NONLOCAL MAXIMUM PRINCIPLES AND APPLICATIONS

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Abstract

We extend a nonlocal maximum principle obtained in [2], which allows us to use a monotone method to find radial solutions of an elliptic problem in the presence of lower and upper solutions.

Keywords and phrases: *Singular second order boundary value problem; Nonlocal; Green's function; Maximum Principles; Lower and Upper solutions; Monotone method.*

AMS subject classification: 34B15, 34A40.

This work was partially supported by Fundação para a Ciência e Tecnologia, projects POCTI/Mat/57258/2004 and POCTI-ISFL-1-209 (Centro de Matemática e Aplicações Fundamentais).

1 Introduction

Boundary value problems where a nonlocal term appears have attracted much attention recently due to their role in many problems of physics and engineering. In [2], where we improved some results of [4], we have dealt with some aspects concerning that kind of problems, namely existence and approximation of radial solutions in a ball of $\mathbb{R}^N$.

There, we observed that there are obvious difficulties in using the method of lower and upper solutions in the presence of nonlocal terms. Nevertheless, we developped a monotone method for problems of the form

$$-u''(t) - \frac{n-1}{t}u'(t) = f\left(u(t), \omega_n \int_0^1 s^{n-1} g\left(u(s)\right)\, ds\right), \quad u'(0) = 0 = u(1), \qquad (1)$$

assuming that $f(u_2, v) - f(u_1, v) \geq -\lambda^2(u_2 - u_1)$ for some $\lambda > 0$, $f(u, v)$ is k_1-Lipschitz in v and g is k_2-Lipschitz (which are similar conditions to those used in [5]). The method is based on a "nonlocal maximum principle" asserting that

$$-u''(t) - \frac{n-1}{t}u'(t) + \lambda^2 u(t) + M \int_0^1 s^{n-1} |u(s)|\, ds \geq 0, \quad u'(0) \leq 0,\, u(1) \geq 0 \qquad (2)$$

implies that $u \geq 0$ in $I = [0,1]$. The fact that we needed the assumption $\lambda^2 + M < 1$ is a limitation in the strength of this principle.

The purpose of this paper is basically to extend the nonlocal maximum principle so as to allow its applicability to a large range of values of $\lambda > 0$ and $M > 0$.

We investigate the admissible range of values in two cases: first we consider a simple model - for which values of $\lambda > 0$ and $M > 0$ do the inequalities

$$-u''(t) + \lambda^2 u(t) + M \int_0^1 |u(s)|\, ds \geq 0, \quad u(0) \geq 0,\, u(1) \geq 0$$

yield a maximum principle? Then we proceed to the inequality (2), related to the important class of radial problems in a ball. It turns out that the two situations may be dealt in a similar way, although some computations are easier in the first case.

In the course of our approach we find it convenient to consider the linear singular differential equation

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \tag{3}$$

and find an expression for one of its solutions as

$$u(t) = \int_0^1 H_\lambda(t,s)h(s)ds,$$

where H_λ is a Green's function. The solution we have in mind exists for a certain class of right-hand sides h, and may satisfy boundary conditions $u'(0) = a$, $u(1) = 0$, where a needs not be zero.

We have organized the paper as follows: In section 2 we collect our remarks concerning the linear equation (3). In section 3 we study a class of nonlocal linear boundary value problems that are useful in the sequel. In section 4 we proceed with the consideration of nonlocal semilinear problems and establish the nonlocal maximum principle. In the final section, we briefly illustrate the use of the principle to establish a monotone method for (1).

2 Some remarks about the solutions of a linear problem

Let us consider the differential equation

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \quad t \in]0,1], \tag{4}$$

where $k > 1$, $\lambda > 0$ and $h \in L^2_{k+2}(0,1) \equiv \left\{ h(t) \text{ measurable: } \int_0^1 \tau^{k+2} h(\tau)^2\, d\tau < \infty \right\}$.

We shall use the Hilbert Spaces

$$H_k(0,1) = \left\{ u \in AC\,]0,1] \colon \int_0^1 \tau^k u'(\tau)^2\, d\tau < \infty,\;\; u(1) = 0 \right\},$$

with the norm $\|u\| = \left(\int_0^1 \tau^k u'(\tau)^2\, d\tau \right)^{1/2}$.

Following [1], for $u \in H_k(0,1)$, with $k > 1$, we have $\left(\int_0^1 \tau^{k-2} u(\tau)^2\, d\tau \right)^{1/2} \leq \frac{2}{k-1}\|u\|$, so the functional

$$J(u) := \int_0^1 \frac{1}{2}\left(t^k u'(t)^2 + \lambda^2 t^k u(t)^2 \right) + t^k h(t)u(t)\, dt$$

is well defined in $H_k(0,1)$, since

$$\int_0^1 t^k h(t)u(t)\,dt \le \left(\int_0^1 t^{k-2}u(t)^2\,dt\right)^{1/2}\left(\int_0^1 t^{k+2}h(t)^2\,dt\right)^{1/2}.$$

It is obvious that $J(u)$ is a coercive strictly convex functional, so that equation (4) has a unique solution in $H_k(0,1)$.

