# Note on the solution of a differential equation (DRAFT)

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**Abstract**

We present a definition for solution of a first-order differential equation which takes undefinedness into consideration.

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While being centuries old, the notion of solution is not clear enough to support work in real recursive function theory, as there is no commonly-accepted way to consider undefinedness. A standard textbook on differential equations ([BDH02]) tells us the following:

> **A solution** of the differential equation is a function of the independent variable that, when substituted into the equation as the dependent variable, satisfies the equation for all values of the independent variable.

This would seem like an acceptable definition if it were not for the issue of undefinedness. Undefinedness is of special importance for classical computability theory, and will also be important for the inductive class of functions we will study bellow. Consider, for instance, $\delta : \mathbb{R} \to \mathbb{R}$ to be the function given by:

$$\delta(x) = \begin{cases} 0 & \text{if } x \neq 0, \\ 1 & \text{if } x = 0. \end{cases}$$

This is Kronecker's $\delta$ function. Now let $q$ be given by $q(x) = \frac{1}{1-\delta(1-x)}$. We can see that $q(x)$ is not defined for $x = 1$ and is equal to 1 everywhere else,

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and this plays a role in differential equations involving $q$. If we look at the equations

$$h(0) = 0 \qquad \partial_x h(x) = q(x),$$

we do not know exactly how to solve them. If we make $h(x) = h_1(x) = x$, then $h$ can be said to be a solution, according to [BDH02], for all values *except for* $x = 1$. But it seems as if the value of $q$ for point 1 does not really matter: the null measure of the set of such points gives us this impression. But in this sense, the solution is not any more suitable than

$$h(x) = h_2(x) = \begin{cases} x & \text{if } x < 1 \\ x + 3 & \text{if } x \geqslant 1 \end{cases},$$

which also satisfies the differential equation for all points except for $x = 1$. Usual methods based on continuity are not applicable here. As we will see, this problem needs to be dealt with. We present three possible approaches, instantiated for this particular initial value problem:

**Cutoff Approach** Force the solution function to be undefined for all points not in a certain open interval containing 0. In the case of the above example, all points greater or equal to 1 are out of the domain.

**Maximal Approach** Allow all *solutions*. In this instance, all functions such as $h_1$ and $h_2$ can be called *solutions* of the problem.

**Minimal Approach** Among all the potential solutions, choose one to be called *the solution* and discard the rest.

Let us see the following initial value problem, in order to experiment with the implications of each approach:

$$h(0) = 0, \qquad \partial_x h(x) = \frac{1}{\sec x} = \frac{1}{\frac{1}{\cos x}}.$$

If we adopt the cutoff approach, then the function will be equal to $\sin x$ for $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and undefined everywhere else. For the maximal approach, we would call solution to all functions that are equal to $\sin x + c_i$, for any $c_i$ where $i$ is an integer, such that $c_0 = 0$ and $x \in \left((2i-1)\frac{\pi}{2}, (2i+1)\frac{\pi}{2}\right)$. Finally, adopting the minimal approach, we could say, for instance, that $h(x) = \sin x$ on the whole axis is a solution.

We feel that there is no obvious choice. Confronting mathematicians working on differential equations with examples similar to the second one, they would usually go for the minimal approach, and having experimented with all of these, we have also come to the conclusion that the minimal approach is the more convenient one. It is convenient in the sense that we can choose either approach and our inductive class of functions remains the same, only it is more complicated to define many of the functions if we adopt any of the other approaches.

We remind the reader that a discontinuity at the point $y_0 \in \mathbb{R}$ in function $g$ is said to be removable if $\lim_{y \to y_0} g(y)$ exists and is finite. The removal of this discontinuity consists in setting the value of the function at point $y_0$ to the value of this limit. Also, a function with non-Zeno discontinuities along $y$ is a function with only a finite number of discontinuities in every interval in $\mathbb{R}$ with finite length.

**Definition 1** *Let $f$ be an $n$-ary function and $g$ be an $(n+2+m)$-ary function, both with $m$ components. Given the initial value problem*

$$\bar{h}(\bar{x}, 0) = \bar{f}(\bar{x}) \qquad \partial_y \bar{h}(\bar{x}, y) = \bar{g}(\bar{x}, y, \bar{h}(\bar{x}, y)),$$

*an $(n+1)$-ary function $\hat{h}$ with $m$ components is said to be the* solution *of the initial value problem in interval $I \subseteq \mathbb{R}$ if $I$ is the largest open interval such that either $I = \emptyset$ or $0 \in I$,*[1] *and, for all $\bar{x}$,*

*a. $\hat{h}$ is continuous and defined exactly in $I$.*
*b. $g(\bar{x}, y, \bar{z})$ is continuous except for non-Zeno removable discontinuities along $y \in I$. We write $\hat{g}$ to stand for the function $\bar{g}$ with all removable discontinuities removed.*
*c. When $\bar{h}$ and $\bar{g}$ are respectively replaced for $\hat{h}$ and $\hat{g}$ the equation is satisfied for all values of $y$ in $I$.*

This definition embodies both the cutoff approach and the minimal approach. The basic idea is to ignore discontinuities in a locally finite number of points, and it will also allow us to do our work more easily, without excessive details. The conditions imposed on $g$ allow us to disregard pathological cases.

For example, let $p$ be the function expressed by:

$$p(z) = \begin{cases} z & \text{if } z > \frac{1}{2} \text{ and } z \neq \mathrm{e}, \\ \frac{1}{2} & \text{if } z < \frac{1}{2}, \\ \perp & \text{otherwise.} \end{cases}.$$

There are removable discontinuities for $x = \mathrm{e}$ and $x = -1$. Now take the following initial value problem:

$$h(0) = 1 \qquad \partial_x h(x) = g(x, h(x)),$$

where $g$ is the function given by $g(x, z) = p(z)$. The only solution of this problem, according to the given definition, is the function $h$ such that:

$$h(x) = \begin{cases} e^x & \text{if } x > \log(\frac{1}{2}), \\ \frac{1}{2}(x + 1 + \log(2)) & \text{if } x \leqslant \log(\frac{1}{2}). \end{cases}.$$

[1] We consider the empty interval so we can have solutions that are everywhere undefined.

## References

[BDH02] P. Blanchard, R. L. Devaney, and G. R. Hall. *Differential Equations*. BROOKS/COLE, 2nd edition, 2002.