Spectral stability of Markov systems

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Abstract *For a class of semigroups of stochastic dynamical systems, $x \mapsto P_x$, where x denotes a state and P_x the state probability transition, we relate its spectral stability with the combinatorial stability of the underlying non-deterministic dynamics, associated to the point-set map $x \mapsto supp(P_x)$.*

1 Introduction

Uniformly hyperbolic (Axiom A) systems were introduced by Smale, see [Sm], in the early sixties, and have been widely studied both from the topological and the ergodic view-point. See e.g. [S] and references therein. Smooth deterministic uniformly hyperbolic systems $f : X \to X$ admit a precise description of its dynamics: the *spectral decomposition theorem* states that there is a decomposition of the non-wandering set $\Omega(f)$ into a finite number of hyperbolic basic sets which are permuted by f. The dynamics of f partially orders the basic set components of $\Omega(f)$, the minimal, or final, elements being the *attractors* of f. As the name indicates this decomposition relates with the spectral decomposition of the linear operator which describes the action of f on the tangent vector fields to X. In the seventies, the ergodic theory of uniformly hyperbolic systems was established by the work of Sinai, Ruelle and Bowen. See [Si1], [Ru], [BR], [Bo]. For these dynamical systems each attractor has a physical measure, whose basin of attraction cover almost every point is state space. Physical measures were first constructed by Sinai [Si1] for Anosov diffeomorphisms. This was extended by Ruelle [Ru] for general hyperbolic (Axiom A) diffeomorphisms, and by Bowen-Ruelle [BR] for Axiom A flows. A *physical measure* of an attractor is one that describes the system time average for a positive volume set of initial states. The set of all such initial states is called the physical measure basin of attraction. A measure on an attractor is said to be *stochastically stable*, concept introduced by Kolmogorov and Sinai, if it is stable under small stochastic perturbations of the deterministic system. More precisely, introducing a random noise, the limit measures of the random perturbations approach the attractor physical measure as the noise level tends to zero. See, e.g., [V1]. The key idea of introducing a random noise in a deterministic system, and then looking at the limit measures as the noise level tends to zero, goes back to Kolmogorov. See [Si2]. Kolmogorov expressed the ideia that zero-noise limits represent measures that yield a certain "physical" insight of the system's behaviour. See [BDV], [Y2]. The effects of small random errors on the asymptotic distribution of points in the basin of a hyperbolic attractor were considered, for different perturbation schemes, by Kifer and Young. They established the stability of uniformly hyperbolic attractors, for different models of random perturbations. See [K1], [K3], [Y1]. See also [K4] and [V1]. But, beyond the Axiom A setting, the problem of existence and finiteness of physical measures and their stability, prevails as a major purpose in dynamical systems. A good comprehension of which dynamical systems admit physical measures was not yet achieved, but some work has been done. See [BDV], [V2], [Y2] for surveys on much of the progress already made. See also [Li] for a recent survey on random dynamical systems. Under very mild conditions, a random noise can have a powerfull simplifying effect on the complexity of the dynamics of a deterministic system. Namely, under arbitrary small random perturbations any deterministic system has finitely many attractors (see, for instance, [A]). The spectrum of the Perron-Frobenius operator, which reflects the action of dynamics upon measures, may also be simplified. In general, the spectrum of this linear operator can be complex, but when we add a random noise this usually makes the operator compact or weakly compact with pure point spectrum. A compact operator can be, spectrally speaking, well approximated by finite-dimension operators. Thus random perturbations of a deterministic system may, just as well, be considered on finite (discrete) approximations of state space. Finite state Markov chains are the stochastic, or random dynamical systems on a finite state space. One may think that these dynamical systems are what we actually see when running computer simulations of deterministic dynamical systems. Each such dynamical system is specified by a *stochastic matrix* with the state probability transitions. The Perron operator of this finite state system is the stochastic matrix. The Markov chain also determines an *oriented graph*, encapsulating some qualitative aspects of the system behaviour. The theory of finite state Markov chains establishes a correspondence between spectral properties of the stochastic matrix on one side, and combinatorial properties of the corresponding graph on the other hand. See, e.g., [B], [D].

This theory can be nicely extended into a theory for continuous, or lower semi-continuous, random dynamical systems of Markov type on a compact manifold X. In a previous article [DT] we developed such a theory in its topological and combinatorial aspects. In this complementary paper we will adress the correspondent spectral theory and its relation with the results obtained there. The main novelty with respect to finite state Markov chain theory is that in this context, because we are dealing with continuous systems, it makes sense defining stability: combinatorial stability or spectral stability. In the previous article [DT] we established and characterized combinatorial stability. The core of the present article will be to establish and characterize spectral stability.

In [DT] we introduced a class of semigroups $\mathcal{O}$ of non-deterministic maps, that we referred as *open maps*. We defined a *non-deterministic dynamical system* on a state space X to be any multi-valued mapping $\varphi : X \to \mathcal{P}(X)$, where $\mathcal{P}(X)$ denotes the set of all X subsets. See [AF] for an overview on multi-valued analysis. The state space X was a compact manifold and we made a fundamental assumption on the state transition sets $\varphi(x)$ $(x \in X)$. They were always non-empty connected open sets. The lower semi-continuity of a non-deterministic dynamical system φ with respect to the Hausdorff distance was then equivalent to the openness of graph (φ). Such systems were referred as *open maps*. Given any open map φ, a sequence $x_0, x_1, \cdots, x_n$ such that $x_i \in \varphi(x_{i-1})$ for all $i = 1, \cdots, n$ was called a $\varphi-$*orbit*, and x_n a φ-*iterate* of x_0. The *recurrent set* of φ was defined by $\Omega(\varphi) = \{x \in X : x \text{ is a } \varphi - \text{iterate of } x\}$. This set splits into equivalence classes, each class being formed by states which are accessible from each other. The dynamics of φ partially orders these equivalence classes, the minimal, or final, classes being the *attractors* of the open map φ. For these dynamical systems we have defined a concept of *combinatorial stability*, which roughly means that the same combinatorics of attractors persists for all nearby systems. In that article we have characterized the combinatorial stability of open maps, and established its genericity.

Given a state space X, any function f that associates to each state $x \in X$ a state probability transition f_x on X will be called a (discrete time) stochastic dynamical system or, simply, a Markov system. Deterministic dynamical systems correspond to such functions when each value $f_x = \delta_{\varphi(x)}$ is a Dirac measure sitting at some point $\varphi(x) \in X$. Here we shall introduce a class of semigroups $\mathcal{H}$ of Markov systems, which is large and natural to make stochastic perturbations of continuous deterministic dynamical systems. A natural homomorphism $\varphi : \mathcal{H} \to \mathcal{O}$ relates each stochastic dynamical system $f \in \mathcal{H}$ with the non-deterministic open map $\varphi_f : x \mapsto \text{int}(\text{supp}(f_x))$.

Our main goal is to study and compare, for generic systems $f \in \mathcal{H}$, the combinatorial stability of φ_f, with the spectral stability of the linear operator $\mathcal{L}_f : \mu \mapsto f_*\mu$, that to each probability distribution μ associates the μ-conditional probability distribution in the next instant, also known as the Perron-Frobenius operator. The fixed points of this linear operator are precisely the system invariant measures. The spectral stability of f relates with the fact that no eigenvalues can enter, or leave, the unit circle.

Our main results are:

Theorem A (Spectral Stability Characterization)
Given a topological semigroup $\mathcal{H}$ of Markov systems, $f \in \mathcal{H}$ is spectrally stable if and only if φ_f is combinatorially stable.

Theorem B (Genericity of Spectral Stability)
Given a topological semigroup $\mathcal{H}$ of Markov systems, the set of spectrally stable systems in $\mathcal{H}$ is open and dense in $\mathcal{H}$.

Palis conjectured [P] that *every dynamical system can be approximated by one with finitely many attractors, each having a stochastically stable physical measure, whose basins of attraction cover almost every point in state space*. See, e.g., [V2]. This conjecture suggested the main motivation for the present study: to understand stochastic stability in the realm of stochastic dynamical systems, at least in a class of Markov chains which is suitable for stochastic perturbations of continuous maps.

2 Markov Systems: some notations and definitions

Throughout this work X will denote a compact Riemannian manifold, d will be the geodesic distance on X and m will be the corresponding normalized Riemmanian volume on X, i.e. $m(X) = 1$. The Banach algebra of all continuous real-valued functions on X, $\psi : X \to \mathbb{R}$, endowed with the uniform proximity norm

