Slip boundary effects on unsteady flows of incompressible viscous heat conducting fluids with a non-linear internal energy-temperature relationship

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We consider unsteady flows of a homogeneous incompressible fluid-like material with the viscosity depending on the temperature and on the shear-rate, and the heat conductivity being a function of the temperature and its gradient. Restricting to the internal flows and assuming Navier's slip at the tangential directions on the boundary, we establish a long-time and large-data existence of suitable weak solutions to the relevant models. A combination of L^∞ truncation method applied to establish the compactness of the velocity gradient and the Lipschitz truncation method applied to establish to obtain compactness of the temperature gradient leads to the existence results valid for range of parameters interesting from the point of view of applications.

Keywords: incompressible fluid, thermodynamics, suitable weak solution, existence

AMS Subject Classification: 35Q, 76N

1. Introduction

We consider a homogeneous incompressible fluid-like material with the nonconstant material moduli: the viscosity depends on the temperature and on the shear-rate and

*Miroslav Bulíček thanks the Jindřich Nečas center for Mathematical Modeling, project LC06052 financed by the Ministry of Education, Youth and Sport, for its support.
†This research was performed during the stay of Luisa Consiglieri at the Jindřich Nečas Center for Mathematical Modeling, project LC06052 at the Charles University in Prague, Czech Republic.
‡Josef Málek has been supported by the project MSM 0021620839 financed by MSMT and the project GACR 201/05/0164.

the heat conductivity coefficient is a function of the temperature and its gradient. Owing to the dependence of the viscosity on the shear-rate, the considered material has the ability to capture shear-thinning or shear-thickening phenomena exhibited by many non-Newtonian fluids. We refer to Málek and Rajagopal [2005] for a recent description of features that cannot be exhibited by a Navier-Stokes (i.e. Newtonian) fluid.

Dealing with a homogeneous incompressible fluid its motion, in our setting, will be captured by $(\mathbf{v}, \pi, e)$, $\mathbf{v}$ being the velocity, π is the mean normal stress (the pressure) and e is the internal energy.

We also assume that there is a one to one (possibly nonlinear) relationship between the temperature θ and the internal energy e. As a consequence, the heat conductivity coefficient $\tilde{k}$ is a function of e and $\nabla_x e$, and the viscosity $\tilde{\nu}$ depends on e and the shear rate that is usually generalized in a full three-dimensional setting, to the quantity $|D\mathbf{v}|^2 = D\mathbf{v} : D\mathbf{v}$ where $D\mathbf{v}$ is the symmetric part of the velocity gradient $\nabla_x \mathbf{v}$.

We are interested in mathematical analysis of the relevant model expressed as the system of nonlinear partial differential equations (that describe the balance of mass[a], the balance of momentum and the balance of energy. We require that the model is thermomechanically consistent, i.e., the second law of thermodynamics expressed in terms of the Clusius-Duhen inequality is met.

We restrict ourselves to internal flows (no flux of linear momentum through the boundary is allowed). A special attention is however devoted to the slip effects on the boundary. Regarding the temperature, we permit all possibilities that gamut the nonhomogeneous Dirichlet and Neumann conditions. These types of boundary conditions and the structural assumptions on the constitutive quantities specified by certain growth, coerciveness and monotone-type conditions include various model parameters.

Our aim in this paper is to establish a long-time and large-data existence result for the largest range of model parameters. We clarify the precise meaning of the word "largest" later after formulating our main result. To achieve this aim we incorporate in our studies the following approaches and tools:

• We deal with the notion of weak solutions as it seems to be a very natural concept of solution for the equations of continuum thermodynamics, as their relevant weak formulation is "equivalent", at least for incompressible fluids, to the original formulation of the balance equations over arbitrary (measurable) control volumes. We may refer to Oseen [1927], Leray [1934] or Feireisl [2004] and many other studies for details. There are natural bounds on certain quantities: the total energy is bounded uniformly w.r.t. time, it means e and $|\mathbf{v}|^2/2$ belong to $L^\infty(0,T;L^1)$, and all dissipative quantities are at least L^1-integrable.

• We prefer to work with the equation for the energy $(e + |\mathbf{v}|^2/2)$ rather than

[a]Since the considered fluid is homogeneous, the density is constant at any spatial point x and any time instant; the balance of mass thus simplifies to $\operatorname{div} \mathbf{v} = 0$.

to use alternative formulations as the equation for the internal energy (or the equation for the temperature) since these alternative formulations are equivalent to the equation for the total energy only if the velocity field $\mathbf{v}$ is smooth (More precisely, only if $\mathbf{v}$ is an admissible test function in the weak formulation of the balance of linear momentum).

The idea to use the equation for $(e + |\mathbf{v}|^2/2)$ (rather than the equation for e or θ) in the existence theory has been put in the place firstly in Feireisl and Málek [2006] and has been successfully incorporated in further recent studies Bulíček et al. [2007a] and Bulíček et al. [2007b]. The advantage of this approach consists in dealing with the quantity $\operatorname{div}(\tau\mathbf{v})$, $\mathbf{T} := -\pi\mathbf{I} + \tau$ being the Cauchy stress, instead of $\tau \cdot \nabla\mathbf{v}$. While $\tau \cdot \nabla\mathbf{v}$ is in general only L^1-integrable quantity, $\tau\mathbf{v}$ is not only L^q-integrable quantity with $q > 1$, but it is also weakly compact.

• We consider the Navier's slip on the boundary as for this type of boundary conditions we know how to introduce the pressure globally as integrable function, at least for $\mathcal{C}^{1,1}$ boundary. We are not able to treat the Dirichlet (no-slip) boundary conditions at this moment.

• We incorporate L^∞-truncation method that goes back to work Boccardo and Murat [1992] in order to establish almost everywhere convergence of the velocity gradient.

• We incorporate the properties of Lipschitz approximations of Sobolev functions in order to establish almost everywhere convergence of the temperature gradient.

Let Ω be a bounded open subset of $\mathbb{R}^n (n \in \mathbb{N})$, $\Omega \in C^{1,1}$, with boundary $\partial\Omega$, $T > 0$ and $Q = \Omega \times]0, T[$. Thermal incompressible viscous flows are governed by the following system of partial differential equations

$$\operatorname{div}\mathbf{v} = \sum_{i=1}^{n} \frac{\partial v_i}{\partial x_i} = 0 \text{ in } Q; \tag{1.1}$$

$$\partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div}\tau(\cdot, e, D\mathbf{v}) = -\nabla\pi + \mathbf{f} \text{ in } Q; \tag{1.2}$$

$$\partial_t \left(\frac{1}{2}|\mathbf{v}|^2 + e\right) - \operatorname{div}(\mathbf{q}(\cdot, e, \nabla e) + \tau(\cdot, e, D\mathbf{v})\mathbf{v}) = \\ = \left(\mathbf{f} - \nabla(\frac{1}{2}|\mathbf{v}|^2 + e + \pi)\right) \cdot \mathbf{v} \text{ in } Q; \tag{1.3}$$

$$\partial_t e + \operatorname{div}(\mathbf{v}e) - \operatorname{div}\mathbf{q}(\cdot, e, \nabla e) \geq \tau(\cdot, e, D\mathbf{v}) : D\mathbf{v} \text{ in } Q, \tag{1.4}$$

where $\mathbf{v}$ is the velocity vector, π denotes the pressure, $\mathbf{f}$ denotes the external forces, and $\tau = (\tau_{ij})$ denotes the viscous part of the Cauchy stress tensor $\mathbf{T}$. The density is constant and it is assumed equal to 1. The internal energy e is a nonlinear invertible function of the temperature θ through the specific heat c_p

$$e = \int^{\theta} c_p(s)ds \Leftrightarrow \theta = \theta(e),$$

the heat flux $(-\mathbf{q})$ is given by a general law, and the external source is only constituted by the dissipative term.

The boundary $\partial\Omega$ is such that $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}$ where Γ_D, Γ are open subsets of $\partial\Omega$ with smooth boundaries and such that $\Gamma_D \cap \Gamma = \emptyset$, and $\mathrm{meas}(\Gamma_D) > 0$. Hencefurther, let us assume the Dirichlet condition

$$e = e_r \quad \text{on } \Gamma_D \times]0,T[; \tag{1.5}$$

for the sake of simplicity $e_r = 0$, and the fluid-boundary interactions Hadrian and Panagiotopoulos [1978]

$$\mathbf{v}_N = 0, \quad \begin{cases} \tau_T + \alpha(\cdot, e)\mathbf{v}_T = 0, & \text{on } \Sigma_D := \Gamma_D \times]0,T[; \\ \tau_T + \varphi(\cdot, e)|\mathbf{v}_T|^{s-2}\mathbf{v}_T = 0, & \text{on } \Sigma := \Gamma \times]0,T[; \end{cases} \tag{1.6}$$

where $\mathbf{v}_N, \mathbf{v}_T$ are the normal and the tangential components of the velocity vector, respectively, $\tau_T = \tau \cdot \mathbf{n} - \tau_N \mathbf{n}$ is the tangential component of τ, which coincides with the tangential stress tensor σ_T, and φ denotes the friction coefficient. Here $\mathbf{n} = (n_i)$ denotes the unit outward normal to $\partial\Omega$. Finally, the radiation heat transfer involving the frictional work Hutter and Rajagopal [1994]

$$\mathbf{q}(\cdot, e, \nabla e) \cdot \mathbf{n} + \gamma(\cdot, e) = \varphi(\cdot, e)|\mathbf{v}_T|^s, \tag{1.7}$$

where the energy dependent function γ represents a general convective effect.

2. Assumptions and main results

Let us define the following Banach spaces, for $p, q > 1$ and $0 \leq k \leq 1$,

$$\begin{aligned} W_N^{k,p} &= \{\mathbf{v} \in \mathbf{W}^{k,p}(\Omega) : \ \mathbf{v}_N = \mathbf{0} \text{ on } \partial\Omega\} \\ W_{N,\mathrm{div}}^{k,p} &= \{\mathbf{v} \in W_N^{k,p} : \ \mathrm{div}\, \mathbf{v} = 0 \text{ in } \Omega\} \\ W_q &= \{e \in W^{1,q}(\Omega) : \quad e = 0 \text{ on } \Gamma_D\}; \end{aligned}$$

endowed with the norms

$$\|\cdot\|_{1,p,\Omega} = \|D\cdot\|_{p,\Omega} + \|\cdot\|_{2,\Gamma_D} \qquad \|\cdot\|_{W_q} = \|\nabla\cdot\|_{q,\Omega}$$

with $\|\cdot\|_{k,p}$ denoting the canonical norms in scalar space $W^{k,p}$ as well as in the vector space $\mathbf{W}^{k,p}$. The Sobolev space $W^{0,p}$ means the Lebesgue space L^p. We will denote by $\|\cdot\|_{p;X}$ the usual norm in the Bochner space $L^p(0,T;X)$ with X a Banach space.

Before we give the precise formulation of that what we mean by weak solution and before we give the main existence theorem we prove two technical lemmas that will be frequently use in what follows.

Let us begin by proving a trace lemma.

Lemma 2.1. *Let $r, c \in [1, \infty[$. Let $v \in L^r(0,T;W^{1,r}(\Omega)) \cap L^\infty(0,T;L^c(\Omega))$. Then there exists a constant $C := C(\|v\|_{r;W^{1,r}}, \|v\|_{\infty;L^c})$ such that*

$$\|v\|_{L^a(0,T;L^b(\partial\Omega))} \leq C \tag{2.1}$$

for all a, b *satisfying the following relation*

$$a \le b\frac{nr+rc-nc}{bn+c-cn}, \qquad \begin{cases} r \le n,\ r < c \le b \le \frac{(n-1)r}{n-r}, \\ r \le n,\ c \le r \le b \le \frac{(n-1)r}{n-r}, \\ r > n,\ r < c \le b \le \infty, \end{cases} \tag{2.2}$$

$$a \le \frac{br^3}{nbc+rc-nrc-brc+br^2} \qquad r > n,\ \ c \le r \le b \le \infty.$$

Proof. The key observation that is used in the proof is that there exists continuous trace operator tr such that

$$tr : W^{1/b,b}(\Omega) \to L^b(\partial\Omega) \tag{2.3}$$

for all $b \in]1, \infty)$. For proof see Triebel [1983][b].

We split the proof into two parts. First one for $r \le n$ and second one for $r > n$. **Part 1** ($r \le n$): Here we restrict ourselves only to the case when $c < nr/(n-r)$. We use the following embeddings

$$W^{1,r} \hookrightarrow W^{l,c} \qquad l = 1 + \frac{n}{c} - \frac{n}{r} \qquad c > r, \tag{2.4}$$

$$W^{m,c} \hookrightarrow W^{\frac{1}{b},b} \qquad m = \frac{n}{c} + \frac{1}{b} - \frac{n}{b} \qquad c \le b \le \frac{(n-1)r}{n-r}, \tag{2.5}$$

$$W^{k,r} \hookrightarrow W^{\frac{1}{b},b} \qquad k = \frac{n}{r} + \frac{1}{b} - \frac{n}{b} \qquad r \le b \le \frac{(n-1)r}{n-r}, \tag{2.6}$$

and the standard interpolation inequalities

$$\|v\|_r \le \|v\|_c^{\alpha} \|v\|_{1,r}^{1-\alpha} \qquad \alpha = \frac{rc}{nr+rc-nc}, \tag{2.7}$$

$$\|v\|_{j,r} \le \|v\|_{q,r}^{\frac{j}{q}} \|v\|_r^{1-\frac{j}{q}} \qquad 0 \le j \le q. \tag{2.8}$$

First note that due to (2.3) it is enough to estimate $\int_0^T \|v\|_{\frac{1}{b},b}^a$. Thus,

$$\int_0^T \|v\|_{\frac{1}{b},b}^a \overset{(2.5)}{\underset{(2.6)}{\le}} \begin{cases} \displaystyle\int_0^T \|v\|_{m,c}^a \overset{(2.4),\,(2.8)}{\le} \int_0^T \|v\|_{1,r}^{\frac{am}{l}} \|v\|_c^{a(1-\frac{m}{l})}, & (c > r) \\ \displaystyle\int_0^T \|v\|_{k,r}^a \overset{(2.8)}{\le} \int_0^T \|v\|_{1,r}^{ak} \|v\|_r^{a(1-k)} \overset{(2.7)}{\le} \int_0^T \|v\|_{1,r}^{\gamma} \|v\|_c^{\delta}, & (c \le r) \end{cases}$$

[b]In fact in Triebel [1983] there is not exactly proved the relation (2.3) but we can get it as a simple consequence of several theorems that are also proved there. First in Subsection 2.2.2 (Remark 3) there is shown that $W^{s,p}(\Omega) = \Lambda_{p,p}^s(\Omega)$ for noninteger $s > 0$ and $1 \le p < \infty$ (the first spaces denotes the Sobolev-Slobodetski space and the second one is the Besov space). These spaces are introduced in the same Section 2.2.2. Then in Subsection 2.3.5 there is proved that $\Lambda_{p,q}^s(\Omega) = B_{p,q}^s(\Omega)$ for $s > 0$, $1 \le p < \infty$, $1 \le q \le \infty$ (the first is again the Besov space and the second one is the Triebel space, introduced in Subsection 2.3.1). Finally, in Subsection 3.3.3 (Trace theorem) the following trace theorem is established $tr : B_{p,q}^s(\Omega) \to B_{p,q}^{s-\frac{1}{p}}(\partial\Omega)$ for $s - \frac{1}{p} > 0$, $p > 1$, $q > 0$. As a simple consequence of these three facts we finally get (2.3).

with $\gamma := a(1-(1-k)\alpha)$ and $\delta := a(1-k)\alpha$. Hence for $c > r$ we obtain that it is enough to have

$$am/l \le r \qquad \Leftrightarrow \qquad a \le b\frac{nr+rc-nc}{bn+c-cn}.$$

For $c \le r$, we have the condition

$$a(1-\alpha(1-k)) \le r \qquad \Leftrightarrow \qquad a \le b\frac{nr+rc-nc}{bn+c-cn}.$$

which are exactly the condition (2.2).
Part 2 ($r > n$): For $c > r$ (and $b \ge c$) we obtain the same result as above. For $c \le r$ instead the interpolation (2.7) we use

$$\|v\|_r \le \|v\|_c^{\frac{c}{r}} \|v\|_{1,r}^{\frac{r-c}{r}}. \tag{2.9}$$

Thus we have

$$\int_0^T \|v\|_{\frac{1}{b},b}^a \le \int_0^T \|v\|_{1,r}^{ak}\|v\|_r^{a(1-k)} \le \int_0^T \|v\|_{1,r}^{ak+a(1-k)\frac{r-c}{r}} \|v\|_c^{a(1-k)\frac{c}{r}}.$$

Thus, we need

$$ak + a(1-k)\frac{r-c}{r} \le r \quad \Leftrightarrow a \le \frac{br^3}{nbc+rc-nrc-brc+br^2}$$

that is again exactly the relation (2.2). □

Let us precise the upper bounds used along the paper at the following lemma. First we define for $p, r > 1$

$$\mathcal{U} := L^p(0,T;W_N^{1,p}) \cap L^\infty(0,T;L_N^2) \tag{2.10}$$
$$\mathcal{E} := L^r(0,T;W_r) \cap L^\infty(0,T;L^1(\Omega)). \tag{2.11}$$

With this notation we introduce the second interpolation lemma that will be again used in the following text.

