)=0,\quad \forall i=1,\dots,k, \tag{3.3}$$

$$J'_\varepsilon(\overline{u}_\varepsilon,\overline{v}_\varepsilon)(\phi,-\phi)=0,\quad \forall i=1,\dots,k\ \ \forall\phi\in H^1_0(\Omega), \tag{3.4}$$

and

$$J_\varepsilon(\overline{u}_\varepsilon,\overline{v}_\varepsilon)=\varepsilon^N(\sum_{i=1}^k c_i+\mathrm{o}(1))\quad \textit{as } \varepsilon\to 0. \tag{3.5}$$

Moreover,

$$\int_{\Lambda_i}(\overline{u}_\varepsilon^2+\overline{v}_\varepsilon^2)\geqslant \eta\varepsilon^N, \tag{3.6}$$

for some $\eta>0$*.*

We will denote by c_ε the infimum

$$c_\varepsilon := \inf_{N_\varepsilon} J_\varepsilon\,.$$

We know from Proposition 3.2 that

$$c_\varepsilon\leqslant\varepsilon^N\left(\sum_{i=1}^k c_i+\mathrm{o}(1)\right)\quad \text{as } \varepsilon\to 0. \tag{3.7}$$

It will follow from Proposition 3.3 below that $c_\varepsilon\geqslant 0$ and that any sublevel set $N_\varepsilon\cap\{(u,v):J_\varepsilon(u,v)\leqslant C\}$ with $C>0$ is weakly closed (for our purposes, thanks to (3.7), we may take, say, $C=2\varepsilon^N\sum_i c_i$).

We will show in Section 4 that c_ε is indeed a critical point of J_ε over the space $H\times H$. Here we make some preliminary considerations and prove two auxiliary results that will be needed in Section 4, namely Propositions 3.3 and 3.7.

Functions in N_ε are zero points of the functional $K_\varepsilon : H\times H\to\mathbb{R}^k\oplus H^-$,

$$K_\varepsilon(u,v):=(J'_\varepsilon(u,v)(\phi_1u,\phi_1v),\dots,J'_\varepsilon(u,v)(\phi_ku,\phi_kv),PJ'_\varepsilon(u,v)),$$

where $P : H\times H\to H^- := \{(\phi,-\phi),\ \phi\in H\}$ is the orthogonal projection. We can identify $\mathbb{R}^k$ with the subspace $\mathrm{span}\{(u\phi_i,v\phi_i),\ i=1,...,k\}$. For any $(u,v)\in N_\varepsilon$, its derivative $K'_\varepsilon(u,v)$ is given by

$$K'_\varepsilon(u,v)(\zeta,\xi)=(\mu_1(\zeta,\xi),\dots,\mu_k(\zeta,\xi),PJ''_\varepsilon(u,v)(\zeta,\xi))\in\mathbb{R}^k\oplus H^-,$$

where $PJ''_\varepsilon(u,v)(\zeta,\xi)$ has a meaning according to Riesz's theorem and

$$\mu_i(\zeta,\xi):=J'_\varepsilon(u,v)(\phi_i\zeta,\phi_i\xi)+J''_\varepsilon(u,v)(\phi_iu,\phi_iv)(\zeta,\xi),\qquad i=1,\dots,k.$$

Let us concentrate on $K'_\varepsilon(u,v)$ restricted to the space $\mathbb{R}^k \oplus H^-$. Similarly to the proof of Proposition 2.1, we can check that $Id - K'_\varepsilon(u,v)$ is compact. Now, for any given $\psi \in H$ and $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$, not all of them zero, by denoting $\Phi := \sum_i \lambda_i \phi_i$, $\overline{u} := \Phi + \psi$, $\overline{v} := \Phi - \psi$, we have that

$$\langle K'_\varepsilon(u,v)(\overline{u},\overline{v}), (\overline{u},\overline{v})\rangle_\varepsilon = J'_\varepsilon(u,v)(\Phi^2 u, \Phi^2 v) + J''_\varepsilon(u,v)(\overline{u},\overline{v})(\overline{u},\overline{v}).$$

We shall prove below that this expression is negative (cf. Proposition 3.7), provided ε is small enough. This shows in particular that $K'_\varepsilon(u,v)$ is one-to-one (and thus an isomorphism) in the space $\mathbb{R}^k \oplus H^-$. As a consequence, the tangent space of the manifold N_ε at the point (u,v) is given by $\mathrm{Ker} K'_\varepsilon(u,v)$. Then, according to the Lagrange multiplier rule, N_ε is a natural constraint for the functional J_ε; namely, if the infimum c_ε is achieved at $(u,v) \in N_\varepsilon$ then there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $\psi \in H$ such that

$$J'_\varepsilon(u,v)(\zeta,\xi) = J'_\varepsilon(u,v)(\Phi\zeta, \Phi\xi) + J''_\varepsilon(u,v)(\overline{u},\overline{v})(\zeta,\xi), \qquad \forall \zeta, \xi \in H.$$

By letting $(\zeta,\xi) = (\overline{u},\overline{v})$, so that $J'_\varepsilon(u,v)(\overline{u},\overline{v}) = 0$, we conclude from the previous observation that we must have $\psi = 0$ and $\lambda_1 = \ldots = \lambda_k = 0$, hence (u,v) is indeed a critical point of J_ε. (It can also be observed that, in fact, we can reduce ourselves to a finite dimensional manifold by working instead with the functional $\tilde{J}_\varepsilon : H \to \mathbb{R}$, $\tilde{J}_\varepsilon(u) := J_\varepsilon((u,u) + (\Psi_{u,u}, -\Psi_{u,u}))$, cf. (2.2); we leave the details for the interested reader.)

We now make this ideas precise. Except when indicated otherwise, all integrals take place over $\mathbb{R}^N$, by extending u and v by zero.

Proposition 3.3. *For every $C_0 > 0$ there exist ε_0, D_0, $\eta_0 > 0$ such that, for every $0 < \varepsilon \leqslant \varepsilon_0$,*

$$(u,v) \in N_\varepsilon, \quad J_\varepsilon(u,v) \leqslant C_0 \varepsilon^N \ \Rightarrow \ ||(u,v)||^2_\varepsilon \leqslant D_0 \varepsilon^N \ \text{ and } \ \int_{\Lambda_i} (u^2 + v^2) \geqslant \eta_0 \varepsilon^N,$$

for every $i = 1, ...k$. Also, $J_\varepsilon(u,v) \geqslant \eta_0 \varepsilon^N$ (whence $c_\varepsilon \geqslant \eta_0 \varepsilon^N$).

Proof. 1. Let $\Phi := \sum_i \phi_i$ and $\xi := 1 - \Phi$, so that $\xi \geqslant 0$ in Ω and $\xi = 0$ in Λ. Since $J'_\varepsilon(u,v)(\Phi u, \Phi v) = 0$, we have that

$$\begin{aligned} \int_\Lambda (f(x,u)u + g(x,v)v) &\leqslant \int (f(x,u)u + g(x,v)v)\,\Phi \\ &= \langle u, \Phi v\rangle_\varepsilon + \langle v, \Phi u\rangle_\varepsilon \qquad (3.8) \\ &= 2\langle u,v\rangle_\varepsilon + \langle v-u, \xi(v-u)\rangle_\varepsilon - \langle u, \xi u\rangle_\varepsilon - \langle v, \xi v\rangle_\varepsilon. \end{aligned}$$

Now, since $\xi \geqslant 0$,

$$\langle u, \xi u\rangle_\varepsilon = \varepsilon^2 \int |\nabla u|^2\xi + \int V(x)u^2\xi + \varepsilon^2 \int u\langle \nabla u, \nabla \xi\rangle \geqslant \mathrm{o}_\varepsilon(1)||u||_\varepsilon^2 \tag{3.9}$$

and similarly for $\langle v, \xi v\rangle_\varepsilon$, with $\mathrm{o}_\varepsilon(1) \to 0$ as $\varepsilon \to 0$. On the other hand, since $-\Delta(v-u)+V(x)(v-u) = f(x,u) - g(x,v)$ in Ω and again by $\xi \geqslant 0$, we have that

$$\langle v-u, (v-u)\xi\rangle_\varepsilon = \int (f(x,u) - g(x,v))(v-u)\xi \leqslant \delta \int (u^2+v^2) \tag{3.10}$$

for a small $\delta > 0$. Finally, since $J_\varepsilon(u,v) \leqslant C_0\varepsilon^N$,

$$\begin{aligned} 2\langle u, v\rangle_\varepsilon &\leqslant 2\int (F(x,u) + G(x,v)) + 2C_0\varepsilon^N \\ &\leqslant 2\int_\Lambda (F(u)+G(v)) + \delta \int (u^2+v^2) + 2C_0\varepsilon^N. \end{aligned} \tag{3.11}$$

Since $f(s)s \geqslant (2+\delta')F(s)$ and similarly for $g(s)$, for some $\delta' > 0$ (as follows from (iii) in Lemma 3.1), we see from (3.8) – (3.11) that

$$\int_\Lambda (f(u)u + g(v)v) \leqslant C(\delta + \mathrm{o}_\varepsilon(1))\, ||(u,v)||_\varepsilon^2 + C_0'\varepsilon^N, \tag{3.12}$$

for some $\delta > 0$ and some $C, C_0' > 0$ (independent of δ and ε). Thus, going back to (3.11),

$$2\langle u,v\rangle_\varepsilon - \langle v, \Phi u\rangle_\varepsilon - \langle u, \Phi v\rangle_\varepsilon \leqslant 2\langle u,v\rangle_\varepsilon \leqslant C(\delta + \mathrm{o}_\varepsilon(1))\, ||(u,v)||_\varepsilon^2 + C_0''\varepsilon^N. \tag{3.13}$$

