A note on existence and localization of solutions for certain fourth order boundary value problems

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Abstract

In this paper, we study the existence of solutions for the differential equation

\[ u^{(4)}(t) = f(t, u(t), u''(t)), \]

where \( f \) satisfies one-sided Lipschitz conditions with respect to \( u \) and \( u'' \), with periodic conditions or the boundary conditions from “simply supported” beam theory. We assume the existence of lower and upper solutions (well-ordered and in some cases reversely ordered) and we make use of a 4th order linear differential operator factorization.

Key words: Beam equation; Fourth order boundary value problem; Maximum Principles; Lower and Upper solutions; Reversed order.

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1 Introduction

It is well known that fourth order boundary value problems are motivated by the theory of beam deflection. However, the corresponding boundary value problems are also mathematically interesting as they are usually not so simple as similar 2nd order problems, and well known results for the second order case do not necessarily hold for the corresponding fourth order problems without some strengthening of assumptions. An example of this is the theorem that states that if a given boundary value problem has well ordered lower and upper solutions, then it has a solution lying between those two functions: roughly, this is true in the second order case, but one needs to add monotonicity assumptions to obtain some true statement in the fourth order case. In this note we propose a step in the direction of establishing this type of results.

Recently, several authors have studied existence and multiplicity of solutions of the equation

\[ u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0, 2\pi] \] (1)

with different boundary conditions. Here we are interested in the periodic boundary conditions

\[ u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi) \]
and the “simply supported” boundary conditions
\[ u(0) = u(\pi) = u''(0) = u''(\pi) = 0. \]

The periodic problem with \( f \) not depending on \( u'' \) has been studied by Cabada [2], via maximum principles and the monotone method. Jiang, Gao and Wan [6] obtained results for the full nonlinear problem using the monotone method. Allowing a linear dependence of \( f \) on \( u'' \), Li [8] and Liu and Li [9] have obtained existence results with fixed point theory.

For the “simply supported” case, Bai and Wang [1] have obtained existence and multiplicity results without dependence on \( u'' \). With linear dependence on \( u'' \), we can find results of existence in Li [7] and existence and multiplicity in Yao [11]. Cabada, Cid and Sanchez [3] obtained results for the periodic case, without dependence on \( u'' \), using upper and lower solutions in reversed order. The superlinear case has been studied by B. R. Rynne [10] using a bifurcation technique.

Our paper deals with fourth order boundary value problems in a way that Gao, Weng, Jiang and Hou [4] did for second order. We consider equation (1) with periodic and lower solutions in reversed order. The superlinear case has been studied by B. R. Rynne [10] using a bifurcation technique.

In section 2 we present some maximum principles, in sections 3 and 4 we prove existence results for the periodic case, respectively, for ordered and reversely ordered upper and lower solutions. In section 5 we deal with the “simply supported” case with ordered upper and lower solutions.

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2 Auxiliary results

In this section we state some simple maximum principles that will be useful in following sections.

Lemma 2.1 (Maximum Principle 1). Let \( L < 0, p, q, r \in \mathbb{R} \) with \( p < r < q \) and \( y \in C[p, q] \cap W^{2,1}(p, r) \cap W^{2,1}(r, q) \) such that
\[ y''(t) + Ly(t) = f(t) \geq 0 \text{ a.e. } (p, q), \quad y(p) = y(q), \quad y'(p) \geq y'(q), \quad y'(r^+) \geq y'(r^-). \]

Then \( y(t) \leq 0 \) for all \( t \in [0, 2\pi] \). Moreover, if \( y''(t) + Ly(t) \neq 0 \), then \( y(t) < 0 \) for all \( t \in [0, 2\pi] \).

