

A note on existence and localization of solutions for certain fourth order boundary value problems

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Abstract

In this paper, we study the existence of solutions for the differential equation

$$u^{(4)}(t) = f(t, u(t), u''(t)),$$

where f satisfies one-sided Lipschitz conditions with respect to u and u'' , with periodic conditions or the boundary conditions from “simply supported” beam theory. We assume the existence of lower and upper solutions (well-ordered and in some cases reversely ordered) and we make use of a 4th order linear differential operator factorization.

Key words: Beam equation; Fourth order boundary value problem; Maximum Principles; Lower and Upper solutions; Reversed order.

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1 Introduction

It is well known that fourth order boundary value problems are motivated by the theory of beam deflection. However, the corresponding boundary value problems are also mathematically interesting as they are usually not so simple as similar 2nd order problems, and well known results for the second order case do not necessarily hold for the corresponding fourth order problems without some strengthening of assumptions. An example of this is the theorem that states that if a given boundary value problem has well ordered lower and upper solutions, then it has a solution lying between those two functions: roughly, this is true in the second order case, but one needs to add monotonicity assumptions to obtain some true statement in the fourth order case. In this note we propose a step in the direction of establishing this type of results.

Recently, several authors have studied existence and multiplicity of solutions of the equation

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0, 2\pi] \quad (1)$$

with different boundary conditions. Here we are interested in the periodic boundary conditions

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi)$$

and the “simply supported” boundary conditions

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0.$$

The periodic problem with f not depending on u'' has been studied by Cabada [2], via maximum principles and the monotone method. Jiang, Gao and Wan [6] obtained results for the full nonlinear problem using the monotone method. Allowing a linear dependence of f on u'' , Li [8] and Liu and Li [9] have obtained existence results with fixed point theory.

For the “simply supported” case, Bai and Wang [1] have obtained existence and multiplicity results without dependence on u'' . With linear dependence on u'' , we can find results of existence in Li [7] and existence and multiplicity in Yao [11]. Cabada, Cid and Sanchez [3] obtained results for the problem without dependence on u'' , using upper and lower solutions in reversed order. The superlinear case has been studied by B. R. Rynne [10] using a bifurcation technique.

Our paper deals with fourth order boundary value problems in a way that Gao, Weng, Jiang and Hou [4] did for second order. We consider equation (1) with periodic as well as “simply supported” boundary conditions, and prove existence results (considering f one-sided Lipschitz in both variables u and u'') if there exist lower and upper solutions (ordered or in reversed order for the periodic case, and ordered in the “simply supported” case).

Habets and Sanchez [5] have obtained results similar to those we obtain here, using Lipschitz conditions. The main difference is that in our case only localization is obtainable, no iterative method is available.

In section 2 we present some maximum principles, in sections 3 and 4 we prove existence results for the periodic case, respectively, for ordered and reversely ordered upper and lower solutions. In section 5 we deal with the “simply supported” case with ordered upper and lower solutions.

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2 Auxiliary results

In this section we state some simple maximum principles that will be useful in following sections.

Lemma 2.1 (Maximum Principle 1). *Let $L < 0$, $p, q, r \in \mathbb{R}$ with $p < r < q$ and $y \in C[p, q] \cap W^{2,1}(p, r) \cap W^{2,1}(r, q)$ such that*

$$y''(t) + Ly(t) = f(t) \geq 0 \text{ a.e. } (p, q), \quad y(p) = y(q), \quad y'(p) \geq y'(q), \quad y'(r^+) \geq y'(r^-).$$

Then $y(t) \leq 0$ for all $t \in [0, 2\pi]$. Moreover, if $y''(t) + Ly(t) \not\equiv 0$, then $y(t) < 0$ for all $t \in]0, 2\pi[$.

Proof. Suppose that $y(t) > 0$ for all $t \in]p, q[$. Then we would have the contradiction

$$0 \geq y'(r^-) - y'(r^+) + y'(q) - y'(p) = \int_p^r f(t) - Ly(t) dt + \int_r^q f(t) - Ly(t) dt > 0.$$

If $y(p) > 0$, then $y(q) > 0$ and therefore there exist two intervals $[p, p_1]$ and $[q_1, q]$ where $y > 0$, $y(p_1) = y(q_1) = 0$, $y'(p_1) \leq 0$ and $y'(q_1) \geq 0$. If r belongs to one of the intervals, then we would have the contradiction

$$0 \geq y'(r^-) - y'(r^+) + y'(q) - y'(p) + y'(p_1) - y'(q_1) = \int_p^{p_1} f(t) - Ly(t) dt + \int_{q_1}^q f(t) - Ly(t) dt > 0,$$

otherwise, the contradiction is the same, without the terms involving r .

