# Existence of local strong solutions for a quasilinear Benney system

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**Abstract**

We prove in this Note the existence and uniqueness of a strong local solution to the Cauchy problem for the quasilinear Benney system (1).

## Existence d'une solution locale forte pour un système de Benney quasilinéaire

**Résumé**

Nous prouvons dans cette Note l'existence et unicité d'une solution locale forte du problème de Cauchy pour le système de Benney quasilinéaire (1).

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### 1. Introduction and main result.

We consider the system introduced by Benney in [1] to study the interaction between short and long waves, for example gravity waves in fluids :

$$\begin{cases} iu_t + u_{xx} = |u|^2 u + vu & (a) \\ & x \in \mathbf{R}, t \geq 0, \\ v_t + [f(v)]_x = |u|^2_x & (b) \end{cases} \qquad (1)$$

where $f$ is a polynomial real function, $u$ and $v$ (real) represent the short and the long wave, respectively.

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In [2] the existence of weak solutions for (1) was proved for $f(v) = av^2 - bv^3$, with $a$ and $b$ real constants, $b > 0$, in the following sense:

**Theorem 1.1** *Given $u_0, v_0 \in H^1(\mathbf{R})$ with $v_0$ real-valued, there exists functions*

$$u \in L^\infty(\mathbf{R}_+; H^1(\mathbf{R})), \qquad v \in L^\infty(\mathbf{R}_+; (L^4 \cap L^2)(\mathbf{R}))$$

*such that*

$$i \int_0^\infty \int_{\mathbf{R}} u \frac{\partial \varphi}{\partial t}\, dx\, dt + \int_0^\infty \int_{\mathbf{R}} \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial x}\, dx\, dt + \int_{\mathbf{R}} u_0(x)\varphi(x,0)\, dx + \int_0^\infty \int_{\mathbf{R}} |u|^2 u \varphi\, dx\, dt + \int_0^\infty \int_{\mathbf{R}} vu\varphi\, dx\, dt = 0,$$

$$\int_0^\infty \int_{\mathbf{R}} v \frac{\partial \psi}{\partial t}\, dx\, dt + \int_0^\infty \int_{\mathbf{R}} f(v) \frac{\partial \psi}{\partial x}\, dx\, dt + \int_{\mathbf{R}} v_0(x)\psi(x,0)\, dx - \int_0^\infty \int_{\mathbf{R}} \frac{\partial}{\partial x} |u|^2 \psi\, dx\, dt = 0,$$

*for all functions $\varphi, \psi \in C_0^1(\mathbf{R} \times [0, +\infty[)$ (i.e. in the class of continuously differentiable functions with compact support), with $\varphi$ being complex-valued and $\psi$ real-valued.*

This result was obtained for this particular system by application of the vanishing viscosity method and we could not extend the necessary estimates to the Burger's case ($a = 1, b = 0$) or to more general cases. Here we will prove the existence of (local) strong solutions to (1) for general $f$, extending previous results in [6, 7] for $f$ linear :

**Theorem 1.2** *Let $(u_0, v_0) \in H^3(\mathbf{R}) \times H^2(\mathbf{R})$ and $f \in C^3(\mathbf{R})$. Then there exists a unique strong solution $(u, v)$ of the Cauchy problem associated to (1), with*

$$(u, v) \in C^j([0,T]; H^{3-2j}(\mathbf{R})) \times C^j([0,T]; H^{2-j}(\mathbf{R})),\ j = 0, 1.$$

*Here, the life-span $T > 0$ depends exclusively on $f$ and on the initial data $(u_0, v_0)$.*

The main difficulty here is the derivative-loss in the right-hand side of equation ($1 - a$). This cannot be handled easily by the Schrödinger kernel, due to its limited smoothing properties. The method employed in [6, 7] for $f$ linear, based in the inhomogeneous smoothing effect of the Schrödinger group, can not be easily implemented for $f$ nonlinear. We will address this problem by introducing some auxiliary functions and rewriting system (1) without derivative loss. A similar technique was introduced in [5] to solve the fully nonlinear wave equation and employed in [4], in the context of the Zakharov-Rubenchik system.

Another interesting open problem is the study of the probable blow-up of the local smooth solutions.