Proof. Suppose that \( y(t) > 0 \) for all \( t \in [p, q] \). Then we would have the contradiction
\[ 0 \geq y'(r^-) - y'(r^+) + y'(q) - y'(p) = \int_p^r f(t) - Ly(t) \, dt + \int_r^q f(t) - Ly(t) \, dt > 0. \]

If \( y(p) > 0 \), then \( y(q) > 0 \) and therefore there exist two intervals \( [p, p_1] \) and \( [q_1, q] \) where \( y > 0 \), \( y(p_1) = y(q_1) = 0 \), \( y'(p_1) \leq 0 \) and \( y'(q_1) \geq 0 \). If \( r \) belongs to one of the intervals, then we would have the contradiction
\[ 0 \geq y'(r^-) - y'(r^+) + y'(q) - y'(p) + y'(p_1) - y'(q_1) = \int_p^{p_1} f(t) - Ly(t) \, dt + \int_{q_1}^q f(t) - Ly(t) \, dt > 0. \]
otherwise, the contradiction is the same, without the terms involving \( r \).

If \( y(p) < 0 \) then there exists an interval \([p_1,q_1]\) where \( y > 0 \) and \( y(p_1) = y(q_1) = 0 \), and we can apply the arguments used in the first case. \( \square \)

**Lemma 2.2 (Maximum Principle 2).** Let \( 0 < L < \frac{1}{2}, \ p < r < q \) with \( q - p \leq 2\pi \) and \( y \in C[p,q] \cap W^{2,1}(p,r) \cap W^{2,1}(r,q) \) such that

\[
y''(t) + Ly(t) = f(t, y) \geq 0 \quad \text{a.e.} \quad (p,q), \quad y(p) = y(q), \quad y'(p) \geq y'(q), \quad y'(r^+) \geq y'(r^-).
\]

Then \( y(t) \geq 0 \) for all \( t \in [0,2\pi] \). Moreover, if \( y''(t) + Ly(t) \not\equiv 0 \), then \( y(t) > 0 \) for all \( t \in ]0,2\pi[ \).

**Proof.** It follows easily combining the arguments used in the proof of the previous lemma and the proof of Proposition 2.3 in \([5]\) . \( \square \)

**Lemma 2.3 (Maximum Principle 3).** Let \( L < 1 \) and \( y \in W^{2,1}(0,\pi) \) such that

\[
y''(t) + Ly(t) \geq 0 \quad \text{a.e.} \quad (0,\pi), \quad y(0) \leq 0, \quad y(\pi) \leq 0.
\]

Then \( y(t) \leq 0 \) for all \( t \in [0,\pi] \). Moreover, if \( y''(t) + Ly(t) \not\equiv 0 \), then \( y(t) < 0 \) for all \( t \in ]0,\pi[ \).

**Lemma 2.4 (Maximum Principle 4).** Let \( L < 1, M \in \mathbb{R} \) and \( y \in W^{2,1}(0,\pi) \) such that

\[
y''(t) + L y^+(t) - M y^-(t) \geq 0 \quad \text{a.e.} \quad (0,\pi), \quad y(0) \leq 0, \quad y(\pi) \leq 0,
\]

where \( y^+, y^- \) are respectively the positive and negative parts of \( y \). Then \( y(t) \leq 0 \) for all \( t \in [0,\pi] \).

### 3 Periodic problem with ordered upper and lower solutions

Let us consider the fourth order equation

\[
u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0,2\pi] \tag{2}
\]

where \( f \) is a \( L^1 \)-Carathéodory function, with periodic boundary conditions

\[
u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi). \tag{3}
\]

We say that \( \alpha \in W^{4,1}(0,2\pi) \) is a lower solution of the boundary value problem (2)–(3) if

\[
\alpha^{(4)}(t) \leq f(t, \alpha(t), \alpha''(t)), \quad t \in [0,2\pi],
\]

\[
\alpha(0) = \alpha(2\pi), \quad \alpha'(0) = \alpha'(2\pi), \quad \alpha''(0) = \alpha''(2\pi), \quad \alpha'''(0) \leq \alpha'''(2\pi).
\]

A function \( \beta \in W^{4,1}(0,2\pi) \) satisfying the reversed inequalities is called an upper solution.

Let \( \alpha, \beta \) be respectively a lower and an upper solution of (2)–(3), such that \( \alpha(t) \leq \beta(t) \) for all \( t \in [0,2\pi] \).