Proposition 2.1. *If $h \in L^2_1(0,1) \equiv \left\{h(t) \text{ measurable: } \int_0^1 \tau h(\tau)^2\,d\tau < \infty\right\}$, then, the unique solution u of (4) in $H_k(0,1)$ is in fact in $C^1[0,1]$ and it satisfies $u'(0)=0$ (note that $L^2_1(0,1) \subset L^2_{k+2}(0,1)$).*

Proof. Equation (4) is obviously equivalent to

$$-\left(t^k u'(t)\right)' + \lambda^2 t^k u(t) = t^k h(t).$$

If $h \in L^2_1(0,1)$, it is easy to verify that $\left|t^k u'(t)\right|$ satisfies Cauchy's condition at $t=0$, therefore there exists $L \in \mathbb{R}$ such that $\lim_{t\to 0}\left|t^k u'(t)\right| = L$. Necessarily $L=0$, because otherwise we would not have $u \in H_k(0,1)$. Applying Cauchy-Schwarz inequality, it is easy to see that

$$\begin{aligned}\left|t^k u'(t)\right| \le& \left|\int_0^t \lambda^2 \tau^k u(\tau)\,d\tau\right| + \left|\int_0^t \tau^k h(\tau)\,d\tau\right| \\ \le& c_1\left(\int_0^t \tau^{k-2}u(\tau)^2\,d\tau\right)^{1/2} t^{\frac{k+3}{2}} + c_2\left(\int_0^t \tau h(\tau)^2\,d\tau\right)^{1/2} t^k,\end{aligned}$$

for some constants $c_1, c_2 > 0$.

If $\frac{k+3}{2} \ge k$ ($k \le 3$), it is obvious that $\lim_{t\to 0} u'(t) = 0$. Otherwise, if $k > 3$, we have

$$\left|t^k u'(t)\right| \le c\,t^{\frac{k+3}{2}}, \tag{5}$$

for some constant $c > 0$.

In general, if we have $\left|t^k u'(t)\right| \le c\,t^\alpha$, then $|u(t)| \le C + Ct^{\alpha-k+1}$, for some $C > 0$, hence, we can conclude that near $t=0$, there exists a constant $c_3 > 0$ such that

$$\left(\int_0^t \tau^{k-2}u(\tau)^2\,d\tau\right)^{1/2} \le c_3\,t^{\min\left(\frac{k-1}{2},\frac{2\alpha-k+1}{2}\right)}.$$

Consequently, for some $c_4 > 0$, we have

$$\left|t^k u'(t)\right| \le c_4\,t^{\min(k+1,\,\alpha+2)} + c_2\left(\int_0^t \tau h(\tau)^2\,d\tau\right)^{1/2} t^k,$$

and setting $\alpha = \frac{k+3}{2}$, it is easy to see that with a finite number of iterations of this process, we will get

$$\left|t^k u'(t)\right| \le c^*\,t^{k^*} + c_2\left(\int_0^t \tau h(\tau)^2\,d\tau\right)^{1/2} t^k,$$

where $k^* > k$, and then the conclusion follows easily. □

It is a standard procedure in the literature to associate solutions of a boundary value problem to fixed points of some functional operator. In our case, the solutions of the second order homogeneous differential equation

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = 0, \tag{6}$$

which is equivalent to $\big(t^k u'(t)\big)' = \lambda^2 t^k u(t)$, with initial conditions $u(0) = 1$, $u'(0) = 0$, may be viewed as fixed points of the operator

$$Tu(t) = 1 + \int_0^t \frac{\lambda^2}{\tau^k} \int_0^\tau s^k u(s)\, ds\, d\tau,$$

defined in some functional space. Considering the space $Z = \{u \in C[0, t_0] : u(0) = 1\}$, for some t_0 small enough, T has a unique fixed point since it is a contraction. The singularity of equation (6) is at the point $t = 0$, so it is obvious that this solution can be extended to the interval $[0, 1]$. Let u_1 be this solution, and consider the function $v_1(t) = u_1(t) \int_t^1 \frac{ds}{s^k u_1(s)^2}$, which is the solution of (6) obtained by the standard method of reducing the order of an ordinary differential equation. The solutions u_1 and v_1 are linearly independent and their associated Wronskian is $W(t) = u_1(t)v_1'(t) - u_1'(t)v_1(t) = -t^{-k}$. Furthermore, they satisfy the following properties, which we shall use in the next proposition: $u_1'(t) \geq 0$, $v_1(1) = 0$, $v_1(t) \sim t^{-(k-1)}$, and $v_1'(t) \sim t^{-k}$ as $t \to 0$ (we write $f(t) \sim g(t)$ as $t \to 0$ if and only if $\lim_{t\to 0} \frac{f(t)}{g(t)} = L \neq 0$).

Proposition 2.2. *Let $h \in X \equiv \big\{h(t)$ measurable: $\exists\, c \in \mathbb{R},\ h_0 \in L_1^2(0,1), h(t) = \frac{c}{t} + h_0(t)\big\}$. Then the boundary value problem*

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \quad t \in]0, 1],\ u(1) = 0, \tag{7}$$

has a unique solution in $C^2]0, 1] \cap C^1[0, 1]$, given by the integral expression

$$u(t) = -u_1(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)}\, ds - v_1(t) \int_0^t \frac{u_1(s)h(s)}{W(s)}\, ds. \tag{8}$$

Proof. Let us first note that $L_1^2(0, 1) \subset X \subset L_{k+2}^2(0, 1)$, so that equation (4) has a unique solution in $H_k(0, 1)$, that satisfies $u(1) = 0$.

Suppose that $h \in L_1^2(0, 1)$, that is, $c = 0$. Applying the method of undetermined coefficients, we see that the unique solution of

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \quad t \in]0, 1], \qquad u'(0) = u(1) = 0, \tag{9}$$

is given by the well defined integral expression

$$u(t) = -u_1(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)}\, ds - v_1(t) \int_0^t \frac{u_1(s)h(s)}{W(s)}\, ds. \tag{10}$$

If we differentiate this expression, we get

$$u'(t) = -u_1'(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)}\, ds - v_1'(t) \int_0^t \frac{u_1(s)h(s)}{W(s)}\, ds$$

from which, after some computation, we can confirm that $u'(0) = 0$.