If $y(p) < 0$ then there exists an interval $]p_1, q_1[$ where $y > 0$ and $y(p_1) = y(q_1) = 0$, and we can apply the arguments used in the first case. \square

Lemma 2.2 (Maximum Principle 2). *Let $0 < L < \frac{1}{4}$, $p < r < q$ with $q - p \leq 2\pi$ and $y \in C[p, q] \cap W^{2,1}(p, r) \cap W^{2,1}(r, q)$ such that*

$$y''(t) + Ly(t) = f(t) \geq 0 \text{ a.e. } (p, q), \quad y(p) = y(q), \quad y'(p) \geq y'(q), \quad y'(r^+) \geq y'(r^-).$$

Then $y(t) \geq 0$ for all $t \in [0, 2\pi]$. Moreover, if $y''(t) + Ly(t) \not\equiv 0$, then $y(t) > 0$ for all $t \in]0, 2\pi[$.

Proof. It follows easily combining the arguments used in the proof of the previous lemma and the proof of Proposition 2.3 in [5]. \square

Lemma 2.3 (Maximum Principle 3). *Let $L < 1$ and $y \in W^{2,1}(0, \pi)$ such that*

$$y''(t) + Ly(t) \geq 0 \text{ a.e. } (0, \pi), \quad y(0) \leq 0, \quad y(\pi) \leq 0.$$

Then $y(t) \leq 0$ for all $t \in [0, \pi]$. Moreover, if $y''(t) + Ly(t) \not\equiv 0$, then $y(t) < 0$ for all $t \in]0, \pi[$.

Lemma 2.4 (Maximum Principle 4). *Let $L < 1$, $M \in \mathbb{R}$ and $y \in W^{2,1}(0, \pi)$ such that*

$$y''(t) + Ly^+(t) - My^-(t) \geq 0 \text{ a.e. } (0, \pi), \quad y(0) \leq 0, \quad y(\pi) \leq 0,$$

where y^+, y^- are respectively the positive and negative parts of y . Then $y(t) \leq 0$ for all $t \in [0, \pi]$.

3 Periodic problem with ordered upper and lower solutions

Let us consider the fourth order equation

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0, 2\pi] \quad (2)$$

where f is a L^1 -Carathéodory function, with periodic boundary conditions

$$u(0) = u(2\pi), \quad u'(0) = u'(2\pi), \quad u''(0) = u''(2\pi), \quad u'''(0) = u'''(2\pi). \quad (3)$$

We say that $\alpha \in W^{4,1}(0, 2\pi)$ is a lower solution of the boundary value problem (2)–(3) if

$$\begin{aligned} \alpha^{(4)}(t) &\leq f(t, \alpha(t), \alpha''(t)), \quad t \in [0, 2\pi] \\ \alpha(0) &= \alpha(2\pi), \quad \alpha'(0) = \alpha'(2\pi), \quad \alpha''(0) = \alpha''(2\pi), \quad \alpha'''(0) \leq \alpha'''(2\pi). \end{aligned}$$

A function $\beta \in W^{4,1}(0, 2\pi)$ satisfying the reversed inequalities is called an upper solution.

Let α, β be respectively a lower and an upper solution of (2)–(3), such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, 2\pi]$.

In the following, let us assume the hypothesis

(H1) there exist constants $C, D > 0$ with $D^2 > 4C$, such that

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -C(u_2 - u_1) + D(v_2 - v_1) \quad (4)$$

for a.e. $t \in [0, 2\pi]$, $\alpha(t) \leq u_1 \leq u_2 \leq \beta(t)$, $v_1 \leq v_2$.

Remark 3.1. If $f(t, u, v)$ is a C^1 function in (u, v) , the inequality in (H1) is equivalent to $\frac{\partial f}{\partial u} \geq -C$ and $\frac{\partial f}{\partial v} \geq D$.

Let $m, M < 0$ be the two roots of the equation $x^2 + Dx + C = 0$ (note that $C = mM$, $D = -(m + M)$).