Lemma 2.2. *Let* $\mathbf{v} \in \mathcal{U}$ *and* $e \in \mathcal{E}$ *then*

$$\mathbf{v} \in \mathbf{L}^{p(n+2)/n}(Q), \tag{2.12}$$

$$\mathbf{v} \in \mathbf{L}^2(\Sigma_D), \qquad p \geq \frac{2(n+1)}{n+2}, \tag{2.13}$$

$$\mathbf{v} \in \mathbf{L}^s(\Sigma), \qquad s \leq \begin{cases} \dfrac{p(n+2)-2}{n} & \text{if } p \leq n, \\ \dfrac{p(p^2+2(n-1))}{p^2-2p+2n} & \text{if } p > n, \end{cases} \tag{2.14}$$

$$e \in L^{r(n+1)/n}(Q), \tag{2.15}$$

$$e \in L^l(\Sigma), \qquad 1 \leq l \leq \begin{cases} \dfrac{r(n+1)-1}{n} & \text{if } r < n, \\ \dfrac{r(r^2+n-1)}{r^2-r+n} & \text{if } r \geq n, \end{cases} \tag{2.16}$$

$$\mathbf{v} \otimes \mathbf{v} \in \mathbf{L}^{p(n+2)/(2n)}(Q), \tag{2.17}$$

$$|\mathbf{v}|^2\mathbf{v} \in \mathbf{L}^{p(n+2)/(3n)}(Q). \tag{2.18}$$

Moreover if

$$r : \begin{cases} \left.\begin{array}{l} 1 < r < q - \dfrac{n}{n+1} \\ q > \dfrac{2n+1}{n+1} \end{array}\right\} & \text{for } p \geq n, \\ \left.\begin{array}{l} \max\left(1, \dfrac{p(n-1)}{(p-1)(n+1)}\right) < r < q - \dfrac{n}{n+1} \\ q > \max\left(\dfrac{2n+1}{n+1}, \dfrac{2np-p-n}{(p-1)(n+1)}\right), \end{array}\right\} & \text{for } \dfrac{3n}{n+2} \leq p < n \end{cases} \tag{2.19}$$

then

$$e\mathbf{v} \in \mathbf{L}^1(Q). \tag{2.20}$$

Proof. The relation (2.12) is standard and well known (see, for instance, Málek et al. [1996]). The assertions (2.15), (2.17) and (2.18) can be also proved by Sobolev embedding. The trace assertions (2.13), (2.14) and (2.16) are consequence of Lemma 2.1. Indeed, for $r = p$, $c = 2$ and $a = b = s$ we obtain (2.14) and for $r = r$, $c = 1$ and $a = b = l$ we obtain (2.16). We prove only the last relation (2.20) that is in some sense nonstandard.

First, if $p \geq n$ then from Sobolev embedding we see that it is enough to have $r > 1$ which is exactly (2.19). Notice that the condition on q in (2.19) is due to the restriction on r.

For $p < n$ we use two standard interpolation inequalities

$$\|\cdot\|_s \leq \|\cdot\|_2^{1-\alpha}\|\cdot\|_{1,p}^{\alpha} \qquad 2 \leq s \leq \frac{np}{n-p}, \quad \alpha := \frac{s-2}{s}\frac{np}{np+2p-2n} \tag{2.21}$$

$$\|\cdot\|_m \leq \|\cdot\|_1^{1-\beta}\|\cdot\|_{1,r}^{\beta} \qquad 1 \leq m \leq \frac{nr}{n-r}, \quad \beta := \frac{m-1}{m}\frac{nr}{nr+r-n}. \tag{2.22}$$

Next, for an arbitrary $s \in]2,(np)/(n-p)[$ we set $m := s'$ in (2.22) and we can compute

$$\int_Q |\mathbf{v}||e| \le \int_0^T \|\mathbf{v}\|_s \|e\|_{s'} \le C \int_0^T \|\mathbf{v}\|_{1,p}^{\alpha} \|e\|_{1,r}^{\beta} \le C \left(\int_0^T \|e\|_{1,r}^{\frac{p}{p-\alpha}\beta} \right)^{\frac{p-\alpha}{p}} \le C \tag{2.23}$$

providing

$$s' \le \frac{nr}{n-r}, \qquad \frac{p}{p-\alpha}\beta \le r. \tag{2.24}$$

Since

$$\begin{aligned} \frac{p}{p-\alpha}\beta &= \frac{s(np+2p-2n)}{s(np+2p-3n)+2n} \frac{nr}{s(nr+r-n)} \\ &= \frac{np+2p-2n}{s(np+2p-3n)+2n} \frac{nr}{nr+r-n} \end{aligned}$$

we have that second condition in (2.24) is equivalent to

$$\frac{np+2p-2n}{s(np+2p-3n)+2n} \le \frac{nr+r-n}{n} = r\frac{n+1}{n} - 1. \tag{2.25}$$

Because we want to have r as small as possible we are led to minimalize left-hand side of (2.25) w.r.t. s. We restrict ourselves only onto the case when $p \ge 3n/(n+2)$ that is exactly the same parameter when $|\mathbf{v}|^3$ is integrable. With this restriction on p we see that we must set s as large as possible. Thus, we choose

$$s := \frac{np}{n-p} \qquad \Longrightarrow \alpha = 1, \quad \beta = \frac{n-p}{p} \frac{r}{nr+r-n}.$$

And we see that the second condition in (2.24) is valid if

$$r \ge \frac{p(n-1)}{(p-1)(n+1)}, \tag{2.26}$$

that is exactly the relation (2.19). It remains to recover that also the first condition in (2.24) is valid with our choice of s. Thus, we get after some computation the condition

$$r \ge \frac{np}{np-n+2p}.$$

However, because $n > p$ we obtain again after some computation that

$$\frac{p(n-1)}{(p-1)(n+1)} > \frac{np}{np-n+2p}.$$

Hence, for (2.19) the first condition in (2.24) is valid. □

Definition 2.1. We say that the problem (1.1)-(1.7) is a $(p-q)$ coupled fluid-energy system if

• $\tau : Q \times \mathbb{R} \times \mathbb{M}_{n\times n} \to \mathbb{M}_{n\times n}$ is a Carathéodory function, that is, measurable with respect to $(x,t) \in Q$ for every $(e, \varkappa) \in \mathbb{R} \times \mathbb{M}_{n\times n}$, and continuous with respect to

$(e,\varkappa) \in \mathbb{R} \times \mathbb{M}_{n\times n}$ for a. a. $(x,t) \in Q$. It satisfies $\tau(\cdot,\cdot,0) = 0$, the $p-$coercivity, the growth condition and the strict monotonicity:

$$\exists p > 1,\ \exists \nu_* > 0:\ \tau(\cdot,e,\varkappa) : \varkappa \geq \nu_*|\varkappa|^p, \tag{2.27}$$
$$\exists \nu^* > 0:\ |\tau(\cdot,e,\varkappa)| \leq \nu^*(|\varkappa|^{p-1}+1), \tag{2.28}$$
$$(\tau(\cdot,e,\varkappa) - \tau(\cdot,e,\zeta)) : (\varkappa - \zeta) > 0,\ \forall \varkappa,\zeta \in \mathbb{M}_{n\times n}, \varkappa \neq \zeta; \tag{2.29}$$

where $\mathbb{M}_{n\times n}$ denotes the set of real symmetric matrices of the type $n \times n$;
• $\mathbf{q} : Q \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function obeying

$$\exists q > 1\ \ \exists \upsilon_* > 0:\ \mathbf{q}(\cdot,e,\zeta)\cdot\zeta \geq \upsilon_*|\zeta|^q; \tag{2.30}$$
$$\exists \upsilon^* > 0:\ |\mathbf{q}(\cdot,e,\zeta)| \leq \upsilon^*(|\zeta|^{q-1}+1); \tag{2.31}$$
$$(\mathbf{q}(\cdot,e,\zeta) - \mathbf{q}(\cdot,e,\varkappa))\cdot(\zeta - \varkappa) > 0, \quad \forall \zeta,\varkappa \in \mathbb{R}^n. \tag{2.32}$$

Next we define what we mean by weak solution to the problem defined in Definition 2.30. Lemmas 2.1 and 2.2 give us the precise bound on parameters p, q, l, s for which it makes a good sense to define weak (distributional) solution.

Definition 2.2. Let $\tau, \mathbf{q}$ satisfy Definition 2.1 with parameters $p > \frac{3n}{n+2}$, $q > \frac{2n+1}{n+1}$. Moreover, let

$$\operatorname*{ess\,inf}_{x\in\Omega} e_0(x) \geq 0; \tag{2.33}$$
$$\mathbf{f} \in L^2(0,T;(W^{1,p}_N)^*), \qquad \mathbf{v}_0 \in L^2_{N,\mathrm{div}}, \qquad e_0 \in L^1(\Omega). \tag{2.34}$$

Let $\alpha : \Sigma_D \times \mathbb{R} \to \mathbb{R}$ and $\varphi, \gamma : \Sigma \times \mathbb{R} \to \mathbb{R}$ be Carathéodory functions such that $\gamma(\cdot,0) = 0$ and

$$\exists \alpha^* > 0:\ 0 \leq \alpha(\cdot,e) \leq \alpha^*, \tag{2.35}$$
$$\exists \varphi^* > 0:\ 0 \leq \varphi(\cdot,e) \leq \varphi^*, \quad \forall e \in \mathbb{R},\ \text{a.e. in } \Sigma; \tag{2.36}$$
$$\exists \gamma^* > 0:\ |\gamma(\cdot,e)| \leq \gamma^*(|e|^l + 1), \quad \forall e \in \mathbb{R},\ \text{a.e. in } \Sigma; \tag{2.37}$$
$$\gamma(\cdot,e)\mathrm{sign}(e) \geq 0, \quad \forall e \in \mathbb{R},\ \text{a.e. in } \Sigma. \tag{2.38}$$

We say that $(\mathbf{v},\pi,e)$ is a weak solution to the $(p-q)$ coupled fluid-energy system (1.1)-(1.7) if $\mathbf{v} \in \mathcal{U}$, $e \in \mathcal{E}$ for some $r > q-1$, and

$$\pi \in \begin{cases} L^{p(n+2)/(2n)}(Q) & \text{if } p < (3n+2)/(n+2) \\ L^{p'}(Q) & \text{if } p \geq (3n+2)/(n+2), \end{cases}$$

$$\partial_t \mathbf{v} \in \mathcal{X} := L^{p'}(0,T;(W^{1,p}_N)') \cap L^{p(n+2)/(2n)}(0,T;(W_N^{1,p(n+2)/[p(n+2)-2n]})')$$

satisfy

$$
\begin{aligned}
&\langle \partial_t \mathbf{v}, \mathbf{w}\rangle - (\mathbf{v}\otimes\mathbf{v}, D\mathbf{w}) + (\tau(\cdot,e,D\mathbf{v}), D\mathbf{w}) + \int_{\Sigma_D} \alpha(\cdot,e)\mathbf{v}_T\cdot\mathbf{w}_T\, dS\, dt \\
&\quad + \int_{\Sigma} \varphi(\cdot,e)|\mathbf{v}_T|^{s-2}\mathbf{v}_T\cdot\mathbf{w}_T\, dS\, dt = \langle \mathbf{f},\mathbf{w}\rangle + \langle \pi, \operatorname{div}\mathbf{w}\rangle \\
&\qquad \text{for all } \mathbf{w}\in L^\infty(0,T;W^{1,\infty}_N),
\end{aligned} \tag{2.39}
$$

$$
\begin{aligned}
&\langle \partial_t e, \phi\rangle - (e\mathbf{v}, \nabla\phi) + (\mathbf{q}(\cdot,e,\nabla e), \nabla\phi) + \int_\Sigma \gamma(\cdot,e)\phi\, dS\, dt \\
&\quad \ge (\tau(\cdot,e,D\mathbf{v}), D\mathbf{v}\phi) + \int_\Sigma \varphi(\cdot,e)|\mathbf{v}_T|^s\phi\, dS\, dt \\
&\qquad \text{for all } \phi\in C^1(\bar{Q}); \phi\ge 0,
\end{aligned} \tag{2.40}
$$

$$
\begin{aligned}
&-\left(\frac{|\mathbf{v}|^2}{2} + e, \partial_t\phi\right) + (\mathbf{q}(\cdot,e,\nabla e) - \tau(\cdot,e,D\mathbf{v})\mathbf{v}, \nabla\phi) + \int_\Sigma \gamma(\cdot,e)\phi\, dS\, dt \\
&\quad = \langle \mathbf{f}, \mathbf{v}\phi\rangle + \left(\left(\frac{|\mathbf{v}|^2}{2} + e + \pi\right)\mathbf{v}, \nabla\phi\right) - \int_{\Sigma_D} \alpha(\cdot,e)|\mathbf{v}_T|^2\phi\, dS\, dt \\
&\quad + \left(\frac{|\mathbf{v}_0|^2}{2} + e_0, \phi(0)\right) \\
&\qquad \text{for all } \phi\in C^1(\bar{Q}); \phi(T)=0,
\end{aligned} \tag{2.41}
$$

completed by initial conditions

$$
\mathbf{v}(\cdot,0) = \mathbf{v}_0, \quad \left(\frac{|\mathbf{v}|^2}{2} + e\right)(\cdot,0) = \frac{|\mathbf{v}_0|^2}{2} + e_0, \tag{2.42}
$$

where $p' = p/(p-1)$ is the conjugate exponent to p, and the symbol $\langle\cdot,\cdot\rangle$ denotes a generic duality pairing, not distinguished between scalar and vector fields.

Theorem 2.1. *Under the assumptions (2.33)-(2.38), if $p > 3n/(n+2)$ and*

$$
1 \le s < \begin{cases} (p(n+2)-2)/n & \text{if } p\le n \\ p(p^2+2(n-1))/(p^2-2p+2n) & \text{if } p > n; \end{cases} \tag{2.43}
$$

$$
1 \le l < \begin{cases} (q-1)(n+1)/n & \text{if } q\le n(n+2)/(n+1) \\ \mathcal{L}_{max} & \text{if } q > n(n+2)/(n+1); \end{cases} \tag{2.44}
$$

where $\mathcal{L}_{max} = \frac{qn+q-n}{n+1} + \frac{(qn+q-n)(qn+q-2n-1)}{(qn+q-n)(qn+q-2n-1)+n(n+1)^2}$, then there exists a weak solution $(\mathbf{v}, e, \pi)$ to the $(p-q)$ coupled fluid-energy system (1.1)-(1.7), for all r satisfying (2.19). *Moreover,*

$$
e(x,t)\ge 0, \quad \text{for a.e. } (x,t)\in Q. \tag{2.45}
$$

It represents the existence of a solution for

$$
\begin{aligned}
n = q = 2: &\quad 1 < r < 4/3 \quad p > 3/2 \\
n = 3, q = 2: &\quad 1 < r < 5/4 \quad p > 9/5.
\end{aligned}
$$

Corollary 2.1. *If the assumptions of Theorem 2.1 are fulfilled with $e_r \neq 0$ and (2.33) replaced by*

$$\exists \bar{e} > 0 : \quad \operatorname*{ess\,inf}_{x\in\Omega} e_0(x) \geq \bar{e}, \tag{2.46}$$

and γ additionally verifies $\gamma(\cdot, \min(\bar{e}, e_r)) = 0$ and the monotonicity property

$$\gamma(\cdot, e)\mathrm{sign}(e - \min(\bar{e}, e_r)) \geq 0, \ \forall e \in \mathbb{R}, \ \text{a.e. in } \Sigma, \tag{2.47}$$

then, $e \geq \min(\bar{e}, e_r)$ a.e. in Q.

3. Approximate results

The proof of Theorem 2.1 will be done by using a sequence of approximative problems. Here, we give all theorems about the existence of approximative problems that will be proved in the next sections. In what follows the symbol $\mathcal{M}_\mu$ is the "Helmholtz-mollification", i.e., we define

$$\mathcal{M}_\mu(\mathbf{v}) := (\chi \mathbf{v}) * \omega - \nabla \eta$$

with ω denoting a mollifier with support in a ball of radii $\frac{1}{\mu}$,

$$\chi(x) = \begin{cases} 0 & \text{if } \mathrm{dist}(x, \partial\Omega) \leq \frac{2}{\mu} \\ 1 & \text{elsewhere,} \end{cases}$$

and η is due to the Helmholtz decomposition, that is,

$$\Delta \eta = \mathrm{div}[(\chi \mathbf{v}) * \omega] \text{ in } \Omega,$$
$$\nabla \eta \cdot \mathbf{n} = 0 \text{ on } \partial\Omega, \qquad \int_\Omega \eta dx = 0.$$

Next Theorem establishes the existence of solution to the problem with mollified convective term.