Suppose now that $h(t) \notin L^2_1(0,1)$, that is, $h(t) = \frac{c}{t} + h_0(t)$ for some $c \neq 0$, $h_0 \in L^2_1(0,1)$. In this case, the integral expression (8) is still well defined, satisfies equation (4), and

$$u'(0) = -\lim_{t\to 0} v_1'(t) \int_0^t \frac{u_1(s)h(s)}{W(s)}\,ds = \lim_{t\to 0} c\, v_1'(t) \int_0^t u_1(s)s^{k-1}\,ds = -\frac{c}{k}.$$

□

Remark 2.3. Expression (8) can obviously be written in the form

$$\int_0^1 H_\lambda(t,s)h(s)\,ds, \tag{11}$$

which allows us to get the explicit form of the Green's function associated to (9). From the expression of H_λ, it is a simple matter to verify that it is continuous in $[0,1]\times[0,1]$ and positive in $]0,1[\times]0,1[$.

From the proof of the previous proposition, we infer that formula (11), where the Green's function H_λ appears, provides us the unique solution of (4) for all the boundary conditions $u'(0) = a \in \mathbb{R}$, $u(1) = 0$, whenever $h(t) + \frac{ka}{t} \in L^2_1(0,1)$.

The boundary value problem

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \qquad u(1) = b,$$

with $b \neq 0$, has also a unique solution in $C^2]0,1] \cap C^1[0,1]$ (if we had two different solutions w_1, w_2, then $w_1 - w_2$ would be the unique solution of the homogeneous problem, which is identically zero), given by $u_0(t) + \frac{b}{u_1(1)}u_1(t)$, where $u_0(t)$ is the unique solution of

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t),$$

in $H_k(0,1)$. Note that for some functions $h(t) \notin X$ we can still obtain a solution of equation (4) via the Green's function, which possibly has infinite derivative at $t = 0$, or simply does not have derivative at $t = 0$, but we will not consider these cases.

Consider now the equation for $k = 1$

$$-u''(t) - \frac{1}{t}u'(t) + \lambda^2 u(t) = h(t), \quad t \in]0,1], \tag{12}$$

where $\lambda > 0$ and $h \in L^q_1(0,1) \equiv \left\{h(t) \text{ measurable: } \int_0^1 t h(t)^q\,dt < \infty\right\}$, for some $1 < q < 2$. Consider also the functional

$$J(u) := \int_0^1 \frac{1}{2}\left(t\,u'(t)^2 + \lambda^2 t\,u^2(t)\right) + t\,h(t)\,u(t)\,dt,$$

defined in $H_1(0,1)$. We have

$$\int_0^1 t\,h(t)\,u(t)\,dt \le \left(\int_0^1 t u(t)^p\,dt\right)^{1/p} \left(\int_0^1 t h(t)^q\,dt\right)^{1/q}.$$

Following [3], since for any $p > 2$ we have $u^p \le Ce^{|u|^{2-\eta}}$, for some C, $\eta > 0$, we know that $\int_0^1 t u(t)^p\,dt < \infty$, and therefore the functional $J(u)$ is well defined in $H_1(0,1)$.

We can state exactly the same results obtained above for $k > 1$ in the case $k = 1$, just noticing that in this case $v_1(t) \sim \ln t$, and $v_1'(t) \sim t^{-1}$. The fact that $1 < q < 2$ allows us to conclude that with a function $h(t) \sim \frac{1}{t}$, $J(u)$ is well defined, and the associated solution is the one obtained via Green function, with non-zero derivative at $t = 0$.

3 Nonlocal Linear Problems

Let us consider the linear boundary value problem in the interval $[0, 1]$

$$-u''(t) + \lambda^2 u(t) = h(t), \quad u(0) = u(1) = 0, \tag{13}$$

where $\lambda > 0$ and $h \in C[0, 1]$.

This problem has a well known Green's function

$$G_\lambda(t,s) = \begin{cases} \frac{\sinh(\lambda)\cosh(\lambda t)\sinh(\lambda s) - \cosh(\lambda)\sinh(\lambda s)\sinh(\lambda t)}{\lambda \sinh(\lambda)}, & t \geq s \\ \frac{\sinh(\lambda)\cosh(\lambda s)\sinh(\lambda t) - \cosh(\lambda)\sinh(\lambda t)\sinh(\lambda s)}{\lambda \sinh(\lambda)}, & t \leq s, \end{cases}$$

and therefore we have

$$u(t) = \int_0^1 G_\lambda(t,s)h(s)\,ds.$$

Proposition 3.1. *Let $w \in C[0, 1] \cap C^2]0, 1[$ be such that*

$$-w''(t) + \lambda^2 w(t) + M \int_0^1 w(\tau)\,d\tau = 0, \quad w(0) = w(1) = 0, \tag{14}$$

for some $\lambda > 0$, $M > 0$. Then we have $w(t) = 0$ for all $t \in [0, 1]$.

Proof. Assume towards a contradiction that there exists $w(t) \neq 0$ satisfying (14).

If $w(t) \gneqq 0$ (by $\gneqq$ we mean $\geq$ and $\not\equiv$), then w reaches a positive maximum for some $t_0 \in]0, 1[$, where we would have the contradiction

$$0 < -w''(t_0) + \lambda^2 w(t_0) + M \int_0^1 w(\tau)\,d\tau = 0.$$

If $w(t) \lneqq 0$, we get a contradiction with a similar argument. So $w(t)$ must have a positive maximum for some $t_1 \in]0, 1[$ and a negative minimum for some $t_2 \in]0, 1[$. With $t = t_1$ in (14) we get $\int_0^1 w(\tau)\,d\tau < 0$, and with $t = t_2$ in (14) we get $\int_0^1 w(\tau)\,d\tau > 0$. The conclusion now follows. □

Lemma 3.2. *Let $u \in C[0, 1] \cap C^2]0, 1[$ be such that*

$$-u''(t) + \lambda^2 u(t) + M \int_0^1 u(\tau)\,d\tau = f(t) \geq 0, \quad u(0) = a \geq 0,\ u(1) = b \geq 0, \tag{15}$$

for some $\lambda > 0$, $M > 0$, and consider the $C^2[0, 1]$ functions U, V, where $U(t)$ is the unique solution of (13) with $h(t) = 1$ and $V(t)$ is the unique solution of $-V''(t) + \lambda^2 V(t) = 0$, with boundary conditions $V(0) = a$, $V(1) = b$ (note that U and V depend on λ).