$$\|\psi\|_\infty = \max\{|\psi(x)| \,:\, x \in X\}$$

is denoted by $C^0(X)$, and $\mathcal{M}(X) = (C^0(X))'$ is the dual Banach space of all finite real measures on X, with its usual total variation norm $\|\mu\|$. We denote by $\mathcal{M}_0(X)$ the closed subspace formed by all measures $\mu \in \mathcal{M}(X)$ with zero mass, i.e. measures such that $\mu(X) = \int_X 1\, d\,\mu = 0$. The set of all probability measures on X, denoted by $\mathcal{M}_{\text{prob}}(X)$, is a closed convex subset of $\mathcal{M}(X)$ which is contained in an affine space paralell to $\mathcal{M}_0(X)$. The space $\mathcal{M}_{\text{prob}}(X)$ is compact w.r.t. the weak-$*$ topology. The weak-$*$ topology in $\mathcal{M}(X)$ is the vector space topology defined by the family of seminorms, one for each test function $\psi \in C^0(X)$, $\|\mu\|_\psi = \left| \int_X \psi\, d\,\mu \right|$. Given a sequence $\{\mu_n\} \subseteq \mathcal{M}(X)$, we shall write $\mu_n \xrightarrow{*} \mu$ to mean that μ_n converges to μ in the weak-$*$ topology, as $n \to \infty$. The Banach space of all m-integrable functions on X with the usual L^1-norm, $\|h\|_1 = \int_X |h(x)|\, d\,m(x)$ will be denoted by $L^1(X, m)$. This space is isometrically embbeded in $\mathcal{M}(X)$ through the inclusion map $L^1(X, m) \hookrightarrow \mathcal{M}(X)$, $h \mapsto h\,m$. We shall denote by $L^1_0(X, m)$ the subspace of all functions $h \in L^1(X, m)$ with zero average, i.e. $\int_X h(x)\, d\,m(x) = 0$. Given $x \in X$, δ_x will denote the Dirac measure sitting at the point x, which is defined by $\int_X \psi(y)\, d\,\delta_x(y) = \psi(x)$, for every test function $\psi \in C^0(X)$. We say that a sequence of functions $\{q_x^\epsilon\}_{\epsilon>0} \subseteq L^1(X, m)$ is a *weak-$*$ approximation of* δ_x if $q_x^\epsilon \xrightarrow{*} \delta_x$ as $\epsilon \to 0$, and for every $\epsilon > 0$: (i) $\text{supp}(q_x^\epsilon) \subseteq B_\epsilon(x)$, (ii) $q_x^\epsilon(y) \geq 0$ for every $y \in X$, and (iii) $\int_X q_x^\epsilon(y)\, d\,m(y) = 1$.

We shall call *Markov system* to any weak-$*$ continuous mapping $p : X \to \mathcal{M}_{\text{prob}}(X)$. The probability measure $p(x) = p_x$ is refered as the transition probability at state $x \in X$. We denote by $\mathcal{MS}(X)$ the set of all Markov systems. A Markov system $p : X \to \mathcal{M}_{\text{prob}}(X)$ will also be refered as a stochastic dynamical system. A Markov system is called deterministic if for some continuous mapping $f : X \to X$, we have $p(x) = \delta_{f(x)}$, for every $x \in X$.

The Perron-Forbenius operator of a Markov system $p : X \to \mathcal{M}_{\text{prob}}(X)$ is the linear operator $\mathcal{L}_p : \mathcal{M}(X) \to \mathcal{M}(X)$, defined by

$$\mathcal{L}_p(\mu) = \int_X p(x)\; d\,\mu(x)\,, \qquad \text{for every } \mu \in \mathcal{M}(X)\;.$$

The integral of the measure-valued function p is well defined, in a sense that can be found, for instance, in [Ru]. The adjoint operator $\mathcal{L}_p^* : C^0(X) \to C^0(X)$, is given by

$$\mathcal{L}_p^*(\psi)(x) = \int_X \psi(y)\; d\,p_x(y)\,, \qquad \text{for every } \psi \in C^0(X)\;.$$

Both $\mathcal{L}_p$ and $\mathcal{L}_p^*$ are bounded linear operators with norms less or equal than 1.

The convolution of two Markov systems $p,\, q \in \mathcal{MS}(X)$ is the Markov system $p * q : X \to \mathcal{M}_{\rm prob}(X)$, defined by

$$(p*q)(x) = \mathcal{L}_p(q_x) = \mathcal{L}_p(\mathcal{L}_q(\delta_x)) \qquad \text{for every } \; x \in X \; .$$

The space $(\mathcal{MS}(X), *)$ is a semigroup with identity, where the identity is the deterministic Markov system $x \mapsto \delta_x$. The map $p \mapsto \mathcal{L}_p$ is a semigroup homomorphism taking $\mathcal{MS}(X)$ into the algebra of bounded linear operators on the Banach space $\mathcal{M}(X)$.

We shall say that a measure $\mu \in \mathcal{M}(X)$ is $p-$invariant when $\mathcal{L}_p\mu = \mu$. We say that a measurable set $A \subseteq X$ is $p-$invariant when $\mathcal{L}_p^*\chi_A = \chi_A$, where χ_A denotes the characteristic function $\chi_A(x) = \begin{cases} 1 & \text{if} \quad x \in A \\ 0 & \text{if} \quad x \in X - A \end{cases}$.

We say that a Markov system $p : X \to \mathcal{M}(X)$ is *absolutely continuous w.r.t.* m if $p_x = f_x\, m$, with $f_x \in L^1(X, m)$, for every $x \in X$. absolutely continuous Markov Chains are defined by *stochastic transition functions* $f : X \times X \to \mathbb{R}$ such that:

(a) $f(x, y)$ is measurable on $X \times X$,

(b) $f(x, y) \geq 0$, for every $(x, y) \in X \times X$,

(c) $\int_X f(x, y)\, dm(y) = 1$, for every $x \in X$,

(d) the real valued function $X \to \mathbb{R}$, $x \mapsto \int f(x, y)\, \psi(y)\, dm(y)$, is continuous for every test function $\psi \in C^0(X)$.

A function $f : X \times X \to \mathbb{R}$ satisfying (a), (b), (d) and

(c') $\int_X f(x, y)\, dm(y) \leq 1$, for every $x \in X$,

is called a *substochastic transition function.*

The subset of all absolutely continuous Markov systems forms a subsemigroup, without indentity, of $\mathcal{MS}(X)$. Given two transition functions $f, g : X \times X \to \mathbb{R}$, the convoluted Markov system is defined by the usual function convolution

$$(f * g)(x, z) = \int_X f(x, y) g(y, z)\, dm(y) \; .$$

From now on we shall identify each absolutely continuous Markov system with its probability transition function $f : X \times X \to \mathbb{R}$. Given any such absolutely continuous

Markov system f, the operator $\mathcal{L}_f$ takes $\mathcal{M}(X)$ into $L^1(X,m)$ and its restriction to $L^1(X,m)$ is given by

$$\mathcal{L}_f(q)(y) = \int_X q(x)\, f(x,y)\, dm(x) \qquad q \in \mathrm{Ł}^1(X,m)\ .$$

The adjoint action on $L^\infty(X,m)$ is given by

$$\mathcal{L}_f^*(g)(x) = \int_X f(x,y)\, g(y)\, dm(y) \qquad g \in \mathrm{Ł}^\infty(X,m)\ .$$

Given a Markov system $p : X \to \mathcal{M}_{\text{prob}}(X)$, $\sigma(\mathcal{L}_p)$ will denote the spectrum of the Perron-Frobenius operator $\mathcal{L}_p$. The *spectral radius* of $\mathcal{L}_p$, i.e. the lowest upper bound for absolute values of elements in $\sigma(\mathcal{L}_f)$, will be denoted by $r(\mathcal{L}_p)$. Of course $r(\mathcal{L}_p) = 1$. The *discrete spectrum* of $\mathcal{L}_p$, i.e. the set of all eigenvalues in $\sigma(\mathcal{L}_p)$ that are isolated and have finite multiplicity, will be denoted by $\sigma_{\text{disc}}(\mathcal{L}_p)$. The complement of $\sigma_{\text{disc}}(\mathcal{L}_p)$ in $\sigma(\mathcal{L}_p)$ is called the *essential spectrum* of $\mathcal{L}_p$, and denoted by $\sigma_{\text{ess}}(\mathcal{L}_p)$. The *essential spectral radius* of $\mathcal{L}_p$, i.e. the lowest upper bound for absolute values of elements in $\sigma_{\text{ess}}(\mathcal{L}_f)$, is denoted by $r_{\text{ess}}(\mathcal{L}_p)$. It is well known, see for instance [W], that the Perron operator $\mathcal{L}_f$ of any absolutely continuous Markov system f is a weakly compact operator. In particular, $r_{\text{ess}}(\mathcal{L}_f) = 0$ and, therefore, the spectrum $\sigma(\mathcal{L}_f)$ is at most countable. All spectrum points in $\sigma(\mathcal{L}_f) - \{0\}$ are isolated eigenvalues with finite multiplicity.

Given an absolutely continuous Markov system f, we can decompose the spectrum of $\mathcal{L}_f$ as:

$$\sigma(\mathcal{L}_f) = \sigma_0(\mathcal{L}_f) \cup \sigma_1(\mathcal{L}_f),$$

where $\sigma_0(\mathcal{L}_f) = \{\lambda \in \sigma(\mathcal{L}_f) : |\lambda| < 1\}$, and $\sigma_1(\mathcal{L}_f) = \sigma(\mathcal{L}_f) - \sigma_0(\mathcal{L}_f)$. Of course $\sigma_1(\mathcal{L}_f)$ is finite while $\sigma_0(\mathcal{L}_f)$ is at most countable but closed for the complex plane topology. Consequently, $\sigma_0(\mathcal{L}_f)$ and $\sigma_1(\mathcal{L}_f)$ are disjoint compact sets and, therefore, there is an associated decomposition of $L^1(X,m)$ into two $\mathcal{L}_f$-invariant subspaces:

$$L^1(X,m) = E_0(f) \oplus E_1(f)\ ,$$

where $E_1(f)$ has finite dimension.

We shall denote by $r_{\text{int}}(\mathcal{L}_f)$ the *interior spectral radius* of $\mathcal{L}_f$, i.e. the lowest upper bound of all absolute values of elements in $\sigma_0(\mathcal{L}_f)$.