2. Next, it follows from our assumptions that $J_\varepsilon'(u,v)(\phi_i v, \phi_i u) = 0$ for every $i = 1, \ldots, k$, that is

$$\langle u, \phi_i u\rangle_\varepsilon + \langle v, \phi_i v\rangle_\varepsilon = \int (f(x,u)v + g(x,v)u)\phi_i. \tag{3.14}$$

Then, thanks to (3.2) and (3.12),

$$\langle u, \Phi u\rangle_\varepsilon + \langle v, \Phi v\rangle_\varepsilon \leqslant \mu||(u,v)||_\varepsilon^2 + C_\mu(\delta + \mathrm{o}_\varepsilon(1))||(u,v)||_\varepsilon^2 + C_0'\varepsilon^N. \tag{3.15}$$

3. We combine the estimates in (3.10), (3.13) and (3.15) to deduce that

$$\begin{aligned} ||(u,v)||_\varepsilon^2 &= \langle u, \Phi u\rangle_\varepsilon + \langle v, \Phi v\rangle_\varepsilon + \langle v-u, \xi(v-u)\rangle_\varepsilon + 2\langle u,v\rangle_\varepsilon - \langle v, \Phi u\rangle_\varepsilon - \langle u, \Phi v\rangle_\varepsilon \\ &\leqslant C(\mu + \delta + \mathrm{o}_\varepsilon(1))||(u,v)||_\varepsilon^2 + C_\mu(\delta + \mathrm{o}_\varepsilon(1))||(u,v)||_\varepsilon^2 + (C_0' + C_0'')\varepsilon^N. \end{aligned} \tag{3.16}$$

By choosing a small $\mu > 0$ and, subsequentely, a small $\delta > 0$ we conclude that

$$||(u,v)||_\varepsilon^2 \leqslant D_0\varepsilon^N \tag{3.17}$$

for every small ε. This proves the first part of Proposition 3.3.

4. Next, we go back to the equation in (3.14). By replacing if necessary ϕ_i by ϕ_i^2 in the definition of the set N_ε, we may assume that $|\nabla\phi_i|^2 \leqslant C\phi_i$. Therefore,

$$\begin{aligned}\langle u, \phi_i u\rangle_\varepsilon &= \varepsilon^2 \int |\nabla u|^2\phi_i + \int V(x)u^2\phi_i + \varepsilon^2 \int u\langle \nabla u, \nabla\phi_i\rangle \\ &\geqslant \frac{1}{2}\left(\varepsilon^2 \int |\nabla u|^2\phi_i + \int V(x)u^2\phi_i\right) - C'\varepsilon^2 \int u^2, \end{aligned} \tag{3.18}$$

while (3.17) and the assumption that $\int_{\Lambda_i}(u^2+v^2) \geqslant \varepsilon^{N+1}$ imply

$$C'\varepsilon^2 \int u^2 \leqslant D_0'\varepsilon^{N+2} \leqslant \delta \int V(x)(u^2+v^2)\phi_i, \tag{3.19}$$

for a small $\delta > 0$, provided $\varepsilon > 0$ is sufficiently small. By proceeding in a similar way with the function v we conclude that

$$\langle u, \phi_i u\rangle_\varepsilon + \langle v, \phi_i v\rangle_\varepsilon \geqslant \frac{1}{2}\left(\varepsilon^2 \int (|\nabla u|^2 + |\nabla v|^2)\phi_i + \int V(x)(u^2+v^2)\phi_i\right). \tag{3.20}$$

As for the right-hand member of (3.14), we recall that (3.1) holds and also that $|f(s)| \leqslant \delta|s| + C_\delta|s|^{2^*-1}$ and so

$$\int |f(x,u)v|\phi_i \leqslant \delta \int (u^2+v^2)\phi_i + C_\delta \int_{\Lambda_i} |u|^{2^*-1}|v|, \tag{3.21}$$

with

$$\begin{aligned}\int_{\Lambda_i} |u|^{2^*-1}|v| &= \int_{\Lambda_i} |u\phi_i|^{2^*-1}|v\phi_i| \\ &\leqslant C\left(\int |u\phi_i|^{2^*} + \int |v\phi_i|^{2^*}\right) \\ &\leqslant C\left(\int (|\nabla u|^2 + |\nabla v|^2)\phi_i\right)^{2^*/2} + C\left(\int (u^2+v^2)\phi_i\right)^{2^*/2}\end{aligned}$$

thanks to Sobolev's inequality. Thus, since $2^*/2 > 1$,

$$\int_{\Lambda_i} |u|^{2^*-1}|v| \leqslant C\left(\int (|\nabla u|^2 + |\nabla v|^2)\phi_i\right)^{2^*/2} + \delta \int V(x)(u^2+v^2)\phi_i. \tag{3.22}$$

If we combine (3.14), (3.20), (3.21), (3.22) we see that, for some small $\delta > 0$,

$$\begin{aligned}\varepsilon^2 \int (|\nabla u|^2 + |\nabla v|^2)\phi_i + \int V(x)(u^2+v^2)\phi_i &\leqslant C\left(\int (|\nabla u|^2 + |\nabla v|^2)\phi_i\right)^{2^*/2} \\ &+ \delta \int V(x)(u^2+v^2)\phi_i \end{aligned} \tag{3.23}$$

if ε is sufficiently small. Since $(u^2+v^2)\phi_i \neq 0$, this implies that

$$0 < \varepsilon^2 \int (|\nabla u|^2 + |\nabla v|^2)\phi_i \leqslant C \left(\int (|\nabla u|^2 + |\nabla v|^2)\phi_i \right)^{2^*/2},$$

and so, since $2^*/2 = N/(N-2)$,

$$\varepsilon^2 \int (|\nabla u|^2 + |\nabla v|^2)\phi_i \geqslant \eta \varepsilon^N, \tag{3.24}$$

for some $\eta > 0$. By combining (3.17), (3.23) and (3.24) we also deduce that

$$\int_{\Lambda_i} (u^2+v^2) \geqslant \eta_0 \varepsilon^N$$

for some $\eta_0 > 0$. At last, it may be observed that the coefficients of the ε^N-term in (3.16) are bounded above by a (fixed) multiple of $J_\varepsilon(u,v)$, and this, together with (3.24), yields that

$$J_\varepsilon(u,v) \geqslant \eta_1 \varepsilon^N,$$

for some $\eta_1 > 0$, which completes the proof of Proposition 3.3.

For further reference, we mention that (3.2), (3.14), (3.20) and (3.24) show that

$$\int_{\Lambda_i} (f(u)u + g(v)v) \geqslant \eta_2 \varepsilon^N, \tag{3.25}$$

for some $\eta_2 > 0$. □

Lemma 3.4. *Under the conditions of Proposition 3.3, let $i \in \{1, \ldots, k\}$ and $\psi \in H$. Then*

$$\alpha_{i,\varepsilon} := \langle (v-u)\phi_i, \psi \rangle_\varepsilon - \langle v-u, \phi_i \psi \rangle_\varepsilon = \mathrm{o}_\varepsilon(1) \varepsilon^{N/2} ||\psi||_\varepsilon,$$

where $\mathrm{o}_\varepsilon(1) \to 0$ as $\varepsilon \to 0$, uniformly in u, v, ψ.

Proof. Clearly,

$$\begin{aligned} |\alpha_{i,\varepsilon}| &= \left| \varepsilon^2 \int (v-u)\langle \nabla \phi_i, \nabla \psi \rangle - \varepsilon^2 \int \psi \langle \nabla \phi_i, \nabla (v-u) \rangle \right| \\ &\leqslant C\varepsilon^2 \left(\int |v-u|\,|\nabla \psi| + \int |\psi|\,|\nabla(v-u)| \right) \leqslant \mathrm{o}_\varepsilon(1) ||(u,v)||_\varepsilon ||\psi||_\varepsilon, \end{aligned}$$

and the conclusion follows from Proposition 3.3. □

Lemma 3.5. *Under the conditions of Proposition 3.3, let $i \in \{1, \ldots, k\}$ and denote*

$$\beta_{i,\varepsilon} := \langle u, \phi_i^2 v \rangle_\varepsilon + \langle v, \phi_i^2 u \rangle_\varepsilon + 2\langle \phi_i u, \phi_i v \rangle_\varepsilon - 2 \int (f(x,u)u + g(x,v)v)\phi_i^2.$$

Then

$$\beta_{i,\varepsilon} \leqslant \mathrm{o}_\varepsilon(1)\varepsilon^N,$$

where $\mathrm{o}_\varepsilon(1) \to 0$ *as* $\varepsilon \to 0$*, uniformly in* u, v*.*