In the following, let us assume the hypothesis

(H1) there exist constants \( C, D > 0 \) with \( D^2 > 4C \), such that

\[
f(t, u_2, v_2) - f(t, u_1, v_1) \geq -C (u_2 - u_1) + D (v_2 - v_1) \tag{4}
\]

for a.e. \( t \in [0,2\pi], \alpha(t) \leq u_1 \leq u_2 \leq \beta(t), v_1 \leq v_2 \).
Remark 3.1. If \( f(t,u,v) \) is a \( C^1 \) function in \((u,v)\), the inequality in (H1) is equivalent to \( \frac{\partial f}{\partial u} \geq -C \) and \( \frac{\partial f}{\partial v} \geq D \).

Let \( m, M < 0 \) be the two roots of the equation \( x^2 + Dx + C = 0 \) (note that \( C = mM, D = -(m+M) \)).

Setting \( a(t) = \alpha''(t) + m\alpha(t) \) and \( b(t) = \beta''(t) + m\beta(t) \), we have the following result.

**Proposition 3.2.** If \( f \) is a \( L^1 \)-Carathéodory function satisfying (H1) for \( \alpha(t), \beta(t) \) lower and upper solutions such that \( \alpha(t) \leq \beta(t) \) for all \( t \in [0,2\pi] \), then \( b(t) \leq a(t) \).

**Proof.** Setting \( y(t) = b(t) - a(t) \), we have \( y(0) = y(2\pi) \) and \( y'(0) \geq y'(2\pi) \). Suppose towards a contradiction that there exists \( t_0 \in [0,2\pi] \) such that \( y(t_0) > 0 \).

If \( y(t) > 0 \) for all \( t \in [0,2\pi] \), we have (noting that \( b(t) - m\beta(t) \geq a(t) - m\alpha(t) \) and \( m^2 + mD + C = 0 \))

\[
y''(t) + My(t) = b''(t) - a''(t) + Mb(t) - Ma(t)
= \beta''(t) + (m + M)\beta'(t) + (m + M)m\beta(t) - m^2\beta(t) -
- (\alpha''(t) + (m + M)\alpha'(t) + (m + M)m\alpha(t) - m^2\alpha(t))
\geq f(t, \beta(t), b(t) - m\beta(t)) - f(t, \alpha(t), a(t) - m\alpha(t)) +
+ (m + M)y(t) - m^2(\beta(t) - \alpha(t)) \geq 0,
\]

and this is a contradiction, since by the Maximum Principle 2.1 we would have \( y(t) \leq 0 \).

Otherwise, considering if necessary the periodic extension of \( y(t) \), there exists an interval \([p,q] \) with \( q - p < 2\pi \) such that \( y(p) = y(q) = 0 \), \( y'(p) \geq 0 \geq y'(q) \), and \( y(t) > 0 \) for \( t \in [p,q] \), possibly with \( p < 0 < q \), in which case \( y'(0^+) \geq y'(0^-) \). Applying Maximum Principle 2.1 again, we reach a contradiction.

\[ \square \]

Let

\[
p(t,x) = \begin{cases} 
b(t), & x < b(t) \\
x, & b(t) \leq x \leq a(t) \\
a(t), & x > a(t). \end{cases}
\]

Consider the boundary value problem

\[
u''(t) + mu(t) = L_m u(t) = q(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),
\]

with \( q \in L^1[0,1] \). Since \( m < 0 \), the operator \( L_m \) is invertible so that we can write its unique solution \( u \) as \( u = L_m^{-1}q \), and by the Maximum principle 2.1 we know that if \( q(t) \geq 0 \) then \( u(t) \leq 0 \). Since \( \alpha(t) = L_m^{-1}a(t) \), \( \beta(t) = L_m^{-1}b(t) \) and \( b(t) \leq p(t, x(t)) \leq a(t) \), we have

\[
\alpha(t) \leq L_m^{-1}p(t, x(t)) \leq \beta(t).
\]

Let us consider the modified problem

\[
x''(t) + Mx(t) = (Fx)(t) \equiv f(t, L_m^{-1}p(t, x(t)), p(t, x(t)) - mL_m^{-1}p(t, x(t))) +
+ (m + M)p(t, x(t)) - m^2L_m^{-1}p(t, x(t)), \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi). \tag{5}
\]

