# Isotropic ppmc immersions

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**Abstract**

Isometric immersions with parallel pluri-mean curvature ("ppmc") in euclidean $n$-space generalize constant mean curvature ("cmc") surfaces to higher dimensional Kähler submanifolds. Like cmc surfaces they allow a one-parameter family of isometric deformations rotating the second fundamental form at each point. If these deformations are trivial the ppmc immersions are called *isotropic*. Our main result drastically restricts the intrinsic geometry of such a submanifold: Locally, it must be a symmetric space or a Riemannian product unless the immersion is holomorphic or a superminimal surface in a sphere. We can give a precise classification if the codimension is less than 7. The main idea of the proof is to show that the tangent holonomy is restricted and to apply the Berger-Simons holonomy theorem.

*Key words:* Kähler submanifolds, pluriharmonic maps, holonomy
*1991 MSC:* 53C21, 53C35, 53C43

## 1 Introduction

Let $M$ be a Kähler manifold (not necessarily complete) and $T = TM$ its tangent bundle. The complex structure of $M$ defines a parallel tensor field $J : T \to T$ with $J^2 = -I$. This belongs to a one-parameter family of parallel rotations

$$R_\theta = (\cos\theta)I + (\sin\theta)J. \tag{1}$$

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Now let $f : M \to \mathbb{R}^n$ be an isometric immersion with normal bundle $N$ and second fundamental form $\alpha : S^2T \to N$ where $S^2$ denote the symmetric tensor product. Using $R_\theta$, we may define a new tensor field $\tilde{\alpha}_\theta : S^2T \to N$,

$$\tilde{\alpha}_\theta(v, w) = \alpha(R_\theta v, R_\theta w) \tag{2}$$

for any $v, w \in T$. We may ask when this happens to be also a second fundamental form, belonging to another isometric immersion $f_\theta : M \to \mathbb{R}^n$. However the normal bundles $N$ and $N_\theta$ of $f$ and $f_\theta$ will be different in general. Thus the best we can ask for is that the second fundamental form $\alpha_\theta$ of $f_\theta$ satisfies

$$\alpha_\theta = \psi_\theta \, \tilde{\alpha}_\theta, \tag{3}$$

where $\psi_\theta : N \to N_\theta$ is a parallel vector bundle isomorphism.

Immersions $f$ which allow such an "associated family" $f_\theta$ have been characterized in (BEFT) in terms of *pluri-mean curvature*. The complexified tangent bundle $T^c$ of a Kähler manifold $M$ splits into the $(\pm i)$-eigenbundles of $J$, and the corresponding bundle decomposition $T^c = T' + T''$ is Levi-Civita parallel. If $f : M \to \mathbb{R}^n$ is an isometric immersion, the restrictions of $\alpha$ to the parallel subbundles $S^2T'$, $T' \otimes T''$, $S^2T'' \subset S^2T$ are called $\alpha^{(2,0)}$, $\alpha^{(1,1)}$, $\alpha^{(0,2)}$, respectively. The component $\alpha^{(1,1)}$ is called *pluri-mean curvature*; it collects the mean curvature vectors of the rectrictions of $f$ to all complex one-dimensional submanifolds (complex curves) in $M$. The immersion $f$ is said to have *parallel pluri-mean curvature* (*ppmc*) if $\alpha^{(1,1)}$ is parallel. It turns out that an associated family of the above type (3) exists if and only if $f$ is a ppmc immersion.

There is a special kind of ppmc immersions: those whose associated family (3) is constant, $f_\theta = f$. Those are called *isotropic*. They are characterized by the condition that the three components of $\alpha$ define a parallel orthogonal decomposition of $N^c = N \otimes \mathbb{C}$ of the form

$$N^c = N' \oplus (N^o)^c \oplus N'' \tag{4}$$

such that $\alpha^{(2,0)}$, $\alpha^{(1,1)}$, $\alpha^{(0,2)}$ take values in these three subbundles, respectively (cf. Theorem 8 of (BEFT)). In particular the real normal bundle splits orthogonally into parallel subbundles

$$N = N^o + N^1 \tag{5}$$

where $N^o$ is the (real) image of $\alpha^{(1,1)}$ while $\alpha^{(2,0)}$ and $\alpha^{(0,2)}$ take value in the complexification of $N^1$.