Theorem 3.1. *Let the assumptions (2.33)-(2.44) be fulfilled under $p > 2(n+1)/(n+2)$. For each $\mu \in \mathbb{N}$, there exists $(\mathbf{v}_\mu, e_\mu, \pi_\mu)$ in $\mathcal{U} \times \mathcal{E} \times L^{p'}(Q)$, $\partial_t \mathbf{v}_\mu \in L^{p'}(0, T; (W^{1,p}_N)')$, satisfying*

$$\mathrm{div}\, \mathbf{v}_\mu = 0, \tag{3.1}$$

$$\begin{aligned} &\langle \partial_t \mathbf{v}_\mu, \mathbf{w}\rangle + (D\mathbf{v}_\mu, \mathcal{M}_\mu(\mathbf{v}_\mu) \otimes \mathbf{w}) + (\tau(e_\mu, D\mathbf{v}_\mu), D\mathbf{w}) - (\pi_\mu, \mathrm{div}\, \mathbf{w}) \\ &+ \int_{\Sigma_D} \alpha(e_\mu)\mathbf{v}_{\mu T} \cdot \mathbf{w}_T \, dS\, dt + \int_\Sigma \varphi(e_\mu)|\mathbf{v}_{\mu T}|^{s-2}\mathbf{v}_{\mu T} \cdot \mathbf{w}_T \, dS\, dt = \langle \mathbf{f}, \mathbf{w}\rangle \\ &\qquad \text{for all } \mathbf{w} \in L^p(0, T; W^{1,p}_N), \end{aligned} \tag{3.2}$$

$$\begin{aligned} &\langle \partial_t e_\mu, \phi\rangle + (\mathcal{M}_\mu(\mathbf{v}_\mu), \nabla e_\mu \phi) + (\mathbf{q}(e_\mu, \nabla e_\mu), \nabla\phi) + \int_\Sigma \gamma(e_\mu)\phi \, dS\, dt \\ &= (\tau(e_\mu, D\mathbf{v}_\mu), D\mathbf{v}_\mu \phi) + \int_\Sigma \varphi(e_\mu)|\mathbf{v}_{\mu T}|^s \phi \, dS\, dt \\ &\qquad \text{for all } \phi \in L^\infty(0, T; W_{r/(r-q+1)}), \end{aligned} \tag{3.3}$$

$$\mathbf{v}_\mu(\cdot, 0) = \mathbf{v}_0, \quad e_\mu(\cdot, 0) = e_0.$$

Next, let us state the quasi-compressible approximative problem.

Theorem 3.2. *Under the assumptions of Theorem 3.1, for each* $\varepsilon > 0$, *there exists* $(\mathbf{v}_\varepsilon, e_\varepsilon, \pi_\varepsilon)$ *in* $\mathcal{U} \times \mathcal{E} \times L^p(0,T; W^{2,p}(\Omega))$, $\partial_t \mathbf{v}_\varepsilon \in L^{p'}(0,T; (W_N^{1,p})')$, *satisfying*

$$\langle \partial_t \mathbf{v}_\varepsilon, \mathbf{w}\rangle + (D\mathbf{v}_\varepsilon, \mathcal{M}_\mu(\mathbf{v}_\varepsilon) \otimes \mathbf{w}) + (\tau(e_\varepsilon, D\mathbf{v}_\varepsilon), D\mathbf{w}) - (\pi_\varepsilon, \operatorname{div} \mathbf{w})$$
$$+ \int_{\Sigma_D} \alpha(e_\varepsilon)\mathbf{v}_{\varepsilon T} \cdot \mathbf{w}_T \, dS\, dt + \int_\Sigma \varphi(e_\varepsilon)|\mathbf{v}_{\varepsilon T}|^{s-2}\mathbf{v}_{\varepsilon T} \cdot \mathbf{w}_T \, dS\, dt = \langle \mathbf{f}, \mathbf{w}\rangle \tag{3.4}$$
for all $\mathbf{w} \in L^p(0,T; W_N^{1,p})$,

$$\langle \partial_t e_\varepsilon, \phi\rangle + (\mathcal{M}_\mu(\mathbf{v}_\varepsilon), \nabla e_\varepsilon \phi) + (\mathbf{q}(e_\varepsilon, \nabla e_\varepsilon), \nabla\phi) + \int_\Sigma \gamma(e_\varepsilon)\phi \, dS\, dt$$
$$= (\tau(e_\varepsilon, D\mathbf{v}_\varepsilon), D\mathbf{v}_\varepsilon \phi) + \int_\Sigma \varphi(e_\varepsilon)|\mathbf{v}_{\varepsilon T}|^s \phi \, dS\, dt \tag{3.5}$$
for all $\phi \in L^\infty(0,T; W_{r/(r-q+1)})$,

$$\varepsilon(\nabla\pi_\varepsilon, \nabla\phi) + (\operatorname{div} \mathbf{v}_\varepsilon, \phi) = 0, \qquad \textit{for all } \phi \in W^{1,p}(\Omega) \textit{ and a. a. } t \in (0,T); \tag{3.6}$$
$$\mathbf{v}_\varepsilon(\cdot, 0) = \mathbf{v}_0, \quad e_\varepsilon(\cdot, 0) = e_0.$$

4. Proof of Theorem 3.2 (μ, ε fixed)

For arbitrary $\mathbf{w} \in W_N^{1,p}$ and $\phi \in W_{r/(r-q+1)}$, there exist regularizing sequences in $W_N^{1,\beta}$ and W_β, respectively, with $\beta > n$.

4.1. *The Faedo-Galerkin approximation scheme*

For $\varepsilon > 0$, consider the solution of the homogeneous Neumann problem for the Laplace equation (see, for instance, Galdi [1994])

$$\begin{aligned} \varepsilon\Delta\pi(t) &= \operatorname{div} \mathbf{v}(t) && \text{in } \Omega \\ \nabla\pi(t) \cdot \mathbf{n} &= 0 && \text{on } \partial\Omega \\ \int_\Omega \pi(t)dx &= 0, \end{aligned}$$

which satisfies

$$\varepsilon\|\pi(t)\|_{2,p} \le C(\Omega, p)\|\mathbf{v}(t)\|_{1,p}, \quad \forall \mathbf{v}(t) \in W_N^{1,p} \tag{4.1}$$
$$\varepsilon\|\pi(t)\|_{1,r} \le C(\Omega, r)\|\mathbf{v}(t)\|_{0,r}, \quad \forall \mathbf{v}(t) \in W_N^{1,p} \cap \mathbf{L}^r(\Omega), \quad \text{a.e. } t \in]0,T[. \tag{4.2}$$

Denote by $\mathcal{F}_\varepsilon : W_N^{1,p} \to W^{2,p}(\Omega)$ the well defined continuous operator such that $\mathcal{F}_\varepsilon(\mathbf{v}) = \pi$, and let $\{(\mathbf{w}^j, w^j)\}_{j\in\mathbb{N}}$ be a basis of $W_N^{1,\beta} \times W_\beta$ with $\beta > n$. From the Carathéodory theory Zeidler [1990], there exists a local-in-time solution

$$\mathbf{v}^{N,M} \in \langle \mathbf{w}^1, \cdots, \mathbf{w}^N\rangle \Leftrightarrow \mathbf{v}^{N,M}(x,t) = \sum_{j=1}^N c_j^{N,M}(t)\mathbf{w}^j(x),$$

$$e^{N,M} \in \langle w^1, \cdots, w^M\rangle \Leftrightarrow e^{N,M}(x,t) = \sum_{j=1}^M d_j^{N,M}(t) w^j(x),$$

to the following system of ordinary differential equations, for every $M, N \in \mathbb{N}$,

$$\begin{aligned}\frac{d}{dt}(\mathbf{v}^{N,M}, \mathbf{w}^j) - (\mathcal{M}_\mu(\mathbf{v}^{N,M}) \otimes \mathbf{v}^{N,M}, \nabla \mathbf{w}^j) + (\tau(e^{N,M}, D\mathbf{v}^{N,M}), D\mathbf{w}^j) + \\ +(\alpha(e^{N,M})\mathbf{v}_T^{N,M}, \mathbf{w}_T^j) + (\varphi(e^{N,M})|\mathbf{v}_T^{N,M}|^{s-2}\mathbf{v}_T^{N,M}, \mathbf{w}_T^j) - \\ -(\mathcal{F}_\varepsilon(\mathbf{v}^{N,M}), \nabla \mathbf{w}^j) = (\mathbf{f}, \mathbf{w}^j), \quad j = 1, \cdots, N; \qquad (4.3)\end{aligned}$$

$$\begin{aligned}\frac{d}{dt}(e^{N,M}, w^j) - (e^{N,M}\mathcal{M}_\mu(\mathbf{v}^{N,M}), \nabla w^j) + (\mathbf{q}(e^{N,M}, \nabla e^{N,M}), \nabla w^j) + \\ +(\gamma(e^{N,M}), w^j) = (\tau(e^{N,M}, D\mathbf{v}^{N,M}) : D\mathbf{v}^{N,M}, w^j) + \\ +(\varphi(e^{N,M})|\mathbf{v}_T^{N,M}|^s, w^j), \ j = 1, \cdots, M, \qquad (4.4)\end{aligned}$$

under the initial conditions $\mathbf{v}_0^N$, $e_0^{N,M}$ given by the projections of $\mathbf{v}_0$ and the mollification e_0^N of e_0 (after extending e_0 by $\bar{e}$ outside Ω), respectively, onto linear hulls of the base's vectors. Note that

$$\begin{aligned}\mathbf{v}_0^N &\to \mathbf{v}_0 \quad \text{strongly in } \mathbf{L}^2(\Omega), \\ e_0^{N,M} &\to e_0^N \quad \text{strongly in } L^2(\Omega), \\ e_0^N &\to e_0 \quad \text{strongly in } L^1(\Omega).\end{aligned}$$

Using the fact that $\operatorname{div} \mathcal{M}_\mu(\mathbf{v}^{N,M}) = 0$, we have the standard estimates, independently of M,

$$\begin{aligned}\sup_{t\in[0,T]} \|\mathbf{v}^{N,M}(t)\|_{2,\Omega}^2 + \|D\mathbf{v}^{N,M}\|_{p,Q}^p + \|\mathbf{v}^{N,M}\|_{2,\Sigma_D}^2 + \|\mathbf{v}^{N,M}\|_{s,\Sigma}^s \\ + \varepsilon\|\nabla\pi^{N,M}\|_{2,Q}^2 \le \|\mathbf{v}_0\|_{2,\Omega}^2 + C\|\mathbf{f}\|_{p',Q}^{p'} := R,\end{aligned} \qquad (4.5)$$

$$\sup_{t\in[0,T]} \|e^{N,M}(t)\|_{2,\Omega}^2 + \|\nabla e^{N,M}\|_{q,Q}^q \le \|e_0^N\|_{2,\Omega}^2 + C(N), \qquad (4.6)$$

$$\|\mathbf{q}(e^{N,M}, \nabla e^{N,M})\|_{q',Q}^{q'} \le C(\|\nabla e^{N,M}\|_{q,Q}^q + 1), \qquad (4.7)$$

$$\left\|\frac{d}{dt}c^{N,M}\right\|_{L^2(0,T)} \le C(N), \qquad (4.8)$$

$$\|\partial_t e^{N,M}\|_{q',(W_q)'} \le C(N,\mu). \qquad (4.9)$$

Hence, using Lemma 2.2, it follows

$$\|e^{N,M}\|_{q(n+2)/n,Q} \le C(N); \qquad (4.10)$$

$$\|e^{N,M}\|_{l+\epsilon,\Sigma} \le C(N), \quad \epsilon > 0. \qquad (4.11)$$

The global-in-time existence of $(\mathbf{v}^N, e^N)$ is a consequence of the above estimates.

4.2. *Passage to the limit as $M \to \infty$ (μ, ε fixed)*

In order to pass to the limit with M, when M tends to infinity (N fixed), we can extract a subsequence, still denoted by $(\mathbf{v}^{N,M}, e^{N,M})$, verifying (4.5)-(4.11) and consequently

$$\begin{aligned}
c^{N,M} &\rightharpoonup c^N && \text{weakly* in } L^\infty(0,T);\\
\frac{d}{dt}c^{N,M} &\rightharpoonup \frac{d}{dt}c^N && \text{weakly in } L^2(0,T);\\
e^{N,M} &\rightharpoonup e^N && \text{weakly* in } L^\infty(0,T;L^2(\Omega));\\
e^{N,M} &\rightharpoonup e^N && \text{weakly in } L^q(0,T;W_q);\\
\partial_t e^{N,M} &\rightharpoonup \partial_t e^N && \text{weakly in } L^{q'}(0,T;(W_q)');\\
e^{N,M} &\to e^N && \text{strongly in } L^m(Q), \quad \text{for } 1\le m< q(n+2)/n;\\
e^{N,M} &\to e^N && \text{strongly in } L^l(\Sigma);\\
\mathbf{q}(e^{N,M},\nabla e^{N,M}) &\rightharpoonup \bar{\mathbf{q}}^N && \text{weakly in } \mathbf{L}^{q'}(Q).
\end{aligned}$$

Since we also have $c^{N,M} \to c^N$ strongly in $C(0,T)$ due to the Arsela-Ascoli Theorem, we obtain the system

$$\begin{aligned}
&\tfrac{d}{dt}(\mathbf{v}^N,\mathbf{w}^j) - (\mathcal{M}_\mu(\mathbf{v}^N)\otimes\mathbf{v}^N,\nabla\mathbf{w}^j) + (\tau(e^N,D\mathbf{v}^N),D\mathbf{w}^j) +\\
&+ (\alpha(e^N)\mathbf{v}^N_T,\mathbf{w}^j_T) + (\varphi(e^N)|\mathbf{v}^N_T|^{s-2}\mathbf{v}^N_T,\mathbf{w}^j_T) - (\mathcal{F}_\varepsilon(\mathbf{v}^N),\nabla\mathbf{w}^j) =\\
&= (\mathbf{f},\mathbf{w}^j), \quad j=1,\cdots,N; \qquad (4.12)\\
&\tfrac{d}{dt}(e^N,\phi) - (e^N\mathcal{M}_\mu(\mathbf{v}^N),\nabla\phi) + (\mathbf{q}(e^N,\nabla e^N),\nabla\phi) + (\gamma(e^N),\phi) =\\
&= (\tau(e^N,D\mathbf{v}^N):D\mathbf{v}^N,\phi) + (\varphi(e^N)|\mathbf{v}^N_T|^s,\phi), \quad \forall\phi\in W_q, \text{ a.e. in }]0,T[, \qquad (4.13)
\end{aligned}$$

providing that

$$\mathbf{q}(e^N,\nabla e^N) = \bar{\mathbf{q}}^N. \qquad (4.14)$$

Indeed, applying the strictly monotone assumption (2.32), using the Galerkin equation (4.4) and passing to the limit as $M\to\infty$, we conclude that

$$\left\langle \bar{\mathbf{q}}^N - \mathbf{q}(e^N,\nabla\phi), \nabla(e^N-\phi)\right\rangle \ge 0,$$

and (4.14) follows by using the Minty method.

4.3. *Minimum principle*

First, we prove that $e^N \ge 0$ a.e. in Q. We use the function $\phi(x,t) := \min(0,e^N(x,t))$ as a test function in (4.13) to get (note that this is a possible test function because $\phi = 0$ on Σ_D)

$$\|\phi\|^2_{2,\Omega} + \int_0^t\int_{\Omega[e^N<0]} \mathbf{q}(e^N,\nabla e^N)\cdot\nabla e^N + \int_0^t\int_{\Gamma[e^N<0]}\gamma(e^N)e^N \le 0.$$

Hence, we get that $\phi\equiv 0$ that directly implies that $e^N\ge 0$.

Hencefurther, the set $\Omega[\mathcal{S}]$ means $\{x\in\Omega:\ \mathcal{S}(x)\}$ with $\mathcal{S}$ denoting a sentence to be point-wisely satisfied. Analogously for $\Gamma[\mathcal{S}]$, $Q[\mathcal{S}]$ or simply $\{\mathcal{S}\}$ whenever the meaning is not ambigous.