Given any absolutely continuous Markov system f a sequence $x_0, x_1, \cdots, x_n$ such that $f(x_{i-1}, x_i) > 0$ for all $i = 1, \cdots, n$ is called an $f-$*orbit*, and we say that x_n is an $f-$*iterate* of x_0. An absolutely continuous Markov system is called *irreducible* if for each pair of points $x, y \in X$ there is some $n \in \mathbb{N}$ such that the probability transition density from x to y in n iterates is positive, i.e. $f^n(x, y) > 0$. Given an absolutely continuous irreducible Markov system f, and a state $x \in X$, the greatest comon divisor d of all times $n \in \mathbb{N}$ such that $f^n(x, x) > 0$ is called the period of f. The period does not depend on the choice of state $x \in X$. An irreducible Markov system f is called *acyclic* if it has period one. The state space X of an irreducible Markov system f of period d can be decomposed into a finite union of f^d-invariant subsets $X = X_0 \cup \cdots \cup X_{d-1}$ such that each restriction $(f^d)_{X_i} : X_i \times X_i \to \mathbb{R}$, is an irreducible acyclic Markov system on X_i.

We shall denote by f_R the restriction to $R \times R$ of a given function $f : X \times X \to \mathbb{R}$, for any subset $R \subseteq X$. If f is stochastic transition function then:

1. f_R is a substochastic transition function.

2. f_R is a stochastic transition function $\Leftrightarrow$ R is f-invariant.

The subspace of all functions $h \in L^1(X, m)$ which vanish outside R, for any given subset $R \subseteq X$, will be denoted by L^1_R. Of course the subspace L^1_R is $\mathcal{L}_f-$invariant if and only if R is $f-$invariant. $L^1_{R,0}$ will denote the subspace $L^1_{R,0} = L^1_R \cap \mathcal{M}_0(X)$. Of course the subspace $L^1_{R,0}$ is $\mathcal{L}_f-$invariant if and only if R is $f-$invariant. The action of $\mathcal{L}_f$ on the invariant subspace $L^1_{R,0}$ will be denoted by $\mathcal{L}_{f_{R,0}}$.

3 Abstract Spectral Bounds

Let $\mathcal{H}(X)$ be the set of all absolutely continuous Markov systems (i.e. probability transition functions) $f : X \times X \to \mathbb{R}$ satisfying the following extra conditions:

(1) f is bounded,

(2) f is lower semi-continuous,

(3) for each $x \in X$, the open set $\varphi_f(x) = \{ y \in X \, : \, f(x, y) > 0 \}$ is connected.

In a previous article [DT] we have defined the space $\mathcal{O}(X)$ of all point-set maps $\varphi : X \to \mathcal{P}(X)$ whose graph $graph\,\varphi = \{(x,y)\in X\times X : y\in\varphi(x)\}$ is open in $X\times X$, and such that $\varphi(x)$ is non-empty and connected for every $x\in X$. For two open maps $\varphi,\psi : X\to\mathcal{P}(X)$ the usual composition product $\varphi\circ\psi : X\to\mathcal{P}(X)$ of φ and ψ at x was defined by

$$(\varphi\circ\psi)(x) = \varphi(\psi(x)) = \cup_{y\in\psi(x)}\varphi(y)\ .$$

With this composition, $\mathcal{O}(X)$ is a semigroup of *non-deterministic* dynamical systems. For an open map $\varphi : X\to\mathcal{P}(X)$ and a subset $A\subseteq X$ the *image* $\varphi(A)\in\mathcal{P}(X)$ was defined by $\varphi(A) = \cup_{x\in A}\varphi(x)$. We said that $A\subseteq X$ is $\varphi-$*invariant* when $\varphi(A)\subseteq A$.

The space $\mathcal{H}(X)$ is a convolution subsemigroup of $\mathcal{MS}(X)$. We note that item 2. in the definition of $\mathcal{H}(X)$ ensures that φ_f is an open map in the semigroup $\mathcal{O}(X)$.

Given $f\in\mathcal{H}(X)$ and an open φ_f-invariant set $R\subseteq X$, let

$$\begin{aligned}\tau_f(R) &= \frac{1}{2}\sup_{x,z\in R}\int_R |f(x,y)-f(z,y)|\,dm(y)\\ &= 1-\min_{x,z\in R}\int_R f(x,y)\wedge f(z,y)\,dm(y)\ .\end{aligned}$$

and

$$\tau_f^*(R) = \inf_{n\geq 1}\,[\,\tau_{f^n}(R)\,]^{1/n}\ .$$

This quantity $\tau_f^*(R)$ will be called here the *mixing rate* in R. Next, we make some remarks on this concept of mixing rate:

1. $\tau_{f^n}(R) = 0 \Leftrightarrow$ the transition probabilities $f_x^n(\cdot) = f^n(x,\cdot)$ do not depend on x, for x over R.

2. If for some pair of points $x,y\in R$, the transition probabilities f_x^n and f_y^n have disjoint supports, then $\tau_{f^n}(R) = 1$.

3. $\tau_{f^n}(R)$ is half the diameter, in $\mathcal{M}_{\text{prob}}(X)$, of the set of probability transitions $\{\,f_x^n\,:\,x\in R\,\}$.

4. If $\tau_f^*(R) < 1$ then the restriction Markov system f_R on R is irreducible and acyclic.

Under the same invariance assumption on $R \subseteq X$, $\varphi_f(R) \subseteq R$, we define

$$\begin{aligned} \beta_f(R) &= 1 - \min_{x \in X} \int_R f(x,y)\, dm(y) \\ &= \sup_{x \in X} \int_{R^c} f(x,y)\, dm(y) \end{aligned}$$

and

$$\beta_f^*(R) = \inf_{n \geq 1} \left[\beta_{f^n}(R) \right]^{1/n} .$$

This quantity $\beta_f^*(R)$ will be called here the *escape rate* to R. Next, we make some remarks on this concept of escape rate:

1. $\beta_{f^n}(R) = 0 \quad \Leftrightarrow \quad \varphi_{f^n}(X) = (\varphi_f)^n(X) \subseteq R$.

2. If for some point $x \in X$, the transition probability f_x^n has support disjoint from R, then $\beta_{f^n}(R) = 1$.

3. $\beta_{f^n}(R)$ is the largest norm, in $\mathcal{M}_{\mathrm{prob}}(X)$, of the set of probability transitions restricted to R^c, $\{ (f_x^n)_{R^c} \, : \, x \in X \}$.

4. If $\beta_f^*(R) < 1$ then every $f-$orbit eventually enters R with probability one.

Assume, as before, that R is φ_f-invariant. Then L^1_R is $\mathcal{L}_f-$invariant. We can decompose $L^1(X,m)$ as $L^1(X,m) = L^1_R \oplus L^1_{R^c}$, where $R^c = X - R$. Given $h \in L^1(X,m)$, denote by $h_R = h \cdot \chi_R$ the function obtained multiplying h by R's characteristic function χ_R. Thus each function $q \in L^1(X,m)$ can be identified with the pair $\begin{bmatrix} q_R \\ q_{R^c} \end{bmatrix}$. Using this notation, if $q' = (\mathcal{L}_f)^n \, q$, for some $q \in L^1(X,m)$, then

$$\begin{bmatrix} q'_R \\ q'_{R^c} \end{bmatrix} = \begin{bmatrix} \mathcal{L}_{f_R} & \star \\ O & \mathcal{L}_{f_{R^c}} \end{bmatrix} \begin{bmatrix} q_R \\ q_{R^c} \end{bmatrix} ,$$

where O denotes the null operator from L^1_R into $L^1_{R^c}$, $\mathcal{L}_{f_R}$ describes the action of $\mathcal{L}_f$ on the invariant subspace L^1_R, and $\mathcal{L}_{f_{R^c}}$ the action of $\mathcal{L}_f$ on the non-invariant subspace $L^1_{R^c}$ composed with the projection onto $L^1_{R^c}$. Notice that $\mathcal{L}_{f_{R^c}}$ is a Perron-Frobenius type of operator associated with the substochastic transition function f_{R^c}. Therefore, we can decompose the spectrum of $\mathcal{L}_f$ as

$$\sigma\left(\mathcal{L}_f\right) = \sigma\left(\mathcal{L}_{f_R}\right) \cup \sigma\left(\mathcal{L}_{f_{R^c}}\right).$$

Proposition 3.1. *Given $f \in \mathcal{H}(X)$ and an open φ_f-invariant set $R \subseteq X$,*

$$\tau_f(R) = \sup_{q \in L^1_{R,0}} \frac{\left\|\mathcal{L}_{f_R}\, q\right\|_1}{\|q\|_1} = \left\|\mathcal{L}_{f_{R,0}}\right\|_1 .$$

In particular $\tau^*_f(R) = \lim_{n\to\infty} \left\|(\mathcal{L}_{f_{R,0}})^n\right\|_1^{1/n} = r\left(\mathcal{L}_{f_{R,0}}\right)$ *is the spectral radius of the operator* $\mathcal{L}_{f_{R,0}}$.