Suppose that

$$\frac{M}{1 + M\int_0^1 U(\tau)\,d\tau} \leq \inf_{0<t,s<1} \frac{G_\lambda(t,s)}{U(t)\,U(s)}, \quad \text{and} \quad \frac{M\,U(t)}{1 + M\int_0^1 U(\tau)\,d\tau} \leq \frac{V(t)}{\int_0^1 V(\tau)\,d\tau}. \tag{16}$$

Then we have $u(t) \geq 0$ for all $t \in [0, 1]$.

Proof. Let v and w be such that

$$\begin{aligned} &-v''(t) + \lambda^2 v(t) = f(t), \qquad v(0) = a,\ v(1) = b, \\ &-w''(t) + \lambda^2 w(t) = \frac{M\int_0^1 v(\tau)\,d\tau}{1 + M\int_0^1 U(\tau)\,d\tau}, \quad w(0) = w(1) = 0. \end{aligned}$$

As $w(t) = \frac{M\int_0^1 v(\tau)\,d\tau}{1+M\int_0^1 U(\tau)\,d\tau}U(t)$, it can be easily verified that $v - w$ satisfies (15). Proposition 3.1 allows us to conclude that $u = v - w$, so we only need to prove that $v \geq w$.

Using the Green's function G_λ defined above and the fact that $G_\lambda(t,s) = G_\lambda(s,t)$, we have

$$
\begin{aligned}
v(t) &= \int_0^1 G_\lambda(t,s)f(s)\,ds + V(t), \quad \text{and} \\
w(t) &= \frac{M}{1+M\int_0^1 U(\tau)\,d\tau}\int_0^1 G_\lambda(t,\sigma)\,d\sigma \int_0^1 \left(\int_0^1 G_\lambda(\tau,s)f(s)\,ds + V(\tau)\right)d\tau \\
&= \frac{M}{1+M\int_0^1 U(\tau)\,d\tau}\left(\int_0^1 U(t)U(s)f(s)\,ds + U(t)\int_0^1 V(\tau)\,d\tau\right),
\end{aligned}
$$

and therefore, if the conditions in (16) are verified, we have $v \geq w$. □

Remark 3.3. The explicit form of U and V is:

$$
\begin{aligned}
U(t) &= -\frac{e^{-\lambda t}\left(-1+e^{\lambda}t\right)\left(-e^{\lambda}+e^{\lambda}t\right)}{\left(1+e^{\lambda}\right)\lambda^2} \\
V(t) &= \frac{e^{-\lambda t}\left(-be^{\lambda}+ae^{2\lambda}-ae^{2\lambda t}+be^{\lambda+2\lambda t}\right)}{-1+e^{2\lambda}}.
\end{aligned}
$$

Let us now consider the linear boundary value problem

$$
-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \quad u(1) = b,\ u \in C^2]0,1] \cap C^1[0,1], \tag{17}
$$

where $k \geq 1$, $\lambda > 0$ and $h \in X$.

As stated before, this problem has a unique solution given by

$$
u(t) = \int_0^1 H_\lambda(t,s)h(s)\,ds + \frac{b}{u_1(1)}\,u_1(t),
$$

where, as before, $u_1(t)$ is the solution of the homogeneous equation with $u_1(0) = 1$, $u_1'(0) = 0$.

Lemma 3.4. *The Green's function $H_\lambda(t,s)$ satisfies the following symmetry property:*

$$
t^k H_\lambda(t,s) = s^k H_\lambda(s,t).
$$

Proof. Let u_1, u_2 be such that

$$
-u_i''(t) - \frac{k}{t}u_i'(t) + \lambda^2 u_i(t) = f_i(t), \quad u_i'(0) = u_i(1) = 0, \quad i = 1,2
$$

for some continuous functions f_1, f_2. The equations above are obviously equivalent to

$$
-\left(t^k u_i'(t)\right)' + \lambda^2 t^k u_i(t) = t^k f_i(t)
$$

Using this form of the equations, integrating by parts we obtain

$$
\int_0^1 t^k f_1(t)u_2(t)\,dt = \int_0^1 t^k f_2(t)u_1(t)\,dt,
$$

and therefore

$$
\int_0^1\int_0^1 t^k f_1(t)H_\lambda(t,s)f_2(s)\,ds\,dt = \int_0^1\int_0^1 t^k f_2(t)H_\lambda(t,s)f_1(s)\,ds\,dt.
$$

Given the arbitrariness of f_1 and f_2, the conclusion follows now easily. □

Proposition 3.5. *Let $w \in C^2[0,1]$ be such that*

$$-w''(t) - \frac{k}{t}w'(t) + \lambda^2 w(t) + M\int_0^1 \tau^k w(\tau)\, ds = 0, \quad w'(0) = w(1) = 0, \tag{18}$$

for some $\lambda > 0$, $M > 0$. Then we have $w(t) = 0$ for all $t \in [0,1]$.