## 2. An equivalent system.

Let us take $(u,v)$ a solution of (1). By setting $F = u_t$, we obtain from $(1-a)$

$$iF + u_{xx} - u = |u|^2 u + u(v-1),$$

and

$$u = (\Delta - 1)^{-1}(|u|^2 u + u(v-1) - iF), \tag{2}$$

with $\Delta = \dfrac{\partial^2}{\partial x^2}$. Also, differentiating $(1-a)$ with respect to $t$ leads to

$$iF_t + F_{xx} = 2|u|^2 F + u^2 \bar{F} + Fv + uv_t,$$

and from $(1-b)$,

$$iF_t + F_{xx} = 2|u|^2 F + u^2 \bar{F} + Fv + u|u|_x^2 - uv_x f'(v). \tag{3}$$

These computations are our motivation to consider the following Cauchy problem:

$$\begin{cases} iF_t + F_{xx} = 2|u|^2 F + u^2 \bar{F} + Fv + u|\tilde{u}|_x^2 - uv_x f'(v) & (a) \\ v_t + [f(v)]_x = |\tilde{u}|_x^2 & (b) \\ F(x,0) = F_0(x) \in H^1(\mathbf{R}),\ v(x,0) = v_0(x) \in H^2(\mathbf{R}) & \end{cases} \tag{4}$$

where $u$ and $\tilde{u}$ are given in terms of $F$ by

$$u(x,t) = u_0 + \int_0^t F(x,s)ds \quad \text{and} \quad \tilde{u}(x,t) = (\Delta - 1)^{-1}(|u|^2 u + u(v-1) - iF). \tag{5}$$

Note that in this system derivative losses do not occur. Indeed, the regularization of $(\Delta - 1)^{-1}$ puts $\tilde{u}$ in $H^3$ and therefore the right-hand side of $(4-a)$ is in $H^1$, like $F$.

We will prove the following lemma:

**Lemma 2.1** *Let $(F_0, v_0) \in H^1(\mathbf{R}) \times H^2(\mathbf{R})$ and $f \in C^3(\mathbf{R})$. Then there exists $T > 0$ and a unique strong solution $(F,v)$ of the Cauchy problem $(4-a,b)$, with*

$$(F,v) \in C^j([0,T]; H^{1-2j}(\mathbf{R})) \times C^j([0,T]; H^{2-j}(\mathbf{R})),\ j = 0,1.$$

*Here, the life-span $T > 0$ depends exclusively on $f$ and on the initial data $(F_0, v_0)$.*

This lemma will be proved in the next section, using the general theory of Kato for quasi-linear equations ([3]).

We now explain why Lemma 2.1 implies our main Theorem 1.2:

If $(F, v)$ is a solution of (4), by differentiating (5) with respect to $t$ we obtain

$$u_t = F.$$

Replacing in $(1-a)$ yields by $(4-b)$

$$\begin{aligned}(iu_t + u_{xx})_t &= 2|u|^2 F + u^2\bar{F} + Fv + u|\tilde{u}|_x^2 - uv_x f'(v) \\ &= 2|u|^2 u_t + u^2\bar{u}_t + u_t v + uv_t\end{aligned}$$

Hence $(iu_t + u_{xx} - |u|^2 u - uv)_t = 0$ and we get

$$iu_t + u_{xx} - |u|^2 u - uv = \phi_0(x),$$

where $\phi_0(x) = iF_0(x) + u_0''(x) - |u_0(x)|^2 u_0(x) - u_0(x)v_0(x)$. By setting

$$F_0(x) = i(u_0''(x) - |u_0(x)|^2 u_0(x) - u_0(x)v_0(x)), \tag{6}$$

we obtain $\phi_0 = 0$ and $(u, v)$ satisfies $(1-a)$. Furthermore, from $(1-a)$,

$$u = (\Delta - 1)^{-1}(|u|^2 u + u(v-1) - iu_t). \tag{7}$$

Therefore $u = \tilde{u}$ and $(u, v)$ satisfies $(1-b)$. Note that $u_t = F \in C([0,T]; H^1(\mathbf{R}))$. Also $u(.,t) = u_0(.) + \int_0^t F(.,s)ds \in C([0,T]; H^1(\mathbf{R}))$, but from (7) we have in fact

$$u \in C([0,T]; H^3(\mathbf{R})).$$

## 3. Proof of Lemma 2.1.

In order to apply a variant of theorem 6 in [3] we need to set the Cauchy problem (4) in the framework of real spaces. We introduce the new variables

$$F_1 = \Re F,\ F_2 = \Im F,\ u_1 = \Re u,\ u_2 = \Im u$$

and with $U = (F_1, F_2, v)$, $F_{10} = \Re F_0$, $F_{20} = \Im F_0$ (4) can be written as follows :

$$\begin{cases} \dfrac{\partial}{\partial t}U + A(U)U = g(t,U) \\ (F_1(x,0), F_2(x,0), v(x,0)) = (F_{10}(x), F_{20}(x), v_0(x)) \in (H^1(\mathbf{R}))^2 \times H^2(\mathbf{R}) \end{cases} \tag{8}$$

where

$$A(U) = \begin{bmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & 0 \\ 0 & 0 & f'(v)\dfrac{\partial}{\partial x} \end{bmatrix}$$