Proof. We may subtract the quantity $2J'_\varepsilon(u,v)(\phi_i u, \phi_i v) = 0$ in the expression of $\beta_{i,\varepsilon}$, the resulting quantity being then given by

$$\begin{aligned} 2\langle u, (\phi_i^2 - \phi_i)u\rangle_\varepsilon + 2\langle v, (\phi_i^2 - \phi_i)v\rangle_\varepsilon &+ 2\langle v - u, (\phi_i - \phi_i^2)(v-u)\rangle_\varepsilon \\ &+ 2\int (f(x,u)u + g(x,v)v)(\phi_i - \phi_i^2) + \mathrm{o}_\varepsilon(1)\varepsilon^N, \end{aligned}$$

where the $\mathrm{o}_\varepsilon(1)\varepsilon^N$ term arises from the quantity $2\langle \phi_i u, \phi_i v\rangle_\varepsilon - \langle u, \phi_i^2 v\rangle_\varepsilon - \langle v, \phi_i^2 u\rangle_\varepsilon = 2\varepsilon^2 \int uv|\nabla\phi_i|^2$. Since $\phi_i - \phi_i^2 \geqslant 0$ in Ω and $\phi_i - \phi_i^2 = 0$ in Λ, and since $-\Delta(v-u) = f(x,u) - g(x,v)$, the third and fourth terms above can be estimated in a similar manner, yielding that

$$\beta_{i,\varepsilon} \leqslant 2\langle u, (\phi_i^2 - \phi_i)u\rangle_\varepsilon + 2\langle v, (\phi_i^2 - \phi_i)v\rangle_\varepsilon + \delta \int (u^2 + v^2)(\phi_i - \phi_i^2) + \mathrm{o}_\varepsilon(1)\varepsilon^N.$$

Moreover, $\langle u, (\phi_i^2 - \phi_i)u\rangle_\varepsilon = \varepsilon^2 \int |\nabla u|^2(\phi_i^2 - \phi_i) + \int V(x)u^2(\phi_i^2 - \phi_i) + \mathrm{o}_\varepsilon(1)\varepsilon^N$ and so

$$\beta_{i,\varepsilon} \leqslant \int (2V(x) - \delta)(u^2 + v^2)(\phi_i^2 - \phi_i) + \mathrm{o}_\varepsilon(1)\varepsilon^N.$$

By letting $\delta \leqslant 2\inf_{\mathbb{R}^N} V$, the conclusion follows. □

The expression for γ_ε which appears in our next lemma is suggested by the one in the proof of Lemma 2.2.

Lemma 3.6. *Under the conditions of Proposition 3.3, let* $\psi \in H$, $\mu_1, \ldots, \mu_k \in \mathbb{R}$ *with* $\sum_i \mu_i^2 = 1$*, and denote* $\Phi := \sum_i \mu_i \phi_i$ *and*

$$\begin{aligned} \gamma_\varepsilon := 2||\psi||_\varepsilon^2 &+ \int \Big(\frac{f(x,u)}{u} + \frac{g(x,v)}{v}\Big)\psi^2 \\ &+ \int \Big(\frac{\partial f}{\partial s}(x,u) - \frac{f(x,u)}{u}\Big)(\Phi u + \psi)^2 + \int \Big(\frac{\partial g}{\partial s}(x,v) - \frac{g(x,v)}{v}\Big)(\Phi v - \psi)^2. \end{aligned}$$

Then there exists $\eta > 0$ *such that*

$$\gamma_\varepsilon \geqslant \eta\varepsilon^N + ||\psi||_\varepsilon^2.$$

Proof. We may assume that $\mu_i = 1$ for some i. By recalling that $f'(s) \geqslant (1+\delta')f(s)/s$ and similarly for $g(s)$ for some $\delta' > 0$ we see that

$$\gamma_\varepsilon - ||\psi||_\varepsilon^2 \geqslant ||\psi||_\varepsilon^2 + \delta' \int_{\Lambda_i} \left(\frac{f(u)}{u}(u+\psi)^2 + \frac{g(v)}{v}(v-\psi)^2\right).$$

We use the change of variables $u^\varepsilon(x) := u(\varepsilon x)$, $v^\varepsilon(x) := v(\varepsilon x)$, $\psi^\varepsilon(x) := \psi(\varepsilon x)$. If the expression above is not bounded below by some $\eta\varepsilon^N$ then there exist sequences $\varepsilon \to 0$ and $||\psi^\varepsilon||_1 \to 0$ such that

$$\int_{\Lambda_i/\varepsilon} (f(u^\varepsilon)u^\varepsilon + 2f(u^\varepsilon)\psi^\varepsilon + g(v^\varepsilon)v^\varepsilon - 2g(v^\varepsilon)\psi^\varepsilon) \to 0.$$

Since $(u^\varepsilon)_\varepsilon$ and $(v^\varepsilon)_\varepsilon$ are bounded in $H^1(\mathbb{R}^N)$, this contradicts (3.25) and proves Lemma 3.6. □

Proposition 3.7. *Under the conditions of Proposition 3.3, there exists $\varepsilon_0 > 0$ such that for any $0 < \varepsilon \leqslant \varepsilon_0$, any $\psi \in H$, and any $i \in \{1, \dots, k\}$,*

$$J'_\varepsilon(u,v)(\phi_i^2 u, \phi_i^2 v) + J''_\varepsilon(u,v)(\phi_i u + \psi, \phi_i v - \psi)(\phi_i u + \psi, \phi_i v - \psi) < 0.$$

Proof. A straightforward computation shows that the above expression is given by $2\alpha_{i,\varepsilon} + \beta_{i,\varepsilon} - \gamma_{i,\varepsilon}$, where these quantities were defined respectively in Lemmas 3.4, 3.5 and 3.6 (here $\gamma_{i,\varepsilon}$ stands for the expression in Lemma 3.5 with $\mu_i = 1$ and $\mu_j = 0$ if $j \neq i$). According to these lemmas, for ε small enough this is bounded above by $o_\varepsilon(1)\varepsilon^{N/2}||\psi||_\varepsilon - \varepsilon^N\eta/2 - ||\psi||_\varepsilon^2$ for some $\eta > 0$, and the conclusion follows. □

We conclude this section with the

Proof of Proposition 3.2.

1. For every $\bar{t} = (t_1, \dots, t_k) \in [0,1] \times \cdots \times [0,1]$, let

$$u_{\varepsilon,\bar{t}} := \sum_i t_i u_{i,\varepsilon}, \qquad v_{\varepsilon,\bar{t}} := \sum_i t_i v_{i,\varepsilon}, \tag{3.26}$$

and let $\Psi_{\varepsilon,\bar{t}}$ be such that

$$J'_\varepsilon((u_{\varepsilon,\bar{t}}, v_{\varepsilon,\bar{t}}) + (\Psi_{\varepsilon,\bar{t}}, -\Psi_{\varepsilon,\bar{t}}))(\phi, -\phi) = 0, \qquad \forall \phi \in H_0^1(\Omega), \tag{3.27}$$

that is, $\Psi_{\varepsilon,\bar{t}} \in H_0^1(\Omega)$ is such that

$$\begin{aligned} -2\varepsilon^2\Delta\Psi_{\varepsilon,\bar{t}} + 2V(x)\Psi_{\varepsilon,\bar{t}} &= -\varepsilon^2\Delta v_{\varepsilon,\bar{t}} + V(x)v_{\varepsilon,\bar{t}} + \varepsilon^2\Delta u_{\varepsilon,\bar{t}} - V(x)u_{\varepsilon,\bar{t}} \\ &\quad -f(x, u_{\varepsilon,\bar{t}} + \Psi_{\varepsilon,\bar{t}}) + g(x, v_{\varepsilon,\bar{t}} - \Psi_{\varepsilon,\bar{t}}). \end{aligned} \tag{3.28}$$

We have that, uniformly in ε and in $\bar{t}$,

$$\int_\Omega \left(\varepsilon^2|\nabla\Psi_{\varepsilon,\bar{t}}|^2 + V(x)\Psi_{\varepsilon,\bar{t}}^2\right) \leqslant C$$

and

$$\int_{\Omega\setminus\Lambda}\left(\varepsilon^2|\nabla\Psi_{\varepsilon,\bar t}|^2+V(x)\Psi^2_{\varepsilon,\bar t}\right)\to 0\quad\text{as }\varepsilon\to 0. \tag{3.29}$$

Indeed, concerning for example the less obvious property (3.29) we observe that

$$-2\varepsilon^2\Delta\Psi_{\varepsilon,\bar t}+2V(x)\Psi_{\varepsilon,\bar t}=-f(\Psi_{\varepsilon,\bar t})+g(-\Psi_{\varepsilon,\bar t})\qquad\text{in }\Omega\setminus\widetilde\Lambda,$$

and this readily implies the estimate in (3.29) over the set $\Omega\setminus\widetilde\Lambda$. As for $\widetilde\Lambda\setminus\Lambda$, in view of the change of variables $y=\varepsilon x+x_i$, let us denote $\phi_i^\varepsilon(x):=\phi_i(\varepsilon x+x_i)$, $V_i^\varepsilon(x):=V_i(\varepsilon x+x_i)$, $\Psi^\varepsilon_{\bar t}(x):=\Psi_{\varepsilon,\bar t}(\varepsilon x+x_i)$, $\Lambda_i^\varepsilon:=\frac{\Lambda_i-x_i}{\varepsilon}$, $f_i^\varepsilon(x,s):=f(\varepsilon x+x_i,s)$ and so on, so that, taking the previous remark into account, (3.29) will be a consequence of showing that, for every $i=1,\dots,k$,

$$\int_{\widetilde\Lambda_i^\varepsilon\setminus\Lambda_i^\varepsilon}\left(|\nabla\Psi^\varepsilon_{\bar t}|^2+V_i^\varepsilon(\Psi^\varepsilon_{\bar t})^2\right)\to 0\qquad\text{as }\varepsilon\to 0,\ \text{uniformly in }\bar t. \tag{3.30}$$