Proof. We obtain $w(t) = 0$ using similar arguments to those used in the proof of Proposition 3.1. □

Lemma 3.6. *Let $u \in C^2[0,1]$ be such that*

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) + M\int_0^1 \tau^k u(\tau)\, ds = f(t) \geq 0, \quad u'(0) = a \leq 0,\ u(1) = b \geq 0, \tag{19}$$

for some $\lambda > 0$, $M > 0$. Suppose that

$$\frac{M}{1 + M\int_0^1 \tau^k U(\tau)\, d\tau} \leq \inf_{0<t,s<1} \frac{H_\lambda(t,s)}{U(t)\,U(s)s^k}, \quad \text{and} \quad \frac{M\,U(t)}{1 + M\int_0^1 \tau^k U(\tau)\, d\tau} \leq \frac{u_1(t)}{\int_0^1 \tau^k u_1(\tau)\, d\tau} \tag{20}$$

where $U(t)$ is the unique solution of (17) with $h(t) = 1$, a, $b = 0$. Then we have $u(t) \geq 0$ for all $t \in [0,1]$.

Proof. Note that $f \in X$. Let v and w be such that

$$\begin{aligned}
&-v''(t) - \frac{k}{t}v'(t) + \lambda^2 v(t) = f(t), \qquad v \in C^2]0,1] \cap C^1[0,1],\ v(1) = b, \\
&-w''(t) - \frac{k}{t}w'(t) + \lambda^2 w(t) = \frac{M\int_0^1 \tau^k v(\tau)\, d\tau}{1 + M\int_0^1 \tau^k U(\tau)\, d\tau}, \quad w'(0) = w(1) = 0.
\end{aligned}$$

As $w(t) = \frac{M\int_0^1 \tau^k v(\tau)\, d\tau}{1+M\int_0^1 \tau^k U(\tau)\, d\tau} U(t)$, it can be easily verified that $v - w$ satisfies (19). Proposition 3.5 allows us to conclude that $u = v - w$, so we only need to prove that $v \geq w$.

Using the Green's function H_λ defined above and the previous lemma, we have

$$\begin{aligned}
v(t) &= \int_0^1 H_\lambda(t,s) f(s)\, ds + \frac{b}{u_1(1)}\, u_1(t), \quad \text{and} \\
w(t) &= \frac{M}{1 + M\int_0^1 \tau^k U(\tau)\, d\tau} \int_0^1 H_\lambda(t,\sigma)\, d\sigma \int_0^1 \tau^k \left(\int_0^1 H_\lambda(\tau,s) f(s)\, ds + \frac{b}{u_1(1)}\, u_1(\tau) \right) d\tau \\
&= \frac{M}{1 + M\int_0^1 \tau^k U(\tau)\, d\tau} \left(\int_0^1 U(t)U(s)s^k f(s)\, ds + \frac{b\,U(t)}{u_1(1)} \int_0^1 \tau^k u_1(\tau)\, d\tau \right),
\end{aligned}$$

and therefore, if the conditions in (20) are verified, we have $v \geq w$. □

Remark 3.7. In the two previous results we do not need to consider $C^2[0,1]$ functions, the same conclusions are valid in $C^1[0,1[\cap C^2\,]0,1[$.

4 Nonlocal Semi-Linear Problems

Consider the boundary value problem

$$-u''(t) + \lambda^2 u(t) + M\int_0^1 |u(\tau)|\, d\tau = f(t), \quad u(0) = a \geq 0,\ u(1) = b \geq 0. \tag{21}$$

Proposition 4.1. *If*

$$M < \min_{\substack{u \in H_0^1(0,1) \\ u \neq 0}} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau}{\left(\int_0^1 |u(\tau)|\, d\tau\right)^2},$$

then the problem (21) has a unique solution.

Proof. We shall consider two cases:

(i) If $f(t) = 0$, and $a = b = 0$, multiplying the equation in (21) by u and integrating by parts, we have

$$\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau = -M \int_0^1 |u(\tau)|\, d\tau \int_0^1 u(\tau)\, d\tau \leq M \left(\int_0^1 |u(\tau)|\, d\tau\right)^2,$$

and the conclusion follows.

(ii) If $f(t) \neq 0$, let u_1, u_2 be such that

$$-u_i''(t) + \lambda^2 u_i(t) + M \int_0^1 |u_i(\tau)|\, d\tau = f(t), \quad u_i(0) = a, \ \ u_i(1) = b, \quad i = 1, 2.$$

Setting $w = u_1 - u_2$, we have

$$-w''(t) + \lambda^2 w(t) + M \int_0^1 \theta(\tau) w(\tau)\, d\tau = 0, \quad w(0) = w(1) = 0,$$

where $\theta(\tau) = \frac{|u_1(\tau)| - |u_2(\tau)|}{u_1(\tau) - u_2(\tau)}$. Since $|\theta(\tau)| \leq 1$, using an argument similar to the one in (i), we get $w(t) = 0$, and therefore there is a unique solution to (21).

□

Proposition 4.2. *We have*

$$\min_{\substack{u \in H_0^1(0,1) \\ u \neq 0}} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau}{\left(\int_0^1 |u(\tau)|\, d\tau\right)^2} = \min_{\substack{u \in H_0^1(0,1) \\ u \neq 0}} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau}{\left(\int_0^1 u(\tau)\, d\tau\right)^2}.$$

Proof. If a function u_0 minimizes the left-hand side, then, since $|u_0| \in H_0^1(0,1)$, the right-hand side has the same value. □

Let

$$l_1 = \min_{\substack{u \in H_0^1(0,1) \\ u \neq 0}} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau}{\left(\int_0^1 u(\tau)\, d\tau\right)^2} = \min_{\substack{u \in H_0^1(0,1) \\ \int_0^1 u(\tau)\, d\tau = 1}} \int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau)\, d\tau.$$