Considering the operator \( \Phi : C[0,2\pi] \to C[0,2\pi] \) with \( \Phi x = L_m^{-1}(Fx) \), since \( p(t, x(t)) \) and \( L_m^{-1}p(t, x(t)) \) are bounded and \( f \) is a Carathéodory function, there exists a \( L^1[0,2\pi] \) function \( g(t) \) such that \( |(Fx)(t)| \leq g(t) \) for a.e. \( t \in [0,2\pi] \). Therefore, applying Leray-Schauder’s fixed point Theorem, we can conclude that \( \Phi \) has a fixed point \( x(t) \) which is a solution of the modified problem (5).
**Proposition 3.3.** Let \( x(t) \) be a solution of the modified problem (5). Assuming (H1), for given lower and upper solutions \( \alpha(t) \) and \( \beta(t) \), with \( \alpha \leq \beta \) for all \( t \in [0, 2\pi] \), we have

\[
b(t) \leq x(t) \leq a(t).
\]

**Proof.** We will only prove that \( x(t) \leq a(t) \), since the other inequality can be obtained with similar arguments.

We have

\[
a''(t) + Ma(t) \leq f \left( t, L_m^{-1} a(t), a(t) - mL_m^{-1} a(t) \right) + (m + M)a(t) - m^2 L_m^{-1} a(t),
\]

\[
a(0) = a(2\pi), \quad a'(0) \leq a'(2\pi).
\]

Setting \( y(t) = x(t) - a(t) \), we have \( y(0) = y(2\pi) \), \( y'(0) \geq y'(2\pi) \). Suppose towards a contradiction that there exists \( t_0 \in [0, 2\pi] \) such that \( y(t_0) > 0 \).

If \( y(t) > 0 \) for all \( t \), we have \( x(t) > a(t) \) and, therefore, \( p(t, x(t)) = a(t) \), so

\[
x''(t) + Mx(t) = f \left( t, L_m^{-1} a(t), a(t) - mL_m^{-1} a(t) \right) + (m + M)a(t) - m^2 L_m^{-1} a(t) \geq a''(t) + Ma(t),
\]

which is a contradiction, since by the Maximum Principle 2.1 we would have \( y(t) \leq 0 \).

Otherwise, considering if necessary the periodic extension of \( y(t) \), there exists an interval \([p, q]\) with \( q - p < 2\pi \) such that \( y(p) = y(q) = 0 \), \( y'(p) \geq 0 \geq y'(q) \), \( y(t) > 0 \) for \( t \in [p, q] \), possibly with \( p < 0 < q \), in which case \( y'(0^+) \geq y'(0^-) \). Then, recalling that \( L_m^{-1} p(t, x(t)) \geq L_m^{-1} a(t) \),

\[
y''(t) + My(t) = x''(t) + Mx(t) - a''(t) - Ma(t) \geq f \left( t, L_m^{-1} p(t, x(t)), p(t, x(t)) - mL_m^{-1} p(t, x(t)) \right) - f \left( t, L_m^{-1} a(t), a(t) - mL_m^{-1} a(t) \right) + (m + M)y(t) - m^2 (L_m^{-1} p(t, x(t)) - a(t)) \geq 0,
\]

so, we have again a contradiction by the Maximum Principle 2.1. \( \square \)

**Theorem 3.4.** Assuming (H1), for given lower and upper solutions \( \alpha \) and \( \beta \), with \( \alpha \leq \beta \), the boundary value problem (2)–(3) has a solution \( u(t) \in W^{4,1}(0, 2\pi) \) such that \( \alpha \leq u \leq \beta \).

**Proof.** Let \( u(t) = L_m^{-1} x(t) \), where \( x(t) \) is a solution of the modified problem (5). Since \( x(t) = u''(t) + mu(t) \), we have \( x''(t) + Mx(t) = u^{(4)}(t) - Du''(t) + Cu(t) \). On the other hand,

\[
x''(t) + Mx(t) = f \left( t, L_m^{-1} p(t, x(t)), p(t, x(t)) - mL_m^{-1} p(t, x(t)) \right) + (m + M)p(t, x(t)) - m^2 L_m^{-1} p(t, x(t)) = f \left( t, u(t), u''(t) \right) - Du''(t) + Cu(t),
\]

so \( u(t) \) satisfies (2)–(3). Moreover, from

\[
\beta'' + m\beta \leq u'' + mu \leq \alpha'' + m\alpha,
\]

taking into account the boundary conditions and Maximum Principle 2.1, we obtain \( \alpha \leq u \leq \beta \). \( \square \)