Setting $a(t) = \alpha''(t) + m\alpha(t)$ and $b(t) = \beta''(t) + m\beta(t)$, we have the following result

Proposition 3.2. *If f is a L^1 -Carathéodory function satisfying (H1) for $\alpha(t), \beta(t)$ lower and upper solutions such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, 2\pi]$, then $b(t) \leq a(t)$.*

Proof. Setting $y(t) = b(t) - a(t)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $t_0 \in [0, 2\pi]$ such that $y(t_0) > 0$.

If $y(t) > 0$ for all $t \in [0, 2\pi]$, we have (noting that $b(t) - m\beta(t) \geq a(t) - m\alpha(t)$ and $m^2 + mD + C = 0$)

$$\begin{aligned} y''(t) + My(t) &= b''(t) - a''(t) + Mb(t) - Ma(t) \\ &= \beta^{(4)}(t) + (m + M)\beta''(t) + (m + M)m\beta(t) - m^2\beta(t) - \\ &\quad - (\alpha^{(4)}(t) + (m + M)\alpha''(t) + (m + M)m\alpha(t) - m^2\alpha(t)) \\ &\geq f(t, \beta(t), b(t) - m\beta(t)) - f(t, \alpha(t), a(t) - m\alpha(t)) + \\ &\quad + (m + M)y(t) - m^2(\beta(t) - \alpha(t)) \geq 0, \end{aligned}$$

and this is a contradiction, since by the Maximum Principle 2.1 we would have $y(t) \leq 0$.

Otherwise, considering if necessary the periodic extension of $y(t)$, there exists an interval $[p, q]$ with $q - p < 2\pi$ such that $y(p) = y(q) = 0$, $y'(p) \geq 0 \geq y'(q)$, and $y(t) > 0$ for $t \in]p, q[$, possibly with $p < 0 < q$, in which case $y'(0^+) \geq y'(0^-)$. Applying Maximum Principle 2.1 again, we reach a contradiction. \square

Let

$$p(t, x) = \begin{cases} b(t), & x < b(t) \\ x, & b(t) \leq x \leq a(t) \\ a(t), & x > a(t). \end{cases}$$

Consider the boundary value problem

$$u''(t) + mu(t) \equiv L_m u(t) = q(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

with $q \in L^1[0, 1]$. Since $m < 0$, the operator L_m is invertible so that we can write its unique solution u as $u = L_m^{-1}q$, and by the Maximum principle 2.1 we know that if $q(t) \geq 0$ then $u(t) \leq 0$. Since $\alpha(t) = L_m^{-1}a(t)$, $\beta(t) = L_m^{-1}b(t)$ and $b(t) \leq p(t, x(t)) \leq a(t)$, we have

$$\alpha(t) \leq L_m^{-1}p(t, x(t)) \leq \beta(t).$$

Let us consider the modified problem

$$\begin{aligned} x''(t) + Mx(t) &= (Fx)(t) \equiv f(t, L_m^{-1}p(t, x(t)), p(t, x(t)) - mL_m^{-1}p(t, x(t))) + \\ &\quad + (m + M)p(t, x(t)) - m^2L_m^{-1}p(t, x(t)), \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi). \end{aligned} \tag{5}$$

Considering the operator $\Phi : C[0, 2\pi] \rightarrow C[0, 2\pi]$ with $\Phi x = L_M^{-1}(Fx)$, since $p(t, x(t))$ and $L_m^{-1}p(t, x(t))$ are bounded and f is a Carathéodory function, there exists a $L^1[0, 2\pi]$ function $g(t)$ such that $|(Fx)(t)| \leq g(t)$ for a.e. $t \in [0, 2\pi]$. Therefore, applying Leray-Schauder's fixed point Theorem, we can conclude that Φ has a fixed point $x(t)$ which is a solution of the modified problem (5).

Proposition 3.3. *Let $x(t)$ be a solution of the modified problem (5). Assuming (H1), for given lower and upper solutions $\alpha(t)$ and $\beta(t)$, with $\alpha \leq \beta$ for all $t \in [0, 2\pi]$, we have*

$$b(t) \leq x(t) \leq a(t).$$

Proof. We will only prove that $x(t) \leq a(t)$, since the other inequality can be obtained with similar arguments.