To find l_1, we need to solve a constrained extrema problem, which we can do using Lagrange Multipliers (the proposition above allows us to use a differentiable restriction). Our minimizer u_0 satisfies

$$-u_0''(t) + \lambda^2 u_0(t) = m, \quad u_0(0) = u_0(1) = 0,$$

where m is the Lagrange Multiplier, so $u_0(t) = mU(t)$. Since $\int_0^1 u_0(\tau)\, d\tau = 1$, we get $m = \left(\int_0^1 U(\tau)\, d\tau\right)^{-1}$, and consequently

$$l_1 = \int_0^1 {u'_0}^2(\tau) + \lambda^2 u_0^2(\tau)\, d\tau = \frac{1}{\int_0^1 U(\tau)\, d\tau}.$$

Theorem 4.3 (Maximum Principle 1)**.** *Let λ, M be positive constants, G_λ the Green's function associated to (13), $U(t) = \int_0^1 G_\lambda(t,s)\, ds$, and $V(t)$ the unique solution of $-V''(t)+\lambda^2 V(t) = 0$, with boundary conditions $V(0) = a \geq 0$, $V(1) = b \geq 0$. Suppose that*

$$\frac{M}{1 + M\int_0^1 U(\tau)\, d\tau} \leq \inf_{0<t,s<1} \frac{G_\lambda(t,s)}{U(t)\, U(s)}, \qquad \frac{V(t)}{\int_0^1 V(\tau)\, d\tau} \geq \frac{M\, U(t)}{1 + M\int_0^1 U(\tau)\, d\tau},$$

and

$$M < \frac{1}{\int_0^1 U(\tau)\, d\tau}.$$

Then, if $u \in C[0,1] \cap C^2]0,1[$ satisfies

$$-u''(t) + \lambda^2 u(t) + M\int_0^1 |u(\tau)|\, d\tau \geq 0, \quad u(0) = a \geq 0,\ u(1) = b \geq 0, \tag{22}$$

we have $u(t) \geq 0$.

Proof. Let $f(t) = -u''(t) + \lambda^2 u(t) + M\int_0^1 |u(\tau)|\, d\tau$. By Lemma (3.2), we know that the linear problem (15) has a nonnegative solution, and therefore, this nonnegative solution has to be the only solution of (21). ☐

Using *Mathematica*, we have the following estimates relative to the first pair of conditions:

$\lambda = 0.2$	$M_{max} \approx 5.98$
$\lambda = 0.5$	$M_{max} \approx 5.92$
$\lambda = 1$	$M_{max} \approx 5.71$
$\lambda = 2$	$M_{max} \approx 4.89$
$\lambda = 4$	$M_{max} \approx 2.74$
$\lambda = 7$	$M_{max} \approx 0.62$
$\lambda = 10$	$M_{max} \approx 0.09$

The last condition is less restrictive, as it is shown by the following graph:

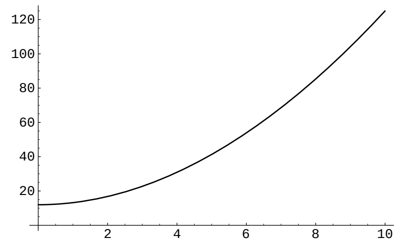

Figure 1: $l_1(\lambda) = \frac{1}{\int_0^1 U(\tau)\, d\tau}$

Using the same technique, we can reach similar results for the boundary value problem

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) + M\int_0^1 \tau^k |u(\tau)| \, d\tau = f(t), \quad u'(0) = a \leq 0, \; u(1) = b \geq 0. \quad (23)$$

Let us consider the Hilbert Space

$$H_k(0,1) = \left\{ u \in AC\,]0,1] \colon \int_0^1 \tau^k u'^2(\tau)\, d\tau < \infty, \;\; u(1) = 0 \right\},$$

with the norm $\|u\| = \left(\int_0^1 \tau^k u'^2(\tau)\, d\tau \right)^{1/2}$. Following [1], for any $u \in H_k(0,1)$ with $k > 1$, we have $\int_0^1 \tau^k u^2 \leq C \|u\|^2$, for some $C > 0$.

Remark 4.4. Note that if $u \in H_k(0,1)$, then $|u| \in H_k(0,1)$.

Proposition 4.5. *If*

$$M < \min_{\substack{u \in H_k(0,1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau) \right) \, d\tau}{\left(\int_0^1 \tau^k |u(\tau)| \, d\tau \right)^2},$$

then the problem (23) has a unique solution.

Proof. As stated before we can write equation (23) in the form

$$-\left(t^k u'(t) \right)' + \lambda^2 t^k u(t) + M t^k \int_0^1 \tau^k |u(\tau)| \, d\tau = t^k f(t)$$

We shall consider two cases:

(i) If $f(t) = 0$, a, $b = 0$, multiplying the equation in (23) by u and integrating by parts, we have

$$\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau) \right) \, d\tau = -M \int_0^1 \tau^k |u(\tau)| \, d\tau \int_0^1 \tau^k u(\tau)\, d\tau \leq M \left(\int_0^1 \tau^k |u(\tau)| \, d\tau \right)^2,$$

and the conclusion follows.

(ii) If $f(t) \neq 0$, let u_1, u_2 be such that

$$-u_i''(t) - \frac{k}{t}u_i'(t) + \lambda^2 u_i(t) + M \int_0^1 \tau^k |u_i(\tau)| \, d\tau = f(t), \quad u_i'(0) = a, \; u_i(1) = b,$$

Setting $w = u_1 - u_2$, we have

$$-\left(t^k w'(t) \right)' + \lambda^2 t^k w(t) + M t^k \int_0^1 |\theta(\tau)| \, \tau^k |w(\tau)| \, d\tau = 0, \quad w'(0) = w(1) = 0,$$

where $\theta(\tau) = \frac{|u_1(\tau)| - |u_2(\tau)|}{u_1(\tau) - u_2(\tau)}$. Since $|\theta(\tau)| \leq 1$, using an argument similar to the one in (i), we get $w(t) = 0$, and therefore there is a unique solution to (21).