We can reach a similar conclusion, assuming the following hypothesis
(H1’) there exist constants $C, D > 0$ with $D < 4C + 1/4$ and $D^2 > 4C$, such that
\[
f(t, u_2, v_1) - f(t, u_1, v_2) \geq -C(u_2 - u_1) - D(v_1 - v_2)
\]
for a.e. $t \in [0, 2\pi]$, $\alpha(t) \leq u_1 \leq u_2 \leq \beta(t)$, $v_1 \leq v_2$.

**Remark 3.5.** If $f(t, u, v)$ is a $C^1$ function in $(u, v)$, the inequality in (H1’) is equivalent to $D_u f \leq -C$ and $D_v f \leq -D$.

Let $0 < m, M < \frac{1}{4}$ be the two roots of the equation $x^2 - Dx + C = 0$ (note that $C = mM, D = m + M$).

Defining $a(t)$ and $b(t)$ as above, we have the following result

**Proposition 3.6.** If $f$ is a $L^1$-Carathéodory function satisfying (H1’) for $\alpha(t), \beta(t)$ lower and upper solutions such that $\alpha(t) \leq \beta(t)$, then $b(t) \geq a(t)$.

**Proof.** Setting $y(t) = b(t) - a(t)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $t_0 \in [0, 2\pi]$ such that $y(t_0) < 0$. We can reach a contradiction with similar arguments from the ones used in Proposition 3.2, using instead the Maximum principle 2.2. \qed

Using the same arguments as in the previous case, we prove the following result

**Theorem 3.7.** Assuming (H1’), for given lower and upper solutions $\alpha$ and $\beta$, with $\alpha \leq \beta$, the boundary value problem (2)–(3) has a solution $u(t) \in W^{4,1}(0, 2\pi)$ such that $\alpha \leq u \leq \beta$.

4 Periodic problem with upper and lower solutions in reversed order

In this section we prove similar results from the ones above, but with lower and upper solutions in reversed order, that is $\beta(t) \leq \alpha(t)$, for all $t \in [0, 2\pi]$.

(H2) there exist constants $C, D$ with $C < 0$ and $D > -4C - 1/4$, such that
\[
f(t, u_2, v_1) - f(t, u_1, v_2) \geq -C(u_1 - u_2) + D(v_2 - v_1)
\]
for a.e. $t \in [0, 2\pi]$, $\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)$, $v_1 \leq v_2$.

**Remark 4.1.** If $f(t, u, v)$ is a $C^1$ function in $(u, v)$, the inequality in (H2) is equivalent to $\frac{\partial f}{\partial u} \leq -C$ and $\frac{\partial f}{\partial v} \geq D$.

Let $M < 0$ and $0 < m < \frac{1}{4}$ be the two roots of the equation $x^2 + Dx + C = 0$ (note that $C = mM, D = -(m + M)$).

Defining $a(t) = \alpha''(t) + m\alpha(t)$ and $b(t) = \beta''(t) + m\beta(t)$, we have the following result

**Proposition 4.2.** If $f$ is a $L^1$-Carathéodory function satisfying (H2) for $\alpha(t), \beta(t)$ lower and upper solutions such that $\beta(t) \leq \alpha(t)$, then $b(t) \leq a(t)$.

**Proof.** Setting $y(t) = b(t) - a(t)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $t_0 \in [0, 2\pi]$ such that $y(t_0) > 0$.

If $y(t) > 0$ for all $t$, we have (noting that $b(t) - m\beta(t) \geq a(t) - m\alpha(t)$)
\[
y''(t) + My(t) \geq f(t, \beta(t), b(t) - m\beta(t)) - f(t, \alpha(t), a(t) - m\alpha(t)) + (m + M)y(t) - m^2(\beta(t) - \alpha(t)) \geq 0,
\]
and this is a contradiction, since by the Maximum Principle 2.1 we would have \( y(t) \leq 0 \).