Proof. Take $q_1, q_2 \in L^1_R$ such that $q_1, q_2 \geq 0$ and $\displaystyle\int_R q_1(y)\, dm(y) = \int_R q_2(y)\, dm(y) = 1$. Clearly, $q = q_1 - q_2 \in L^1_{R,0}$. By definition of $\tau_f(R)$ we have that for every $x, z \in R$,

$$\int_X |f(x,y) - f(z,y)|\, dm(y) \;\leq\; 2\,\tau_f(R),$$

and so, averaging we have

$$\int_X \left| \int_R q_1(x) f(x,y)\, dm(x) - \int_R q_2(z) f(z,y)\, dm(z) \right| dm(y) \;\leq\; 2\,\tau_f(R).$$

Hence,

$$\int_X |\mathcal{L}_f\, q_1(y) - \mathcal{L}_f\, q_2(y)|\, dm(y) \;\leq\; 2\,\tau_f(R),$$

that is,

$$\|\mathcal{L}_f\, q_1 - \mathcal{L}_f\, q_2\|_1 \;\leq\; 2\,\tau_f(R).$$

If q_1 and q_2 have disjoit supports, then $\|q_1 - q_2\|_1 = 2$ and, therefore,

$$\|\mathcal{L}_f\, q_1 - \mathcal{L}_f\, q_2\|_1 \;\leq\; \tau_f(R)\; \|q_1 - q_2\|_1 \,.$$

In general, there is $\alpha \geq 1$ and $q \in L^1_R$ such that

$$q_1 - q_2 = \alpha^{-1}(q^+ - q^-) = \alpha^{-1} q \;\text{ with }\; \int_R q^+\, dm = 1 = \int_R q^-\, dm.$$

Then,

$$\begin{aligned}
\|\mathcal{L}_f\, q_1 - \mathcal{L}_f\, q_2\|_1 &= \alpha^{-1} \left\|\mathcal{L}_f\, q^+ - \mathcal{L}_f\, q^-\right\|_1 \\
&\leq \alpha^{-1}\, \tau_f(R)\; \left\|q^+ - q^-\right\|_1 \\
&= \tau_f(R)\; \|\, q_1 - q_2\,\|_1 \,.
\end{aligned}$$

Therefore, $\left\|\mathcal{L}_{f_{R,0}}\right\|_1 \leq \tau_f(R)$. Conversely, let $\{q^\epsilon_x\}$ denote a weak-$*$ approximation of δ_x in $L^1(X, m)$, for each $x \in X$. Given $x, z \in R$,

$$\begin{aligned}
\int_X |f(x,y) - f(z,y)|\; dm(y) &= \|\mathcal{L}_{f_R}\, \delta_x - \mathcal{L}_{f_R}\, \delta_z\|_1 \\
&= \lim_{\epsilon \to 0} \|\mathcal{L}_{f_R}\, q^\epsilon_x - \mathcal{L}_{f_R}\, q^\epsilon_z\|_1 \\
&\leq \lim_{\epsilon \to 0} \left\|\mathcal{L}_{f_{R,0}}\right\|_1\; \|q^\epsilon_x - q^\epsilon_z\|_1 \\
&\leq 2 \left\|\mathcal{L}_{f_{R,0}}\right\|_1 .
\end{aligned}$$

This implies that $\tau_f(R) \leq \left\|\mathcal{L}_{f_{R,0}}\right\|_1$, which completes the proof. □

Proposition 3.2. *Given $f \in \mathcal{H}(X)$ and an open set $R \subseteq X$,*

$$\beta_f(R) = \sup_{q \in L^1(X,m)} \frac{\|\mathcal{L}_{f_{R^c}}\, q\|_1}{\|q\|_1} = \|\mathcal{L}_{f_{R^c}}\|_1 \;\; .$$

In particular $\beta^*_f(R) = \lim_{n\to\infty} \|(\mathcal{L}_{f_{R^c}})^n\|_1^{\,1/n} = r\left(\mathcal{L}_{f_{R^c}}\right)$ *is the spectral radius of the operator* $\mathcal{L}_{f_{R^c}}$.

Corollary 3.3. *Given $f \in \mathcal{H}(X)$ and an open φ_f-invariant set $R \subseteq X$,*

$$r_{\rm int}\,(\mathcal{L}_f) \leq \max\{\tau_f^*(R), \beta_f^*(R)\}.$$

We shall say that an open φ_f-invariant set $R \subseteq X$ is an *acyclical spectral attractor* for $f \in \mathcal{H}(X)$ if and only if R is connected and $\tau_f^*(R) < 1$. When the set R splits as a disjoint union of d connected sets,

$$R \,=\, R_0 \,\cup\, \varphi_f(R_0) \,\cup\, \cdots \,\cup\, \varphi_f^d(R_0),$$

such that R_0 is an acyclic spectral attractor for f^d we say that R is a periodic spectral attractor of period d.

We shall say that an open φ_f-invariant set $R \subseteq X$ is *spectrally attractive* for $f \in \mathcal{H}(X)$ if and only if $\beta_f^*(R) < 1$.

We now state two propositions characterizing the uniform asymptotic behaviour of $f^n(x, y)$ as $n \to \infty$.

Proposition 3.4. *Let $R \subseteq X$ be a periodic spectral attractor of period d for $f \in \mathcal{H}(X)$ with connected components $R_0, \cdots, R_{d-1}$. For each component R_i, $(0 \leq i \leq d-1)$ there exists a unique $\mathcal{L}_{f^d}-$invariant probability density q_i supported in R_i such that for every $(x, y) \in R_i \times R_i$,*

$$q_i(y) = \lim_{n\to\infty} f^{nd}(x, y).$$

Furthermore, given $\tau_{f^d}^*(R_i) < r < 1$ there is $C_r(f) > 0$ such that for all $(x, y) \in R_i \times R_i$ and $n \geq 1$

$$\left|f^{nd}(x, y) - q_i(y)\right| \leq C_r\, r^n.$$

Moreover q_i is lower semi-continuous and is continuous when f is continuous.

Proposition 3.5. *Let $R \subseteq X$ be spectrally attractive for $f \in \mathcal{H}(X)$. For every $(x, y) \in X \times R^c$*

$$\lim_{n\to\infty} f^n(x, y) = 0.$$

Furthermore, given $\beta_f^*(R) < r < 1$ there is $C_r(f) > 0$ such that for every $(x, y) \in X \times R^c$

$$0 \leq f^n(x, y) \leq C_r\, r^n.$$

Corollary 3.6. *If $\tau_f^*(R) < 1$ and $\beta_f^*(R) < 1$ then f has a unique invariant measure supported in R, $\mu = \mathcal{L}_f(\mu)$, which is a hyperbolic attractor for the the affine action of operator $\mathcal{L}_f$ on $\mathcal{M}_{\text{prob}}(X)$.*

Proof. of proposition 3.4 Let R_i, $(0 \le i \le d-1)$, be a connected component of R and take $\tau_{f^d}^*(R_i) < r < 1$. It follows from definition that there is $N \in \mathbb{N}$ such that $\tau_{f^{Nd}}(R_i) < r$. Let $y \in R_i$ and define

$$\overline{M}_n(y) = \sup_{x \in R_i} f^{nNd}(x,y), \qquad \underline{M}_n(y) = \inf_{x \in R_i} f^{nNd}(x,y).$$

We have

$$\begin{aligned}\overline{M}_{n+1}(y) &= \sup_{x \in R_i} \int_{R_i} f^{Nd}(x,z)\, f^{nNd}(z,y)\, dm(z) \\ &\le \sup_{x \in R_i} \int_{R_i} f^{Nd}(x,z)\, \overline{M}_n(y)\, dm(z) \le \overline{M}_n(y).\end{aligned}$$

Similarly,

$$\underline{M}_{n+1}(y) \ge \underline{M}_n(y).$$

Therefore there exist the following limits

$$\overline{M}(y) = \lim_{n \to \infty} \overline{M}_n(y) \ge \underline{M}(y) = \lim_{n \to \infty} \underline{M}_n(y).$$

For $x, z \in R_i$ let

$$B_+ = \left\{ v \in R_i : f^{Nd}(x,v) - f^{Nd}(z,v) > 0 \right\}$$

and

$$B_- = \left\{ v \in R_i : f^{Nd}(x,v) - f^{Nd}(z,v) < 0 \right\}.$$

Since

$$\int_{R_i} f^{Nd}(x,v)\, dm(v) = \int_{R_i} f^{Nd}(z,v)\, dm(v) = 1,$$

we have

$$\int_{B_+} \left(f^{Nd}(x,v) - f^{Nd}(z,v) \right) dm(v) = -\int_{B_-} \left(f^{Nd}(x,v) - f^{Nd}(z,v) \right) dm(v).$$

Thus

$$\begin{aligned} f^{(n+1)Nd}(x,y)-f^{(n+1)Nd}(z,y) &= \int_{R_i} \big(f^{Nd}(x,v)-f^{Nd}(z,v)\big)\ f^{nNd}(v,y)\,dm(v) \\ &\le \overline{M}_n(y)\int_{B_+}\big(f^{Nd}(x,v)-f^{Nd}(z,v)\big)\ dm(v) \\ &\quad + \underline{M}_n(y)\int_{B_-}\big(f^{Nd}(x,v)-f^{Nd}(z,v)\big)\ dm(v) \\ &= \big(\overline{M}_n(y)-\underline{M}_n(y)\big)\int_{B_+}\big(f^{Nd}(x,v)-f^{Nd}(z,v)\big)\ dm(v). \end{aligned}$$

Because

$$\int_{B_+}\big(f^{Nd}(x,v)-f^{Nd}(z,v)\big)\ dm(v) = \frac{1}{2}\int_{R_i}\big|f^{Nd}(x,v)-f^{Nd}(z,v)\big|\ dm(v),$$

it follows from definition of $\tau_{f^{Nd}}(R_i)$ that

$$\int_{B_+}\big(f^{Nd}(x,v)-f^{Nd}(z,v)\big)\ dm(v) \le \tau_{f^{Nd}}(R_i).$$

Since x and z were arbitrary, we have that

$$\overline{M}_{n+1}(y)-\underline{M}_{n+1}(y) \le \tau_{f^{Nd}}(R_i)\ \big(\overline{M}_n(y)-\underline{M}_n(y)\big)\,.$$

Hence

$$\overline{M}(y)=\underline{M}(y)=\lim_{n\to\infty} f^{nNd}(x,y)=q_i(y).$$

Notice that

$$\overline{M}_1(y)-\underline{M}_1(y)\ \le\ \big\|f^{Nd}\big\|_\infty\,.$$

Therefore, for every $(x,y)\in R_i\times R_i$,

$$\big|f^{nNd}(x,y)-q_i(y)\big|\ \le\ \big\|f^{Nd}\big\|_\infty\ r^{(n-1)Nd},\ \ n=1,2,\cdots.$$

This implies that

$$\big|f^{nd}(x,y)-q_i(y)\big|\ \le C_r(f)\,r^n,\ \ n=1,2,\cdots,$$

where $C_r(f)=\dfrac{\|f^d\|_\infty}{r^{Nd}}$. In particular,

$$\lim_{n\to\infty} f^{nd}(x,y)=q_i(y).$$

It is clear from definition that q_i is a probability density supported in R_i. Furthermore, for any $(x,y) \in R_i \times R_i$,

$$f^{(n+1)d}(x,y) = \int_{R_i} f^{nd}(x,z)\, f^d(z,y)\, dm(z).$$

When $n \to \infty$, we get

$$q_i(y) = \int_{R_i} q_i(z)\, f^d(z,y)\, dm(z),$$

that is, q_i is $\mathcal{L}_{f^d}-$invariant. Moreover, since f is lower semi-continuous so is q_i. When f is continuous, q_i is continuous.