4.4. *Estimates independently of N*

The existence of the pressure $\pi^N \in W^{2,p}(Q)$ such that it belongs to a bounded set independent on N is due to (4.1)-(4.5), and the following relation holds

$$\int_\Omega \{\partial_t \mathbf{v}^N + (\mathcal{M}_\mu(\mathbf{v}^N)\cdot\nabla)\mathbf{v}^N - \operatorname{div}\tau(e^N, D\mathbf{v}^N)\}\cdot\mathbf{v} = \int_\Omega \{\mathbf{f} - \nabla\pi^N\}\cdot\mathbf{v}$$
$$\text{a.e. in }]0,T[\quad \forall \mathbf{v} \in \langle \mathbf{w}^1, \cdots, \mathbf{w}^N\rangle.$$

Arguing as in Consiglieri [2000], we get

$$\|\gamma(e^N)\|_{1,\Sigma} \leq \|e_0\|_{1,\Omega} + I_N; \tag{4.15}$$
$$\|e^N\|_{\infty,L^1(\Omega)} \leq I_N + T\|e_0\|_{1,\Omega} + |Q|/2; \tag{4.16}$$
$$\|\nabla e^N\|_{r,Q}^r \leq C\{I_N + \|e_0\|_{1,\Omega}\} \times \|e^N\|_{\infty,L^1(\Omega)}^{r(q-r)/(qn)}; \tag{4.17}$$
$$I_N := \|\tau(e^N, D\mathbf{v}^N) : D\mathbf{v}^N\|_{1,Q} + \|\varphi(e^N)|\mathbf{v}_T^N|^s\|_{1,\Sigma}$$

for every exponent $1 < r < q - n/(n+1)$ (cf. Boccardo et al. [1997] or Boccardo and Gallouët [1989]). From (4.5), the standard energy estimates hold

$$\|\mathbf{v}^N\|_{\infty;\mathbf{L}^2(\Omega)} \leq C; \tag{4.18}$$
$$\|\mathbf{v}^N\|_{p;W_N^{1,p}} \leq C; \tag{4.19}$$
$$\|\tau(e^N, D\mathbf{v}^N)\|_{p',Q}^{p'} \leq C(\|D\mathbf{v}^N\|_{p,Q}^p + 1). \tag{4.20}$$

Using Lemma 2.2, we can deduce from (4.15)-(4.17) that

$$\|\mathbf{v}^N\|_{p(n+2)/n,Q} \leq C; \tag{4.21}$$
$$\|e^N\|_{r(n+1)/n,Q} \leq C; \tag{4.22}$$
$$\|\mathbf{v}^N \otimes \mathbf{v}^N\|_{p(n+2)/(2n),Q} \leq C. \tag{4.23}$$

Next, using the standard procedure (see for example Consiglieri [2006]) we can estimate time derivatives such that for $p \geq 2n/(n+1)$:

$$\int_0^T \|\partial_t \mathbf{v}^N\|_{(W_N^{1,p})'}^{p'} dt \leq C\left(\mu + \frac{1}{\varepsilon} + 1\right)\|\mathbf{v}^N\|_{p;W_N^{1,p}}^p + C\|\mathbf{f}\|_{p',Q}^{p'}; \tag{4.24}$$
$$\|\partial_t e^N\|_{1,(W_{r/(r-q+1)})'} = \int_0^T \sup_{\|\phi\|_{W_{r/(r-q+1)}}\leq 1} |\langle \partial_t e^N, \phi\rangle| dt \leq$$
$$\leq \|\mathbf{q}(e^N, \nabla e^N)\|_{r/(q-1),Q} + C(\mu)\|e^N\|_{r/(q-1),Q} + \|\gamma(e^N)\|_{1,\Sigma} + I_N. \tag{4.25}$$

4.5. *Passage to the limit as $N \to \infty$ (μ, ε fixed)*

In order to pass to the limit with N, when N tends to infinity, we can extract a subsequence, still denoted by $(\mathbf{v}^N, e^N, \pi^N)$, verifying (4.15)-(4.25) and consequently

$$\begin{aligned}
\mathbf{v}^N &\rightharpoonup \mathbf{v} && \text{weakly* in } L^\infty(0,T;L^2_N);\\
\mathbf{v}^N &\rightharpoonup \mathbf{v} && \text{weakly in } L^p(0,T;W^{1,p}_N);\\
\partial_t\mathbf{v}^N &\rightharpoonup \partial_t\mathbf{v} && \text{weakly in } L^{p'}(0,T;(W^{1,p}_N)');\\
\mathbf{v}^N &\to \mathbf{v} && \text{strongly in } \mathbf{L}^m(Q), \quad \text{for } 1 \le m < p(n+2)/n;\\
\mathbf{v}^N &\to \mathbf{v} && \text{strongly in } \mathbf{L}^2(\Sigma_D);\\
\mathbf{v}^N &\to \mathbf{v} && \text{strongly in } \mathbf{L}^s(\Sigma), \quad \text{for } s \text{ fulfiling } (2.44);\\
e^N &\rightharpoonup e && \text{weakly in } L^r(0,T;W_r), \quad \text{for } 1 < r < q - n/(n+1);\\
e^N &\to e && \text{strongly in } L^m(Q), \quad \text{for } 1 \le m < r(n+1)/n;\\
e^N &\to e && \text{strongly in } L^l(\Sigma), \quad \text{for } l \text{ fulfiling } (2.44);\\
\pi^N &\rightharpoonup \pi && \text{weakly in } L^p(0,T;W^{2,p}(\Omega));\\
\tau(e^N, D\mathbf{v}^N) &\rightharpoonup \bar{\tau} && \text{weakly in } \mathbf{L}^{p'}(Q);\\
\mathbf{q}(e^N, \nabla e^N) &\rightharpoonup \bar{\mathbf{q}} && \text{weakly in } \mathbf{L}^{r/(q-1)}(Q).
\end{aligned}$$

Since we can compute

$$\|\nabla(\pi^N - \pi)\|^2_{2,Q} = \frac{1}{\varepsilon}(\mathbf{v}^N, \nabla\pi^N) - (\nabla\pi^N, \nabla\pi) - (\nabla\pi, \nabla(\pi^N - \pi)) \qquad (4.26)$$

we can conclude the strong convergence

$$\nabla\pi^N \to \nabla\pi \text{ in } \mathbf{L}^2(Q). \qquad (4.27)$$

Moreover, because of validity of the following energy equalities

$$\begin{aligned}
\|\mathbf{v}^N(T)\|^2_{2,\Omega} + \int_Q \tau(e^N, D\mathbf{v}^N) : D\mathbf{v}^N + \int_{\Sigma_D} \alpha(e^N)|\mathbf{v}^N_T|^2 + \int_\Sigma \varphi(e^N)|\mathbf{v}^N_T|^s +&\\
+\varepsilon\|\nabla\pi^N\|^2_{2,Q} = \int_Q \mathbf{f}\cdot\mathbf{v}^N + \|\mathbf{v}_0\|^2_{2,\Omega};&\\
\|\mathbf{v}(T)\|^2_{2,\Omega} + \int_Q \bar{\tau} : D\mathbf{v} + \int_{\Sigma_D} \alpha(e)|\mathbf{v}_T|^2 + \int_\Sigma \varphi(e)|\mathbf{v}_T|^s +&\\
+\varepsilon\|\nabla\pi\|^2_{2,Q} = \int_Q \mathbf{f}\cdot\mathbf{v} + \|\mathbf{v}_0\|^2_{2,\Omega}.&
\end{aligned}$$

we can apply the strict monotone assumption (2.29), pass to the limit as $N \to \infty$, and use the Minty method (we already have point-wise convergence of e^N) to conclude that

$$\tau(e, D\mathbf{v}) = \bar{\tau}. \qquad (4.28)$$

Moreover, we can easily obtain obtain the strong convergences[c]

$$\tau(e^N, D\mathbf{v}^N) : D\mathbf{v}^N \to \tau(e, D\mathbf{v}) : D\mathbf{v} \text{ in } L^1(Q) \tag{4.29}$$

$$\varphi(e^N)|\mathbf{v}_T^N|^s \to \varphi(e)|\mathbf{v}_T|^s \text{ in } L^1(\Sigma). \tag{4.30}$$

We argue as in Boccardo et al. [1997] to pass to the limit with the nonlinear term $\mathbf{q}(e^N, \nabla e^N)$ in the energy equation and to conclude that

$$\mathbf{q}(e, \nabla e) = \bar{\mathbf{q}}. \tag{4.31}$$

Indeed, using the continuity of $\mathbf{q}(x, t, \cdot, \cdot)$ and Vitali Theorem, it is sufficient to prove that ∇e^N a.e. converges to ∇e (cf. Section 4.6).

Then $(\mathbf{v}, e, \pi)$ is the corresponding limit to solution (3.4)-(3.6).

4.6. *Almost everywhere convergence of* ∇e^N

By the strict monotonicity assumption (2.32), it is sufficient to prove that, for some $\theta > 0$ and for some subsequence still denoted by e^N,

$$\lim_{N\to+\infty} \int_Q [(\mathbf{q}(e^N, \nabla e^N) - \mathbf{q}(e^N, \nabla e)) \cdot \nabla(e^N - e)]^\theta = 0. \tag{4.32}$$

Let us decompose the integral as

$$\begin{aligned} \int_Q [(\mathbf{q}(e^N, \nabla e^N) - \mathbf{q}(e^N, \nabla e)) \cdot \nabla(e^N - e)]^\theta &= \int_{\{|e|\geq k\}} [\cdots]^\theta + \\ + \int_{\{|e|<k:\ |e^N - T_{k,\nu}(e)|>\delta\}} [\cdots]^\theta &+ \int_{\{|e|<k:\ |e^N - T_{k,\nu}(e)|\leq\delta\}} [\cdots]^\theta \\ &:= I_1^{N,k} + I_2^{N,k,\nu,\delta} + I_3^{N,k,\nu,\delta}, \end{aligned}$$

where $k, \nu, \delta > 0$ are constants independent on N, T_k is the k-truncation,

$$T_{k,\nu}(e) = \max(-k, \min(k, T_\nu(e))), \tag{4.33}$$

and T_ν is the time regularization

$$T_\nu(e) = \nu \int_{-\infty}^t \tilde{e}(x, \varsigma) \exp[\nu(\varsigma - t)] d\varsigma \tag{4.34}$$

with $\tilde{e}$ denoting the zero extension of e outside $[0, T]$. Here we use the notation introduced in Section 4.3.

Using Hölder inequality, the assumption (2.31) and the estimate (4.17), we obtain for $\theta < r/q < 1$

$$\begin{aligned} I_1^{N,k} &\leq \left(\|\nabla e^N\|_{r,Q}^r + \|\nabla e\|_{r,Q}^r\right)^{\theta q/r} [\text{meas}(\{|e| \geq k\})]^{1-\theta q/r} \\ &\leq C\, [\text{meas}(\{|e| \geq k\})]^{1-\theta q/r} \underset{N,k\to+\infty}{\longrightarrow} 0. \end{aligned} \tag{4.35}$$

[c]Indeed, to show (4.29) it is enough to take into account the weak convergence $\tau(e^N, D\mathbf{v}^N) \rightharpoonup \tau(e, D\mathbf{v})$ the strict monotonicity of τ and the fact that $(\tau(e^N, D\mathbf{v}^N), D\mathbf{v}^N) \to (\tau(e, D\mathbf{v}), D\mathbf{v})$.

Since the integrand is positive, we can argue as before

$$
\begin{aligned}
I_2^{N,k,\nu,\delta} &\leq \int_{\{|e^N - T_{k,\nu}(e)|>\delta\}} \left[(\mathbf{q}(e^N, \nabla e^N) - \mathbf{q}(e^N, \nabla T_k(e))) \cdot \nabla(e^N - T_k(e)) \right]^\theta \\
&\leq C \left[\mathrm{meas}(\{|e^N - T_{k,\nu}(e)| > \delta\}) \right]^{1-\theta q/r} \underset{N,k,\nu\to+\infty}{\longrightarrow} 0,
\end{aligned} \tag{4.36}
$$

taking into account that the characteristic function verifies the following properties [Boccardo et al., 1997, Lemma 3.2], for almost every $\delta > 0$,

$$
\begin{aligned}
\lim_{N\to+\infty} \chi_{\{|e^N - T_{k,\nu}(e)|>\delta\}} &= \chi_{\{|e-T_{k,\nu}(e)|>\delta\}}, \quad \forall k, \nu > 0, \\
\lim_{\nu\to+\infty} \chi_{\{|e-T_{k,\nu}(e)|>\delta\}} &= \chi_{\{|e-T_k(e)|>\delta\}}, \quad \forall k > 0.
\end{aligned}
$$

As done to $I_2^{N,k,\nu,\delta}$, now using Hölder inequality (with exponents $1/\theta$ and $1/(1-\theta)$) we obtain

$$
\begin{aligned}
I_3^{N,k,\nu,\delta} &\leq \int_{\{|e^N - T_{k,\nu}(e)|\leq\delta\}} [(\mathbf{q}(e^N, \nabla e^N) - \mathbf{q}(e^N, \nabla T_k(e))) \cdot \nabla(e^N - T_k(e))]^\theta \\
&\leq \left(\int_{\{|e^N - T_{k,\nu}(e)|\leq\delta\}} [\cdots] \right)^\theta [\mathrm{meas}(Q)]^{1-\theta} \\
&:= \left(I_4^{N,k,\nu,\delta} + I_5^{N,k,\nu,\delta} \right)^\theta [\mathrm{meas}(Q)]^{1-\theta}.
\end{aligned}
$$

We can write, using the fact that $|T_{k,\nu}(e)| \leq k$,

$$
\begin{aligned}
I_5^{N,k,\nu,\delta} &:= \int_{\{|e^N - T_{k,\nu}(e)|\leq\delta\}} \mathbf{q}(e^N, \nabla T_k(e)) \cdot \nabla(e^N - T_k(e)) \\
&= \int_{\{|e-T_{k,\nu}(e)|\leq\delta\}} \mathbf{q}(T_{k+\delta}(e), \nabla T_k(e))) \cdot \nabla(T_{k+\delta}(e) - T_k(e)) + o(1) \\
&= \int_{\{|e-T_{k,\nu}(e)|\leq\delta\}} \mathbf{q}(e, \nabla T_k(e)) \cdot \nabla(e - T_k(e)) + o(1) = o(1),
\end{aligned} \tag{4.37}
$$

where $o(1)$ vanishes as $N \to +\infty$. When we look for

$$
\begin{aligned}
I_4^{N,k,\nu,\delta} &:= \int_{\{|e^N - T_{k,\nu}(e)|\leq\delta\}} \mathbf{q}(e^N, \nabla e^N) \cdot \nabla(e^N - T_{k,\nu}(e) + T_{k,\nu}(e) - T_k(e)) \\
&= I_6^{N,k,\nu,\delta} + \int_{\{|e^N - T_{k,\nu}(e)|\leq\delta\}} \mathbf{q}(e^N, \nabla e^N) \cdot \nabla(T_{k,\nu}(e) - T_k(e)) \\
&\leq I_6^{N,k,\nu,\delta} + \|\mathbf{q}(T_{k+\delta}(e), \nabla T_{k+\delta}(e))\|_{q',Q} \|\nabla(T_{k,\nu}(e) - T_k(e))\|_{q,Q},
\end{aligned}
$$

we take $\phi = T_k(e^N) \in W_q$ as a test function in (4.13) and recalling (2.31) we can rewrite the estimates (4.6)-(4.7) as

$$
\begin{aligned}
\|\nabla T_k(e^N)\|_{q,Q}^q &\leq (\|e_0\|_{1,\Omega} + C)\, k; \\
\|\mathbf{q}(T_k(e^N), \nabla T_k(e^N))\|_{q',Q}^{q'} &\leq C(k+1).
\end{aligned}
$$

Then, for every $k > 0$,

$$\nabla T_k(e^N) \underset{N\to+\infty}{\rightharpoonup} \nabla T_k(e) \text{ in } \mathbf{L}^q(Q),$$

and we can conclude the strong convergence of $T_{k,\nu}(e)$ to $T_k(e)$ in $L^q(0,T;W_q)$ as ν tends to infinity.

Now, it remains to show that

$$I_6^{N,k,\nu,\delta} = \int_{\{|e^N - T_{k,\nu}(e)|\leq\delta\}} \mathbf{q}(e^N,\nabla e^N)\cdot\nabla(e^N - T_{k,\nu}(e)) \to 0.$$

Thus, taking $\phi = T_\delta(e^N - T_{k,\nu}(e))$ as a test function in (4.13) we have

$$\begin{aligned} I_6^{N,k,\nu,\delta} &= -\langle \partial_t e^N, T_\delta(e^N - T_{k,\nu}(e))\rangle - \left(\nabla e^N \cdot \mathcal{M}_\mu(\mathbf{v}^N), T_\delta(e^N - T_{k,\nu}(e))\right) \\ &\quad - \int_\Sigma \gamma(e^N) T_\delta(e^N - T_{k,\nu}(e))\, dS\, dt + \int_\Sigma \varphi(e^N)|\mathbf{v}_T^N|^s T_\delta(e^N - T_{k,\nu}(e))\, dS\, dt \\ &\qquad + \left(\tau(e^N, D\mathbf{v}^N) : D\mathbf{v}^N, T_\delta(e^N - T_{k,\nu}(e))\right) \\ &\leq \Big(C(\mu)\|\nabla e^N\|_{r,Q}^r + \|\gamma(e^N)\|_{1,\Sigma} + C\|D\mathbf{v}^N\|_{p,Q}^p + \varphi^*\|\mathbf{v}^N\|_{s,\Sigma}^s\Big)\,\delta, \end{aligned}$$

applying the following property [Boccardo et al., 1997, Lemma 3.1]:

$$\langle \partial_t e^N, T_\delta(e^N - T_{k,\nu}(e))\rangle \geq o(1), \quad \forall k,\nu,\delta. \tag{4.38}$$

Thus recalling the estimates (4.17), (4.15), (4.19), (4.20) and taking $\delta \to 0$, we conclude (4.32).