and

$$g(t,U) = \begin{bmatrix} 2|u^2|F_2 - (u_1^2 - u_2^2)F_2 + 2u_1u_2F_1 + F_2v + u_2|\tilde{u}|_x^2 - u_2v_xf'(v) \\ -2|u^2|F_1 - (u_1^2 - u_2^2)F_1 - 2u_1u_2F_2 - F_1v - u_1|\tilde{u}|_x^2 + u_1v_xf'(v) \\ |\tilde{u}|_x^2 \end{bmatrix}$$

which is a non-local source term.
Now we set $X = (H^{-1}(\mathbf{R}))^2 \times L^2(\mathbf{R})$, $Y = (H^1(\mathbf{R}))^2 \times H^2(\mathbf{R})$ and introduce $S : Y \longrightarrow X$ defined by $S = (1-\Delta)I$, which is an isomorphism. Moreover $A : U = (F_1, F_2, v) \in W \longrightarrow G(X, 1, \beta)$, where $W$ is an open ball in $Y$ centered at the origin and with radius $R$ and $G(X, 1, \beta)$ denotes the set of all linear operators $D$ in $X$ such that $-D$ generates a $C_0$ - semigroup $\{e^{-tD}\}$ with

$$\left\|e^{-tD}\right\| \leq e^{\beta t}, \ t \in [0, +\infty[,$$

$$\beta = \frac{1}{2} \sup_{x\in\mathbf{R}} |f''(v(x))\, v_x(x)| \leq c\, R\, \alpha(R),$$

where $c > 0$ is a numerical constant and $\alpha(R)$ is a continuous function (cf.[3], §8). It is easy to see that $g$ verifies, for fixed $T > 0$,

$$\|g(t, U)\|_Y \leq \lambda, \quad t \in [0, T], \ U \in W.$$

Now, with $B_0(v) \in \mathcal{L}(L^2(\mathbf{R}))$, $v$ in a ball $W_1$ in $H^2(\mathbf{R})$, $B_0(v)$ defined by (8.7) in [3]

$$B_0(v) = -[f''(v)v_{xx} + f'''(v)v_x^2]\frac{\partial}{\partial x}(1-\Delta)^{-1} - 2f'(v)v_x\frac{\partial^2}{\partial x^2}(1-\Delta)^{-1},$$

we introduce an operator $B(U) \in \mathcal{L}(X)$, $U = (F_1, F_2, v) \in W$, defined by

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & B_0(v) \end{bmatrix}$$

In [3], §8, Kato proved that for $v \in W_1$ we have

$$(1-\Delta)\left(f'(v)\frac{\partial}{\partial x}\right)(1-\Delta)^{-1} = f'(v)\frac{\partial}{\partial x} + B_0(v).$$

Hence, we easily derive for $U \in W$

$$SA(U)S^{-1} = A(U) + B(U).$$

Now, for each pair $(U, U^*)$, $U = (F_1, F_2, v)$ and $U^* = (F_1^*, F_2^*, v^*)$ in $W$ we will prove that

$$\|g(t,U) - g(t,U^*)\|_{L^1(0,T';X)} \leq c(T') \sup_{0\leq t\leq T'} \|U(t) - U^*(t)\|_X \tag{9}$$

for $T' \in [0,T]$ where $c(T')$ is a continuous increasing function such that $c(0) = 0$. Let us point out that if $h \in L^2(\mathbf{R})$ and $w \in H^1(\mathbf{R})$ we easily derive

$$\|hw\|_{H^{-1}} \leq \|h\|_{H^{-1}} \|w\|_{H^1}.$$

Hence, for example, we get, with an obvious notation,

$$\|F_1 u_1 (u_1^* - u_1)\|_{H^{-1}} \leq \|F_1\|_{H^1} \|u_1\|_{H^1} \|u_1^* - u_1\|_{H^{-1}}$$

and, for $t \leq T'$

$$\left\| f'(v) v_x \left( \int_0^t F_2 d\tau - \int_0^t F_2^* d\tau \right) \right\|_{H^{-1}} \leq \|f'(v) v_x\|_{H^1} \int_0^t \|F - F^*\|_{H^{-1}} d\tau$$

$$\leq c(T') \sup_{0 \leq t \leq T'} \|U(t) - U^*(t)\|_X$$

where $c(T')$ is a continuous increasing function such that $c(0) = 0$. Now, Lemma 2.1 is an easy consequence of Theorem 6 in [3], where the local condition (7.7) is replaced by (9) which is sufficient for the proof of this theorem.

**Acknowledgements:** This research was partially supported by FCT under program POCI 2010 (Portugal/FEDER-EU).

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