Now suppose q_i' is another $\mathcal{L}_{f^d}-$invariant probability density supported in R_i. By the invariance of q_i', for every $y \in R_i$,

$$\begin{aligned} q_i'(y) &= \int_{R_i} q_i'(z)\, f^{nd}(z,y)\, dm(z) \\ &\underset{n\to\infty}{\longrightarrow} \int_{R_i} q_i'(z)\, q_i(y)\, dm(z) = q_i(y). \end{aligned}$$

Therefore, q_i is the unique $\mathcal{L}_{f^d}-$invariant probability density supported in R_i. □

Proof. of proposition 3.5 Take $\beta_f^*(R) < r < 1$. It follows from definition that there is $N \in \mathbb{N}$ such that $\beta_{f^N}(R) < r$. Let $y \in R^c$ and define

$$M_n(y) = \sup_{x\in X} f^{nN}(x,y).$$

It follows from definitions that,

$$\begin{aligned} M_{n+1}(y) &= \sup_{x\in X} f^{(n+1)N}(x,y) \\ &= \sup_{x\in X} \int_X f^N(x,z)\, f^{nN}(z,y)\, dm(z) \\ &= \sup_{x\in X} \int_{R^c} f^N(x,z)\, f^{nN}(z,y)\, dm(z) \\ &\le M_n(y) \sup_{x\in X} \int_{R^c} f^N(x,z)\, dm(z) \\ &= M_n(y)\, \beta_{f^N}(R). \end{aligned}$$

Notice that $M_1(y) \leq \left\|f^N\right\|_\infty$. Hence, for any $(x,y) \in X \times R^c$,

$$0 \leq f^{nN}(x,y) \;\leq\; \left\|f^N\right\|_\infty r^{(n-1)N}, \;\; n = 1, 2, \cdots .$$

This implies that

$$0 \leq f^n(x,y) \;\leq\; C_r(f)\, r^n, \;\; n = 1, 2, \cdots ,$$

where $C_r(f) = \dfrac{\|f\|_\infty}{r^N}$. □

Given $f \in \mathcal{H}(X)$, let us considerer the associated open map $\varphi_f \in \mathcal{O}(X)$ defined in item (3) in the definition of the space $\mathcal{H}(X)$ by $\varphi_f(x) = \{\, y \in X \,:\, f(x,y) > 0 \,\}$ $(x \in X)$. In [DT] the dynamics of open maps was characterized. Briefly, given any open map φ, a sequence $x_0, x_1, \cdots, x_n$ such that $x_i \in \varphi(x_{i-1})$ for all $i = 1, \cdots, n$ is called a $\varphi-$*orbit*, and we say that x_n is a $\varphi-$*iterate* of x_0. If for every $\epsilon > 0$, y is a φ_ϵ^*-iterate of x, where φ_ϵ^* is the open map whose graph is an $\epsilon-$radius ball of graph(φ), we say that y is a $\varphi-$*pseudo-iterate* of x. The *recurrent* and *chain-recurrent sets* of φ are defined respectively by $\Omega(\varphi) = \{x \in X \,:\, x \text{ is a } \varphi - \text{iterate of } x\}$ and $R(\varphi) = \{x \in X : x \text{ is a } \varphi\text{−pseudo-iterate of } x\}$. Both these sets split into equivalence classes, each class being formed by states which are accessible from each other. The set of all these classes is then partially ordered by the dynamics of φ. At the bottom of this hierarchy are two special limit sets: the *final recurrent* and the *final chain-recurrent sets*, denoted respectively by $\Omega_{\text{final}}(\varphi)$ and $R_{\text{final}}(\varphi)$, of all states $x \in \Omega(\varphi)$ $(x \in R(\varphi))$ such that every iterate (pseudo-iterate) of x still has some iterate (pseudo-iterate) which comes back to x. These limit sets contain all the asymptotic dynamical behaviour of φ. They both decompose into a finite number of equivalence classes, called respectively $\Omega-$final and $R-$final classes. We denote by $\Lambda^{\Omega}_{\text{final}}(\varphi)$ respectively $\Lambda^{R}_{\text{final}}(\varphi)$ the set of all equivalence classes of the limit sets $\Omega_{\text{final}}(\varphi)$ and $R_{\text{final}}(\varphi)$. Each $\Omega-$final and $R-$final class decomposes into a finite number of connected pieces, called respectively $\Omega-$final and $R-$final components, which are permuted by φ. See Theorems 5.1 and 5.2 of [DT]. The restriction of φ to each of these pieces is in some sense irreducible. We call period of a final class to the number of its connected components. The period of a connected component is the period of its class. We denote by $\Sigma^{\Omega}_{\text{final}}(\varphi)$ respectively $\Sigma^{R}_{\text{final}}(\varphi)$ the set of connected pieces of the limit sets $\Omega_{\text{final}}(\varphi)$ and $R_{\text{final}}(\varphi)$. The dynamics of φ on these limit sets is given by the following theorems.

Proposition 3.7. *Given $f \in \mathcal{H}(X)$, each $\Omega-$final class of period d is a periodic spectral attractor of period d for f.*

Proposition 3.8. *Given $f \in \mathcal{H}(X)$, $\Omega_{\text{final}}(\varphi_f)$ is spectrally attractive for f.*

Corollary 3.9. *Given $f \in \mathcal{H}(X)$, let $\Sigma^{\Omega}_{\text{final}}(\varphi_f) = \{R_1, R_2, \cdots, R_s\}$. Let κ_f be the maximum between $\beta^*_f(R)$ and $\tau^*_f(R_i)$, for $i = 1, \cdots, s$. Then*

$$r_{\text{int}}(\mathcal{L}_f) \leq \kappa_f.$$

Corollary 3.10. *Given $f \in \mathcal{H}(X)$, let $\Sigma^{\Omega}_{\text{final}}(\varphi_f) = \{R_1, R_2, \cdots, R_s\}$, where each component R_i is $\varphi_f^{d_i}-$invariant for some power $d_i \geq 1$. Then there is a f^{d_i}-invariant measure supported on R_i, $\mu_i = \mathcal{L}_{f^{d_i}}\, \mu_i$, for each $i = 1, \cdots, s$, such that:*

1. *The sum $E_1(f)$ of all generalized eigenspaces associated with eigenvalues in the unit circle is the $s-$dimensional space spanned by the measures $\mu_1, \cdots, \mu_s$.*
2. *The action of $\mathcal{L}_f$ on the invariant subspace $E_1(f)$ w.r.t. the basis $\{\mu_1, \cdots, \mu_s\}$ is represented by the permutation matrix associated with the permutation π_f.*
3. *The eigenvalues of $\mathcal{L}_f$ in the unit circle are, with multiplicity, the $d-$unity roots $\mathbb{U}^d = \{\lambda \in \mathbb{C} : \lambda^d = 1\}$, counted for every cycle of length d in permutation π_f.*
4. *The subspace $E_1(f)$ is normally hyperbolic (contractive).*

To prove Proposition 3.7 and Proposition 3.8 above we will use the concept of final kernel that we defined in a previous article. In [DT] we called the *thickness* of φ_f the smallest volume ($m-$measure) of all components in $\Sigma^{\Omega}_{\text{final}}(\varphi_f)$. We said that an open set $K \subseteq \Omega_{\text{final}}(\varphi_f)$ is a *final kernel* of φ_f if and only if there is a one-to-one correspondence $R \mapsto K_R$, between components $R \in \Sigma^{\Omega}_{\text{final}}(\varphi)$ and connected components K_R of K, such that $\overline{K_R} \subseteq R$ for every $R \in \Sigma^{\Omega}_{\text{final}}(\varphi_f)$. We said that K is a *final kernel with finite order* N if and only if K is a final kernel of φ_f, and furthermore

(1) For each component $R \in \Sigma^{\Omega}_{\text{final}}(\varphi_f)$ of period d, the only connected component K_R of K contained in R satisfies $\overline{R \times K_R} \subseteq \text{graph}\left(\varphi_f^{N\, d}\right)$.

(2) For each $x \in X$, $\varphi_f^N(x)$ contains at least the closure of one of K's connected components.

We called the *thickness* of the final kernel K the smallest volume of all connected components of K. We proved that every final kernel K of φ_f is a final kernel with some finite order $N \in \mathbb{N}$. In particular, φ_f admits finite-order final kernels, whose thickness is arbitrarily close to the thickness of φ_f. See Lemma 5.20 of [DT].