We have

$$\begin{aligned} a''(t) + Ma(t) &\leq f(t, L_m^{-1}a(t), a(t) - mL_m^{-1}a(t)) + (m + M)a(t) - m^2L_m^{-1}a(t), \\ a(0) &= a(2\pi), \quad a'(0) \leq a'(2\pi). \end{aligned}$$

Setting $y(t) = x(t) - a(t)$, we have $y(0) = y(2\pi)$, $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $t_0 \in [0, 2\pi]$ such that $y(t_0) > 0$.

If $y(t) > 0$ for all t , we have $x(t) > a(t)$ and, therefore, $p(t, x(t)) = a(t)$, so

$$\begin{aligned} x''(t) + Mx(t) &= f(t, L_m^{-1}a(t), a(t) - mL_m^{-1}a(t)) + (m + M)a(t) - m^2L_m^{-1}a(t) \geq \\ &\geq a''(t) + Ma(t), \end{aligned}$$

which is a contradiction, since by the Maximum Principle 2.1 we would have $y(t) \leq 0$.

Otherwise, considering if necessary the periodic extension of $y(t)$, there exists an interval $[p, q]$ with $q - p < 2\pi$ such that $y(p) = y(q) = 0$, $y'(p) \geq 0 \geq y'(q)$, $y(t) > 0$ for $t \in]p, q[$, possibly with $p < 0 < q$, in which case $y'(0^+) \geq y'(0^-)$. Then, recalling that $L_m^{-1}p(t, x(t)) \geq L_m^{-1}a(t)$,

$$\begin{aligned} y''(t) + My(t) &= x''(t) + Mx(t) - a''(t) - Ma(t) \geq \\ &\geq f(t, L_m^{-1}p(t, x(t)), p(t, x(t)) - mL_m^{-1}p(t, x(t))) - f(t, L_m^{-1}a(t), a(t) - mL_m^{-1}a(t)) + \\ &\quad + (m + M)y(t) - m^2(L_m^{-1}p(t, x(t)) - \alpha(t)) \geq 0, \end{aligned}$$

so, we have again a contradiction by the Maximum Principle 2.1. \square

Theorem 3.4. *Assuming (H1), for given lower and upper solutions α and β , with $\alpha \leq \beta$, the boundary value problem (2)–(3) has a solution $u(t) \in W^{4,1}(0, 2\pi)$ such that $\alpha \leq u \leq \beta$.*

Proof. Let $u(t) = L_m^{-1}x(t)$, where $x(t)$ is a solution of the modified problem (5). Since $x(t) = u''(t) + mu(t)$, we have $x''(t) + Mx(t) = u^{(4)}(t) - Du''(t) + Cu(t)$. On the other hand,

$$\begin{aligned} x''(t) + Mx(t) &= f(t, L_m^{-1}p(t, x(t)), p(t, x(t)) - mL_m^{-1}p(t, x(t))) + \\ &\quad + (m + M)p(t, x(t)) - m^2L_m^{-1}p(t, x(t)) = \\ &= f(t, u(t), u''(t)) - Du''(t) + Cu(t), \end{aligned}$$

so $u(t)$ satisfies (2)–(3). Moreover, from

$$\beta'' + m\beta \leq u'' + mu \leq \alpha'' + m\alpha,$$

taking into account the boundary conditions and Maximum Principle 2.1, we obtain $\alpha \leq u \leq \beta$. \square

We can reach a similar conclusion, assuming the following hypothesis

(H1') there exist constants $C, D > 0$ with $D < 4C + 1/4$ and $D^2 > 4C$, such that

$$f(t, u_2, v_1) - f(t, u_1, v_2) \geq -C(u_2 - u_1) - D(v_1 - v_2) \quad (6)$$

for a.e. $t \in [0, 2\pi]$, $\alpha(t) \leq u_1 \leq u_2 \leq \beta(t)$, $v_1 \leq v_2$.

Remark 3.5. If $f(t, u, v)$ is a C^1 function in (u, v) , the inequality in (H1') is equivalent to $\frac{\partial f}{\partial u} \geq -C$ and $\frac{\partial f}{\partial v} \leq -D$.

Let $0 < m, M < \frac{1}{4}$ be the two roots of the equation $x^2 - Dx + C = 0$ (note that $C = mM$, $D = m + M$).