To that purpose, let us fix $\omega_i\subset\Lambda_i$ and let ξ_i be a cut-off function such that $\xi_1=1$ in $\widetilde\Lambda_i\setminus\Lambda_i$, $\xi_i=0$ in ω_i and $\xi_i=0$ outside a small neighborhood of $\widetilde\Lambda_i$. Then, over the support of $\xi_i^\varepsilon(x):=\xi_i(\varepsilon x+x_i)$, the function $\Psi^\varepsilon_{\bar t}$ satisfies

$$\begin{aligned}-2\Delta\Psi^\varepsilon_{\bar t}+2V_i^\varepsilon\Psi^\varepsilon_{\bar t}&=t_if_i^\varepsilon(x,u_i)-f_i^\varepsilon(x,t_iu_i+\Psi^\varepsilon_{\bar t})\\&\quad-t_ig_i^\varepsilon(x,v_i)+g_i^\varepsilon(x,t_iv_i-\Psi^\varepsilon_{\bar t})+\mathrm{o}(1),\end{aligned} \tag{3.31}$$

since $||(1-\phi_i^\varepsilon)u_i||_{H^1(\mathbb R^N)}\to 0$ and $||(1-\phi_i^\varepsilon)v_i||_{H^1(\mathbb R^N)}\to 0$ as $\varepsilon\to 0$. If we multiply this equation by $\Psi^\varepsilon_{\bar t}\xi_i^\varepsilon$ and integrate by parts we see that

$$\begin{aligned}2\int\left(|\nabla\Psi^\varepsilon_{\bar t}|^2+V_i^\epsilon(\Psi^\varepsilon_{\bar t})^2\right)\xi_i^\varepsilon&\leqslant\int\left(t_if_i^\varepsilon(x,u_i)-f_i^\varepsilon(x,t_iu_i)\right)\Psi^\varepsilon_{\bar t}\,\xi_i^\varepsilon\\&\quad+\int\left(g_i^\varepsilon(x,t_iv_i)-t_ig_i^\varepsilon(x,v_i)\right)\Psi^\varepsilon_{\bar t}\,\xi_i^\varepsilon+\mathrm{o}(1)\\&\leqslant C\int_{\mathbb R^N\setminus\Lambda_i^\varepsilon}\left(|u_i|+|v_i|\right)|\Psi^\varepsilon_{\bar t}|+\mathrm{o}(1)=\mathrm{o}(1),\end{aligned}$$

since $u_i,v_i\in L^2(\mathbb R^N)$ and since $\int_{\mathbb R^N}|\Psi^\varepsilon_{\bar t}|^2$ is bounded uniformly in ε and in $\bar t$. This proves (3.30) and establishes (3.29).

2. Now, let us introduce the continuous function

$$\theta_{i,\varepsilon}(\bar t):=J'_\varepsilon((u_{\varepsilon,\bar t},v_{\varepsilon,\bar t})+(\Psi_{\varepsilon,\bar t},-\Psi_{\varepsilon,\bar t}))(u_{\varepsilon,\bar t}\phi_i,v_{\varepsilon,\bar t}\phi_i), \tag{3.32}$$

where $u_{\varepsilon,\bar{t}}$, $v_{\varepsilon,\bar{t}}$ and $\Psi_{\varepsilon,\bar{t}}$ were defined in (3.26) and (3.27). We claim that there exist $\varepsilon_0, \mu > 0$ such that, for any $0 < \varepsilon \leqslant \varepsilon_0$ and every points $t_j \in [0,1]$, $j \neq i$,

$$\theta_{i,\varepsilon}(t_1, \dots, 1-\mu, \dots, t_k) > 0 > \theta_{i,\varepsilon}(t_1, \dots, 1+\mu, \dots, t_k). \tag{3.33}$$

We observe that, by Miranda's theorem and by the definition in (3.27), this yields the desired conclusion (3.3) (and also (3.4), thanks again to (3.27)).

3. In order to prove (3.33), and in view of Proposition 2.4, let us consider $\Psi_{t_i} \in H^1(\mathbb{R}^N)$ such that

$$\begin{aligned} -2\Delta\Psi_{t_i} + 2V(x_i)\Psi_{t_i} &= t_i f(u_i) - f(t_i u_i + \Psi_{t_i}) \\ &\quad + g(t_i v_i - \Psi_{t_i}) - t_i g(v_i) \quad \text{ in } \mathbb{R}^N. \end{aligned} \tag{3.34}$$

Since Ψ_{t_i} is a continuous map as a function of t_i, we can use Lebegue's dominated convergence theorem to deduce that

$$\int_{\widetilde{\Lambda}_i^\varepsilon} (V(x_i) - V(\varepsilon x + x_i))\, \Psi_{t_i}^2 \to 0 \quad \text{ as } \varepsilon \to 0, \text{ uniformly in } t_i,$$

and thus also

$$\int_{\widetilde{\Lambda}_i^\varepsilon} (V(x_i) - V(\varepsilon x + x_i))\, \Psi_{t_i}\, (\Psi_{\bar{t}}^\varepsilon - \Psi_{t_i}) \to 0 \quad \text{ as } \varepsilon \to 0, \text{ uniformly in } \bar{t}. \tag{3.35}$$

Similarly, we have that

$$\int_{\mathbb{R}^N \setminus \Lambda_i^\varepsilon} \left(|\nabla \Psi_{t_i}|^2 + V(x_i)\Psi_{t_i}^2\right) \to 0 \quad \text{ as } \varepsilon \to 0, \text{ uniformly in } t_i. \tag{3.36}$$

Now, by comparing (3.31) and (3.34) we see that

$$\begin{aligned} -2\Delta(\Psi_{\bar{t}}^\varepsilon - \Psi_{t_i}) &+ 2V_i^\varepsilon(\Psi_{\bar{t}}^\varepsilon - \Psi_{t_i}) = 2(V(x_i) - V_i^\varepsilon)\Psi_{t_i} \\ &+ t_i(f_i^\varepsilon(u_i) - f(u_i)) + f(t_i u_i + \Psi_{t_i}) - f_i^\varepsilon(t_i u_i + \Psi_{\bar{t}}^\epsilon) \\ &+ t_i(g(v_i) - g_i^\varepsilon(v_i)) + g_i^\varepsilon(t_i v_i - \Psi_{\bar{t}}^\varepsilon) - g(t_i v_i - \Psi_{t_i}) + \mathrm{o}(1). \end{aligned}$$

By recalling that $f_i^\varepsilon(x,s) = f(x)$ if $x \in \Lambda_i^\varepsilon$ and by adding $\pm(f_i^\varepsilon(x, t_i u_i + \Psi_{t_i}^\varepsilon) - g_i^\varepsilon(x, t_i v_i - \Psi_{t_i}^\varepsilon))$ in the right-hand member of the above equation, we conclude from (3.30), (3.35) and (3.36) that

$$\int_{\widetilde{\Lambda}_i} |\nabla(\Psi_{\bar{t}}^\varepsilon - \Psi_{t_i})|^2 + V(\varepsilon x + x_i)(\Psi_{\bar{t}}^\varepsilon - \Psi_{t_i})^2 \to 0 \quad \text{ as } \varepsilon \to 0, \text{ uniformly in } \bar{t}. \tag{3.37}$$

4. At last, let us consider the map introduced in Proposition 2.4, namely

$$\theta_i(t_i) := I'_{V(x_i)}(t_i(u_i,v_i) + (\Psi_{t_i}, -\Psi_{t_i}))(u_i,v_i).$$

Taking (3.32) and (3.37) into account we see that, for any $\bar{t} = (t_1,\dots,t_k)$,

$$\varepsilon^{-N}\theta_{i,\varepsilon}(\bar{t}) = \theta_i(t_i) + \mathrm{o}_\varepsilon(1) = \delta_i(1-t_i) + \mathrm{o}_{t_i}(1-t_i) + \mathrm{o}_\varepsilon(1), \tag{3.38}$$

where

$$\frac{\mathrm{o}_{t_i}(1-t_i)}{1-t_i} \to 0 \ \text{ as } t_1 \to 1 \quad \text{ and } \quad \mathrm{o}_\varepsilon(1) \to 0 \ \text{ as } \varepsilon \to 0, \text{ uniformly in } \bar{t}.$$

Clearly, this implies the desired conclusion (3.33).

