NONLOCAL MAXIMUM PRINCIPLES
AND APPLICATIONS

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Abstract

We extend a nonlocal maximum principle obtained in [2], which allows us to use
a monotone method to find radial solutions of an elliptic problem in the presence of
lower and upper solutions.

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1 Introduction

Boundary value problems where a nonlocal term appears have attracted much attention
recently due to their role in many problems of physics and engineering. In [2], where we
improved some results of [4], we have dealt with some aspects concerning that kind of
problems, namely existence and approximation of radial solutions in a ball of \( \mathbb{R}^N \).

There, we observed that there are obvious difficulties in using the method of lower
and upper solutions in the presence of nonlocal terms. Nevertheless, we developed a
monotone method for problems of the form

\[-u''(t) - \frac{n-1}{t} u'(t) = f \left( u(t), \omega_n \int_0^1 s^{n-1} g(u(s)) \, ds \right), \quad u'(0) = 0 = u(1), \quad (1)\]

assuming that \( f(u_2, v) - f(u_1, v) \geq -\lambda^2 (u_2 - u_1) \) for some \( \lambda > 0 \), \( f(u, v) \) is \( k_1 \)-Lipschitz
in \( v \) and \( g \) is \( k_2 \)-Lipschitz (which are similar conditions to those used in [5]). The method
is based on a “nonlocal maximum principle” asserting that

\[-u''(t) - \frac{n-1}{t} u'(t) + \lambda^2 u(t) + M \int_0^1 s^{n-1} |u(s)| \, ds \geq 0, \quad u'(0) \leq 0, \quad u(1) \geq 0 \quad (2)\]
implies that \( u \geq 0 \) in \( I = [0,1] \). The fact that we needed the assumption \( \lambda^2 + M < 1 \) is a limitation in the strength of this principle.

The purpose of this paper is basically to extend the nonlocal maximum principle so as to allow its applicability to a large range of values of \( \lambda > 0 \) and \( M > 0 \).

We investigate the admissible range of values in two cases: first we consider a simple model - for which values of \( \lambda > 0 \) and \( M > 0 \) do the inequalities

\[
-u''(t) + \lambda^2 u(t) + M \int_0^1 |u(s)| \, ds \geq 0, \quad u(0) \geq 0, \quad u(1) \geq 0
\]

yield a maximum principle? Then we proceed to the inequality (2), related to the important class of radial problems in a ball. It turns out that the two situations may be dealt in a similar way, although some computations are easier in the first case.

In the course of our approach we find it convenient to consider the linear singular differential equation

\[
-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = h(t),
\]

and find an expression for one of its solutions as

\[
u(t) = \int_0^1 H_\lambda(t,s)h(s)ds,
\]

where \( H_\lambda \) is a Green’s function. The solution we have in mind exists for a certain class of right-hand sides \( h \), and may satisfy boundary conditions \( u'(0) = a, u(1) = 0 \), where \( a \) needs not be zero.

We have organized the paper as follows: In section 2 we collect our remarks concerning the linear equation (3). In section 3 we study a class of nonlocal linear boundary value problems that are useful in the sequel. In section 4 we proceed with the consideration of nonlocal semilinear problems and establish the nonlocal maximum principle. In the final section, we briefly illustrate the use of the principle to establish a monotone method for (1).

## 2 Some remarks about the solutions of a linear problem

Let us consider the differential equation

\[
-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = h(t), \quad t \in [0,1],
\]

where \( k > 1, \lambda > 0 \) and \( h \in L_{k+2}^2(0,1) \equiv \left\{ h(t) \text{ measurable: } \int_0^1 \tau^{k+2} h(\tau)^2 \, d\tau < \infty \right\} \).

We shall use the Hilbert Spaces

\[
H_k(0,1) = \left\{ u \in AC[0,1]: \int_0^1 \tau^k u'(\tau)^2 \, d\tau < \infty, \quad u(1) = 0 \right\},
\]

with the norm \( \|u\| = \left( \int_0^1 \tau^k u'(\tau)^2 \, d\tau \right)^{1/2} \).