Defining $a(t)$ and $b(t)$ as above, we have the following result

Proposition 3.6. *If f is a L^1 -Carathéodory function satisfying (H1') for $\alpha(t), \beta(t)$ lower and upper solutions such that $\alpha(t) \leq \beta(t)$, then $b(t) \geq a(t)$.*

Proof. Setting $y(t) = b(t) - a(t)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $t_0 \in [0, 2\pi]$ such that $y(t_0) < 0$. We can reach a contradiction with similar arguments from the ones used in Proposition 3.2, using instead the Maximum principle 2.2. \square

Using the same arguments as in the previous case, we prove the following result

Theorem 3.7. *Assuming (H1'), for given lower and upper solutions α and β , with $\alpha \leq \beta$, the boundary value problem (2)–(3) has a solution $u(t) \in W^{4,1}(0, 2\pi)$ such that $\alpha \leq u \leq \beta$.*

4 Periodic problem with upper and lower solutions in reversed order

In this section we prove similar results from the ones above, but with lower and upper solutions in reversed order, that is $\beta(t) \leq \alpha(t)$, for all $t \in [0, 2\pi]$.

(H2) there exist constants C, D with $C < 0$ and $D > -4C - 1/4$, such that

$$f(t, u_1, v_2) - f(t, u_2, v_1) \geq -C(u_1 - u_2) + D(v_2 - v_1) \quad (7)$$

for a.e. $t \in [0, 2\pi]$, $\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)$, $v_1 \leq v_2$.

Remark 4.1. If $f(t, u, v)$ is a C^1 function in (u, v) , the inequality in (H2) is equivalent to $\frac{\partial f}{\partial u} \leq -C$ and $\frac{\partial f}{\partial v} \geq D$.

Let $M < 0$ and $0 < m < \frac{1}{4}$ be the two roots of the equation $x^2 + Dx + C = 0$ (note that $C = mM$, $D = -(m + M)$).

Defining $a(t) = \alpha''(t) + m\alpha(t)$ and $b(t) = \beta''(t) + m\beta(t)$, we have the following result

Proposition 4.2. *If f is a L^1 -Carathéodory function satisfying (H2) for $\alpha(t), \beta(t)$ lower and upper solutions such that $\beta(t) \leq \alpha(t)$, then $b(t) \leq a(t)$.*

Proof. Setting $y(t) = b(t) - a(t)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $t_0 \in [0, 2\pi]$ such that $y(t_0) > 0$.

If $y(t) > 0$ for all t , we have (noting that $b(t) - m\beta(t) \geq a(t) - m\alpha(t)$)

$$\begin{aligned} y''(t) + My(t) &\geq f(t, \beta(t), b(t) - m\beta(t)) - f(t, \alpha(t), a(t) - m\alpha(t)) + \\ &\quad + (m + M)y(t) - m^2(\beta(t) - \alpha(t)) \geq 0, \end{aligned}$$

and this is a contradiction, since by the Maximum Principle 2.1 we would have $y(t) \leq 0$.

Otherwise, considering if necessary the periodic extension of $y(t)$, there exists an interval $[p, q]$ with $q - p < 2\pi$ such that $y(p) = y(q) = 0$, $y'(p) \geq 0 \geq y'(q)$, $y(t) > 0$ for $t \in]p, q[$, possibly with $p < 0 < q$, in which case $y'(0^+) \geq y'(0^-)$. Applying Maximum Principle 2.1 again, we reach a contradiction. \square

Let

$$p(t, x) = \begin{cases} b(t), & x < b(t) \\ x, & b(t) \leq x \leq a(t) \\ a(t), & x > a(t). \end{cases}$$

Consider the boundary value problem

$$u''(t) + mu(t) \equiv L_m u(t) = q(t), \quad u(0) = u(2\pi), \quad u'(0) = u'(2\pi),$$

with $q \in L^1[0, 1]$. Since $m < 1$, the operator L_m is invertible so that we can write its unique solution u as $u = L_m^{-1}q$, and by the Maximum principle 2.2 we know that if $q(t) \geq 0$, then $u(t) \geq 0$. Since $\alpha(t) = L_m^{-1}a(t)$, $\beta(t) = L_m^{-1}b(t)$ and $b(t) \leq p(t, x(t)) \leq a(t)$, we have

$$\beta(t) \leq L_m^{-1}p(t, x(t)) \leq \alpha(t).$$

Let us consider the modified problem

$$\begin{aligned} x''(t) + Mx(t) = (Fx)(t) \equiv & f(t, L_m^{-1}p(t, x(t)), p(t, x(t)) - mL_m^{-1}p(t, x(t))) + \\ & + (m + M)p(t, x(t)) - m^2L_m^{-1}p(t, x(t)), \quad x(0) = x(2\pi), \quad x'(0) = x'(2\pi). \end{aligned} \quad (8)$$