Otherwise, considering if necessary the periodic extension of \( y(t) \), there exists an interval \([p, q]\) with \( q - p < 2\pi\) such that \( y(p) = y(q) = 0\), \( y'(p) \geq 0 \geq y'(q)\), \( y(t) > 0\) for \( t \in [p, q]\), possibly with \( p < 0 < q\), in which case \( y'(0^+) \geq y'(0^-)\). Applying Maximum Principle 2.1 again, we reach a contradiction.

\[ \square \]

Let
\[
p(t, x) = \begin{cases} b(t), & x < b(t) \\ x, & b(t) \leq x \leq a(t) \\ a(t), & x > a(t). \end{cases}
\]

Consider the boundary value problem
\[
u''(t) + mu(t) \equiv L_m u(t) = q(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),
\]
with \( q \in L^1[0, 1] \). Since \( m < 1 \), the operator \( L_m \) is invertible so that we can write its unique solution \( u \) as \( u = L_m^{-1} q \), and by the Maximum principle 2.2 we know that if \( q(t) \geq 0 \), then \( u(t) \geq 0 \). Since \( \alpha(t) = L_m^{-1} a(t), \beta(t) = L_m^{-1} b(t) \) and \( b(t) \leq p(t, x(t)) \leq a(t) \), we have
\[
\beta(t) \leq L_m^{-1} p(t, x(t)) \leq \alpha(t).
\]

Let us consider the modified problem
\[
x''(t) + Mx(t) = (F(x))(t) \equiv f(t, L_m^{-1} p(t, x(t)), p(t, x(t))) - mL_m^{-1} p(t, x(t)) + (m + M)p(t, x(t)) - m^2 L_m^{-1} p(t, x(t)), \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi).
\]

(8)

Considering the operator \( \Phi : C[0, 2\pi] \to C[0, 2\pi] \) with \( \Phi x = L_m^{-1}(F(x)) \) since \( p(t, x(t)) \) and \( L_m^{-1} p(t, x(t)) \) are bounded and \( f \) is a Carathéodory function, there exists a \( L^1[0, 2\pi] \) function \( g(t) \) such that \( |(F(x))(t)| \leq g(t) \) for a.e. \( t \in [0, 2\pi] \). Therefore, applying Leray-Schauder’s fixed point Theorem, we can conclude that \( \Phi \) has a fixed point \( x(t) \) which is a solution of the modified problem (8).

**Proposition 4.3.** Let \( x(t) \) be a solution of the modified problem (8). Assuming (H2), for given lower and upper solutions \( \alpha \) and \( \beta \), with \( \alpha \leq \beta \), we have
\[
b(t) \leq x(t) \leq a(t).
\]

**Proof.** The proof is similar to the one of proposition 3.3. \( \square \)

**Theorem 4.4.** Assuming (H2), for given lower and upper solutions \( \alpha \) and \( \beta \), with \( \beta \leq \alpha \), the boundary value problem (2)–(3) has a solution \( u(t) \in W^{4,1}(0, 2\pi) \) such that \( \beta \leq u \leq \alpha \).

**Proof.** The proof is similar to the one of theorem 3.4. \( \square \)

We can reach a similar conclusion, assuming the following hypothesis

(H2\textsuperscript{′}) there exist constants \( C, D \) with \( C < 0 \) and \( D < 4C + 1/4 \) such that
\[
f(t, u_1, v_1) - f(t, u_2, v_2) \geq -C(u_1 - u_2) - D(v_1 - v_2)
\]
for a.e. \( t \in [0, 2\pi] \), \( \beta(t) \leq u_1 \leq u_2 \leq \alpha(t) \), \( v_1 \leq v_2 \).

**Remark 4.5.** If \( f(t, u, v) \) is a \( C^1 \) function in \( (u, v) \), the inequality in (H2\textsuperscript{′}) is equivalent to \( \frac{\partial f}{\partial u} \leq -C \) and \( \frac{\partial f}{\partial v} \leq -D \).
Let $m < 0$ and $0 < M < \frac{1}{4}$ be the two roots of the equation $x^2 - Dx + C = 0$ (note that $C = mM$, $D = m + M$).