Proof. of proposition 3.7 Let $R_1, \cdots, R_s$ be the $\Omega-$final components of φ_f. Take some final kernel $K \subseteq \Omega_{\text{final}}(\varphi_f)$ and choose $N \in \mathbb{N}$ such that K is a finite kernel of order N. Let $K_1, \cdots, K_s$ be the connected components of K corresponding to $R_1, \cdots, R_s$. Item 1. in the definition of finite kernel of order N implies that for each component R_i of period d_i and for all $(x, y) \in \overline{R_i \times K_i}$, one has $f^{N d_i}(x, y) > 0$. Moreover, because f is lower semi-continuous, there is $c_i > 0$ such that for all $(x, y) \in \overline{R_i \times K_i}$, $f^{N d_i}(x, y) \geq c_i$. Therefore, $\tau_{f^{N d_i}}(R_i) \leq 1 - c_i < 1$ which implies that $\tau^*_{f^{d_i}}(R_i) < 1$. □

Proof. of proposition 3.8 Take some final kernel $K \subseteq \Omega_{\text{final}}(\varphi_f)$ and choose $N \in \mathbb{N}$ such that K is a finite kernel of order N. Item 2. in the definition of finite kernel of order N implies that for each $x \in X$, $\int_K f^N(x, y)\, dm(y) > 0$. Thus, since $K \subseteq \Omega_{\text{final}}(\varphi_f)$, one has $\int_{\Omega_{\text{final}}(\varphi_f)} f^N(x, y)\, dm(y) > 0$ for each $x \in X$. Moreover, because f is lower semi-continuous, there is $\alpha_0 > 0$ such that for all $x \in X$, $\int_{\Omega_{\text{final}}(\varphi_f)} f^N(x, y)\, dm(y) \geq \alpha_0$. Therefore, $\beta_{f^N}(\Omega_{\text{final}}(\varphi_f)) \leq 1 - \alpha_0 < 1$ and, consequently, $\beta^*_f(\Omega_{\text{final}}(\varphi_f)) < 1$. □

4 Topological Semigroups of Markov Systems

In a previous article [DT] we defined several topological spaces of open sets. See [N] for an overview on topological spaces of sets. We also introduced and topologized some semigroups of open maps. Any open map φ can be identified with its graph graph(φ) and, therefore, seen as an open set in $X \times X$. Thus, semigroups of open maps can be given topologies as subspaces of topological spaces of open sets. The key concept of *topological semigroup of open maps*, to which the main Theorems that characterize the cominatorial stability of open maps and establish its genericity apply, was defined.

Consider any subsemigroup of open maps $\mathcal{O}_1 \subseteq \mathcal{O}(X)$, endowed with some topology.

Definition 4.1. We say that $\mathcal{O}_1$ is a *topological semigroup of open maps* if and only if:

(1) the Hausdorff distance between open map graphs is continuous;

(2) for each $\varphi \in \mathcal{O}_1$, there is a family of open maps $\{\tilde{\varphi}_\epsilon\}_{\epsilon>0}$ in $\mathcal{O}_1$ such that

  (a) $\overline{\text{graph}\,(\varphi)} = \bigcap_{\epsilon>0} \text{graph}\,(\tilde{\varphi}_\epsilon)$;

  (b) for all ϵ_1, ϵ_2, if $\epsilon_1 > \epsilon_2 > 0$ then $\overline{\text{graph}\,(\tilde{\varphi}_{\epsilon_2})} \subseteq \text{graph}\,(\tilde{\varphi}_{\epsilon_1})$; and

  (c) $\lim_{\epsilon\to 0^+} \tilde{\varphi}_\epsilon = \varphi$ w.r.t. $\mathcal{O}_1$ topology.

(3) given $\epsilon > 0$, an integer $N \in \mathbb{N}$, and non-empty open subsets U, $V \subseteq X$ such that $\overline{U \times V} \subseteq \text{graph}\left(\varphi^N\right)$, there is a neighbourhood $\mathcal{N}$ of φ in $\mathcal{O}_1$ such that for all $\psi \in \mathcal{N}$ and $x \in \overline{U}$, $m(V \setminus \widehat{\psi^N}(x)) < \epsilon$, where $\widehat{\psi} : X \to \mathcal{P}(X)$ is defined by setting $\text{graph}\left(\widehat{\psi}\right) = \left(\overline{\text{graph}\,(\psi)}\right)^\circ$.

Given open maps $\varphi, \psi : X \to \mathcal{P}(X)$, we will write $\varphi \leq \psi$ to mean that $\text{graph}\,(\varphi) \subseteq \text{graph}\,(\psi)$, and $\varphi \prec \psi$ to say that $\overline{\text{graph}\,(\varphi)} \subseteq \text{graph}\,(\psi)$.

Consider any subsemigroup of Markov systems $\mathcal{H}_1 \subseteq \mathcal{H}(X)$, endowed with some topology.

Definition 4.2. We say that $\mathcal{H}_1$ is a *topological semigroup of Markov systems* over a topological semigroup of open maps $\mathcal{O}_1$ if and only if for any $f \in \mathcal{H}_1$:

(1) $\varphi_f \in \mathcal{O}_1$;

(2) The map $f \mapsto \varphi_f$ is continuous for the topology of $\mathcal{O}_1$;

(3) $\mathcal{H}_1$ admits outer approximations in the sense that given $f \in \mathcal{H}_1$, for every neighbourhood $\mathcal{N}$ of f in $\mathcal{H}_1$ there is $g \in \mathcal{N}$ such that $\varphi_f \prec \varphi_g$;

(4) $\lim_{g\to f} \left\|\mathcal{L}_f^*\,\varphi - \mathcal{L}_g^*\,\varphi\right\|_\infty = 0$ for all $\varphi \in C^0(X)$;

(5) $\tau_f(R)$ and $\beta_f(R)$ vary upper semicontinuously with f for any set $R \subseteq X$.

Given $f, g \in \mathcal{H}(x)$ we say that f *is dominated by* g when $\varphi_f \prec \varphi_g$.

We now topologize the semigroup $\mathcal{H}(X)$ turning it into a topological semigroup of Markov systems.

Consider

$$d_\infty(f, g) = \max_{(x,y)\in X\times X} |f(x, y) - g(x, y)|$$

and

$$d_1(f, g) = \max_{x\in X} \int_X |f(x, y) - g(x, y)|\, dm(y).$$

Remark 4.3. Given $f, g \in \mathcal{H}$ and $N \in \mathbb{N}$,

1. $d_\infty(f^N, g^N) \leq d_\infty(f, g)(1 + N \|f\|_\infty)$.
2. $d_1(f^N, g^N) \leq d_1(f, g)(1 + N \|f\|_\infty)$.
3. $\left\|f^N\right\|_\infty \leq \|f\|_\infty$
4. $\left\|f^N\right\|_1 \leq \|f\|_1$

Proposition 4.4. *Let $(\mathcal{O}(X), \rho)$ be a topological semigroup of open maps and consider*

$$\rho_\infty(f, g) = \max\{d_\infty(f, g), \rho(\varphi_f, \varphi_g\}$$

and

$$\rho_1(f, g) = \max\{d_1(f, g), \rho(\varphi_f, \varphi_g\}.$$

With both the topology associated to ρ_∞ and ρ_1, $\mathcal{H}(X)$ is a topological semigroup of Markov systems over $\mathcal{O}(X)$.

Proof. Definition 4.2(4) is clear. We prove now Definition 4.2(3). Let $f \in \mathcal{H}(X)$. For each $\epsilon > 0$ let φ^*_ϵ be, as before, the open map whose graph is an $\epsilon-$radius ball of graph(φ_f). Clearly, $\varphi_f \prec \varphi^*_\epsilon$. Let $\{h_\epsilon\}_{\epsilon>0}$, $h_\epsilon : X \times X \to [0, 1]$, be a family of continuous maps defined by:

(i) graph(φ_{h_ϵ}) = graph(φ^*_ϵ), and

(ii) $h_\epsilon(x,y) = 1$ for every $(x,y) \in \mathrm{graph}\,(\varphi_f)$.

For each map h_ϵ and $x \in X$, let

$$\tilde{h}_\epsilon(x) = \int_X h_\epsilon(x,y)\, dm(y),$$

and define, for $x, y \in X$,

$$f_\epsilon(x,y) = \frac{h_\epsilon(x,y)}{\tilde{h}_\epsilon(x)}.$$

For every $x \in X$, $\tilde{h}_\epsilon(x) \geq m(\varphi_f(x)) \geq c_0 > 0$, where c_0 denotes the volume (m measure) of a ball of radius $\xi_0 = \xi_0(\varphi_f) > 0$ for some $\xi_0 > 0$. (See Lemma 2.7 in [DT] where we prove that given $\varphi \in \mathcal{O}(X)$ there exist a map $F : X \to X$ and $\xi_0 > 0$ such that $\mathcal{B}_{\xi_0}(\mathrm{graph}\,(F)) \subseteq \mathrm{graph}\,(\varphi)$). Therefore, f_ϵ is bounded and continuous. We define the family $\{g_\epsilon\}_{\epsilon>0}$ by

$$g_\epsilon = (1-\epsilon) f + \epsilon\, f_\epsilon.$$

It is easy to see that $g_\epsilon \in \mathcal{H}(X)$, for every $\epsilon > 0$ sufficiently small. Clearly,

$$\lim_{\epsilon \to 0} d_\infty(f, g_\epsilon) = 0,$$

which implies that

$$\lim_{\epsilon \to 0} d_1(f, g_\epsilon) = 0.$$

Moreover, φ_{g_ϵ} is an open map whose graph coincides with the graph of φ^*_ϵ. Therefore,

$$\lim_{\epsilon \to 0} \rho(\varphi_f, \varphi_{g_\epsilon}) = 0.$$

Thus,

$$\lim_{\epsilon \to 0} \rho_\infty(f, g_\epsilon) = \lim_{\epsilon \to 0} \rho_1(f, g_\epsilon) = 0.$$