As for the estimate (3.5), we observe that it follows from (3.29), (3.36) and (3.37) that

$$\varepsilon^{-N} J_\varepsilon(\overline{u}_\varepsilon, \overline{v}_\varepsilon) = \sum_i I_{V(x_i)}(t_{i,\varepsilon}(u_i,v_i) + (\Psi_{t_{i,\varepsilon}}, -\Psi_{t_{i,\varepsilon}})) + \mathrm{o}(1).$$

On the other hand, it follows from (3.38) that $t_{i,\varepsilon} \to 1$ (and hence $\Psi_{t_{i,\varepsilon}} \to 0$) as $\varepsilon \to 0$, and (3.5) follows. At last,

$$\int_{\Lambda_i} (\overline{u}_\varepsilon^2 + \overline{v}_\varepsilon^2) \geqslant t_{i,\varepsilon}^2\, \varepsilon^N \int_{\Lambda_i^\varepsilon} (u_i^2 + v_i^2) \geqslant \eta\varepsilon^N,$$

for some $\eta > 0$, and this concludes the proof of Proposition 3.2. □

4 Proof of Theorem 1.1

We use the same notations as in the previous section.We recall that $c_\varepsilon := \inf_{N_\varepsilon} J_\varepsilon$ and we denote

$$J_\varepsilon^i(u,v) := \int_{\widetilde{\Lambda}_i} \left(\varepsilon^2\langle\nabla u, \nabla v\rangle + V(x)uv - F(x,u) - G(x,v)\right).$$

Our main result in this section is the following.

Theorem 4.1. *For every small $\varepsilon > 0$ there exists $(u_\varepsilon, v_\varepsilon) \in N_\varepsilon$ such that*

$$J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon \qquad \text{and} \qquad J'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0.$$

Moreover,

$$J_\varepsilon^i(u_\varepsilon, v_\varepsilon) = \varepsilon^N(c_i + \mathrm{o}(1)) \quad \forall i \qquad \text{and} \qquad c_\varepsilon = \varepsilon^N\left(\sum_{i=1}^k c_i + \mathrm{o}(1)\right)$$

as $\varepsilon \to 0$.

Proof. 1. By the considerations at the beginning of Section 3, for any small $\varepsilon > 0$ there exists a constrained Palais-Smale sequence for the functional J_ε at the level c_ε, that is there exist sequences $(u_n, v_n) \in N_\varepsilon$, $\lambda_i^n \in \mathbb{R}$ $(i = 1, \dots, k)$ and $\psi_n \in H$ such that, for any $\zeta, \xi \in H$,

$$J'_\varepsilon(u_n, v_n)(\zeta, \xi) = J'_\varepsilon(u_n, v_n)(\Phi_n\zeta, \Phi_n\xi) + J''_\varepsilon(u_n, v_n)(\Phi_n u_n + \psi_n, \Phi_n v_n - \psi_n)(\zeta, \xi) + \mathrm{o}_n(1)$$

where $\mathrm{o}_n(1) \to 0$ uniformly for bounded ζ, ξ as $n \to \infty$, and $\Phi_n := \sum_i \lambda_i^n \phi_i$; moreover, $J_\varepsilon(u_n, v_n) \to c_\varepsilon$. This can be checked in [27] (or in e.g. [16, p. 207 & 219] if one works instead with the reduced functional mentioned just before Proposition 3.3). Let $\lambda_n := (\sum_i (\lambda_i^n)^2)^{1/2}$. We claim that

$$\lambda_n \to 0 \qquad \text{and} \qquad \psi_n \to 0. \tag{4.1}$$

Indeed, we let $\zeta = (\Phi_n u_n + \psi_n)/\lambda_n, \xi = (\Phi_n v_n - \psi_n)/\lambda_n$ so that $J'_\varepsilon(u_n, v_n)(\zeta, \xi) = 0$. Similarly to Proposition 3.7 we find that, for some $\eta > 0$,

$$\eta\lambda_n^2 + ||\phi_n||_\varepsilon^2 \leqslant \mathrm{o}_n(1)\,(|\lambda_n| + ||\phi_n||_\varepsilon)$$

with $\mathrm{o}_n(1) \to 0$ as $n \to \infty$. Then (4.1) follows. (In case $\lambda_n = 0$ by letting $(\zeta, \xi) = (\psi_n, -\psi_n)$ we immediately get that $\psi_n \to 0$.)

2. It follows from (4.1) that (u_n, v_n) is a (bounded) Palais-Smale sequence for J_ε, namely $J_\varepsilon(u_n, v_n) \to c_\varepsilon$ and $J'_\varepsilon(u_n, v_n) \to 0$ as $n \to \infty$. Up to a subsequence, let $(u_\varepsilon, v_\varepsilon)$ be a weak limit of the sequence $(u_n, v_n)_n$. Of course, $J'_\varepsilon(u_\varepsilon, v_\varepsilon) = 0$ and $(u_\varepsilon, v_\varepsilon) \in N_\varepsilon$. Moreover, since $2J_\varepsilon(u_n, v_n) = 2J_\varepsilon(u_n, v_n) - J'_\varepsilon(u_n, v_n)(u_n, v_n) + \mathrm{o}_n(1) = \int (f(x, u_n)u_n + g(x, v_n)v_n) + \mathrm{o}_n(1)$, we can use Fatou's lemma to deduce that $J_\varepsilon(u_\varepsilon, v_\varepsilon) \leqslant c_\varepsilon$, so that actually $J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon$.

3. Let ξ be a cut-off function in $\mathbb{R}^N$ such that $\xi = 0$ in Λ and $\xi = 1$ in $\Omega \setminus \Lambda'$. By testing $J'_\varepsilon(u_\varepsilon, v_\varepsilon)(\xi v_\varepsilon, \xi u_\varepsilon) = 0$ we see that

$$\int_{\Omega\setminus\Lambda'} (\varepsilon^2|\nabla u_\varepsilon|^2 + V(x)u_\varepsilon^2 + \varepsilon^2|\nabla v_\varepsilon|^2 + V(x)v_\varepsilon^2) = \mathrm{o}_\varepsilon(\varepsilon^N) \qquad \text{as } \varepsilon \to 0. \tag{4.2}$$

This, in turn, readily implies that

$$J_\varepsilon^i(u_\varepsilon, v_\varepsilon) = J_\varepsilon(\phi_i u_\varepsilon, \phi_i v_\varepsilon) + \mathrm{o}_\varepsilon(\varepsilon^N) \qquad \forall i \in \{1, \dots, k\}, \tag{4.3}$$

and that

$$J_\varepsilon(u_\varepsilon, v_\varepsilon) = \sum_{i=1}^k J_\varepsilon(\phi_i u_\varepsilon, \phi_i v_\varepsilon) + \mathrm{o}_\varepsilon(\varepsilon^N). \tag{4.4}$$

Also, $\langle(\phi_i - 1)u_\varepsilon, \phi_i v_\varepsilon\rangle_\varepsilon + \langle\phi_i u_\varepsilon, (\phi_i - 1)v_\varepsilon\rangle_\varepsilon = \mathrm{o}_\varepsilon(\varepsilon^N)$ and therefore

$$\begin{aligned} 2J_\varepsilon(\phi_i u_\varepsilon, \phi_i v_\varepsilon) &= \langle u_\varepsilon, \phi_i v_\varepsilon\rangle_\varepsilon + \langle v_\varepsilon, \phi_i u_\varepsilon\rangle_\varepsilon + \mathrm{o}_\varepsilon(\varepsilon^N) - 2\int(F(x, \phi_i u_\varepsilon) + G(x, \phi_i v_\varepsilon)) \\ &\geqslant \int_{\Lambda_i'}(f(u_\varepsilon)u_\varepsilon - 2F(u_\varepsilon) + g(v_\varepsilon)v_\varepsilon - 2G(v_\varepsilon)) + \mathrm{o}_\varepsilon(\varepsilon^N) \\ &\geqslant \delta'\int_{\Lambda_i'}(f(u_\varepsilon)u_\varepsilon + g(v_\varepsilon)v_\varepsilon) + \mathrm{o}_\varepsilon(\varepsilon^N) \\ &\geqslant \eta\varepsilon^N, \end{aligned} \tag{4.5}$$

as follows from (3.25).

4. Since $u_\varepsilon \to u$ in $L^2(\Lambda_i')$, we may assume that $|u_\varepsilon|$ remains bounded by a function in $L^2(\Lambda_i')$ as $\varepsilon \to 0$, and similarly for $|v_\varepsilon|$. So, by letting $u_i^\varepsilon(x) := u_\varepsilon(\varepsilon x + x_i)\phi_i(\varepsilon x + x_i)$, $v_i^\varepsilon(x) := v_\varepsilon(\varepsilon x + x_i)\phi_i(\varepsilon x + x_i)$, Lebesgue's dominated convergence theorem implies that

$$\int |V(x_i) - V(\varepsilon x + x_i)|((u_i^\varepsilon)^2 + (v_i^\varepsilon)^2) \to 0 \qquad \text{as } \varepsilon \to 0.$$

Therefore, thanks again to (4.2),

$$J_\varepsilon(\phi_i u_\varepsilon, \phi_i v_\varepsilon) = \varepsilon^N I_{V(x_i)}(u_i^\varepsilon, v_i^\varepsilon) + \mathrm{o}_\varepsilon(\varepsilon^N), \tag{4.6}$$

in particular (cf. (4.5)) $0 < \liminf_{\varepsilon\to 0} I_{V(x_i)}(u_i^\varepsilon, v_i^\varepsilon) \leqslant \limsup_{\varepsilon\to 0} I_{V(x_i)}(u_i^\varepsilon, v_i^\varepsilon) < \infty$, and also

$$\sup\{|I'_{V(x_i)}(u_i^\varepsilon, v_i^\varepsilon)(\phi, \psi)|,\ \phi, \psi \in H^1(\mathbb{R}^N),\ ||\phi|| + ||\psi|| \leqslant 1\} \to 0.$$