Considering the operator $\Phi : C[0, 2\pi] \rightarrow C[0, 2\pi]$ with $\Phi x = L_M^{-1}(Fx)$ since $p(t, x(t))$ and $L_m^{-1}p(t, x(t))$ are bounded and f is a Carathéodory function, there exists a $L^1[0, 2\pi]$ function $g(t)$ such that $|(Fx)(t)| \leq g(t)$ for a.e. $t \in [0, 2\pi]$. Therefore, applying Leray-Schauder's fixed point Theorem, we can conclude that Φ has a fixed point $x(t)$ which is a solution of the modified problem (8).

Proposition 4.3. *Let $x(t)$ be a solution of the modified problem (8). Assuming (H2), for given lower and upper solutions α and β , with $\alpha \leq \beta$, we have*

$$b(t) \leq x(t) \leq a(t).$$

Proof. The proof is similar to the one of proposition 3.3. \square

Theorem 4.4. *Assuming (H2), for given lower and upper solutions α and β , with $\beta \leq \alpha$, the boundary value problem (2)–(3) has a solution $u(t) \in W^{4,1}(0, 2\pi)$ such that $\beta \leq u \leq \alpha$.*

Proof. The proof is similar to the one of theorem 3.4. \square

We can reach a similar conclusion, assuming the following hypothesis

(H2') there exist constants C, D with $C < 0$ and $D < 4C + 1/4$ such that

$$f(t, u_1, v_1) - f(t, u_2, v_2) \geq -C(u_1 - u_2) - D(v_1 - v_2) \quad (9)$$

for a.e. $t \in [0, 2\pi]$, $\beta(t) \leq u_1 \leq u_2 \leq \alpha(t)$, $v_1 \leq v_2$.

Remark 4.5. If $f(t, u, v)$ is a C^1 function in (u, v) , the inequality in (H2') is equivalent to $\frac{\partial f}{\partial u} \leq -C$ and $\frac{\partial f}{\partial v} \leq -D$.

Let $m < 0$ and $0 < M < \frac{1}{4}$ be the two roots of the equation $x^2 - Dx + C = 0$ (note that $C = mM$, $D = m + M$).

Defining $a(t)$ and $b(t)$ as above, we have the following result

Proposition 4.6. *If f is a L^1 -Carathéodory function satisfying (H2') for $\alpha(t), \beta(t)$ lower and upper solutions such that $\beta(t) \leq \alpha(t)$, then $b(t) \geq a(t)$.*

Proof. Setting $y(t) = b(t) - a(t)$, we have $y(0) = y(2\pi)$ and $y'(0) \geq y'(2\pi)$. Suppose towards a contradiction that there exists $t_0 \in [0, 2\pi]$ such that $y(t_0) < 0$. We can reach a contradiction with similar arguments from the ones used in Proposition 4.2, using instead the Maximum principle 2.2. \square

Using the same arguments as in the previous case, we prove the following result

Theorem 4.7. *Assuming (H2'), for given lower and upper solutions α and β , with $\beta \leq \alpha$, the boundary value problem (2)–(3) has a solution $u(t) \in W^{4,1}(0, 2\pi)$ such that $\beta \leq u \leq \alpha$.*

5 “Simply supported” problem with ordered upper and lower solutions

Let us now consider the fourth order equation

$$u^{(4)}(t) = f(t, u(t), u''(t)), \quad t \in [0, \pi] \quad (10)$$

where f is a L^1 -Carathéodory function, with the boundary conditions

$$u(0) = u(\pi) = u''(0) = u''(\pi) = 0. \quad (11)$$

We say that $\alpha \in W^{4,1}(0, \pi)$ is a lower solution of the boundary value problem (10)–(11) if

$$\begin{aligned} \alpha^{(4)}(t) &\leq f(t, \alpha(t), \alpha''(t)), \quad t \in [0, \pi] \\ \alpha(0) &\leq 0, \quad \alpha(\pi) \leq 0, \quad \alpha''(0) \geq 0, \quad \alpha''(\pi) \geq 0. \end{aligned}$$

A function $\beta \in W^{4,1}(0, \pi)$ satisfying the reversed inequalities is called an upper solution.