5. Proof of Theorem 3.1 (μ fixed)

In order to pass to the limit, when ε tends to zero, the estimate (4.24) is no more valid. To estimate π_ε independently of ε we choose $\mathbf{w} = \nabla\eta_\varepsilon$ as a test function in (3.4), where η_ε is the solution of the following homogeneous Neumann problem for the Laplace equation (see, for instance, Galdi [1994])

$$\begin{aligned} \Delta\eta_\varepsilon(t) &= |\pi_\varepsilon(t)|^{p'-2}\pi_\varepsilon(t) - \frac{1}{|\Omega|}\int_\Omega |\pi_\varepsilon(t)|^{p'-2}\pi_\varepsilon(t)dx \text{ in } \Omega \\ \nabla\eta_\varepsilon(t)\cdot\mathbf{n} &= 0 \text{ on } \partial\Omega \\ \int_\Omega \eta_\varepsilon(t)dx &= 0, \end{aligned}$$

which satisfies

$$\|\eta_\varepsilon(t)\|_{2,p}^p \leq C(\Omega,p)\|\pi_\varepsilon(t)\|_{0,p'}^{p'}, \qquad \text{a.e. } t \in]0,T[. \tag{5.1}$$

Then it results

$$\|\pi_\varepsilon\|_{p',Q}^{p'} = \sum_{i=1}^6 I_i$$

where, using embedding, Hölder and Young inequalities,

$$
\begin{aligned}
I_1 &:= \langle \partial_t \mathbf{v}_\varepsilon, \nabla \eta_\varepsilon \rangle \le 0^1 \\
I_2 &:= -\int_Q \mathcal{M}_\mu(\mathbf{v}_\varepsilon) \otimes \mathbf{v}_\varepsilon : D\nabla \eta_\varepsilon \le C \|\mathcal{M}_\mu(\mathbf{v}_\varepsilon) \otimes \mathbf{v}_\varepsilon\|_{p',Q}^{p'} + \frac{1}{6}\|\pi_\varepsilon\|_{p',Q}^{p'}; \\
I_3 &:= \int_Q \tau(e_\varepsilon, D\mathbf{v}_\varepsilon) : D\nabla\eta_\varepsilon \le C\|\tau(e_\varepsilon, D\mathbf{v}_\varepsilon)\|_{p',Q}^{p'} + \frac{1}{6}\|\pi_\varepsilon\|_{p',Q}^{p'}; \\
I_4 &:= \int_{\Sigma_D} \alpha(e_\varepsilon)\mathbf{v}_{\varepsilon T} \cdot \nabla \eta_\varepsilon; \\
I_5 &:= \int_\Sigma \varphi(e_\varepsilon)|\mathbf{v}_{\varepsilon T}|^{s-2}\mathbf{v}_{\varepsilon T} \cdot \nabla\eta_\varepsilon; \\
I_6 &:= -\int_Q \mathbf{f} \cdot \nabla \eta_\varepsilon \le \|\mathbf{f}\|_{p',Q}^{p'} + \frac{1}{6}\|\pi_\varepsilon\|_{p',Q}^{p'}.
\end{aligned}
$$

For $p \ge 2(n+1)/(n+2)$, the integrals I_2, I_3 and I_6 are estimated independently of ε due to (4.21), (4.20) and (2.34), respectively. To estimate the remaining boundary integrals, i.e., terms I_4, I_5 it is enough to combine Lemma 2.1 with our assumptions on s-(2.43):

$$
I_4 \quad \le \begin{cases} C\int_0^T \|\mathbf{v}_\varepsilon\|_{\frac{p(n-1)}{n(p-1)},\Gamma_D}^{p'} dt + \frac{1}{6}\|\pi_\varepsilon\|_{p',Q}^{p'} & p < n; \\ C\int_0^T \|\mathbf{v}_\varepsilon\|_{1,\Gamma_D}^{p'} dt + \frac{1}{6}\|\pi_\varepsilon\|_{p',Q}^{p'} & p \ge n; \end{cases}
$$

$$
I_5 \le \begin{cases} C\int_0^T \|\mathbf{v}_\varepsilon\|_{\frac{(s-1)p(n-1)}{n(p-1)},\Gamma}^{p'(s-1)} dt + \frac{1}{6}\|\pi_\varepsilon\|_{p',Q}^{p'}, & p < n; \\ C\int_0^T \|\mathbf{v}_\varepsilon\|_{(s-1),\Gamma}^{p'(s-1)} dt + \frac{1}{6}\|\pi_\varepsilon\|_{p',Q}^{p'}, & p \ge n. \end{cases}
$$

We can extract a subsequence, still denoted by $(\mathbf{v}_\varepsilon, e_\varepsilon, \pi_\varepsilon)$, verifying (4.15)-(4.22)

[a]We give only a formal proof of this inequality. We denote by f_ε function solving the equation

$$\triangle f_\varepsilon = \operatorname{div} \mathbf{v}_\varepsilon = \frac{1}{\varepsilon}\triangle \pi_\varepsilon$$

with homogeneous Neumann boundary condition. Then we obtain that

$$I_1 = \langle \partial_t \nabla f_\varepsilon, \nabla\eta_\varepsilon\rangle = -\frac{1}{\varepsilon}\int_Q \partial_t\pi_\varepsilon \cdot \triangle\eta_\varepsilon = -\frac{1}{2p'\varepsilon}(\|\pi_\varepsilon(T)\|_{p'}^{p'} - \|\pi_\varepsilon(0)\|_{p'}^{p'}) \le 0$$

because $\pi_\varepsilon(0) \equiv 0$ since $\operatorname{div} \mathbf{v}_\varepsilon(0) = 0$.

such that

$$
\begin{aligned}
\mathbf{v}_\varepsilon &\rightharpoonup \mathbf{v} && \text{weakly* in } L^\infty(0,T;L^2_N);\\
\mathbf{v}_\varepsilon &\rightharpoonup \mathbf{v} && \text{weakly in } L^p(0,T;W^{1,p}_N);\\
\partial_t\mathbf{v}_\varepsilon &\rightharpoonup \partial_t\mathbf{v} && \text{weakly in } L^{p'}(0,T;(W^{1,p}_N)');\\
\mathbf{v}_\varepsilon &\to \mathbf{v} && \text{strongly in } \mathbf{L}^m(Q), \quad \text{for } 1\le m< p(n+2)/n;\\
\mathbf{v}_\varepsilon &\to \mathbf{v} && \text{strongly in } \mathbf{L}^2(\Sigma_D);\\
\mathbf{v}_\varepsilon &\to \mathbf{v} && \text{strongly in } \mathbf{L}^s(\Sigma), \quad \text{for } s \text{ fulfilling } (2.44);\\
e_\varepsilon &\rightharpoonup e && \text{weakly in } L^r(0,T;W_r), \quad \text{for } 1<r<q-n/(n+1);\\
e_\varepsilon &\to e && \text{strongly in } L^m(Q), \quad \text{for } 1\le m< r(n+1)/n;\\
e_\varepsilon &\to e && \text{strongly in } L^l(\Sigma), \quad \text{for } l \text{ fulfilling } (2.44);\\
\pi_\varepsilon &\rightharpoonup \pi && \text{weakly in } L^{p'}(Q);\\
\tau(e_\varepsilon, D\mathbf{v}_\varepsilon) &\rightharpoonup \bar\tau && \text{weakly in } \mathbf{L}^{p'}(Q);\\
\mathbf{q}(e_\varepsilon, \nabla e_\varepsilon) &\rightharpoonup \bar{\mathbf{q}} && \text{weakly in } \mathbf{L}^{r/(q-1)}(Q).
\end{aligned}
$$

Furthermore $(\mathbf{v}, e, \pi)$ is the limit solution given by

$$
\begin{aligned}
&\langle \partial_t\mathbf{v}, \mathbf{w}\rangle + (D\mathbf{v}, \mathcal{M}_\mu(\mathbf{v})\otimes\mathbf{w}) + (\tau(e,D\mathbf{v}), D\mathbf{w}) + \int_{\Sigma_D} \alpha(e)\mathbf{v}_T\cdot\mathbf{w}_T\, dS\, dt\\
&\quad + \int_\Sigma \varphi(e)|\mathbf{v}_T|^{s-2}\mathbf{v}_T\cdot\mathbf{w}_T\, dS\, dt = \langle \mathbf{f}, \mathbf{w}\rangle + (\pi, \operatorname{div}\mathbf{w}) \qquad (5.2)\\
&\quad\quad \text{for all } \mathbf{w}\in L^p(0,T;W^{1,p}_N),\\
&\langle \partial_t e, \phi\rangle + (\mathcal{M}_\mu(\mathbf{v})\cdot\nabla e, \phi) + (\mathbf{q}(\mathbf{e}, \nabla\mathbf{e}), \nabla\phi) + \int_\Sigma \gamma(e)\phi\, dS\, dt\\
&\quad = (\tau(e,D\mathbf{v}), D\mathbf{v}\phi) + \int_\Sigma \varphi(e)|\mathbf{v}_T|^s\phi\, dS\, dt \qquad (5.3)\\
&\quad\quad \text{for all } \phi\in L^\infty(0,T;W_{r/(r-q+1)}),\\
&\operatorname{div}\mathbf{v} = 0, \qquad \mathbf{v}(\cdot,0) = \mathbf{v}_0, \quad e(\cdot,0) = e_0,
\end{aligned}
$$

providing that $\bar\tau = \tau(e, D\mathbf{v})$, $\bar{\mathbf{q}} = \mathbf{q}(e,\nabla e)$ and

$$\mathbf{v}_\varepsilon \rightharpoonup \mathbf{v} \quad \text{strongly in } L^p(0,T;W^{1,p}_N).$$

Arguing as in the section 4.5 we use the monotonicity assumption (2.29) to get for all $\varphi\in L^p(0,T;W^{1,p}_N)$

$$0 \le \langle \tau(e_\varepsilon, D\mathbf{v}_\varepsilon) - \tau(e_\varepsilon, D\varphi), D\mathbf{v}_\varepsilon - D\varphi\rangle \qquad (5.4)$$

and taking $\mathbf{w} := \mathbf{v}_\varepsilon$ as a test function in (3.4) we obtain after passing to the limit as $\varepsilon\to 0$ and using (5.2)

$$(\bar\tau - \tau(e, D\varphi), D(\mathbf{v}-\varphi)) \ge \lim_{\varepsilon\to 0} -(\pi_\varepsilon, \operatorname{div}\mathbf{v}_\varepsilon) = \lim_{\varepsilon\to 0} \varepsilon\|\nabla\pi_\varepsilon\|^2_{2,Q} \ge 0$$

and we can conclude that $\bar\tau = \tau(e, D\mathbf{v})$ by using the Minty method. Note that it is not required the strong convergence of the pressure since (3.6) holds for $\phi = \pi_\varepsilon$.

Finally, it remains to prove that $\bar{\mathbf{q}} = \mathbf{q}(e, \nabla e)$. Again using the continuity of $\mathbf{q}(x, t, \cdot, \cdot)$ and Vitali Theorem, it is sufficient to prove that ∇e_ε a.e. converges to ∇e. Indeed, the argument used for N in Section 4.6 can be repeated for ε.

Then $(\mathbf{v}, e, \pi)$ is the corresponding limit solution (3.2)-(3.3).

6. Proof of Theorem 2.1

First, we rewrite the system (3.2)-(3.3) into the form of equations (2.39)-(2.41). To do it, we set in (3.2) $\mathbf{w} := \mathbf{v}_\mu \phi$ add the result equation to (3.3) and obtain

$$\begin{aligned}
&\langle \partial_t \mathbf{v}_\mu, \mathbf{w} \rangle + (\mathbf{v}_\mu \otimes \mathcal{M}_\mu(\mathbf{v}_\mu), D\mathbf{w}) + (\tau(e_\mu, D\mathbf{v}_\mu), D\mathbf{w}) \\
&\quad + \int_{\Sigma_D} \alpha(e_\mu) \mathbf{v}_{\mu T} \cdot \mathbf{w}_T \, dS \, dt + \int_\Sigma \varphi(e_\mu) |\mathbf{v}_{\mu T}|^{s-2} \mathbf{v}_{\mu T} \cdot \mathbf{w}_T \, dS \, dt \\
&\quad = \langle \mathbf{f}, \mathbf{w} \rangle + (\pi_\mu \operatorname{div} \mathbf{w}) \\
&\qquad \text{for all } \mathbf{w} \in L^p(0, T; W^{1,p}_N)
\end{aligned} \tag{6.1}$$

$$\begin{aligned}
&- \left\langle \frac{|\mathbf{v}_\mu|^2}{2} + e_\mu, \partial_t \phi \right\rangle + (\mathbf{q}(\cdot, e_\mu, \nabla e_\mu) - \tau(\cdot, e_\mu, D\mathbf{v}_\mu) \mathbf{v}_\mu, \nabla \phi) \\
&\quad = \langle \mathbf{f}, \mathbf{v}_\mu \phi \rangle + \int_\Sigma \varphi(\cdot, e_\mu) |\mathbf{v}_{\mu T}|^s \phi \, dS \, dt - \int_\Sigma \gamma(\cdot, e_\mu) \phi \, dS \, dt \\
&\quad + \left(\left(\frac{|\mathbf{v}_\mu|^2}{2} + e_\mu + \pi_\mu \right) \mathbf{v}_\mu, \nabla \phi \right) + \left(\frac{|\mathbf{v}_0|^2}{2} + e_0, \phi(0) \right) \\
&\qquad \text{for all } \phi \in C^1(\bar{Q}) : \phi(T) = 0,
\end{aligned} \tag{6.2}$$

$$\begin{aligned}
&\langle \partial_t e_\mu, \phi \rangle - (e_\mu \mathcal{M}_\mu(\mathbf{v}_\mu), \nabla \phi) + (\mathbf{q}(e_\mu, \nabla e_\mu), \nabla \phi) + \int_\Sigma \gamma(e_\mu) \phi \, dS \, dt \\
&\quad = (\tau(e_\mu, D\mathbf{v}_\mu), D\mathbf{v}_\mu \phi) + \int_\Sigma \varphi(e_\mu) |\mathbf{v}_{\mu T}|^s \phi \, dS \, dt \\
&\qquad \text{for all } \phi \in L^\infty(0, T; W_{r/(r-q+1)}).
\end{aligned} \tag{6.3}$$

We decompose the pressure π_μ such that $\pi_\mu := \pi_{\mu,1} + \pi_{\mu,2}$ where the two particular pressures, $\pi_{\mu,1}$ and $\pi_{\mu,2}$, belong to bounded sets of $L^{p(n+2)/(2n)}(Q)$ and $L^{p'}(Q)$, respectively, independently of μ. For each $t \in]0, T[$, let us introduce $\pi_{\mu,1}$ as the unique solution to the problem (for details see Bulíček et al. [2007b])

$$\begin{aligned}
&-\langle \pi_{\mu,1}(t), \Delta \phi \rangle = \langle \mathbf{v}_\mu \otimes \mathcal{M}_\mu(\mathbf{v}_\mu)(t), D\nabla \phi \rangle, \ \forall \phi \in W^{2,p}(\Omega) : \ \nabla \phi \in W^{1,p}_N, \\
&\int_\Omega \pi_{\mu,1}(t) dx = 0,
\end{aligned} \tag{6.4}$$

and define $\pi_{\mu,2} := \pi_\mu - \pi_{\mu,1}$. Since $\operatorname{div} \mathbf{v}_\mu = 0$, $\pi_{\mu,2}$ solves at each time level

$$\begin{aligned}
\langle \pi_{\mu,2}, \Delta \phi \rangle &= \int_\Omega \tau(e_\mu, D\mathbf{v}_\mu) : D\nabla \phi + \int_{\Gamma_D} \alpha(e_\mu) \mathbf{v}_{\mu T} \cdot \nabla \phi + \\
&+ \int_\Gamma \varphi(e_\mu) |\mathbf{v}_{\mu T}|^{s-2} \mathbf{v}_{\mu T} \cdot \nabla \phi - \int_\Omega \mathbf{f} \cdot \nabla \phi, \ \forall \phi \in W^{2,p}(\Omega) : \ \nabla \phi \in W^{1,p}_N.
\end{aligned} \tag{6.5}$$

Using (4.23) and (4.20), it follows that

$$\|\pi_{\mu,1}\|_{p(n+2)/(2n),Q} \leq C;$$
$$\|\pi_{\mu,2}\|_{p',Q} \leq C.$$

Thus, we conclude the following uniform estimates

$$\|\partial_t \mathbf{v}_\mu\|_{\mathcal{X}} \leq C\left(1 + \|\mathbf{v}_\mu\|^p_{p,W^{1,p}_N} + \|\mathbf{v}_\mu\|^2_{p(n+2)/n,Q}\right);$$
$$\|e_\mu \mathbf{v}_\mu\|_{1,(W_\beta)'} \leq C;$$

for $1/\beta = 1 - n/[r(n+1)] - n/[p(n+2)] > 0$,

$$\|\partial_t e_\mu\|_{1,(W_\beta)'} \leq C.$$

In order to pass to the limit in (3.2) and (6.2)-(6.3) when μ tends to infinity, we can extract a subsequence, still denoted by $(\mathbf{v}_\mu, e_\mu, \pi_\mu)$, verifying (4.15)-(4.22) such that

$$\begin{aligned}
\mathbf{v}_\mu &\rightharpoonup \mathbf{v} && \text{weakly* in } L^\infty(0,T;L^2_N); \\
\mathbf{v}_\mu &\rightharpoonup \mathbf{v} && \text{weakly in } L^p(0,T;W^{1,p}_N); && (6.6)\\
\partial_t\mathbf{v}_\mu &\rightharpoonup \partial_t\mathbf{v} && \text{weakly in } \mathcal{X}; \\
\mathbf{v}_\mu &\to \mathbf{v} && \text{strongly in } \mathbf{L}^m(Q), \quad \text{for } 1 \leq m < p(n+2)/(2n); && (6.7)\\
\mathbf{v}_\mu &\to \mathbf{v} && \text{strongly in } \mathbf{L}^2(\Sigma_D); && (6.8)\\
\mathbf{v}_\mu &\to \mathbf{v} && \text{strongly in } \mathbf{L}^s(\Sigma), \quad \text{for } s \text{ fulfilling (2.44)}; && (6.9)\\
e_\mu &\rightharpoonup e && \text{weakly in } L^r(0,T;W_r), \quad \text{for } 1 < r < q - n/(n+1); \\
e_\mu &\to e && \text{strongly in } L^m(Q), \quad \text{for } 1 \leq m < r(n+1)/n; && (6.10)\\
e_\mu &\to e && \text{strongly in } L^l(\Sigma), \quad \text{for } l \text{ fulfilling (2.44)}; \\
\pi_\mu &\rightharpoonup \pi && \text{weakly in } L^{p(n+2)/(2n)}(Q); \\
\pi_{\mu,1} &\to \pi_1 && \text{strongly in } L^m(Q), \text{ for } 1 \leq m < p(n+2)/(2n); && (6.11)\\
\pi_{\mu,2} &\rightharpoonup \pi_2 && \text{weakly in } L^{p'}(Q); \\
\tau(e_\mu, D\mathbf{v}_\mu) &\rightharpoonup \bar{\tau} && \text{weakly in } \mathbf{L}^{p'}(Q); \\
\mathbf{q}(e_\mu, \nabla e_\mu) &\rightharpoonup \bar{\mathbf{q}} && \text{weakly in } \mathbf{L}^{r(q-1)}(Q).
\end{aligned}$$

Note that (6.11) is consequence of (6.4) together with (6.7). Then $(\mathbf{v}, e, \pi)$ is the limit solution (2.39)-(2.41), if we prove that $D\mathbf{v}_\mu$ is a.e. convergent to $D\mathbf{v}$ (see Section 6.1), and as in the sections 4 and 5, we need at least pointwise convergence of ∇e_μ to ∇e. However the method that has been successfully used in the section 4.6 cannot be used here because the term $\mathbf{u}\nabla e$ is not integrable function anymore. We introduce the method of Lipschitz truncation function (cf. Section 6.2). This method was firstly used by Kinnunen and Lewis in Kinnunen and Lewis [2002] to improve integrability of very weak solution to the incompressible Navier-Stokes equations with power-law relationship. For proof of existence of solution to Navier-Stokes equation it was firstly used by Diening, Růžička and Wolf in Diening et al.