Defining $a(t)$ and $b(t)$ as above, we have the following result

**Proposition 4.6.** If $f$ is a $L^1$-Carathéodory function satisfying (H2') for $\alpha(t), \beta(t)$ lower and upper solutions such that $\beta(t) \leq \alpha(t)$, then $b(t) \geq a(t)$.

**Proof.** Setting $y(t) = b(t) - a(t)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $t_0 \in [0, 2\pi]$ such that $y(t_0) < 0$. We can reach a contradiction with similar arguments from the ones used in Proposition 4.2, using instead the Maximum principle 2.2. 

Using the same arguments as in the previous case, we prove the following result

**Theorem 4.7.** Assuming (H2'), for given lower and upper solutions $\alpha$ and $\beta$, with $\beta \leq \alpha$, the boundary value problem (2)–(3) has a solution $u(t) \in W^{4,1}(0, 2\pi)$ such that $\beta \leq u \leq \alpha$.

5 “Simply supported” problem with ordered upper and lower solutions

Let us now consider the fourth order equation

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0, \pi]$$

where $f$ is a $L^1$-Carathéodory function, with the boundary conditions

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0.$$  \hspace{1cm} (10)

We say that $\alpha \in W^{4,1}(0, \pi)$ is a lower solution of the boundary value problem (10)–(11) if

$$\alpha^{(4)}(t) \leq f(t, \alpha(t), \alpha''(t)), \quad t \in [0, \pi]$$

$$\alpha(0) \leq 0, \quad \alpha(\pi) \leq 0, \quad \alpha''(0) \geq 0, \quad \alpha''(\pi) \geq 0.$$  \hspace{1cm} (11)

A function $\beta \in W^{4,1}(0, \pi)$ satisfying the reversed inequalities is called an upper solution.

Let $\alpha, \beta$ be respectively a lower and an upper solution of (2)–(3), such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, \pi]$.

In the following, let us assume the hypothesis

(H3) there exist constants $C, D$ such that $C < 0$ or $D > 0$, with $D > -C - 1$, $D^2 > 4C$ , and

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -C(u_2 - u_1) + D(v_2 - v_1)$$

for a.e. $t \in [0, 2\pi]$, $\alpha(t) \leq u_1 \leq u_2 \leq \beta(t)$, $v_1 \leq v_2$.  \hspace{1cm} (12)

**Remark 5.1.** If $f(t, u, v)$ is a $C^1$ function in $(u, v)$, the inequality in (H3) is equivalent to $\frac{\partial f}{\partial u} \geq -C$ and $\frac{\partial f}{\partial v} \geq D$.

Let $m < 0$ and $M < 1$ be the two roots of the equation $x^2 + Dx + C = 0$ (note that $C = mM$, $D = -(m + M)$).

Defining $a(t) = \alpha''(t) + m\alpha(t)$ and $b(t) = \beta''(t) + m\beta(t)$, we have the following result

**Proposition 5.2.** If $f$ is a $L^1$-Carathéodory function satisfying (H3) for $\alpha(t), \beta(t)$ lower and upper solutions such that $\alpha(t) \leq \beta(t)$, then $b(t) \leq a(t)$. 

\hspace{1cm} 8
Proof. Setting \( y(t) = b(t) - a(t) \), we have \( y(0) \leq 0 \) and \( y(\pi) \leq 0 \). Suppose towards a contradiction that there exists \( t_0 \in [0, \pi] \) such that \( y(t_0) > 0 \). The result follows using similar arguments to the ones of Proposition 3.2 (using Maximum principle 2.3 instead).

Let
\[
p(t, x) = \begin{cases} 
b(t), & x < b(t) \\ x, & b(t) \leq x \leq a(t) \\ a(t), & x > a(t). \end{cases}
\]

Consider the boundary value problem
\[
u''(t) + mu(t) \equiv L_m u(t) = q(t), \quad u(0) = 0, \quad u(\pi) = 0,
\]
with \( q \in L^1[0, 1] \). Since \( m < 0 \), the operator \( L_m \) is invertible so that we can write its unique solution \( u \) as \( u = L_m^{-1} q \).