Let α, β be respectively a lower and an upper solution of (2)–(3), such that $\alpha(t) \leq \beta(t)$ for all $t \in [0, \pi]$.

In the following, let us assume the hypothesis

(H3) there exist constants C, D such that $C < 0$ or $D > 0$, with $D > -C - 1$, $D^2 > 4C$, and

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq -C(u_2 - u_1) + D(v_2 - v_1) \quad (12)$$

for a.e. $t \in [0, 2\pi]$, $\alpha(t) \leq u_1 \leq u_2 \leq \beta(t)$, $v_1 \leq v_2$.

Remark 5.1. If $f(t, u, v)$ is a C^1 function in (u, v) , the inequality in (H3) is equivalent to $\frac{\partial f}{\partial u} \geq -C$ and $\frac{\partial f}{\partial v} \geq D$.

Let $m < 0$ and $M < 1$ be the two roots of the equation $x^2 + Dx + C = 0$ (note that $C = mM$, $D = -(m + M)$).

Defining $a(t) = \alpha''(t) + m\alpha(t)$ and $b(t) = \beta''(t) + m\beta(t)$, we have the following result

Proposition 5.2. *If f is a L^1 -Carathéodory function satisfying (H3) for $\alpha(t), \beta(t)$ lower and upper solutions such that $\alpha(t) \leq \beta(t)$, then $b(t) \leq a(t)$.*

Proof. Setting $y(t) = b(t) - a(t)$, we have $y(0) \leq 0$ and $y(\pi) \leq 0$. Suppose towards a contradiction that there exists $t_0 \in [0, \pi]$ such that $y(t_0) > 0$. The result follows using similar arguments to the ones of Proposition 3.2 (using Maximum principle 2.3 instead). \square

Let

$$p(t, x) = \begin{cases} b(t), & x < b(t) \\ x, & b(t) \leq x \leq a(t) \\ a(t), & x > a(t). \end{cases}$$

Consider the boundary value problem

$$u''(t) + mu(t) \equiv L_m u(t) = q(t), \quad u(0) = 0, \quad u(\pi) = 0,$$

with $q \in L^1[0, 1]$. Since $m < 0$, the operator L_m is invertible so that we can write its unique solution u as $u = L_m^{-1}q$.

Let us define $\tilde{a}(t)$ such that $\tilde{a}'' + m\tilde{a} = 0$, $\tilde{a}(0) = \alpha(0)$, $\tilde{a}(\pi) = \alpha(\pi)$, and $\tilde{b}(t)$ such that $\tilde{b}'' + m\tilde{b} = 0$, $\tilde{b}(0) = \beta(0)$, $\tilde{b}(\pi) = \beta(\pi)$. It is obvious that $\tilde{a}(t) \leq 0$ and $\tilde{b}(t) \geq 0$ for all $t \in [0, \pi]$.

We have $\alpha(t) = L_m^{-1}a(t) + \tilde{a}(t)$ and $\beta(t) = L_m^{-1}b(t) + \tilde{b}(t)$, then, by the Maximum principle 2.3

$$\alpha(t) \leq L_m^{-1}p(t, x(t)) \leq \beta(t).$$

Proceeding in a similar way as in the previous cases, we can reach an analogue conclusion:

Theorem 5.3. *Assuming (H3), for given lower and upper solutions α and β , with $\alpha \leq \beta$, the boundary value problem (10)–(11) has a solution $u(t) \in W^{4,1}(0, \pi)$ such that $\alpha \leq u \leq \beta$.*

Let us now consider an hypothesis somehow different from the ones considered above. Suppose that

(H4) there exist constants C, D with $C > 0$, $0 < D < 1$, and

$$f(t, u_2, v_2) - f(t, u_1, v_1) \geq C(u_2 - u_1) - D|v_2 - v_1| \quad (13)$$

for a.e. $t \in [0, 2\pi]$, $\alpha(t) \leq u_1 \leq u_2 \leq \beta(t)$, $v_1, v_2 \in \mathbb{R}$.