[2006]. The authors were able to establish the existence of solution for all $p > \frac{2n}{n+2}$ by using this method. However this method cannot be simply used also on heat equation because of L^1 term on the right-hand side. To be more concrete, the most important ingredient of this method is the fact that dissipative term has some $q-1$ growth and apriori estimates gives boundedness in some q space. In our setting this is not true because our estimates on the ∇e are only in the space L^r $(r < q)$ and the growth of $\mathbf{q}$ is $q-1$. However this not true for $T_k(e)$. For this function we have desired estimates and growth. Hence, the main idea is to used Lipschitz truncation method onto this function. The second problem is that for using this method we need to say something about time derivative of $T_k(e)$. However, our limit function do not have time derivative, hence we are led to use this method only for double sequence (e_n, e_m). Moreover, it will be clear from the proof (cf. Section 6.3), we in fact do not know anything also about time derivative of $T_k(e_n)$ but we have some information about this derivative for some mollified function $\mathcal{T}_{k,\delta}(e_m)$.

6.1. *Almost everywhere convergence of* $D\mathbf{v}_\mu$

As in Section 4.6 by the monotonicity assumption (2.29), it is sufficient to prove that, for some $\theta > 0$ and for some subsequence,

$$\lim_{k\to+\infty} \int_Q [(\tau(e_k, D\mathbf{v}_k) - \tau(e_k, D\mathbf{v})) : D(\mathbf{v}_k - \mathbf{v})]^\theta = 0. \tag{6.12}$$

We adapt the argument described in Frehse et al. [2000] for flows with shear dependent viscosity. Let $\mu \in \mathbb{N}$ and set

$$g_\mu := (|\nabla \mathbf{v}_\mu| + |\nabla \mathbf{v}|)^p + (|\tau(e_\mu, D\mathbf{v}_\mu)| + |\tau(e, D\mathbf{v})|)(|D\mathbf{v}_\mu| + |D\mathbf{v}|).$$

From the estimates (4.19) and (4.20), there exists a constant $K \geq 1$ such that

$$0 \leq \int_Q g_\mu dx\ dt \leq K.$$

Let $\delta > 0$ be arbitrary and fixed, there exists $L \leq \delta^{p/\theta}/K$ and a subsequence $\{\mathbf{v}_k\}_{k\in\mathbb{N}} \subset \{\mathbf{v}_\mu\}_{\mu\in\mathbb{N}}$ that satisfy (for details see Frehse et al. [2000])

$$E_k = \{(x,t) \in Q :\ L^2 \leq |(\mathbf{v}_k - \mathbf{v})(x,t)| < L\} :\ \int_{E_k} g_k dx\ dt \leq \delta^{p/\theta}.$$

Thus, we can decompose the following integral as

$$\int_Q [(\tau(e_k, D\mathbf{v}_k) - \tau(e_k, D\mathbf{v})) : D(\mathbf{v}_k - \mathbf{v})]^\theta =$$
$$= \int_{\{|\mathbf{v}_k - \mathbf{v}| \geq L\}} [\cdots]^\theta + \int_{\{|\mathbf{v}_k - \mathbf{v}| < L\}} [\cdots]^\theta := I_1^{k,L} + I_2^{k,L}.$$

Using Hölder inequalities for $\theta < 1$ and (6.6)-(6.9) we get

$$\begin{aligned}
I_1^{k,L} &\le \left(\int_Q g_k\right)^\theta [\text{meas}(\{|\mathbf{v}_k - \mathbf{v}| \ge L\})]^{1-\theta} \\
&\le K^\theta [\text{meas}(\{|\mathbf{v}_k - \mathbf{v}| \ge L\})]^{1-\theta} \underset{k\to+\infty}{\longrightarrow} 0, \\
I_2^{k,L} &\le (I_3^{k,L} + I_4^{k,L})^\theta [\text{meas}(\{|\mathbf{v}_k - \mathbf{v}| < L\})]^{1-\theta} \\
I_3^{k,L} &:= \int_{\{|\mathbf{v}_k - \mathbf{v}| < L\}} \tau(e_k, D\mathbf{v}_k) : D(\mathbf{v}_k - \mathbf{v}), \\
I_4^{k,L} &:= \int_{\{|\mathbf{v}_k - \mathbf{v}| < L\}} \tau(e_k, D\mathbf{v}) : D(\mathbf{v} - \mathbf{v}_k) \underset{k\to+\infty}{\longrightarrow} 0.
\end{aligned}$$

In order to estimate $I_3^{k,L}$, we define

$$\mathbf{w}_k = (\mathbf{v}_k - \mathbf{v}) \left(1 - \min\left(\frac{|\mathbf{v}_k - \mathbf{v}|}{L}, 1\right)\right).$$

By using (6.6)-(6.9) and $|\mathbf{w}_k| \le L$ a.e. in $\bar{Q}$, we get

$$\begin{aligned}
&\mathbf{w}_k \rightharpoonup 0 \quad \text{weakly in} \quad L^p(0, T; W_N^{1,p}); \\
&\mathbf{w}_k \to 0 \quad \text{strongly in} \quad \mathbf{L}^m(Q), \quad \text{for } 1 \le m < +\infty; \\
&\mathbf{w}_k \to 0 \quad \text{strongly in} \quad \mathbf{L}^m(\Sigma_D); \\
&\mathbf{w}_k \to 0 \quad \text{strongly in} \quad \mathbf{L}^m(\Sigma).
\end{aligned} \tag{6.13}$$

Next, we decompose $\mathbf{w}_k$ as

$$\mathbf{w}_k = \mathbf{w}_{k,\text{div}} + \nabla \eta_k \ ,$$

due to L^m-theory for the Laplace operator, it follows that

$$\nabla \eta_k \to 0 \quad \text{strongly in} \quad \mathbf{L}^m(Q), \quad \text{for all } 1 \le m < +\infty, \tag{6.14}$$

and consequently

$$\mathbf{w}_{k,\text{div}} \to 0 \quad \text{strongly in} \quad \mathbf{L}^m(Q), \quad \text{for all } 1 \le m < +\infty. \tag{6.15}$$

On the other hand, using the L^p-regularity for the Laplace operator it follows that

$$\begin{aligned}
\int_0^T \|\nabla \eta_k(t)\|^p_{p(n-1)/(n-p),\Gamma} dt &\le C(\Omega, p) \|\nabla^2 \eta_k\|^p_{p,Q} \\
&\le C(\Omega, p) \|\operatorname{div} \mathbf{w}_k\|^p_{p,Q} \\
\le C \left(\int_{E_k} |\nabla(\mathbf{v}_k - \mathbf{v})|^p + L \int_{\{|\mathbf{v}_k - \mathbf{v}| < L^2\}} |\nabla(\mathbf{v}_k - \mathbf{v})|^p\right) \\
\le C \left(\int_{E_k} g_k dx\, dt + \frac{\delta}{K} \int_Q g_k dx\, dt\right) &\le C\delta^{p/\theta}.
\end{aligned} \tag{6.16}$$

Then

$$\mathbf{w}_{k,\mathrm{div}} \rightharpoonup 0 \quad \text{weakly in} \quad L^p(0,T;W_N^{1,p}); \tag{6.17}$$
$$\mathbf{w}_{k,\mathrm{div}} \to 0 \quad \text{strongly in} \quad \mathbf{L}^2(\Sigma_D); \tag{6.18}$$
$$\mathbf{w}_{k,\mathrm{div}} \to 0 \quad \text{strongly in} \quad \mathbf{L}^s(\Sigma). \tag{6.19}$$

Next, after some calculations we can decompose the integral $I_3^{k,L}$ as

$$\begin{aligned}
I_3^{k,L} &= \int_Q \tau(e_k, D\mathbf{v}_k) : D\mathbf{w}_{k,\mathrm{div}} + I_5^{k,L} + I_6^{k,L} + I_7^{k,L},\\
I_5^{k,L} &:= \int_{\{|\mathbf{v}_k-\mathbf{v}|<L\}} \tau(e_k, D\mathbf{v}_k) : D\nabla\eta_k,\\
I_6^{k,L} &:= \int_{\{|\mathbf{v}_k-\mathbf{v}|<L\}} \tau(e_k, D\mathbf{v}_k) : D(\mathbf{v}_k-\mathbf{v})\frac{|\mathbf{v}_k-\mathbf{v}|}{L},\\
I_7^{k,L} &:= \int_{\{|\mathbf{v}_k-\mathbf{v}|<L\}} \tau(e_k, D\mathbf{v}_k) : \frac{(\mathbf{v}_k-\mathbf{v})}{L} \otimes \nabla(|\mathbf{v}_k-\mathbf{v}|).
\end{aligned} \tag{6.20}$$

From (6.16) and the uniform estimates (4.19)-(4.20) we have

$$|I_5^{k,L}|^\theta \le \|\tau(e_k, D\mathbf{v}_k)\|_{p',Q}^\theta C\delta \le C\delta.$$

For the integrals $I_6^{k,L}$ and $I_7^{k,L}$, we can argue as in the estimate (6.16) concluding that

$$\begin{aligned}
|I_6^{k,L}| &\le \int_{E_k} g_k dx\ dt + L\int_Q g_k dx\ dt \le 2\delta^{p/\theta} < 2\delta^{1/\theta}\ ;\\
|I_7^{k,L}| &\le \int_{E_k} g_k dx\ dt + L\int_Q g_k dx\ dt \le 2\delta^{p/\theta} < 2\delta^{1/\theta},
\end{aligned}$$

for $0<\delta<1$ and $p>1$.

Finally to study the remaining integral in (6.20), we choose $\mathbf{w}=\mathbf{w}_{k,\mathrm{div}}$ (divergenceless function) as a test function in (3.2) and we pass to the limit as $k\to+\infty$ to obtain (6.12). Indeed we use the convergences (6.15), (6.18)-(6.19) and observe that

$$\begin{aligned}
\langle \partial_t \mathbf{v}_k, \mathbf{w}_{k,\mathrm{div}}\rangle &= \langle \partial_t(\mathbf{v}_k-\mathbf{v}), \mathbf{w}_k\rangle + \langle \partial_t(\mathbf{v}_k-\mathbf{v}), \nabla\eta_k\rangle + \langle \partial_t\mathbf{v}, \mathbf{w}_{k,\mathrm{div}}\rangle\\
&\ge \langle \partial_t(\mathbf{v}_k-\mathbf{v}), \nabla\eta_k\rangle + \langle \partial_t\mathbf{v}, \mathbf{w}_{k,\mathrm{div}}\rangle \underset{k\to+\infty}{\longrightarrow} 0,
\end{aligned}$$

applying the convergences (6.14) and (6.17).

6.2. *Lipschitz truncation*

Here, we give several lemmas that will be needed in what follows. Let us begin with introductory remarks. We denote by the symbol d_α the modified parabolic metric, that is defined on $\mathbb{R}^{n+1}$ such that

$$d_\alpha(X,Y) := \max\left(|x-y|, \frac{|t-s|^{1/2}}{\alpha^{1/2}}\right), \quad \alpha>0,$$

where $X, Y \in \mathbb{R}^{n+1}$, $X := (x,t)$, $Y := (y,s)$. We also define the so-called parabolic cube Q_R^α by

$$Q_R^\alpha(X) := \left\{Y \in \mathbb{R}^{n+1}; d_\alpha(X,Y) < R\right\}, \quad R > 0.$$

Next, we introduce an important covering lemma.

Lemma 6.1. *Let $E \subset \mathbb{R}^{n+1}$ be an open bounded set. Then there exists family of cubes $\{Q_{R_i}^\alpha(X_i)\}_{i\in\mathbb{N}}$ and family of smooth functions $\{\psi_i\}_{i\in\mathbb{N}}$ such that*

$$\begin{aligned}
&\bigcup_{i=1}^\infty Q_{R_i/2}^\alpha = \bigcup_{i=1}^\infty Q_{R_i}^\alpha = E\\
&4R_i \le d_\alpha(X_i, \partial E) \le 8R_i, \quad \forall i \in \mathbb{N},\ \textit{with}\ 0 < R_i < 1\\
&R_j > 2R_i \Rightarrow Q_{R_i}^\alpha(X_i) \cap Q_{R_j}^\alpha(X_j) = \emptyset\\
&Q_{R_i/4}^\alpha(X_i) \cap Q_{R_j/4}^\alpha(X_j) = \emptyset \quad \forall i,j \in \mathbb{N}, i \neq j\\
&\mathrm{card}(A_i) \le C(n),\ \forall i \in \mathbb{N}\ \textit{with}\ A_i := \{j \in \mathbb{N}:\ Q_{\frac{2R_i}{3}}^\alpha(X_i) \cap Q_{\frac{2R_j}{3}}^\alpha(X_j) \neq \emptyset\}\\
&\psi_i \in \mathcal{C}_0^\infty(Q_{2R_i/3}^\alpha(X_i)), \quad \forall i \in \mathbb{N}\\
&\alpha R_i^2|\partial_t \psi_i| + R_i|\nabla \psi_i| \le C(n)\ \textit{in}\ \mathbb{R}^{n+1} \quad \forall i \in \mathbb{N}\\
&\sum_{i=1}^\infty \psi_i(X) = 1, \quad \forall X \in E.
\end{aligned} \tag{6.21}$$

Moreover,

$$Q_{R_j}^\alpha(X_j) \subset Q_{4R_i}^\alpha(X_i) \subset E, \quad \forall j \in A_i\ . \tag{6.22}$$

The proof can be found in [Diening et al., 2006, Lemma 3.1, Remark 3.8 and Proposition 3.4].

For sake of completness, we also introduce the notation of some types of the maximal function. All properties of such function that will we described bellow are also proved in [Diening et al., 2006, (see Appendix A)].

We define $\mathcal{M}$ for some $g \ge 0$, $g \in L^a(0,\infty; L^a(\mathbb{R}^n))$, as the parabolic maximal function

$$\mathcal{M}(g)(x,t) := \sup_{0<\rho<\infty} \fint_{(t-\rho,t+\rho)} \left(\sup_{0<R<\infty} \fint_{B_R(x)} g(y,s)\ dy\right)\ ds.$$

Moreover, we also define $\mathcal{M}^\alpha$ as

$$\mathcal{M}^\alpha(g)(x,t) := \sup_{Q_R^\alpha(x,t)} \fint_{Q_R^\alpha(x,t)} g(y,s)\ dy\ ds.$$

The following properties are valid

$$\begin{aligned}
\|\mathcal{M}(g)\|_{a;L^a} &\le \|g\|_{a;L^a}\\
\mathcal{M}^\alpha(g) &\le \mathcal{M}(g) \quad \text{in } \mathbb{R}^{n+1}.
\end{aligned}$$

Note that we use the following notation for mean value over the set Q_R^α for an integrable function u:

$$\overline{u}_{Q_R^\alpha} := \fint_{Q_R^\alpha} u \, dx \, dt.$$

Next lemma is an important Poincaré inequality.