To prove Definition 4.2(4) we will show that given $f, g \in \mathcal{H}(X)$, for all $\varphi \in C^0(X)$:

$$\begin{aligned} \left\| \mathcal{L}^*_f \varphi - \mathcal{L}^*_g \varphi \right\|_\infty &\leq \max_{y \in X} |\varphi(y)|\, d_1(f,g) \\ &\leq \max_{y \in X} |\varphi(y)|\, d_\infty(f,g). \end{aligned}$$

We have that

$$
\begin{aligned}
\left\|\mathcal{L}_f^* \varphi - \mathcal{L}_g^* \varphi\right\|_\infty &= \max_{x\in X}\left|\mathcal{L}_f^* \varphi(x) - \mathcal{L}_g^* \varphi(y)\right| \\
&= \max_{x\in X}\left|\int_X f(x,y)\varphi(y)\,dm(y) - \int_X g(x,y)\varphi(y)\,dm(y)\right| \\
&\leq \max_{x\in X}\int_X |f(x,y)-g(x,y)|\,|\varphi(y)|\ dm(y) \\
&\leq \max_{y\in X}|\varphi(y)|\max_{x\in X}\int_X |f(x,y)-g(x,y)|\ dm(y) \\
&= \max_{y\in X}|\varphi(y)|\, d_1(f,g) \\
&\leq \max_{y\in X}|\varphi(y)|\, d_\infty(f,g).
\end{aligned}
$$

Finally we prove Definition 4.2(5). Given $f, g \in \mathcal{H}(X)$ and open φ_f-invariant set $R \subseteq X$ we have that:

$$
\begin{aligned}
|\tau_f(R) - \tau_g(R)| &= \left|\min_{x,z\in R}\int_R f(x,y)\wedge f(z,y)\,dm(y) - \min_{x,z\in R}\int_R f(x,y)\wedge g(z,y)\,dm(y)\right| \\
&\leq \left|\min_{x\in R}\int_R f(x,y)\,dm(y) - \min_{x\in R}\int_R g(x,y)\,dm(y)\right| \\
&\leq \max_{x\in R}\int_R |f(x,y)-g(x,y)|\ dm(y) \\
&= d_1(f,g) \leq d_\infty(f,g)
\end{aligned}
$$

and

$$\begin{aligned} |\beta_f(R) - \beta_g(R)| &= \left| \min_{x\in X} \int_R f(x,y)\, dm(y) - \min_{x\in X} \int_R g(z,y)\, dm(y) \right| \\ &\le \max_{x\in X} \int_R |f(x,y) - g(x,y)|\ dm(y) \\ &= d_1(f,g) \le d_\infty(f,g). \end{aligned}$$

□

5 Spectral Stability

Definition 5.1. Given a topological semigroup $\mathcal{H}_1 \subseteq \mathcal{H}(X)$, we say that $f \in \mathcal{H}(X)$ is *spectrally stable in* $\mathcal{H}_1$ if and only if there is a neighbourhood $\mathcal{U}$ of f in $\mathcal{H}_1$ and there is $0 < k < 1$ such that for all $g \in \mathcal{U}$:

(1) There is a map $h_g : E_1(f) \to E_1(g)$ that conjugates $\mathcal{L}_f\,|_{E_1(f)}$ to $\mathcal{L}_g\,|_{E_1(g)}$.

(2) The map h_g depends continuously on f w.r.t. the topology in $\mathcal{H}_1$, in the sense that for any $\varphi \in C^0(X)$, $\lambda_\varphi \circ h_g$ converges to λ_φ as g tends to f in $\mathcal{H}_1$, where $\lambda_\varphi : L^1(X,m) \to \mathbb{R}$ is defined by $\lambda_\varphi(\mu) = \int \varphi\, d\,\mu$.

(3) $\sigma_0(\mathcal{L}_g) \cap \{\lambda \in \mathbb{C} : k < |\lambda| < 1\} = \varnothing$.

We note that item 2. above is equivalent to say that the invariant measures of $\mathcal{L}_f$ vary continuously with f w.r.t. the weak-$*$ topology.

In a previous article [DT] we have defined a concept of *combinatorial stability* for open maps, which roughly means that the same combinatorics of attractors persists for all nearby systems. In that article we have characterized the combinatorial stability of open maps, and established its genericity. A natural homomorphism $\varphi : \mathcal{H}(X) \to \mathcal{O}(X)$ relates each stochastic dynamical system $f \in \mathcal{H}(X)$ with the nondeterministic open map $\varphi_f : x \mapsto \text{int}(\text{supp}(f_x))$. Our main goal in this section is to study and compare, for generic systems $f \in \mathcal{H}(X)$, the combinatorial stability of φ_f, with the spectral stability of f. We begin by briefly revieing the concept of combinatorial stability. Recall that each open map $\varphi \in \mathcal{O}(X)$ induces a permutation π_φ on the set $\Sigma^{\Omega}_{\text{final}}(\varphi)$ of $\Omega-$final components. Let $\varphi, \psi \in \mathcal{O}(X)$. We say that φ is *combinatorially equivalent* to ψ if and only if the permutations π_φ and π_ψ are conjugated, that is, there is a bijective map $h : \Sigma^{\Omega}_{\text{final}}(\varphi) \to \Sigma^{\Omega}_{\text{final}}(\psi)$ such that $\pi_\psi \circ h = h \circ \pi_\varphi$.

Given a topological subsemigroup $\mathcal{O}_1 \subseteq \mathcal{O}(X)$, we say that $\varphi \in \mathcal{O}(X)$ is *combinatorially stable in* $\mathcal{O}_1$ if and only if there is a neighbourhood $\mathcal{U}$ of φ in $\mathcal{O}_1$ such that all $\psi \in \mathcal{U}$ are combinatorially equivalent to φ.

We prove that, for a topological semigroup of open maps $\mathcal{O}_1$, a map $\varphi \in \mathcal{O}_1$ is combinatorially stable in $\mathcal{O}_1$ if and only if $\left|\Lambda^{\Omega}_{\text{final}}(\varphi)\right| = \left|\Lambda^{R}_{\text{final}}(\varphi)\right|$ and $\left|\Sigma^{\Omega}_{\text{final}}(\varphi)\right| = \left|\Sigma^{R}_{\text{final}}(\varphi)\right|$. See Theorem 5.3 in [DT]. We also prove there that the set of $\mathcal{O}_1$-combinatorially stable maps is open and dense in the semigroup $\mathcal{O}_1$. See Theorem 5.4 in [DT].

Proposition 5.2. *Let $\mathcal{H}_1$ be any topological semigroup of Markov systems. Given $f \in \mathcal{H}_1$ with φ_f combinatorially stable, let $R_1, \cdots, R_s$ be the $\Omega-$final components of φ_f. There is an open neighbourhood $\mathcal{N}$ of f in $\mathcal{H}_1$ and there are open neighbourhoods $U_1, \cdots, U_s$ of $R_1, \cdots, R_s$, respectively, such that*

(1) *Each U_i, $(1 \le i \le s)$, is an acyclical spectral attractor for f^{d_i} for some $d_i \ge 1$.*

(2) *For all $g \in \mathcal{N}$, $\varphi_g^{d_i}(U_i) \subseteq U_i$ $(1 \le i \le s)$.*

Proof. Since φ_f is combinatorially stable there is a neighbourhood $\mathcal{U}$ of φ_f in $\mathcal{O}_1$ such that every $\phi \in \mathcal{U}$ is combinatorially equivalent to φ_f. By Definition 4.2(2) the map $f \mapsto \varphi_f$ is continuous and, therefore, there is a neighbourhood $\mathcal{N}$ of f in $\mathcal{H}_1$ such that for every $h \in \mathcal{N}$, $\varphi_h \in \mathcal{U}$. Furthermore, Definition 4.2(3) ensures that there is $h \in \mathcal{N}$ such that $\varphi_f \prec \varphi_h$. Let $U_1, \cdots, U_s$ be the $\Omega-$final components of φ_h, i.e. $\Sigma^{\Omega}_{\text{final}}(\varphi_h) = \{U_1, U_2, \cdots, U_s\}$. Because φ_f is combinatorially equivalent to φ_h and $\varphi_f \prec \varphi_h$ we have that $R_i \subseteq U_i$, for $i = 1, \cdots, s$. Take some final kernel $K \subseteq \Omega_{\text{final}}(\varphi_h)$ and choose $N \in \mathbb{N}$ such that K is a finite kernel of order N. Let $K_1, \cdots, K_s$ be the connected components of K corresponding to $U_1, \cdots, U_s$. Item 1. in the definition of finite kernel of order N implies that that for all $(x, y) \in \overline{U_i \times K_i}$, one has $f^{Nd_i}(x, y) > 0$. Moreover, because f is lower semi-continuous, there is $c_i > 0$ such that for all $(x, y) \in \overline{U_i \times K_i}$, $f^{Nd_i}(x, y) \ge c_i$. Therefore, $\tau_{f^{Nd_i}}(U_i) \le 1 - c_i < 1$ which implies that $\tau^*_{f^{d_i}}(U_i) < 1$.