Remark 5.4. If $f(t, u, v)$ is a C^1 function in (u, v) , the inequality in (H4) is equivalent to $\frac{\partial f}{\partial u} \geq C$ and $\left| \frac{\partial f}{\partial v} \right| \leq D$.

Let $m < 0$ be such that $C + Dm - m^2 > 0$ and $D - m < 1$. Defining $a(t) = \alpha''(t) + m\alpha(t)$ and $b(t) = \beta''(t) + m\beta(t)$, we have the following result

Proposition 5.5. *If f is a L^1 -Carathéodory function satisfying (H4) for $\alpha(t), \beta(t)$ lower and upper solutions such that $\alpha(t) \leq \beta(t)$, then $b(t) \leq a(t)$.*

Proof. Setting $y(t) = b(t) - a(t)$, we have $y(0) \leq 0$, $y(\pi) \leq 0$ and

$$\begin{aligned} y''(t) &= \beta^{(4)}(t) - \alpha^{(4)}(t) + m(\beta''(t) - \alpha''(t)) + m^2(\beta(t) - \alpha(t)) - m^2(\beta(t) - \alpha(t)) \geq \\ &\geq f(t, \beta(t), b(t) - m\beta(t)) - f(t, \alpha(t), a(t) - m\alpha(t)) + my(t) - m^2(\beta(t) - \alpha(t)) \geq \\ &\geq C(\beta(t) - \alpha(t)) - D|y(t) - m(\beta(t) - \alpha(t))| + my(t) - m^2(\beta(t) - \alpha(t)) \geq \\ &\geq (C + Dm - m^2)(\beta(t) - \alpha(t)) - D|y(t)| + my(t). \end{aligned}$$

In order to apply Maximum principle 2.4, we can rewrite the previous inequality as

$$y''(t) + (D - m)y^+(t) + (D + m)y^-(t) \geq 0$$

and conclude that $y(t) \leq 0$. □

Proceeding in a similar way as above, we can reach an analogue conclusion:

Theorem 5.6. *Assuming (H4), for given lower and upper solutions α and β , with $\alpha \leq \beta$, the boundary value problem (10)–(11) has a solution $u(t) \in W^{4,1}(0, \pi)$ such that $\alpha \leq u \leq \beta$.*

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References

- [1] **Zhanbing Bai and Haiyan Wang**, *On positive solutions of some nonlinear fourth order beam equations*, J. Math. Anal. Appl. 270 (2002) 357-368.
- [2] **A. Cabada**, *The method of lower and upper solutions for n th order periodic boundary value problems*, J. Appl. Math. Stochastic Anal. 7, No.1, 33-47 (1994).
- [3] **A. Cabada, J. Á. Cid and L. Sanchez**, *Positivity and lower and upper solutions for fourth order boundary value problems*, to appear in Nonlinear Analysis T.M.A..
- [4] **H. Gao, S. Weng, D. Jiang, X. Hou** *On second order periodic boundary value problems with upper and lower solutions in the reversed order*, Electronic Journal of Diff. Eq., 25 (2006) 1–8.
- [5] **P. Habets, L. Sanchez** *A monotone method for fourth order boundary value problems involving a factorizable linear operator*, Portugaliae Mathematica 64 (2007) (to appear).
- [6] **Daqing Jiang, Wenjie Gao and Aying Wan**, *A monotone method for constructing extremal solutions to fourth-order periodic boundary value problems*, Appl. Math. Comput. 132 (2002), 411-421.
- [7] **Yongxiang Li**, *Positive solutions of fourth order boundary value problems with two parameters*, J. Math. Anal. Appl. 281 (2003), 477-484.
- [8] **Yongxiang Li**, *Positive solutions of fourth-order periodic boundary value problems*, Nonlinear Analysis 54 (2003), 1069-1078.
- [9] **Xi-Lan Liu and Wan-Tong Li**, *Positive solutions of the nonlinear fourth order beam equation with three parameters*, J. Math. Anal. Appl. 303 (2005), 150-163.
- [10] **B. P. Rynne**, *Infinitely many solutions of superlinear fourth order boundary value problems*, Top. Meth. Nonl. Anal. 19 (2002), 303-312.
- [11] **Qingliu Yao**, *On the positive solutions of a nonlinear fourth order boundary value problem with two parameters*, Applicable Analysis 83 (2004), 97-107.