Lemma 6.2. *Let $u, f \in L^1(Q_R^\alpha)$ and $\nabla u, \mathbf{q} \in \mathbf{L}^1(Q_R^\alpha)$ satisfy*

$$-\int_{Q_R^\alpha} u \partial_t \phi = \int_{Q_R^\alpha} \mathbf{q} \cdot \nabla \phi + \int_{Q_R^\alpha} f \phi \quad \forall \phi \in \mathcal{C}_0^\infty(Q_R^\alpha). \tag{6.23}$$

Then

$$\int_{Q_R^\alpha} |u - \overline{u}_{Q_R^\alpha}| \le CR \left(\int_{Q_R^\alpha} |\nabla u| + \alpha |\mathbf{q}| + \alpha R |f| \right). \tag{6.24}$$

Proof. The proof for Q_1^1 is the same as the proof of Theorem 5.1 in Diening et al. [2006]. For general Q_R^α we assume that it is a cube with center $\mathbf{0}$. Let u be defined on Q_R^α. Then we define the function v on Q_1^1 as $v(x,t) := u(Rx, \alpha R^2 t)$. Then we can derive that

$$\partial_t v(x,t) = \alpha R^2 \partial_{\alpha R^2 t} u(Rx, \alpha R^2 t) \overset{(6.23)}{=} \alpha \left(R \operatorname{div}_x \mathbf{q}(Rx, \alpha R^2 t) + R^2 f(Rx, \alpha R^2 t) \right).$$

Thus, we are in position to apply our estimate for Q_1^1 to obtain

$$\int_{Q_1^1} |v(x,t) - \overline{v}_{Q_1^1}| \le C \left(\int_{Q_1^1} |\nabla v(x,t)| + \alpha R |\mathbf{q}(Rx, \alpha R^2 t)| + \alpha R^2 |f(Rx, \alpha R^2 t)| \right).$$

After standard substitution, we easily obtain (6.24). □

Finally, let $E \subset Q$ be an open set and $u \in L^1(Q)$. Let $\{Q_{R_i}^\alpha\}$ be the covering of E from Lemma 6.1 and $\{\psi_i\}$ be the corresponding partition of unity. Then we introduce the following truncation operator $\mathcal{L}_E^\alpha$ such that

$$\mathcal{L}_E^\alpha u(x,t) := \begin{cases} u(x,t) & \text{if } (x,t) \in Q \setminus E \\ \displaystyle\sum_{i=1}^\infty \psi_i \overline{u}_{Q_{R_i}^\alpha} & \text{if } (x,t) \in E \end{cases}$$

It is also proved in Diening et al. [2006] that

$$\|\mathcal{L}_E^\alpha u\|_{a,L^a} \le c \|u\|_{a,L^a}.$$

The last lemma of this subsection deals with the most important behavior of the Lipschitz truncation and say something about its time derivative.

Lemma 6.3. *Let Ω be an open bounded set in $\mathbb{R}^n$, $0 < T < \infty$, $Q := \Omega \times]0,T[$. Let $u \in L^\infty(0,T; L^2(\Omega)) \cap L^r(0,T; W^{1,r}(\Omega))$, $f \in L^1(Q)$ and $\mathbf{q} \in L^{r'}(0,T; (W_r)')$, $(1 < r < \infty)$, be such that*

$$\partial_t u = \operatorname{div} \mathbf{q} + f$$

in sense of distribution. Moreover, let $E \subset\subset Q$ be an open set such that

$$\mathcal{M}^\alpha(|\nabla u|) + \alpha\mathcal{M}^\alpha(|\mathbf{q}|) + \alpha\mathcal{M}^\alpha(|f|) \le C < +\infty, \quad a.e.\ in\ Q \setminus E. \tag{6.25}$$

Then there holds

$$\begin{aligned} \nabla\mathcal{L}^\alpha_E u &\in L^\infty(0,T;L^\infty(\Omega)) \\ \partial_t\left(\mathcal{L}^\alpha_E u\right)\left(\mathcal{L}^\alpha_E u - u\right) &\in L^1\left(Q \setminus E\right) \end{aligned} \tag{6.26}$$

and for all $\phi_1 \in \mathcal{C}^\infty_0(\Omega)$ and all $\phi_2 \in \mathcal{C}^\infty_0(0,T)$ we have

$$\begin{aligned} \int_0^T \langle \partial_t u, T_\varepsilon(\mathcal{L}^\alpha_E u)\phi_1\rangle \phi_2\, dt &= -\int_Q \left(u - \mathcal{L}^\alpha_E u\right)\partial_t\left(T_\varepsilon(\mathcal{L}^\alpha_E u)\right)\phi_1\phi_2\, dx\, dt - \\ -\int_Q \partial_t[\mathcal{T}_\varepsilon(\mathcal{L}^\alpha_E u)]\phi_1\phi_2\, dx\, dt &- \int_Q \left(u - \mathcal{L}^\alpha_E u\right) T_\varepsilon(\mathcal{L}^\alpha_E u)\phi_1\left(\partial_t\phi_2\right)\, dx\, dt \end{aligned} \tag{6.27}$$

where T_ε is the usual truncation function and

$$\mathcal{T}_\varepsilon(v) := \int_0^v T_\varepsilon(s)\, ds.$$

The proof can be found in [Diening et al., 2006, (see Theorem 3.9)]. To be correct there is proved this conclusion without the truncation function T_ε.

6.3. *Pointwise convergence of energy gradient*

First, we relabel $e_\mu = e_{\mu(n)} = e_n$ and we define cut-off function $\zeta \in \mathcal{C}^\infty_0(Q)$ such that

$$\zeta(x,t) := \zeta_1(t)\zeta_2(x),$$

where $\zeta_1 \in \mathcal{C}^\infty_0(0,T)$ and $\zeta_2 \in \mathcal{C}^\infty_0(\Omega)$ such that $\zeta(x,t) = 1$ if $\operatorname{dist}(x,\partial\Omega) > \eta > 0$, $\eta < t < T - \eta$. Thus, we are in the position to apply local theory of monotone operators. Indeed it is enough to show that for some $\theta > 0$ it holds

$$\lim_{n,m\to\infty} \int_Q \left[\zeta(\mathbf{q}(e_n,\nabla e_n) - \mathbf{q}(e_n,\nabla e_m))\cdot\nabla(e_n - e_m)\right]^\theta = 0. \tag{6.28}$$

We can split the integral appearing in (6.28) such (after using monotonocity of $\mathbf{q}$) that

$$\begin{aligned} 0 &\le \int_Q \left[\zeta(\mathbf{q}(e_n,\nabla e_n) - \mathbf{q}(e_n,\nabla e_m))\cdot\nabla(e_n - e_m)\right]^\theta \\ &= \int_{\{|e_n - e_m| \le \varepsilon\}} [\dots]^\theta + \int_{\{|e_n - e_m| > \varepsilon\}} [\dots]^\theta := I_1 + I_2, \quad \text{for } \varepsilon > 0. \end{aligned} \tag{6.29}$$

For $\theta < r/q$, we have

$$\begin{aligned} I_2 &\le C\int_{\{|e_n - e_m| > \varepsilon\}} |\nabla e_m|^{\theta q} + |\nabla e_n|^{\theta q} \\ &\le C\left[\operatorname{meas}(\{(x,t) \in Q:\ |e_n(x,t) - e_m(x,t)| > \varepsilon\})\right]^{\frac{r-\theta q}{r}} \overset{n,m\to\infty}{\longrightarrow} 0. \end{aligned}$$

In order to study the integral I_1 we decompose

$$I_1 = \int_Q [\zeta(\mathbf{q}(e_n, \nabla e_n) - \mathbf{q}(e_n, \nabla e_m)) \cdot \nabla T_\varepsilon(e_n - e_m)]^\theta$$
$$= \int_{\{|e_m|>k\}} [\dots]^\theta + \int_{\{|e_m|\le k\}} [\dots]^\theta := I_3 + I_4, \quad \text{for } k > 0.$$

For I_3 we obtain $I_3 \le \left(\frac{C}{k}\right)^{\frac{r-\theta q}{r}}$. For I_4, considering the minimum principle 4.3, we can compute

$$I_4 = \int_{\{e_m \le k\}} \big[\zeta(\mathbf{q}(e_n, \nabla e_n)\mathcal{T}'_{k+\varepsilon,\delta}(e_n) - \mathbf{q}(e_n, \nabla e_m)\mathcal{T}'_{k,\delta}(e_m))$$
$$\cdot\nabla T_\varepsilon(\mathcal{T}_{k+\varepsilon,\delta}(e_n) - \mathcal{T}_{k,\delta}(e_m))\big]^\theta, \quad \text{for } \delta > 0,$$

where

$$\mathcal{T}_{k,\delta}(e) := \begin{cases} e & \text{if } e \le k \\ k + \delta/2 & \text{if } e > k + \delta \end{cases}$$

is nondecreasing smooth function such that $\mathcal{T}'_{k,\delta} \le 1$.

Next, denoting

$$g_{mn} := |\nabla \mathcal{T}_{k+\varepsilon,\delta}(e_n)|^q + |\nabla \mathcal{T}_{k,\delta}(e_m)|^q,$$

we know (it is consequence of apriori estimates) that for all $\nu > 0$ there exists a sequence $\{\lambda_{mn}\} \subset (2^{2^\nu}, 2^{2^{\nu+1}})$ such that (for details see Diening et al. [2006])

$$\lambda^q_{mn}\text{meas}(\{\mathcal{M}(g_{mn}) > \lambda^q_{mn}\}) \le Ck2^{-\nu}. \tag{6.30}$$

Moreover, if we denote by the symbol D_{mn} the following set

$$D_{mn} := \{(x, t) : \ \mathcal{M}(g_{mn}) > \lambda^q_{mn}\},$$

then we have

$$I_4 = \int_{\{e_m \le k\}\cap D_{mn}} [\dots]^\theta + \int_{\{e_m \le k\}\setminus D_{mn}} [\dots]^\theta =: I_5 + I_6.$$

The integral I_5 can be estimated as $I_5 \le \left(\frac{ck}{\lambda^r_{mn}}\right)^{\frac{r-\theta q}{r}}$. Using the fact that for $e_m \ge k + \delta$ the term in the integral has the correct sign we can compute

$$I_6 \le \int_{(\{e_m \le k\}\cup\{e_m \ge k+\delta\})\setminus D_{mn}} [\dots]^\theta \le C \left(\int [\dots]\right)^\theta$$

Thus, we have

$$CI_6^{\frac{1}{\theta}} \le \int_{Q\setminus D_{mn}} [\dots] - \int_{\{k<e_m<k+\delta\}\setminus D_{mn}} [\dots]$$
$$=: I_7 + I_8$$

But for I_8 we have the estimate (for a.a. k and δ)

$$\begin{aligned}|I_8| &\le C\lambda_{mn}^q \operatorname{meas}(\{k < e_m < k+\delta\}) \\ &\overset{m\to\infty}{\le} C\left(\sup_{mn}\lambda_{mn}^q\right)\operatorname{meas}(\{k < e < k+\delta\}) \overset{\delta\to 0}{\longrightarrow} 0.\end{aligned}$$

To simplify the notation, hencefurther we denote by ω_{mn} the function

$$\omega_{mn} := \mathcal{T}_{k+\varepsilon,\delta}(e_n) - \mathcal{T}_{k,\delta}(e_m).$$

Thus, we apply Hölder inequality after we see that I_7 can be rewritten in the following form

$$\begin{aligned}I_7 = \int_{Q\setminus D_{mn}} &[\zeta(\mathbf{q}(e_n,\nabla e_n)\mathcal{T}'_{k+\varepsilon,\delta}(e_n) - \mathbf{q}(e_m,\nabla e_m)\mathcal{T}'_{k,\delta}(e_m) \\ &+ \mathbf{q}(e_m,\nabla e_m)\mathcal{T}'_{k,\delta}(e_m) - \mathbf{q}(e_n,\nabla e_m)\mathcal{T}'_{k,\delta}(e_m))\cdot\nabla T_\varepsilon(\omega_{mn})] \qquad (6.31)\\ \le CI_9 + &\int_{Q\setminus D_{mn}} [\zeta\mathcal{T}'_{k,\delta}(e_m)(\mathbf{q}(e_m,\nabla e_m) - \mathbf{q}(e_n,\nabla e_m))\cdot\nabla T_\varepsilon(\omega_{mn})].\end{aligned}$$

First, we observe that the last integral in (6.31) tends to zero as $m,n\to+\infty$. Indeed, we have

$$\begin{aligned}&\int_{Q\setminus D_{mn}} [\zeta\mathcal{T}'_{k,\delta}(e_m)(\mathbf{q}(e_m,\nabla e_m) - \mathbf{q}(e_n,\nabla e_m))\cdot\nabla T_\varepsilon(\omega_{mn})] \\ &\quad\le \lambda_{mn}\int_{\{|\nabla e_m|\le\lambda_{mn}\}\cap\{|e_m|+|e_n|\le 2(k+\delta+\varepsilon)\}} |\mathbf{q}(e_m,\nabla e_m) - \mathbf{q}(e_n,\nabla e_m)|\end{aligned}$$

and continuity of $\mathbf{q}$ w.r.t. the first variable together with strong convergence (6.10) imply that this integral vanishes as $m,n\to\infty$.

To show that I_9 also tends to zero we denote

$$\begin{aligned}G_{mn} &:= \{(x,t):\ \mathcal{M}(|\nabla\mathbf{v}_n|^p) + \mathcal{M}(|\nabla\mathbf{v}_m|^p) + \\ &\quad + \mathcal{M}(|\mathcal{T}''_{k+\varepsilon,\delta}(e_n)\mathbf{q}(e_n,\nabla e_n)\cdot\nabla e_n|) + \mathcal{M}(|\mathcal{T}''_{k,\delta}(e_m)\mathbf{q}(e_m,\nabla e_m)\cdot\nabla e_m|) > \Lambda\},\\ H_{mn} &:= \{(x,t):\ \mathcal{M}(|\mathbf{v}_n\mathcal{T}_{k+\varepsilon,\delta}(e_n) - \mathbf{v}_m\mathcal{T}_{k,\delta}(e_m)|) > 1\}.\end{aligned}$$

Thus, we have for the term I_9 that

$$\begin{aligned}I_9 &\le \int_{Q\setminus(D_{mn}\cup G_{mn}\cup H_{mn})} + \left|\int_{(Q\setminus D_{mn})\cap G_{mn}}\right| + \left|\int_{(Q\setminus D_{mn})\cap H_{mn}}\right| = \\ &=: I_{10} + I_{11} + I_{12},\end{aligned}$$

and for I_{11}, I_{12} we obtain that

$$\begin{aligned}&I_{11} \le Ck\lambda_{mn}^q\operatorname{meas}(G_{mn}) \le \frac{Ck}{\delta\Lambda}\sup_{mn}\lambda_{mn}^q \overset{\Lambda\to\infty}{\longrightarrow} 0;\\ &I_{12} \le Ck\lambda_{mn}^q\operatorname{meas}(H_{mn}) \overset{m,n\to\infty}{\longrightarrow} \\ &\qquad\left(\sup_{mn} Ck\lambda_{mn}^q\right)\cdot\operatorname{meas}\{|\mathbf{v}||T_{k+\varepsilon,\delta}(e) - T_{k,\delta}(e)| > 1\} \overset{\varepsilon\to 0}{\longrightarrow} 0.\end{aligned}$$

Finally, we define the open set E_{mn} as

$$E_{mn} := D_{mn} \cup G_{mn} \cup H_{mn}$$

and, since E_{mn} is such that (6.25) holds, we can introduce the Lipschitz truncation operator $\mathcal{L}^{\alpha_{mn}}_{E_{mn}}$. Hence the integral I_{10} can be rewritten into the form

$$\begin{aligned} I_{10} = \int_{Q\setminus E_{mn}} \zeta(\mathbf{q}(e_n, \nabla e_n)\mathcal{T}'_{k+\varepsilon,\delta}(e_n) - \mathbf{q}(e_m, \nabla e_m)\mathcal{T}'_{k,\delta}(e_m)) \cdot \nabla T_\varepsilon(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn})) \\ = \int_Q - \int_{E_{mn}} =: I_{13} + I_{14}. \end{aligned}$$

To bound the remaining integral, we use the estimates that come from parabolic truncation. As it was already mentioned, we need to know something about the time derivative of truncated function. In our setting, it means that we need to have the equation for $\partial_t(\mathcal{T}_{k+\varepsilon,\delta}(e_n) - \mathcal{T}_{k,\delta}(e_m))$. But because the truncation function $\mathcal{T}_{k,\delta}$ is smooth and every term in the equation (5.3) have good meaning, we can (formally but rigorously) multiply the equation for internal energy by $\mathcal{T}'_{k,\delta}$. More precisely, we multiply the equation (5.3) for e_n by $\mathcal{T}'_{k+\varepsilon,\delta}(e_n)$, the equation (5.3) for e_m by $\mathcal{T}'_{k,\delta}(e_m)$, subtracting the resulting equations then leads to the relation (in sense of distribution)

$$\begin{aligned} &\partial_t \omega_{mn} + \operatorname{div}\big(\mathbf{v}_n \mathcal{T}_{k+\varepsilon,\delta}(e_n) - \mathbf{v}_m \mathcal{T}_{k,\delta}(e_m)\big) \\ &\quad - \operatorname{div}\big(\mathbf{q}(e_n, \nabla e_n)\mathcal{T}'_{k+\varepsilon,\delta}(e_n) - \mathbf{q}(e_m, \nabla e_m)\mathcal{T}'_{k,\delta}(e_m)\big) \\ &\quad + \mathcal{T}''_{k+\varepsilon,\delta}(e_n)\mathbf{q}(e_n, \nabla e_n) \cdot \nabla e_n - \mathcal{T}''_{k,\delta}(e_m)\mathbf{q}(e_m, \nabla e_m) \cdot \nabla e_m \\ &\quad = \mathcal{T}'_{k+\varepsilon,\delta}(e_n)\tau(e_n, D\mathbf{v}_n) : D\mathbf{v}_n - \mathcal{T}'_{k,\delta}(e_m)\tau(e_m, D\mathbf{v}_m) : D\mathbf{v}_m \end{aligned} \tag{6.32}$$