□

Proposition 4.6. *We have*

$$\min_{\substack{u \in H_k(0,1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau}{\left(\int_0^1 \tau^k |u(\tau)| \, d\tau\right)^2} = \min_{\substack{u \in H_k(0,1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau}{\left(\int_0^1 \tau^k u(\tau) \, d\tau\right)^2}.$$

Proof. If a function u_0 minimizes the left-hand side, then, since $|u_0| \in H_k(0,1)$, the right-hand side has the same value. □

Let

$$l_2 = \min_{\substack{u \in H_k(0,1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau}{\left(\int_0^1 \tau^k u(\tau) \, d\tau\right)^2} = \min_{\substack{u \in H_k(0,1) \\ \int_0^1 \tau^k u(\tau)\, d\tau = 1}} \int_0^1 \tau^k \left(u'^2(\tau) + \lambda^2 u^2(\tau)\right) d\tau.$$

So, to find l_2, we need to solve another constrained extrema problem. Our minimizer u_0 satisfies

$$-u_0''(t) - \frac{k}{t} u_0'(t) + \lambda^2 u_0(t) = m, \quad u_0'(0) = u_0(1) = 0,$$

where m is the Lagrange Multiplier, so $u_0(t) = mU(t)$. Since $\int_0^1 \tau^k u_0(\tau)\, d\tau = 1$, we get $m = \left(\int_0^1 \tau^k U(\tau)\, d\tau\right)^{-1}$, and consequently

$$l_2 = \int_0^1 \tau^k \left(u_0'^2(\tau) + \lambda^2 u_0^2(\tau)\right) d\tau = \frac{1}{\int_0^1 \tau^k U(\tau)\, d\tau}.$$

Theorem 4.7 (Maximum Principle 2)**.** *Let λ, M be positive constants, H_λ the Green's function associated to (17), and $U = \int_0^1 H_\lambda(t,s)\, ds$. Suppose that*

$$\frac{M}{1 + M\int_0^1 \tau^k U(\tau)\, d\tau} \le \inf_{0<t,s<1} \frac{H_\lambda(t,s)}{U(t)\, U(s) s^k}, \qquad \frac{M\, U(t)}{1 + M\int_0^1 U(\tau)\, d\tau} \le \frac{u_1(t)}{\int_0^1 \tau^k u_1(\tau)\, d\tau},$$

and

$$M < \frac{1}{\int_0^1 \tau^k U(\tau)\, d\tau}.$$

Then, if for $0 < t \le 1$, $u \in C^2[0,1]$ satisfies

$$-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) + M \int_0^1 \tau^k |u(\tau)| \, ds \ge 0, \quad u'(0) = a \le 0,\, u(1) = b \ge 0, \quad (24)$$

we have $u(t) \ge 0$.

Proof. Let $f(t) = -u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) + M\int_0^1 \tau^k |u(\tau)|\, ds$. By Lemma (3.6), we know that the linear problem (19) has a nonnegative solution, and therefore, this nonnegative solution has to be the only solution of (23). □

We have the following estimates relative to the cases $k = 1, 2, 3$:

(i) k=1:

$\lambda = 0.25$	$M_{max} \approx 15.95$
$\lambda = 1$	$M_{max} \approx 15.30$
$\lambda = 5$	$M_{max} \approx 5.71$

(ii) k=2:

$$\lambda = 0.25 \qquad M_{max} \approx 29.9$$
$$\lambda = 1 \qquad M_{max} \approx 28.9$$
$$\lambda = 5 \qquad M_{max} \approx 12.2$$

(iii) k=3:

$$\lambda = 0.25 \qquad M_{max} \approx 47.9$$
$$\lambda = 0.5 \qquad M_{max} \approx 47.5$$
$$\lambda = 1 \qquad M_{max} \approx 46.5$$
$$\lambda = 3 \qquad M_{max} \approx 36.0$$
$$\lambda = 5 \qquad M_{max} \approx 21.5$$
$$\lambda = 10 \qquad M_{max} \approx 2.2$$

The last condition is also less restrictive. We present here the graph of $l_2(\lambda)$ in the case $k = 3$:

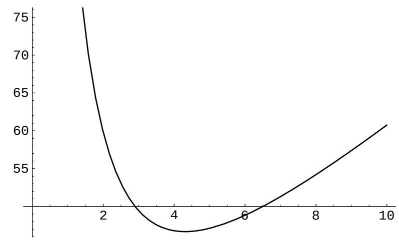

Figure 2: $l_2(\lambda) = \frac{1}{\int_0^1 \tau^3 U(\tau)\, d\tau}$

5 Lower and Upper solutions and the monotone method

Consider the boundary value problem

$$-u''(t) - \frac{n-1}{t}u'(t) = f\left(u(t), \omega_n \int_0^1 s^{n-1} g\left(u(s)\right)\, ds\right) \text{ for } 0 < t \leq 1, \tag{25}$$

and

$$u'(0) = 0 = u(1), \tag{26}$$

where $n \in \mathbb{N}$, f, g are continuous functions, and ω_n is the superficial measure of the unit sphere in $\mathbb{R}^n$. The solutions of this problem are radial solutions of

$$-\Delta u = f\left(u, \int_U g\left(u\right)\right) \tag{27}$$

$$u|_{\partial U} = 0, \tag{28}$$

where $U = B(0,1)$ is the unit sphere in $\mathbb{R}^n$ (see [4]).

We say that $\alpha(t)$ is a *lower solution* of (25)–(26) if

$$-\alpha''(t) - \frac{n-1}{t}\alpha'(t) \leq f\left(\alpha(t), \omega_n \int_0^1 s^{n-1} g\left(\alpha\left(s\right)\right) ds\right), \quad \text{for } 0 < t \leq 1,$$

$$\alpha'(0) \geq 0 \text{ and } \alpha(1) \leq 0.$$

A function β satisfying the reversed inequalities is called an *upper solution*.