In the present paper, we wish to show that such submanifolds must be very special: If $f$ is not holomorphic and $M$ does not locally split as a Riemannian product, it must be locally symmetric or a minimal surface in a sphere. The local symmetry is obtained by applying the holonomy theorem of Berger and Simons (S).

In the locally symmetric case we also get some information about the extrinsic geometry and we conjecture that $f$ is in fact *extrinsically symmetric*, i.e. the whole second fundamental form $\alpha$ is parallel, not only its $(1,1)$-part.

*Acknowledgement:* Part of this paper was prepared during a stay of the first and the third named author at Abdus Salam ICTP Trieste. It is a pleasure for us to thank for this hospitality and financial support.

## 2 The holonomy group

**Theorem 1** *Let $M$ be a locally irreducible Kähler manifold (not necessarily complete) and $f : M \to \mathbb{R}^n$ an isotropic ppmc immersion. Then $M$ is locally symmetric unless $f$ is holomorphic with values in some $\mathbb{C}^k \subset \mathbb{R}^n$ or it is an (isotropic) minimal surface in a sphere $S^{n-1} \subset \mathbb{R}^n$.*

**Proof.** We shall apply the holonomy theorem of Berger and Simons (S). Recall that the holonomy group $H$ of $M$ consists of the parallel displacements $\tau_\gamma$ along all closed curves $\gamma$ in $M$ starting and ending at a point $p \in M$ fixed once and for all. Clearly $H$ is a subgroup of the orthogonal group on the tangent space $T_p = T$ at $p$. Since the complex structure $J$ is parallel, it is preserved by $H$ and thus $H$ is a subgroup of the unitary group on the complex vector space $(T, J)$. Using a unitary basis we may identify $T$ with $\mathbb{C}^m$. Local irreducibilty means that there are locally no parallel subbundles of $T$ which means that the identity component of $H$ acts irreducibly on $T$. In order to show that $M$ is locally symmetric, according to (S) we just have to prove that $H$ does not act transitively on the unit sphere in $T$.

For any $\zeta \in N$ let $A_\zeta = -(\partial\zeta)_T$ be the Weingarten map. Note that $A_\zeta$ commutes (anticommutes) with $J$ for $\zeta \in N^o$ (resp. $\zeta \in N^1$). In fact, if $\zeta \in N^o$, then $\langle A_\zeta T', T'\rangle = \langle \alpha(T', T'), \zeta\rangle \subset \langle N', N^o\rangle = 0$. Since $T'$ is maximal isotropic,[1] this implies that $A_\zeta T' \subset T'$. Likewise, for $\zeta \in N^1$ we have $\langle A_\zeta T', T''\rangle = \langle \alpha(T', T''), \zeta\rangle \in \langle N^o, N^1\rangle = 0$, hence $A_\zeta T' \subset T''$. Thus $A_\zeta$ preserves (resp. reverses) the eigenspaces of $J$ which proves the statement. Let us agree that $\xi$ and $\eta$ always denote vectors in $N^1$ and $N^o$, respectively.

[1] This means that $T'$ is maximal among all those linear subspaces of $T^c$ where the complexified metric $\langle\ ,\ \rangle$ vanishes

We first consider three special cases. If $A_\eta = 0$ for all $\eta \in N^o$, then $\alpha^{(1,1)} = 0$, so $f$ is pluriharmonic and isotropic, hence holomorphic (cf. (ET)). More generally, if $A_\eta$ is a multiple of the identity for all $\eta \in N^o$, then $A_\eta = 0$ for all $\eta \perp \eta_o$ where $\eta_o = \operatorname{trace} \alpha$ is the mean curvature vector of $f$. Hence $f$ is a pluriharmonic immersion into a sphere in $\mathbb{R}^n$, and by (DR) this must be a minimal surface (in particular $\dim M = 2$).

Further, if $A_\xi = 0$ for all $\xi \in N^1$ , then $\alpha^{(2,0)} = 0 = \alpha^{(0,2)}$ since $\alpha^{(2,0)}$ and $\alpha^{(0,2)}$ take values in $N^1 \otimes \mathbb{C}$. This implies that $f$ is extrinsic hermitian symmetric, i.e. a *standard embedding* of some hermitian symmetric space (cf. (F)).[2] In fact, in order to show $\nabla\alpha = 0$ it is enough to compute $(\nabla_Z \alpha)(X, Y)$ for vector fields $X, Y, Z$ taking values in $T' \cup T''$. At least two of these vectors have the same type, say $T'$. Since $\nabla\alpha$ is symmetric (Codazzi), we may assume $X, Y \in T'$. Now $(\nabla_Z \alpha)(X, Y) = \nabla_Z(\alpha(X, Y)) - \alpha(\nabla_Z X, Y) - \alpha(X, \nabla_Z Y)$ vanishes since $X, Y, \nabla_Z X, \nabla_Z Y \in T'$. This shows that $f$ is extrinsic symmetric. Further, from $\alpha^{(2,0)} = 0$ we get $\alpha(JX, JY) = \alpha(X, Y)$, hence by Ferus (F) we see that $f$ is the standard embedding of a hermitian symmetric space.