To prove item 2. choose some neighbourhood $\mathcal{N}$ of f in $\mathcal{H}_1$ such that for all $g \in \mathcal{N}$, $\operatorname{graph}(\varphi_g) \subseteq \operatorname{graph}(\varphi_h)$. Such a neighbourhood exists by Definition 4.1(1) and Definition 4.2(2). Therefore $\varphi_g^{d_i}(U_i) \subseteq \varphi_h^{d_i}(U_i) \subseteq U_i$. $\square$

Lemma 5.3. *Given $\varphi, \psi \in \mathcal{O}(X)$, such that $\varphi \prec \psi$, if φ is combinatorially equivalent to ψ then φ is combinatorially stable.*

Proof. Since $\varphi \prec \psi$, it is a straightforward consequence of the definitions that we have:

$$\left|\Lambda^{\Omega}_{\text{final}}(\varphi)\right| \geq \left|\Lambda^{R}_{\text{final}}(\varphi)\right| \geq \left|\Lambda^{\Omega}_{\text{final}}(\psi)\right|$$

and

$$\left|\Sigma^{\Omega}_{\text{final}}(\varphi)\right| \geq \left|\Sigma^{R}_{\text{final}}(\varphi)\right| \geq \left|\Sigma^{\Omega}_{\text{final}}(\psi)\right|.$$

Because φ is combinatorially equivalent to ψ we have that $\left|\Lambda^{\Omega}_{\text{final}}(\varphi)\right| = \left|\Lambda^{\Omega}_{\text{final}}(\psi)\right|$ and $\left|\Sigma^{\Omega}_{\text{final}}(\varphi)\right| = \left|\Sigma^{\Omega}_{\text{final}}(\psi)\right|$. Therefore, $\left|\Lambda^{\Omega}_{\text{final}}(\varphi)\right| = \left|\Lambda^{R}_{\text{final}}(\varphi)\right|$ and $\left|\Sigma^{\Omega}_{\text{final}}(\varphi)\right| = \left|\Sigma^{R}_{\text{final}}(\varphi)\right|$, that is, φ satisfies the combinatorially stability condition, which implies that φ is combinatorially stable. $\square$

Proof. of Theorem A (Spectral Stability Characterization) Given $f \in \mathcal{H}_1$ assume φ_f is combinatorially stable. By Definition 4.2(2) there is a neighbourhood $\mathcal{N}$ of f in $\mathcal{H}_1$ such that for all $g \in \mathcal{N}$, φ_g is combinatorially equivalent to φ_f. Consequently it follows from Corollary 3.10 that there is a map $h_g : E_1(f) \to E_1(g)$ that conjugates $\mathcal{L}_f|_{E_1(f)}$ to $\mathcal{L}_g|_{E_1(g)}$, which proves Definition 5.1(1).

To prove Definition 5.1(2) we just need to note that Definition 4.2(4) easily implies that the invariant measures of $\mathcal{L}_f$ vary continuously with f w.r.t. the weak-$*$ topology. Indeed, given $f \in \mathcal{H}$ with φ_f combinatorially stable, let $\mathcal{N}$ and U_i, $1 \leq i \leq s$, be as in Proposition 5.2. We have that every invariant measure of $\mathcal{L}_g$ with $g \in \mathcal{N}$ is supported in some $\overline{U_i}$. Take $\mu_f \in E_1(f)$ and $\mu_g \in E_1(g)$ with $\mu_f(U_i) = \mu_g(U_i) = 1$. Choose $N \in \mathbb{N}$ such that $\left\|\mathcal{L}_{f^N_{U_i,0}}\right\|_1 < 1$. For any $\varphi \in C^0(X)$,

$$\begin{aligned}
|\langle \mu_f - \mu_g, \varphi\rangle| &= \left|\left\langle \mathcal{L}_{f^N}\mu_f - \mathcal{L}_{g^N}\mu_g, \varphi\right\rangle\right| \\
&\leq \left|\left\langle \mathcal{L}_{f^N}\mu_f, \varphi\right\rangle - \left\langle \mathcal{L}_{f^N}\mu_g, \varphi\right\rangle\right| + \left|\left\langle \mathcal{L}_{f^N}\mu_g, \varphi\right\rangle - \left\langle \mathcal{L}_{g^N}\mu_g, \varphi\right\rangle\right| \\
&\leq \left|\left\langle \mathcal{L}_{f^N}(\mu_f - \mu_g), \varphi\right\rangle\right| + \left|\left\langle \left(\mathcal{L}_{f^N} - \mathcal{L}_{g^N}\right)\mu_g, \varphi\right\rangle\right|
\end{aligned}$$

Definition 4.2(4) implies that

$$\lim_{g \to f} \left| \left\langle \left(\mathcal{L}_{f^N} - \mathcal{L}_{g^N} \right) \mu_g, \varphi \right\rangle \right| = 0.$$

Therefore,

$$\lim_{g \to f} \left| \langle \mu_f - \mu_g, \varphi \rangle \right| = 0$$

which proves Definition 5.1(2).

Let $R_1, \cdots, R_s$ be the $\Omega-$final components of φ_f in $\Sigma^{\Omega}_{\text{final}}(\varphi_f)$. By Proposition 5.2(1) for each $\Omega-$component R_i, $(1 \le i \le s)$, there is an open neighbourhood U_i of R_i such that U_i is an acyclical spectral attractor for f^{d_i} for some power $d_i \ge 1$. In particular, $\tau^*_{f^{d_i}}(U_i) < 1$ which implies that $\tau^*_f(U_i) < 1$. Set $U = \cup_{i=1}^s U_i$. It is clear that $\Omega_{\text{final}}(\varphi_f) \subseteq U$. Furthermore, because $\Omega_{\text{final}}(\varphi_f)$ is spectrally attractive, one has $\beta^*_f(\Omega_{\text{final}}(\varphi)) < 1$. These two facts together imply that $\beta^*_f(U) < 1$. By Corollary 3.9 we have that $r_{\text{int}}(\mathcal{L}_f) < 1$.

By Proposition 5.2(2) we can make $\mathcal{N}$ smaller so that for every $g \in \mathcal{N}$, one has $\varphi_{g^{d_i}}(U_i) \subseteq U_i$, $(1 \le i \le s)$. Thus, it follows immediatly from Corollary 3.3 that

$$r_{\text{int}}(\mathcal{L}_g) \le \max \left\{ \left\{ \tau^*_g(U_i),\ 1 \le i \le s \right\} \cup \left\{ \beta^*_g(U) \right\} \right\}.$$

By Definition 4.2(5) $\tau^*_f(U_i)$ and $\beta^*_f(U)$ vary upper semicontinuously with f. Therefore we can make $\mathcal{N}$ even smaller so that there is $k < 1$ such that for every $g \in \mathcal{N}$, one has $r_{\text{int}}(\mathcal{L}_g) \le k$. This proves Definition 5.1(3) that there is a spectral gap of size k isolating $\sigma_1(\mathcal{L}_g)$ and $\sigma_0(\mathcal{L}_g)$. Therefore, f is spectrally stable.

Assume now that f is $\mathcal{H}_1-$spectrally stable and let $\mathcal{N}$ be a neighbourhood of f in $\mathcal{H}_1$ where all the systems are spectrally equivalent. In particular, by Definition 5.1(1), all maps φ_g with $g \in \mathcal{N}$ are combinatorially equivalent. By Definition 4.2(3) there is $g \in \mathcal{N}$ such that $\varphi_f \prec \varphi_g$. Therefore, by Lemma 5.3, φ_f is combinatorially stable. □

Proof. of Theorem B (Generacity of Spectral Stability) By definition, spectral stability is an open property. Thus it is enough to prove density. Let $\mathcal{N}_0$ be an arbitrary neighbourhood of f. By Definition 4.2(3) there is $g_0 \in \mathcal{N}_0$ such that $\varphi_f \prec \varphi_{g_0}$. If φ_f is combinatorially equivalent to φ_{g_0}, Lemma 5.3 ensures that φ_f is combinatorially stable. Otherwise, let $\mathcal{N}_1 \subset \mathcal{N}_0$ be a neighbourhood of f where all maps are dominated by g_0. Such a neighbourhood exists by Definition 4.1(1) and Definition 4.2(2). By Definition 4.2(3) there is $g_1 \in \mathcal{N}_1$ such that $\varphi_f \prec \varphi_{g_1}$. If φ_{g_1} is combinatorially equivalent to φ_{g_0}, Lemma 5.3 ensures that φ_{g_1} is combinatorially stable. Otherwise we repeat the process considering a sequence of dominated maps

$$\varphi_f \prec \cdots \prec \varphi_{g_n} \prec \cdots \prec \varphi_{g_1} \prec \varphi_{g_0}.$$

It is a straightforward consequence of the definitions that we have:

$$\left|\Lambda^{\Omega}_{\text{final}}\left(\varphi_f\right)\right| \geq \left|\Lambda^{\Omega}_{\text{final}}\left(\varphi_{g_n}\right)\right| \geq \left|\Lambda^{\Omega}_{\text{final}}\left(\varphi_{g_{n-1}}\right)\right| \geq \left|\Lambda^{\Omega}_{\text{final}}\left(\varphi_{g_0}\right)\right|$$

and

$$\left|\Sigma^{\Omega}_{\text{final}}\left(\varphi_f\right)\right| \geq \left|\Sigma^{\Omega}_{\text{final}}\left(\varphi_{g_n}\right)\right| \geq \left|\Sigma^{\Omega}_{\text{final}}\left(\varphi_{g_{n-1}}\right)\right| \geq \left|\Sigma^{\Omega}_{\text{final}}\left(\varphi_{g_0}\right)\right|.$$

Furthermore, since each open map has a finite number of connected components, we must find an n such that φ_{g_n} is combinatorially equivalent to $\varphi_{g_{n-1}}$. By Lemma 5.3, φ_{g_n} is combinatorially stable, which proves the Theorem. □

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