For a given function $u(t) \in C[0,1]$, consider the boundary value problem

$$-v''(t) - \frac{n-1}{t}v'(t) + \lambda^2 v(t) = f\left(u(t), \omega_n \int_0^1 s^{n-1} g\left(v(s)\right) ds\right) + \lambda^2 u(t),$$

with $v'(0) = 0 = v(1)$. Using the operator $Lu = -u'' - \frac{n-1}{t}u' + \lambda^2 u$, in the space $C^* = \left\{u \in C^2[0,1] \colon u'(0) = u(1) = 0\right\}$ this equation is equivalent to the fixed point equation

$$v = L^{-1}\left(f\left(u, \omega_n \int_0^1 s^{n-1} g\left(v\left(s\right)\right) ds\right) + \lambda^2 u\right) \equiv \Phi_u v. \tag{29}$$

It turns out that it is advantageous to look at Φ_u as an operator from $L^2_{n-1}(0,1)$ into itself. Noticing that L^{-1} is a compact self-adjoint operator in this space with norm $\|L^{-1}\| = (\xi_n^2 + \lambda^2)^{-1}$ where ξ_n is the first positive zero of the Bessel function $J_{\frac{n-2}{2}}$, it is easy to see that if $f(u,v)$ is k_1-Lipschitz in v, g is k_2-Lipschitz, then Φ_u is Lipschitz with constant $\frac{\omega_n k_1 k_2}{(\xi_n^2+\lambda^2)n}$. In particular, when the condition

$$\frac{\omega_n k_1 k_2}{(\xi_n^2 + \lambda^2)n} < 1 \tag{30}$$

is satisfied, Φ_u is a contraction mapping, and therefore has a unique fixed point.

Using maximum principle 4.7, we get the following improved version of Theorem 4.10 in [2]:

Theorem 5.1. *Suppose that $f(u,v)$ is k_1-Lipschitz in v, g is k_2-Lipschitz. Suppose that $M \equiv k_1 k_2 \omega_n$ and λ are in the conditions of the Maximum Principle 4.7, (30) holds and*

$$f(u_2, v) - f(u_1, v) \geq -\lambda^2(u_2 - u_1),$$

for all $v \in \mathbb{R}$, and $u_1 \leq u_2$. Let α_0 and β_0 be a lower and an upper solution of (25)–(26) respectively, with $\alpha_0 \leq \beta_0$ in $[0,1]$. If we take $(\alpha_n)_{n\in\mathbb{N}_0}$ and $(\beta_n)_{n\in\mathbb{N}_0}$ such that,

$$\alpha_{n+1} = \Phi_{\alpha_n}\alpha_{n+1} \quad \textit{and} \quad \beta_{n+1} = \Phi_{\beta_n}\beta_{n+1}, \textit{ for all } n \in \mathbb{N}_0,$$

we obtain

$$\alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0.$$

The monotone bounded sequences $(\alpha_n)_{n\in\mathbb{N}_0}$, $(\beta_n)_{n\in\mathbb{N}_0}$ defined above are convergent in $C[0,1]$ to solutions of (25)–(26).

The main step of the proof consists in verifying that indeed a monotone sequence is obtained. This is a consequence of the following fact: Let $u_1(r) \leq u_2(r)$ be two given functions defined in $[0,1]$ and $v_1(r)$, $v_2(r)$ the two respective solutions of (29). Then $v_1(r) \leq v_2(r)$.

Let us recall the argument (see [2]):

$$
\begin{aligned}
&-(v_2-v_1)'' - \frac{n-1}{r}(v_2-v_1)' + \lambda^2(v_2-v_1) = \\
&= \lambda^2(u_2-u_1) + f\left(u_2, \omega_n \int_0^1 s^{n-1} g(v_2)\, ds\right) - f\left(u_1, \omega_n \int_0^1 s^{n-1} g(v_2)\, ds\right) + \\
&\qquad + f\left(u_1, \omega_n \int_0^1 s^{n-1} g(v_2)\, ds\right) - f\left(u_1, \omega_n \int_0^1 s^{n-1} g(v_1)\, ds\right) \geq \\
&\geq -k_1 k_2 \omega_n \int_0^1 s^{n-1} |v_2-v_1|\, ds.
\end{aligned}
$$

It sufices then to invoke the maximum principle 4.7 to obtain the conclusion.

Example 5.2. Let us consider the non-local differential equation

$$
-u''(t) - \frac{2}{t}u'(t) = f\left(u, 4\pi \int_0^1 s^2 \left(\frac{u(s)^2+1}{3}\right) ds\right) \tag{31}
$$

where

$$
f(u,v) = \begin{cases} (\sqrt{u}+1)(\sin v + 1) + 4, & u \leq 1 \\ \left(\frac{1}{u}+1\right)(\sin v + 1) + 4, & u \geq 1, \end{cases}
$$

with boundary conditions $u'(0) = u(1) = 0$.

Consider $\alpha_0 = 1 - t^2$ and $\beta_0 = \frac{4}{3}\left(1-t^2\right)$. After some computation, we can verify that α_0, β_0 are respectively a lower and an upper solution of (31), both satisfying the considered boundary conditions. Since $0 \leq \alpha_0(t) \leq \beta_0(t) \leq \frac{4}{3}$, for all $t \in [0,1]$, we can consider $k_1 = 2$, $k_2 = \frac{8}{9}$, and $\lambda = 1$. Moreover $\xi_3 = \pi$. Setting $M = \frac{64\pi}{9}$, the conditions of theorem 5.1 are satisfied, and therefore, using the described iterative method, we can aproximate a solution $u(t)$ of (31) satisfying $u'(0) = u(1) = 0$ and $1 - t^2 \leq u(t) \leq \frac{4}{3}\left(1-t^2\right)$.

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