Thus from now on we may assume that there are normal vectors $\eta \in N^o$ and $\xi \in N^1$ such that $A_\xi \neq 0$ and $A_\eta$ has at least two different eigenvalues. Since $N^o, N^1$ are parallel subbundles of $N$, the Weingarten maps $A_\eta$ and $A_\xi$ commute by the Ricci equation, and hence they have a compatible eigenspace decomposition. Let $F \subset T$ be an eigenspace of $A_\xi$ corresponding to some nonzero eigenvalue $\lambda$. Further let

$$T = E_1 \oplus \ldots \oplus E_r \tag{6}$$

be the eigenspace decomposition with respect to $A_\eta$. Due to the compatibility we obtain a decomposition

$$F = F \cap E_1 \oplus \ldots \oplus F \cap E_r \tag{7}$$

with $r \geq 2$. We will show next that (7) still holds when $F$ is replaced by the space $hF$ for any $h \in H$.

In fact, let $h \in H$ correspond to the parallel displacement along a curve $\gamma$ on $M$ starting and ending at $p$. Let $\tilde{h}$ be the parallel displacement in $N^o$ along the same curve $\gamma$. Since $\alpha^{(1,1)}$ is parallel and $\alpha^{(2,0)}, \alpha^{(0,2)}$ take values in $(N^1)^c$, the linear map $A = (\eta \mapsto A_\eta) : N^o \to \operatorname{End}(TM)$ is also parallel. In fact, let $\eta$ be a parallel normal field and $v, w$ parallel tangent fields along some curve $c$

[2] If $M$ is a hermitian symmetric space, the complex structure $J_p$ in every tangent space $T_p$ is a derivation of the curvature tensor and hence can be considered as an element of the Lie algebra $\mathfrak{g}$ of the isometry group of $M$. The *standard embedding* is the map $f : M \to \mathfrak{g}$, $p \mapsto J_p \in \mathfrak{g}$.

in $M$. Then $\langle A_\xi v, w\rangle = \langle \alpha_{vw}, \xi\rangle = \langle \alpha_{vw}^{(1,1)}, \xi\rangle$ is constant, hence $A_\xi v$ is parallel along $c$. Thus $A$ intertwines the parallel displacements of $N^o$ and End $(T)$. Therefore $A_{\tilde{h}\eta} = hA_\eta h^{-1}$, and the eigenspace decomposition corresponding to $A_{\tilde{h}\eta}$ is $T = hE_1 \oplus \ldots \oplus hE_r$. Replacing $\eta$ by $\tilde{h}\eta$ in (7), we get a decomposition $F = F\cap hE_1 + \ldots + F \cap hE_r$. Hence putting $\tilde{F} = h^{-1}F$, we obtain

$$\tilde{F} = \tilde{F} \cap E_1 \oplus \ldots \oplus \tilde{F} \cap E_r. \tag{8}$$

We call a subspace $\tilde{F} \subset T$ *split* if (8) holds. We just have shown that all $hF$, $h \in H$, are split.

Since the complex structure $J$ anticommutes with $A_\xi$, the nonzero eigenvalues of $A_\xi$ come in pairs $\pm\lambda$ and the corresponding eigenspaces $F_\lambda$ and $F_{-\lambda}$ are interchanged by $J$. Hence $\hat{F} = F_\lambda + F_{-\lambda}$ is a complex subspace which is also split, and the same holds for $h\hat{F}$ for any $h \in H$. Now we have to consider two cases: $\hat{F} \neq T$ and $\hat{F} = T$.