Then, by Proposition 2.5 and the Remark following it,

$$I_{V(x_i)}(u_i^\varepsilon, v_i^\varepsilon) \geqslant c_i + \mathrm{o}_\varepsilon(1), \qquad \text{as } \varepsilon \to 0.$$

Thus, by (4.6),

$$J_\varepsilon(\phi_i u_\varepsilon, \phi_i v_\varepsilon) \geqslant \varepsilon^N(c_i + \mathrm{o}_\varepsilon(1)). \tag{4.7}$$

5. At last, we observe that, thanks to (4.7), (4.4) and (3.7),

$$\varepsilon^N(\sum_i c_i + \mathrm{o}_\varepsilon(1)) \leqslant \sum_i J_\varepsilon(\phi_i u_\varepsilon, \phi_i v_\varepsilon) + \mathrm{o}_\varepsilon(\varepsilon^N) = J_\varepsilon(u_\varepsilon, v_\varepsilon) = c_\varepsilon \leqslant \varepsilon^N(\sum_i c_i + \mathrm{o}_\varepsilon(1)),$$

and so equality holds, that is,

$$\sum_i J_\varepsilon(\phi_i u_\varepsilon, \phi_i v_\varepsilon) = \varepsilon^N(\sum_i c_i + \mathrm{o}_\varepsilon(1)). \tag{4.8}$$

Combining (4.7) and (4.8) leads to the conclusion that, for every $i \in \{1, \ldots, k\}$,

$$J_\varepsilon(\phi_i u_\varepsilon, \phi_i v_\varepsilon) = \varepsilon^N(c_i + \mathrm{o}_\varepsilon(1)).$$

Taking (4.3) and (4.4) into account, this completes the proof of Theorem 4.1. □

From now on we consider the positive functions $u_\varepsilon > 0$, $v_\varepsilon > 0$ given by Theorem 4.1, which satisfy

$$-\varepsilon^2 \Delta u_\varepsilon + V(x) u_\varepsilon = g(x, v_\varepsilon), \quad -\varepsilon^2 \Delta v_\varepsilon + V(x) v_\varepsilon = f(x, u_\varepsilon), \quad u_\varepsilon, v_\varepsilon \in H^1(\mathbb{R}^N). \quad (4.9)$$

At this point we need to show that $(u_\varepsilon, v_\varepsilon)$ solves our original problem (1.1) and that the properties (i)-(iv) of Theorem 1.1 hold true. Most of this follows from standard arguments and therefore we will be sketchy.

Lemma 4.2. *The pair* $(u_\varepsilon, v_\varepsilon)$ *solves* (1.1) *and moreover:*

$$\lim_{\varepsilon\to 0} \sup_{\Omega\setminus\Lambda} \{u_\varepsilon, v_\varepsilon\} = 0 \quad \text{and} \quad \liminf_{\varepsilon\to 0} \min\{\sup_{\Lambda_i} u_\varepsilon, \sup_{\Lambda_i} v_\varepsilon\} > 0, \quad (4.10)$$

for every $i = 1, \dots, k$.

Proof. The crucial step in the argument consists in showing that for given points $z_\varepsilon \in \overline{\Lambda}_i$ the following holds:

$$\liminf_{\varepsilon\to 0} \min\{u_\varepsilon(z_\varepsilon), v_\varepsilon(z_\varepsilon)\} > 0 \quad \Rightarrow \quad \lim_{\varepsilon\to 0} V(z_\varepsilon) = \inf_{\Lambda_i} V. \quad (4.11)$$

Indeed, up to subsequences we have that $z_\varepsilon \to \overline{z} \in \overline{\Lambda}$ and that the functions $\overline{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x + z_\varepsilon)$, $\overline{v}_\varepsilon(x) := v_\varepsilon(\varepsilon x + z_\varepsilon)$ (which are bounded in $H^1(\mathbb{R}^N)$, thanks to Proposition 3.3) converge weakly in $H^1(\mathbb{R}^N)$ and in $C^2_{\text{loc}}(\mathbb{R}^N)$ to a nonzero solution $(\overline{u}, \overline{v})$ of the system

$$-\Delta \overline{u} + V(\overline{z})\overline{u} = g(\overline{v}), \quad -\Delta \overline{v} + V(\overline{z})\overline{v} = f(\overline{u}), \quad \overline{u}, \overline{v} \in H^1(\mathbb{R}^N) \quad (4.12)$$

(c.f. [13, Lemma 2.3]); moreover, simple arguments (as in e.g. [11, Lemma 2.2]) imply that $I_{V(\overline{z})}(\overline{u}, \overline{v}) \leqslant \liminf_{\varepsilon\to 0} \frac{1}{\varepsilon^N} J^i_\varepsilon(u_\varepsilon, v_\varepsilon)$, where $I_{V(\overline{z})}$ denotes the energy functional associated to (4.12). By denoting by $c_{V(\overline{z})}$ the corresponding ground-state critical level, we deduce from Theorem 4.1 that $c_{V(\overline{z})} \leqslant c_i$. Thus, as mentioned in the second Remark following Proposition 2.5, we must have that $V(\overline{z}) \leqslant \inf_{\Lambda_i} V$, whence $V(\overline{z}) = \inf_{\Lambda_i} V$, and this proves (4.11).

Now, it follows from (4.11) and our basic assumption $\inf_{\Lambda_i} V < \inf_{\partial\Lambda_i} V$ that $\sup_{\partial\Lambda_i} u_\varepsilon \to 0$ and $\sup_{\partial\Lambda_i} v_\varepsilon \to 0$, for every $i = 1, \dots, k$. Since, according to (4.9), $-\Delta(u_\varepsilon + v_\varepsilon) \leqslant 0$ over $\Omega \setminus \Lambda$, the first conclusion in (4.10) follows from the maximum principle. By recalling that $f(x, u) = f(u)$ if either $x \in \Lambda$ or $x \in \Omega \setminus \Lambda$ and u is small, and similarly for $g(x, v)$, we have that $(u_\varepsilon, v_\varepsilon)$ solves (1.1).

As for the second conclusion in (4.10), suppose that, say, $\sup_{\Lambda_i} u_\varepsilon \to 0$ for some $i \in \{1, \ldots, k\}$ and let $u^\varepsilon(x) := u(\varepsilon x)$, $v^\varepsilon(x) := v(\varepsilon x)$. Then we have that $\sup_{y \in \Lambda_i/\varepsilon} \int_{B_S(y)} (u^\varepsilon)^2 \to 0$ for every $S > 0$ and therefore $\int_{N_R(\Lambda_i/\varepsilon)} (u^\varepsilon)^p \to 0$ as $\varepsilon \to 0$, according to P.L. Lions's concentration lemma (see e.g. [29, Th. 1.34]), for any neighborhood $N_R(\Lambda_i/\varepsilon) := \{x \in \mathbb{R}^N : \text{dist}(x, \Lambda_i/\varepsilon) < R\}$. It then follows as in (4.2) that $\int_{\Lambda_i/\varepsilon}(|\nabla u^\varepsilon|^2 + V(\varepsilon x)(u^\varepsilon)^2 + |\nabla v^\varepsilon|^2 + V(\varepsilon x)(v^\varepsilon)^2) \leqslant \mathrm{o}_R(1)$ as $\varepsilon \to 0$, where $\mathrm{o}_R(1)$ is small for large values of R. This contradicts the fact that, according to Proposition 3.3, $\liminf_{\varepsilon \to 0} \int_{\Lambda_i/\varepsilon}((u^\varepsilon)^2 + (v^\varepsilon)^2) > 0$. □

Our next result concludes the proof of Theorem 1.1 (see also Section 5).

Proposition 4.3. *The functions u_ε, v_ε satisfy the properties (i)-(iv) stated in Theorem 1.1.*

Proof. (sketch) It follows from our previous lemma that for every $i = 1, \ldots, k$ there exist $x_{i,\varepsilon}, y_{i,\varepsilon} \in \Lambda_i$ such that $u(x_{i,\varepsilon}) = \max_{\Lambda_i} u_\varepsilon$ and $v(y_{i,\varepsilon}) = \max_{\Lambda_i} v_\varepsilon$. Moreover (cf. (4.10) –(4.11)) $u(x_{i,\varepsilon}), v(y_{i,\varepsilon}) \geqslant b > 0$ and $\lim V(x_{i,\varepsilon}) = \lim V(y_{i,\varepsilon}) = \inf_{\Lambda_i} V$ as $\varepsilon \to 0$. We claim that $x_{i,\varepsilon} = y_{i,\varepsilon}$ if ε is sufficiently small. Indeed, we let $\overline{u}_\varepsilon(x) := u_\varepsilon(\varepsilon x + x_{i,\varepsilon})$ and $\overline{v}_\varepsilon(x) := v_\varepsilon(\varepsilon x + x_{i,\varepsilon})$, to arrive at a limit problem as in (4.12) with $V(\overline{z}) = \inf_{\Lambda_i} V$. We know from [4] that $\overline{u}$ and $\overline{v}$ are radially symmetric with respect to the origin and that they are decreasing functions. Let $\xi_{i,\varepsilon} := (x_{i,\varepsilon} - y_{i,\varepsilon})/\varepsilon$. In case $|\xi_{i,\varepsilon}| \to \infty$ it follows as in [11, p. 13] and from the estimates in Theorem 4.1 that $2c_i \leqslant c_i$, a contradiction. Thus $(\xi_{i,\varepsilon})$ is bounded and since $\xi_{i,\varepsilon}$ is a local maximum of $\overline{v}_\varepsilon$ it follows from our previous observation that $\xi_{i,\varepsilon} \to 0$. By applying then the argument in [25, p. 3276], which is based on the fact that either $\overline{u}''(0) \neq 0$ or $\overline{v}''(0) \neq 0$, we conclude that $\xi_{i,\varepsilon} = 0$ for small values of ε, thus proving the claim.