Let us define \( \tilde{a}(t) \) such that \( \tilde{a}'' + m\tilde{a} = 0, \tilde{a}(0) = \alpha(0), \tilde{a}(\pi) = \alpha(\pi) \), and \( \tilde{b}(t) \) such that \( \tilde{b}'' + mb = 0, \tilde{b}(0) = \beta(0), \tilde{b}(\pi) = \beta(\pi) \). It is obvious that \( \tilde{a}(t) \leq 0 \) and \( \tilde{b}(t) \geq 0 \) for all \( t \in [0, \pi] \).

We have \( \alpha(t) = L_m^{-1} a(t) + \tilde{a}(t) \) and \( \beta(t) = L_m^{-1} b(t) + \tilde{b}(t) \), then, by the Maximum principle 2.3
\[
\alpha(t) \leq L_m^{-1} p(t, x(t)) \leq \beta(t).
\]

Proceeding in a similar way as in the previous cases, we can reach an analogue conclusion:

**Theorem 5.3.** Assuming (H3), for given lower and upper solutions \( \alpha \) and \( \beta \), with \( \alpha \leq \beta \), the boundary value problem (10)-(11) has a solution \( u(t) \in W^{4,1}(0, \pi) \) such that \( \alpha \leq u \leq \beta \).

Let us now consider an hypothesis somehow different from the ones considered above. Suppose that

(H4) there exist constants \( C, D \) with \( C > 0, 0 < D < 1 \), and
\[
|f(t, u, v_2) - f(t, u, v_1)| \geq C |v_2 - v_1| - D |v_2 - v_1| \quad (13)
\]
for a.e. \( t \in [0, 2\pi] \), \( \alpha(t) \leq u_1 \leq u_2 \leq \beta(t), \) \( v_1, v_2 \in \mathbb{R} \).

**Remark 5.4.** If \( f(t, u, v) \) is a \( C^4 \) function in \((u, v)\), the inequality in (H4) is equivalent to \( \frac{\partial f}{\partial u} \geq C \) and \( \frac{\partial f}{\partial v} \leq D \).

Let \( m < 0 \) be such that \( C + Dm - m^2 > 0 \) and \( D - m < 1 \). Defining \( a(t) = \alpha''(t) + mA(t) \) and \( b(t) = \beta''(t) + mB(t) \), we have the following result

**Proposition 5.5.** If \( f \) is a \( L^1 \)-Carathéodory function satisfying (H4) for \( \alpha(t) \), \( \beta(t) \) lower and upper solutions such that \( \alpha(t) \leq \beta(t) \), then \( b(t) \leq a(t) \).

**Proof.** Setting \( y(t) = b(t) - a(t) \), we have \( y(0) \leq 0, y(\pi) \leq 0 \) and
\[
y''(t) = \beta^{(4)}(t) - \alpha^{(4)}(t) + m(\beta''(t) - \alpha''(t)) + m^2(\beta(\alpha) - \alpha(t)) - m^2(\beta(t) - \alpha(t)) \geq \\
\geq f(t, \beta(t), b(t) - m\beta(t)) - f(t, \alpha(t), a(t) - mA(t)) + my(t) - m^2(\beta(t) - \alpha(t)) \geq \\
\geq C(\beta(t) - \alpha(t)) - D |y(t) - m(\beta(t) - \alpha(t))| + my(t) - m^2(\beta(t) - \alpha(t)) \geq \\
\geq (C + Dm - m^2)(\beta(t) - \alpha(t)) - D |y(t)| + my(t).
\]
In order to apply Maximum principle 2.4, we can rewrite the previous inequality as

\[ y''(t) + (D - m)y^+(t) + (D + m)y^-(t) \geq 0 \]

and conclude that \( y(t) \leq 0 \).

Proceeding in a similar way as above, we can reach an analogue conclusion:

**Theorem 5.6.** Assuming (H4), for given lower and upper solutions \( \alpha \) and \( \beta \), with \( \alpha \leq \beta \), the boundary value problem (10)–(11) has a solution \( u(t) \in W^{4,1}(0, \pi) \) such that \( \alpha \leq u \leq \beta \).

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**References**