**Case 1:** $\hat{F} \neq T$. Then it is an element of some complex Grassmannian $P = G_k(T)$ where $k = \dim_{\mathbb{C}} \hat{F}$. The $H$-orbit of $\hat{F}$ is contained in a connected component of the set of split spaces. This is a proper totally geodesic submanifold $Q \subset P$, more precisely the Riemannian product of $r$ Grassmannians $G_{k_j}(E_j)$ with $k_j = \dim_{\mathbb{C}}(\hat{F} \cap E_j)$. Let $S$ be the smallest totally geodesic submanifold of $Q$ containing the $H$-orbit $H\hat{F} = \{h\hat{F};\ h \in H\}$. Clearly, $S$ is invariant under $H$. Let $G = U(m)$ be the unitary group on $T = \mathbb{C}^m$ which acts as the transvection group on the Grassmannian $P$, and let $G_S = \{g \in G;\ gS = S\}$ be the subgroup leaving $S$ invariant. The induced action of $G_S$ on $S$ (which need not be effective) contains the full transvection group of $S$; this must be a subgroup of $U(E_1) \times \ldots \times U(E_r)$ since $S$ is totally geodesic in $Q$ which is a product of $r$ Grassmannians. Thus the action of the holonomy group $H$ on $S$ induces a Lie group homomorphism $\phi : H \to U(E_1) \times \ldots \times U(E_r)$. This is trivial only if $S = H\hat{F} = \{\hat{F}\}$ which is impossible since $H$ acts irreducibly on $T$.

**Case 2:** $\hat{F} = T$. Then $A_\xi$ has just two eigenspaces $F$ and $JF$, and $T = F \oplus JF$. Thus $F$ belongs to the set of maximal totally real subspaces of $T = \mathbb{C}^m$. These form another symmetric space $P' = U(m)/O(m)$. In fact, $F$ lies in the totally geodesic subspace $Q' \subset P'$ consisting of the split spaces (8); we have $Q' = Q_1 \times \ldots \times Q_r$ where $Q_i = U(m_i)/O(m_i)$ with $m_i = \dim_{\mathbb{C}} E_i$. Since the full $H$-orbit of $F$ is contained in $Q'$, there is again a totally geodesic subspace $S' \subset Q'$ which is preserved by $H$ and contains $F$, and as above we obtain a nontrivial Lie group homomorphism $\phi : H \to U(E_1) \times \ldots \times U(E_r)$.

Now if $H \subset U(m)$ acts transitively on the unit sphere, its identity component $H_o$ is one of the three subgroups $U(m)$, $SU(m)$, $Sp(m/2)$. But $U(m)$ acts transitively on both $P$ and $P'$ and hence it cannot preserve a proper totally geodesic subspace. The other two are simple groups. Since the homomorphism $\phi$ is nontrivial, one of its components $\phi_i : H_o \to U(E_i)$ must be nontrivial and hence injective. But there are no representations of $SU(m)$ or $Sp(m/2)$ with degree $< m$. Hence $\dim E_i = m$ and thus $E_i$ is the whole space $T$ in contradiction to our assumption that $A_\eta$ has at least two different eigenvalues. Thus $H$ does not act transitively on the sphere and $M$ is locally symmetric. □

**Remark:** The only known isotropic ppmc immersions (besides holomorphic maps and isotropic minimal surfaces in spheres) are the so called *extrinsic symmetric* ones, those with $\nabla\alpha = 0$. They split into two subclasses: the standard embeddings of hermitian symmetric spaces where $\alpha^{(2,0)} = 0$ and the Grassmannian $G_2(\mathbb{R}^{m+2})$ of 2-planes in $\mathbb{R}^{m+2}$, doubly covered by the complex quadric $Q^m$ (the space of *oriented* 2-planes) and embedded as symmetric rank 2 projection matrices into the euclidean space of all symmetric endomorphisms with trace 2 on $\mathbb{R}^{m+2}$. This is an example for the second case $\hat{F} = T$ of the previous proof, and it is the only known case (besides surfaces) where both $N^o$ and $N^1$ are nontrivial. We conjecture that there are no other examples. This would require to prove $\nabla\alpha = 0$ for locally symmetric isotropic ppmc immersions. In the following we give some evidence for this conjecture.