First, we bound the integral I_{14}. From apriori estimates, we obtain

$$I_{14} \le Ck \left(\int_{E_{mn}} |\nabla(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))|^q \right)^{\frac{1}{q}}.$$

Next, let $\{Q_{R_i}(X_i)\}$ be the covering from Lemma 6.1. Then we decompose the set E_{mn} into two parts such that

$$\begin{aligned} E^\rho_{mn,1} &:= \{X = (x,t) \in E_{mn} :\ \exists R_i > \rho, X \in Q_{R_i}(X_i)\} \\ E^\rho_{mn,2} &:= E_{mn} \setminus E^\rho_{mn,1}, \quad \text{for } \rho > 0. \end{aligned}$$

If $X \in E^\rho_{mn,1}$ then it follows from the definition of the parabolic truncation $\mathcal{L}^{\alpha_{mn}}_{E_{mn}}$ and the properties of covering that

$$\begin{aligned} \left|\nabla_x(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))(X)\right| &= \left| \nabla \sum_{j\in A_i} \psi_j(X) \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{R_j}} \right| \\ &\le C \sum_{j\in A_i} \frac{1}{R_j} \fint |\omega_{mn}| \le \frac{C}{\rho^{n+3}} \|\omega_{mn}\|_{1;L^1}. \end{aligned} \tag{6.33}$$

Secondly for $X \in E^{\rho}_{mn,2}$ we use again the properties of our covering to deduce

$$
\begin{aligned}
\left|\nabla_x(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))(X)\right| &= \left|\nabla \sum_{j\in A_i} \psi_j(X)\overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{R_j}}\right| \\
&= \left|\nabla \sum_{j\in A_i} \psi_j(X)(\overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{R_j}} - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}) + \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}\right| \\
&\le \sum_{j\in A_i} \frac{C}{R_j}\left|\overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{R_j}} - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}\right| \\
&= \sum_{j\in A_i} \frac{C}{R_j}\left|\ \fint_{Q^{\alpha_{mn}}_{R_j}} \omega_{mn}(Y) - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}\, dY\right| \\
&\le \frac{C}{R_i} \fint_{Q^{\alpha_{mn}}_{4R_i}} |\omega_{mn}(Y) - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}|\, dY.
\end{aligned}
\tag{6.34}
$$

Next, if we denote

$$
\begin{aligned}
&\mathbf{Q} := \mathbf{q}(e_n, \nabla e_n)\mathcal{T}'_{k+\varepsilon,\delta}(e_n) - \mathbf{q}(e_m, \nabla e_m)\mathcal{T}'_{k,\delta}(e_m) \\
&\mathbf{H} := \mathbf{v}_n\mathcal{T}_{k+\varepsilon,\delta}(e_n) - \mathbf{v}_m\mathcal{T}_{k,\delta}(e_m) \\
&G := -\mathcal{T}''_{k+\varepsilon,\delta}(e_n)\mathbf{q}(e_n, \nabla e_n)\cdot\nabla e_n + \mathcal{T}''_{k,\delta}(e_m)\mathbf{q}(e_m, \nabla e_m)\cdot\nabla e_m \\
&\qquad + \mathcal{T}'_{k+\varepsilon,\delta}(e_n)\tau(e_n, D\mathbf{v}_n) : D\mathbf{v}_n - \mathcal{T}'_{k,\delta}(e_m)\tau(e_m, D\mathbf{v}_m) : D\mathbf{v}_m
\end{aligned}
\tag{6.35}
$$

and use this notation in (6.32) we see that ω_{mn} solves in the sense of distribution the equation

$$
\partial_t\omega_{mn} - \operatorname{div}(\mathbf{Q} - \mathbf{H}) = G. \tag{6.36}
$$

Applying Poincaré inequality (6.24) for ω_{mn} to the estimate (6.34) together with the fact ($R_i \le \rho$) we have for all $X \in E^{\rho}_{mn,2}$

$$
\left|\nabla_x(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))(X)\right| \le C \fint_{Q^{\alpha_{mn}}_{4R_i}} |\nabla\omega_{mn}| + \alpha_{mn}(|\mathbf{Q}| + |\mathbf{H}| + \rho|G|)dY.
$$

Moreover, from the property (6.21) of the covering, that there exists $Z \in Q \setminus E_{mn}$ such that $Q^{\alpha_{mn}}_{4R_i}(X_i) \subset Q^{\alpha_{mn}}_{12R_i}(Z)$, thus

$$
\left|\nabla_x(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))(X)\right| \le C(\lambda_{mn} + \alpha_{mn}(\lambda^{q-1}_{mn} + 1 + \rho\Lambda)). \tag{6.37}
$$

Finally, using (6.37) and (6.33) we can easily bound the integral I_{14} such that

$$
\begin{aligned}
I^q_{14} \le Ck^q(\rho^{-n-3}\|\omega_{mn}\|_{1;L^1} + \lambda_{mn} + \alpha_{mn}(\lambda^{q-1}_{mn} + 1 + \rho\Lambda)) \\
\cdot\,(\operatorname{meas}(D_{mn}) + \operatorname{meas}(G_{mn}) + \operatorname{meas}(H_{mn})).
\end{aligned}
$$

The last step is to bound the integral I_{13}. We use $\zeta T_\varepsilon(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))$ as a test function in (6.32) to obtain that

$$I_{13} \le \frac{Ck\varepsilon}{\delta} + \max\{\lambda_{mn}, \alpha_{mn}\Lambda\}\|\omega_{mn}\|_{1;L^1} - \int_0^T \langle \partial_t \omega_{mn}, T_\varepsilon(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))\rangle. \quad (6.38)$$

For the integral with time derivative we use (6.27) to get

$$\begin{aligned} -\int_0^T \langle \partial_t \omega_{mn}, T_\varepsilon(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))\rangle\, dt \le C\|\omega_{mn}\|_{1;L^1} + \\ + \int_{E_{mn}} |\partial_t(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))| \cdot |\omega_{mn} - \mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn})|\, dx\, dt. \end{aligned}$$

Next, we estimate the time derivative in the last integral. Let $X \in Q^{\alpha_{mn}}_{R_i}$ then

$$\begin{aligned} \partial_t(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))(X) &= \partial_t \left(\sum_{j\in A_i} \psi_j(X)(\overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{R_j}} - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}) + \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}} \right) \\ &= \sum_{j\in A_i} \partial_t \psi_j(X) \fint_{Q^{\alpha_{mn}}_{R_j}} \omega_{mn}(Y) - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}\, dY. \end{aligned}$$

Thus, using the properties of ψ_j and our covering we have

$$|\partial_t(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))(X)| \le C\alpha_{mn}^{-1}R_i^{-2} \fint_{Q^{\alpha_{mn}}_{4R_i}} |\omega_{mn}(Y) - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}|\, dY. \quad (6.39)$$

Moreover, for the second term in the integral we have

$$\begin{aligned} (\omega_{mn} - \mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))(X) &= \sum_{j\in A_i} \psi_j(X)(\omega_{mn}(X) - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{R_j}}) \\ &= \sum_{j\in A_i} \psi_j(X) \fint_{Q^{\alpha_{mn}}_{R_j}} (\omega_{mn}(X) - \omega_{mn}(Y))\, dY. \end{aligned}$$

Integrating this relation w.r.t. X adding and subtracting $\overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}$ and using the property (6.22) of covering, we are led to the following inequality

$$\int_{Q^{\alpha_{mn}}_{R_i}} |(\omega_{mn} - \mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))(X)|\, dX \le C \int_{Q^{\alpha_{mn}}_{4R_i}} |\omega_{mn}(Y) - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}|\, dY. \quad (6.40)$$

Combining the estimates (6.39) and (6.40), we finally have

$$\begin{aligned} &\int_{Q^{\alpha_{mn}}_{R_i}} |\partial_t(\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}))||\mathcal{L}^{\alpha_{mn}}_{E_{mn}}(\omega_{mn}) - \omega_{mn}| \\ &\le C\alpha_{mn}^{-1}R_i^{-2}\mathrm{meas}(Q^{\alpha_{mn}}_{R_i}) \left(\fint_{Q^{\alpha_{mn}}_{4R_i}} |\omega_{mn} - \overline{\omega_{mn}}_{Q^{\alpha_{mn}}_{4R_i}}|\, dY \right)^2. \end{aligned} \quad (6.41)$$

Thus, we obtain

$$\int_{E_{mn}} \leq \sum_{i:R_i \geq \rho} \int_{Q^{\alpha_{mn}}_{4R_i}} + \sum_{i:R_i \leq \rho} \int_{Q^{\alpha_{mn}}_{4R_i}} . \tag{6.42}$$

The first sum in (6.42) can be estimated with help of (6.41) as

$$\sum_{i:R_i \geq \rho} \int_{Q^{\alpha_{mn}}_{4R_i}} \leq C\alpha_{mn}^{-1}\rho^{-2n-6}\mathrm{meas}(E_{mn})\|\omega_{mn}\|_{1;L^1}. \tag{6.43}$$

To estimate the second sum we use the same trick as in (6.37), we apply Poincaré inequality to get

$$\begin{aligned} \sum_{i:R_i \leq \rho} \int_{Q^{\alpha_{mn}}_{4R_i}} &\leq C \sum_{i:R_i \leq \rho} \alpha_{mn}^{-1}\mathrm{meas}(Q^{\alpha_{mn}}_{R_i})(\lambda_{mn} + \alpha_{mn}\lambda_{mn}^{q-1} + \alpha_{mn} + \rho\alpha_{mn}\Lambda)^2 \\ &\leq \mathrm{meas}(E_{mn})\left(\alpha_{mn}^{-1}\lambda_{mn}^2 + \alpha_{mn}(\lambda_{mn}^{2q-2} + 1 + \rho^2\Lambda^2)\right). \end{aligned} \tag{6.44}$$

Next, we have already all prepared to prove our conclusion. Indeed having the sequence of λ_{mn} as in (6.30) we define the sequence $\{\alpha_{mn}\}$ such that

$$\alpha_{mn} := \lambda_{mn}^{2-q}.$$

Finally, we begin to pass to the limit in all relations that we have to show that (6.29) converges to zero.

• First, we pass to the limit with m, n, hence $I_2 \to 0$ and

$$\omega_{mn} \to \omega = T_{k+\varepsilon,\delta}(e) - T_{k,\delta}(e)$$

strongly in $L^1(Q)$.

• Secondly, we set $\varepsilon \to 0$. Thus, $I_{12} \to 0$, $\mathrm{meas}(H_{mn}) \to 0$ and $\omega \to 0$. Moreover, we also know that $\mathrm{meas}(G_{mn}) \leq \frac{Ck}{\Lambda\delta}$. Thus, the relation for I_{14} is reduced to

$$I_{14}^q \leq Ck^{q+1}(2^{-\nu} + C(\nu,\delta)(\rho\Lambda + \frac{1}{\Lambda})).$$

The relation for I_{13} is also simplified. We see that the first two terms in (6.38) tends to zero and for the remaining term we use (6.44). Hence we have

$$I_{13} \leq Ck2^{-\nu} + \frac{C(k,\delta,\nu)}{\Lambda} + \rho C(\delta,\Lambda,\nu,k).$$

• Next, passing to the limit with $\rho \to 0$ and then with $\Lambda \to \infty$, we obtain that $I_{11} \to 0$ and also that

$$I_{14}^q + I_{13} \leq Ck2^{-\nu}.$$

• Passing to the limit with $\delta \to 0$ shows that I_8 tends to zero.

• Next setting $\nu \to \infty$ (i.e., $\sup_{mn}\lambda_{mn} \to \infty$) then implies that $I_5,\ |I_{13}|+|I_{14}| \to 0$.

• Finally, passing to the limit with $k \to \infty$ shows that $I_3 \to 0$, and consequently the whole right-hand side of (6.29) tends to zero.

Thus, using strict monotonocity, we obtain point-wise convergence of energy gradient ∇e_n that completes the proof. □

7. Proof of Corollary 2.1

Let us prove that $e^N \geq \min(\overline{e}, e_r)$ a.e. in Q. For each $t \in]0,T[$, let us choose $\phi(x) = \min(0, e^N(x,t) - \min(\overline{e}, e_r)) \leq 0$ as a test function in (4.13) and integrating over $]0,t[$, we obtain

$$\|\phi\|_{2,\Omega}^2 + \int_0^t \int_{\Omega[e^N<\min(\overline{e},e_r)]} \mathbf{q}(e^N, \nabla e^N) \cdot \nabla e^N + \\ + \int_0^t \int_{\Gamma[e^N<\min(\overline{e},e_r)]} \gamma(e^N)(e^N - \min(\overline{e}, e_r)) \leq 0,$$

taking into account the assumption (2.33). Using (2.47) and (2.30), we conclude that $e^N \geq \min(\overline{e}, e_r)$ a.e. in Q.

References

L. Boccardo and T. Gallouët. Nonlinear elliptic and parabolic equations involving measure data. *J. Funct. Anal.*, 87(1):149–169, 1989.

L. Boccardo and F. Murat. Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations. *Nonlinear Anal.*, 19(6):581–597, 1992.

L. Boccardo, A. Dall'Aglio, T. Gallouët, and L. Orsina. Nonlinear parabolic equations with measure data. *J. Funct. Anal.*, 147(1):237–258, 1997.

M. Bulíček, E. Feireisl, and J. Málek. Navier-stokes-fourier system for incompressible fluids with temperature dependent material coefficients. to appear, 2007a.

M. Bulíček, J. Málek, and K. R. Rajagopal. Navier's slip and evolutionary navier-stokes-fourier-like systems with pressure and shear-rate and temperature dependent viscosity. to appear, 2007b.

L. Consiglieri. Weak solutions for a class of non-Newtonian fluids with energy transfer. *J. Math. Fluid Mech.*, 2(3):267–293, 2000.

L. Consiglieri. Friction boundary conditions on thermal incompressible viscous flows. preprint CMAFUL -015, 2006.

L. Diening, M. Růžička, and J. Wolf. Existence of weak solutions to the equations of non-stationary motion of non-Newtonian fluids with shear rate dependent viscosity. to appear, 2006.

E. Feireisl. *Dynamics of viscous compressible fluids*, volume 26 of *Oxford Lecture Series in Mathematics and its Applications*. Oxford University Press, Oxford, 2004.

E. Feireisl and J. Málek. On the Navier-Stokes equations with temperature-dependent transport coefficients. *Differ. Equ. Nonlinear Mech.*, pages Art. ID 90616, 14 pp. (electronic), 2006.

J. Frehse, J. Málek, and M. Steinhauer. On existence results for fluids with shear dependent viscosity—unsteady flows. In *Partial differential equations (Praha, 1998)*, volume 406 of *Chapman & Hall/CRC Res. Notes Math.*, pages 121–129. Chapman & Hall/CRC, Boca Raton, FL, 2000.

G. P. Galdi. *An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I*, volume 38 of *Springer Tracts in Natural Philosophy.* Springer-Verlag, New York, 1994. Linearized steady problems.

U. Hadrian and P.D. Panagiotopoulos. Superpotential flow problems and application to metal forming processes with friction boundary conditions. *Mech. Res. Comm.*, 5:257–267, 1978.

K. Hutter and K. R. Rajagopal. On flows of granular materials. *Contin. Mech. Thermodyn.*, 6(2):81–139, 1994.

J. Kinnunen and J. L. Lewis. Very weak solutions of parabolic systems of p-Laplacian type. *Ark. Mat.*, 40(1):105–132, 2002.

J. Leray. Sur le mouvement d'un liquide visquex emplissant l'espace. *Acta Math.*, 63:193–248, 1934.

J. Málek and K. R. Rajagopal. Mathematical issues concerning the Navier-Stokes equations and some of its generalizations. In *Evolutionary equations. Vol. II*, Handb. Differ. Equ., pages 371–459. Elsevier/North-Holland, Amsterdam, 2005.

J. Málek, J. Nečas, M. Rokyta, and M. Růžička. *Weak and measure-valued solutions to evolutionary PDEs.* Chapman & Hall, London, 1996.

C. W. Oseen. *Hydrodynamik.* Akademische Verlagsgesellschaft, Leipzig, 1927. in German.

H. Triebel. *Theory of function spaces*, volume 78 of *Monographs in Mathematics.* Birkhäuser Verlag, Basel, 1983.

E. Zeidler. *Nonlinear functional analysis and its applications. II/B.* Springer-Verlag, New York, 1990. Nonlinear monotone operators, Translated from the German by the author and Leo F. Boron.