Similarly, if $z_{i,\varepsilon} \in \Lambda_i$ is a given local maximum point of, say, u_ε such that $0 < \limsup_{\varepsilon \to 0} u_\varepsilon(z_{i,\varepsilon})$ then the preceding argument yields that $z_{i,\varepsilon} = y_{i,\varepsilon}$ $(= x_{i,\varepsilon})$ for small values of ε and this, together with the first statement in (4.10), establishes property (iv) of Theorem 1.1

As for property (iii) and keeping the same notations as above, we may assume that b is so small that $c := \inf_\Omega V - f(b)/b - g(b)/b > 0$. Since the limit functions $\overline{u}$ and $\overline{v}$ are radially symmetric and thanks also to the first statement in (4.10), we can choose $R > 0$ so large and ε so small that $\overline{u}_\varepsilon(x) + \overline{v}_\varepsilon(x) \leqslant b \; \forall x \in \partial\omega$, where $\omega := \mathbb{R}^N \setminus (\cup_{j \neq i}(\frac{\Lambda_j - x_{i,\varepsilon}}{\varepsilon}) \cup B_R(0))$. The uniqueness of the maximum points implies that in fact $\overline{u}_\varepsilon(x) + \overline{v}_\varepsilon(x) \leqslant b \; \forall x \in \omega$. As a consequence, $\frac{f(\overline{u}_\varepsilon)}{\overline{u}_\varepsilon+\overline{v}_\varepsilon}+\frac{g(\overline{v}_\varepsilon)}{\overline{u}_\varepsilon+\overline{v}_\varepsilon}\leqslant\frac{f(\overline{u}_\varepsilon)}{\overline{u}_\varepsilon}+\frac{g(\overline{v}_\varepsilon)}{\overline{v}_\varepsilon}\leqslant\frac{f(b)}{b}+\frac{g(b)}{b}<\inf_\Omega V$ in ω and so $-\Delta(\overline{u}_\varepsilon+\overline{v}_\varepsilon)+c(\overline{u}_\varepsilon+\overline{v}_\varepsilon)\leqslant 0$ in ω. The conclusion also holds for a slightly smaller $\Lambda_j''\Subset\Lambda_j$ and $\omega'':=\mathbb{R}^N\setminus(\cup_{j\neq i}(\frac{\Lambda_j''-x_{i,\varepsilon}}{\varepsilon})\cup B_R(0))$; we fix Λ_j'', small numbers $0<\delta'<\delta<\sqrt{c}$ and a finite number of points $y_1,\dots,y_{\ell_0}$ in such a way that $\partial(\cup_{j\neq i}\Lambda_j'')\subset\cup_{\ell=1}^{\ell_0}(B_\delta(y_\ell)\setminus B_{\delta'}(y_\ell))\subset\cup_{j\neq i}\Lambda_j$ and $|x-y_\ell|>2\delta\ \forall\ell\ \forall x\in\Omega\setminus\cup_{j\neq i}\Lambda_j$. Let $w\in H^1(\mathbb{R}^N\setminus B_1(0))$ be such that $-\Delta w+\delta^2 w=0$ and $a_1e^{-2\delta|x|}\leqslant w(x)\leqslant a_2e^{-\delta|x|}\ \forall|x|\geqslant 1$, for some $a_1,a_2>0$. It follows from the maximum principle that $\overline{u}_\varepsilon(x)+\overline{v}_\varepsilon(x)\leqslant\lambda_0 w(x)+\sum_{\ell=1}^{\ell_0}\lambda_\ell w(x-\frac{y_\ell-x_{i,\varepsilon}}{\varepsilon})$ in ω'', for some $\lambda_0>0$ and some $0<\lambda_\ell\leqslant Ce^{2\delta^2/\varepsilon}$. Hence $u_\varepsilon(x)+v_\varepsilon(x)\leqslant C(e^{-\frac{|x-x_{i,\varepsilon}|}{\varepsilon}}+\sum_{\ell=1}^{\ell_0}e^{\frac{2\delta^2-\delta|x-y_\ell|}{\varepsilon}})$ over $\Omega\setminus\cup_{j\neq i}\Lambda_j''$. Since $\delta|x-y_\ell|-2\delta^2\geqslant\mu|x-x_{i,\varepsilon}|$ for every $x\in\Omega\setminus\cup_{j\neq i}\Lambda_j$ and a small $\mu>0$, (iii) follows. □

5 The case $p\neq q$

In Section 4 we have proved Theorem 1.1 except that we have worked with a truncated problem, as explained at the end of Section 1. The full statement of Theorem 1.1 will be established once we prove uniform bounds in $L^\infty(\Omega)$ of the solutions constructed so far. So, let us suppose that $p,q>2$ are such that $1/p+1/q>(N-2)/N$ with, say, $2<p<2^*=2N/(N-2)$ and $p<q$.

Given $n\in\mathbb{N}$, we consider the functions f_n and g_n already defined in (1.3). Then, for a fixed n, thanks to Theorem 1.1 there exists $\varepsilon_{0,n}>0$ such that for $0<\varepsilon<\varepsilon_{0,n}$ there are positive solutions $u_\varepsilon,v_\varepsilon$ of the problem

$$-\varepsilon^2\Delta u+V(x)u=g_n(v),\quad -\varepsilon^2\Delta v+V(x)v=f_n(u)\ \text{ in }\Omega,\qquad u,v\in H_0^1(\Omega),\tag{5.1}$$

satisfying the conclusions of that theorem; at this point, all the quantities appearing in the theorem depend of n. Moreover, by Theorem 4.1 we have

$$I_\varepsilon^n(u_\varepsilon,v_\varepsilon)=\varepsilon^N\left(\sum_{i=1}^k c_{i,n}+o_n(1)\right)\qquad\text{as }\varepsilon\to 0,$$

where $I_\varepsilon^n(u_\varepsilon,v_\varepsilon)=\int\{\varepsilon^2\langle\nabla u,\nabla v\rangle+V(x)uv-F_n(u)-G_n(v)\}$ with obvious notations, and $c_{i,n}$ is the ground-state critical level of

$$-\Delta u+V(x_i)u=g_n(v),\quad -\Delta v+V(x_i)v=f_n(u)\qquad\text{in }\mathbb{R}^N.\tag{5.2}$$

Therefore, given n we can consider $\varepsilon_{0,n}>0$ small enough such that for $0<\varepsilon<\varepsilon_{0,n}$ we have $I_\varepsilon^n(u_\varepsilon,v_\varepsilon)\leqslant 2\varepsilon^N\sum_{i=1}^k c_{i,n}$.

Lemma 5.1. *For every $n \in \mathbb{N}$ there exists $\varepsilon_{0,n} > 0$ such that, for every $0 < \varepsilon < \varepsilon_{0,n}$ we have*

$$I_\varepsilon^n(u_\varepsilon, v_\varepsilon) \leqslant C\varepsilon^N,$$

for some $C > 0$ independent of n and ε. In particular, also

$$\int (f_n(u_\varepsilon)u_\varepsilon + g_n(v_\varepsilon)v_\varepsilon) \leqslant C\varepsilon^N. \tag{5.3}$$

Proof. We only have to prove that $c_{i,n} \leqslant C$, with $C > 0$ independent of n, for any fixed $i = 1, \ldots, k$. We recall that from our assumptions on f we have that $f'(s) \geqslant f(s)/s \geqslant f(1)s^{\delta'}$ for some $0 < \delta' < p - 2$. We set

$$h_f(s) := \begin{cases} f(s) & , \ s \leqslant 1 \\ \frac{f'(1)}{1+\delta'} s^{1+\delta'} + f(1) - \frac{f'(1)}{1+\delta'} & , \ s > 1. \end{cases}$$

Then, for a small $\lambda > 0$ (namely, $\lambda < (1 + \delta')f(1)/f'(1)$) it follows easily that $\lambda h_f' \leqslant f_n'$, thus also $\lambda h_f \leqslant f_n$. We proceed in a similar way with the function g, yielding some function h_g such that $\lambda h_g \leqslant g_n$. Then, according to the second Remark following Proposition 2.5, we conclude that $c_{i,n} \leqslant c_{\lambda h_f, \lambda h_g}$, where the latter quantity refers to the ground-state critical level associated to the problem

$$-\Delta u + V(x_i)u = \lambda h_g(v), \qquad -\Delta v + V(x_i)v = \lambda h_f(u), \qquad u, v \in H^1(\mathbb{R}^N).$$

The final conclusion follows from the fact that the left-hand side of (5.3) is bounded by $\frac{2(2+\delta')}{\delta'} I_\varepsilon^n(u_\varepsilon, v_\varepsilon)$, according to our assumption (fg3). □

We denote by $x_{i,\varepsilon}$ the maximum points of u_ε and v_ε over Λ_i, as mentioned in Theorem 1.1.