## 3 Extrinsic geometry

**Theorem 2** *Let $M$ be a locally symmetric Kähler manifold and $f : M \to \mathbb{R}^n$ an isotropic ppmc immersion. Then at every point $p \in M$, the values of $\nabla\alpha$ are perpendicular to the first normal space spanned by the values of $\alpha$.*

**Proof.** Let $X, Y, Z, W, V \in T'$. From the Gauss equation we get

$$\langle R_{X\bar{Y}}Z, \bar{W}\rangle = \langle \alpha_{X\bar{W}}, \alpha_{\bar{Y}Z}\rangle - \langle \alpha_{XZ}, \alpha_{\bar{Y}\bar{W}}\rangle. \tag{9}$$

Taking covariant differentiation on both sides we get from $\nabla_V R = 0$:

$$0 = -\langle (\nabla_V \alpha)_{XZ}, \alpha_{\bar{Y}\bar{W}}\rangle, \tag{10}$$

recall that by Codazzi equations, $(\nabla_A\alpha)_{BC} = 0$ for all $A, B, C \in T' \cup T''$ unless $A, B, C$ have the same type. Thus the values of $(\nabla\alpha)^{(3,0)}$ are perpendicular to the values of $\alpha^{(2,0)}$ (with respect to the hermitian inner product $(A, B) = \langle A, \bar{B}\rangle$). On the other hand let us recall that the three components of $\alpha$ take values in mutual orthogonal parallel subbundles of $N^c$, hence the values of $(\nabla\alpha)^{(3,0)}$ are also perpendicular to those of $\alpha^{(1,1)}$ and $\alpha^{(0,2)}$ □

**Theorem 3** *Let $M$ be a locally irreducible Kähler manifold and $f : M \to \mathbb{R}^n$ an isotropic ppmc immersion of codimension $\leq 6$. Then $f(M)$ is extrinsically symmetric or a minimal surface in a sphere or $f$ is holomorphic.*

**Proof.** If $f$ is a minimal immersion, then it is pluriminimal, i.e. $\alpha^{(1,1)} = 0$ (cf. (FT)) and moreover isotropic and hence holomorphic (cf. (ET)). Thus we may assume $\eta_o :=$ trace $\alpha \neq 0$. Since $\eta_o = \sum \alpha(E_i, \bar{E}_i)$ for some unitary basis $E_1, \dots, E_m$ of $T'$, we see that $\eta_o =$ trace $\alpha^{(1,1)} \in N^o$ is parallel. But for any parallel section $\eta$ of $N^o$, the corresponding Weingarten map $A_\eta$ is parallel too since $\langle A_\eta v, w\rangle = \langle \alpha_{vw}, \eta\rangle = \langle \alpha_{vw}^{(1,1)}, \eta\rangle$ for any two tangent vectors $v, w$ (the other components of $\alpha_{vw}$ are perpendicular to $\eta$). The eigendistributions of $A_\eta$ would give a product decomposition of $M$. Thus by irreducibility, $A_\eta = \lambda \cdot I$ for some $\lambda \in \mathbb{R}$. For $\eta = \eta_o$, this constant is nonzero by assumption which shows that $f(M)$ lies in a sphere $S^{n-1}$ of radius $r = 1/|\lambda|$. If dim $N^o = 1$, then $f$ is pluriminimal or (1,1)-geodesic in a sphere and hence a minimal surface, cf. (DT). If dim $N^o = 2$, the same conclusion holds: There is (up to multiples) just one other parallel section $\eta \perp \eta_o$ in $N^o$, thus $A_\eta = \lambda \cdot I$. But this time we have $\lambda = 0$ since trace $A_\eta = \langle \text{trace } \alpha, \eta\rangle = \langle \eta_o, \eta\rangle = 0$. Thus $f$ is again pluriminimal in a sphere and thus a minimal surface.

Hence we may assume dim $N^o \geq 3$. By Theorem 1 we know that $M$ is locally symmetric. We consider the decomposition (4) of the complexified normal bundle $N^c$. By Theorem 2, the subbundle $N'$ contains two mutually orthogonal subbundes $N'_1$ and $N'_2$ containing the values of $\alpha^{(2,0)}$ and $(\nabla\alpha)^{(3,0)}$, respectively. If both tensors are nonzero, the dimension of $N'$ is at least 2, and since the same holds for $N'' = \overline{N'}$, the dimension of $N^c$ (the codimension of $f$) must be at least $3 + 2 + 2 = 7$. Otherwise either $\alpha^{(2,0)} = 0$ and $f$ is a standard emmbedding of an hermitian symmetric space, cf. (F), or $(\nabla\alpha)^{(3,0)} = 0$ and hence $\nabla\alpha = 0$ (by the vanishing of $\nabla(\alpha^{(1,1)})$ and Codazzi) and $f$ is extrinsic symmetric. □

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