Lemma 5.2. *Given $\rho > 0$, $i \in \{1, \ldots, k\}$ and $n \in \mathbb{N}$, there exists $\varepsilon_{0,n}$ such that for $0 < \varepsilon < \varepsilon_{0,n}$ we have $u_\varepsilon(x), v_\varepsilon(x) \leqslant 1$, for all $x \in \Omega \setminus \cup_{j \neq i} \Lambda_j$ such that $|x - x_{i,\varepsilon}| \geqslant \rho$.*

Proof. According to Theorem 1.1 we have $u_\varepsilon(x), v_\varepsilon(x) \leqslant \gamma_n e^{-\frac{\beta_n}{\varepsilon}|x - x_{i,\varepsilon}|}$, for all $x \in \Omega \setminus \cup_{j \neq i} \Lambda_j$. Then we just have to choose $\varepsilon_{0,n} \leqslant \rho\beta_n / \log \gamma_n$. □

Taking the previous lemma into account, we are left to the analysis of the behavior of $u_\varepsilon(x), v_\varepsilon(x)$ over small neighborhoods of the points $x_{i,\varepsilon}$. To that purpose, we introduce a cutoff function ϕ_i such that $\phi_i = 1$ in $B_{2\rho}(x_{i,\varepsilon})$, $\phi_i = 0$ in $\mathbb{R}^N \setminus B_{3\rho}(x_{i,\varepsilon})$, and denote $\phi_{i,\varepsilon}(x) = \phi_i(\varepsilon x + x_{i,\varepsilon})$. We also consider the functions

$$\bar{u}_\varepsilon(x) = u_\varepsilon(\varepsilon x + x_{i,\varepsilon}), \qquad \bar{v}_\varepsilon(x) = v_\varepsilon(\varepsilon x + x_{i,\varepsilon}),$$

which satisfy, in the whole space $\mathbb{R}^N$,

$$\begin{cases} -\Delta(\bar{u}_\varepsilon \phi_{i,\varepsilon}) &=& -V(\varepsilon x + x_{i,\varepsilon})\bar{u}_\varepsilon \phi_{i,\varepsilon} + g_n(\bar{v}_\varepsilon)\phi_{i,\varepsilon} - \bar{u}_\varepsilon \Delta\phi_{i,\varepsilon} - 2\langle \nabla \bar{u}_\varepsilon, \nabla \phi_{i,\varepsilon}\rangle \\ -\Delta(\bar{v}_\varepsilon \phi_{i,\varepsilon}) &=& -V(\varepsilon x + x_{i,\varepsilon})\bar{v}_\varepsilon \phi_{i,\varepsilon} + f_n(\bar{u}_\varepsilon)\phi_{i,\varepsilon} - \bar{v}_\varepsilon \Delta\phi_{i,\varepsilon} - 2\langle \nabla \bar{v}_\varepsilon, \nabla \phi_{i,\varepsilon}\rangle \end{cases} \tag{5.4}$$

We now use the same variational setting as in [26]. We define s, t such that $s + t = 2$, $s, t < \frac{N}{2}$ and $p < \frac{2N}{N-2s}$, $q < \frac{2N}{N-2t}$ (see [26, p. 1453]). This implies the following continuous injections $H^s(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$, $H^t(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$. The linear map $A^s : H^s \to L^2$ will denote the canonical isomorphism $A^s(u) := ((1 + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)|)^\vee$, where $\hat{}$ is the Fourier transform and $^\vee$ is its inverse. To be precise, in our next lemma we take instead A^s given by $A^s(u) := (\alpha + |\xi|^2)^{\frac{s}{2}} |\hat{u}(\xi)|)^\vee$, $\alpha := \inf_\Omega V$.

Lemma 5.3. *Given $n \in \mathbb{N}$, there exists $\varepsilon_{0,n}$ such that for $0 < \varepsilon < \varepsilon_{0,n}$ we have*

$$\|\bar{u}_\varepsilon \phi_{i,\varepsilon}\|_{H^s} + \|\bar{v}_\varepsilon \phi_{i,\varepsilon}\|_{H^t} \leqslant C, \qquad \text{with } C > 0 \text{ independent of } n \text{ and } \varepsilon.$$

Proof. We know from (5.3) that $\int (f_n(\bar{u}_\varepsilon)\bar{u}_\varepsilon + g_n(\bar{u}_\varepsilon)\bar{u}_\varepsilon) \leqslant C$. Also, for every $s \geqslant 1$, $|f_n(s)| \leqslant C|s|^{p-1}$ and $|g_n(s)| \leqslant C|s|^{q-1}$ with $1/p + 1/q > (N-2)/N$. Therefore our argument is quite similar to the one in [26, p. 1457] and so we only stress the differences. We first add on both sides of the first equation in (5.4) the term $V(x_{i,\varepsilon})\bar{u}_\varepsilon \phi_{i,\varepsilon}$ and then use the test-functions $A^{-t}A^s(\bar{u}_\varepsilon \phi_{i,\varepsilon})$. With respect to the computations in [26], we now have addicional terms

$$\int (V(x_{i,\varepsilon}) - V(\varepsilon x + x_{i,\varepsilon}))\bar{u}_\varepsilon \phi_{i,\varepsilon} A^{-t}A^s(\bar{u}_\varepsilon \phi_{i,\varepsilon}) - \int \bar{u}_\varepsilon A^{-t}A^s(\bar{u}_\varepsilon \phi_{i,\varepsilon})\Delta\phi_{i,\varepsilon} - 2\int \langle \nabla \bar{u}_\varepsilon, \nabla \phi_{i,\varepsilon}\rangle A^{-t}A^s(\bar{u}_\varepsilon \phi_{i,\varepsilon}).$$

Since V is α-Hölder continuous over $B_{3\rho}(x_{i,\varepsilon})$ (for some $\alpha > 0$), we see that

$$\int (V(x_{i,\varepsilon}) - V(\varepsilon x + x_{i,\varepsilon}))\bar{u}_\varepsilon \phi_{i,\varepsilon} A^{-t}A^s(\bar{u}_\varepsilon \phi_{i,\varepsilon}) \leqslant \rho^\alpha C \|\bar{u}_\varepsilon \phi_{i,\varepsilon}\|^2_{H^s}.$$

Also, for some positive constant C_n depending on n but not on ε,

$$-\int \bar{u}_\varepsilon A^{-t}A^s(\bar{u}_\varepsilon \phi_{i,\varepsilon})\Delta\phi_{i,\varepsilon} - 2\int \langle \nabla \bar{u}_\varepsilon, \nabla \phi_{i,\varepsilon}\rangle A^{-t}A^s(\bar{u}_\varepsilon \phi_{i,\varepsilon}) \leqslant \varepsilon\, C_n \, \|\bar{u}_\varepsilon \phi_{i,\varepsilon}\|_{H^s}.$$

Therefore, proceeding similarly with the second equation in (5.4) and by choosing $\varepsilon_{0,n}$ small enough, it follows as in [26] that

$$\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|^2_{H^s} + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|^2_{H^t} \leqslant \rho^\alpha C(\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|^2_{H^s} + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|^2_{H^t}) + C(\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^s} + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^t}).$$

So, provided $\rho > 0$ is chosen sufficiently small, the conclusion follows. □

Lemma 5.4. *Given $n \in \mathbb{N}$, there exists $\varepsilon_{0,n}$ such that for $0 < \varepsilon < \varepsilon_{0,n}$ we have*

$$\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \leqslant C, \qquad \text{with } C > 0 \text{ independent of } n \text{ and } \varepsilon.$$

Proof. Thanks to Lemma 5.3, we can bootstrap similarly to [26, p. 1450 & 1451]. After a finite number of steps and by taking if necessary a smaller $\varepsilon_{0,n}$, we conclude that $\|\bar{u}_\varepsilon\phi_{i,\varepsilon}\|_{H^{N/2}} + \|\bar{v}_\varepsilon\phi_{i,\varepsilon}\|_{H^{N/2}} \leqslant C$. The conclusion follows from the imbedding $H^{\frac{N}{2}}(\mathbb{R}^N) \hookrightarrow L^\infty(\mathbb{R}^N)$. □

Our final result completes the proof of Theorem 1.1 in its full generality.

Proposition 5.5. *There exist $n_0 \in \mathbb{N}$ and $\varepsilon_0 > 0$ such that for every $0 < \varepsilon < \varepsilon_0$ our solutions $u_\varepsilon, v_\epsilon$ of problem (5.1) with $n = n_0$ satisfy all the assertions of Theorem 1.1.*

Proof. According to Lemma 5.4 we may choose $n_0 \in \mathbb{N}$ large enough so that $\|u_\varepsilon\|_\infty + \|v_\varepsilon\|_\infty \leqslant n_0$ for every $0 < \varepsilon < \varepsilon_{0,n_0}$ and in this way we solve the original problem (1.1). The conclusion follows then from Proposition 4.3. □

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