Following [1], for \( u \in H_k(0,1) \), with \( k > 1 \), we have \( \left( \int_0^1 \tau^{k-2} u(\tau)^2 \, d\tau \right)^{1/2} \leq \frac{2}{k-1} \|u\| \), so the functional

\[
J(u) := \int_0^1 \left( \frac{1}{2} \left( t^k u'(t)^2 + \lambda^2 t^k u(t)^2 \right) + t^k h(t) u(t) \right) dt
\]
is well defined in $H_k(0, 1)$, since

$$\int_0^1 t^k h(t) u(t) \, dt \leq \left( \int_0^1 t^{k-2} u(t)^2 \, dt \right)^{1/2} \left( \int_0^1 t^{k+2} h(t)^2 \, dt \right)^{1/2}.$$  

It is obvious that $J(u)$ is a coercive strictly convex functional, so that equation (4) has a unique solution in $H_k(0, 1)$.

**Proposition 2.1.** If $h \in L^2_k(0, 1) \equiv \left\{ h(t) \text{ measurable: } \int_0^1 \tau h(\tau)^2 \, d\tau < \infty \right\}$, then, the unique solution $u$ of (4) in $H_k(0, 1)$ is in fact in $C^1[0, 1]$ and it satisfies $u'(0) = 0$ (note that $L^2_1(0, 1) \subset L^2_{k+2}(0, 1)$).

**Proof.** Equation (4) is obviously equivalent to

$$-\left( t^k u'(t) \right)' + \lambda^2 t^k u(t) = t^k h(t).$$

If $h \in L^2_1(0, 1)$, it is easy to verify that $|t^k u'(t)|$ satisfies Cauchy’s condition at $t = 0$, therefore there exists $L \in \mathbb{R}$ such that $\lim_{t \to 0} |t^k u'(t)| = L$. Necessarily $L = 0$, because otherwise we would not have $u \in H_k(0, 1)$. Applying Cauchy-Schwarz inequality, it is easy to see that

$$|t^k u'(t)| \leq \left| \int_0^t \lambda^2 \tau t^k u(\tau) \, d\tau \right| + \left| \int_0^t \tau t^k h(\tau) \, d\tau \right| \leq c_1 \left( \int_0^t \tau^{k-2} u(\tau)^2 \, d\tau \right)^{1/2} t^{k+3} + c_2 \left( \int_0^t \tau h(\tau)^2 \, d\tau \right)^{1/2} t^k,$$

for some constants $c_1, c_2 > 0$.

If $\frac{k+3}{2} \geq k$ ($k \leq 3$), it is obvious that $\lim_{t \to 0} u'(t) = 0$. Otherwise, if $k > 3$, we have

$$|t^k u'(t)| \leq c t^{k+3},$$

for some constant $c > 0$.

In general, if we have $|t^k u'(t)| \leq c t^\alpha$, then $|u(t)| \leq C + C t^{\alpha-k+1}$, for some $C > 0$, hence, we can conclude that near $t = 0$, there exists a constant $c_3 > 0$ such that

$$\left( \int_0^t \tau^{k-2} u(\tau)^2 \, d\tau \right)^{1/2} \leq c_3 t^{\min \left( \frac{k+1}{2}, \frac{2\alpha-k+1}{2} \right)}.$$

Consequently, for some $c_4 > 0$, we have

$$|t^k u'(t)| \leq c_4 t^{\min(k+1, \alpha+2)} + c_2 \left( \int_0^t \tau h(\tau)^2 \, d\tau \right)^{1/2} t^k,$$

and setting $\alpha = \frac{k+3}{2}$, it is easy to see that with a finite number of iterations of this process, we will get

$$|t^k u'(t)| \leq c^* t^{k^*} + c_2 \left( \int_0^t \tau h(\tau)^2 \, d\tau \right)^{1/2} t^k,$$

where $k^* > k$, and then the conclusion follows easily.  

\[\square\]
It is a standard procedure in the literature to associate solutions of a boundary value problem to fixed points of some functional operator. In our case, the solutions of the second order homogeneous differential equation

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = 0,$$

which is equivalent to \((t^k u'(t))' = \lambda^2 t^k u(t)\), with initial conditions \(u(0) = 1\), \(u'(0) = 0\), may be viewed as fixed points of the operator

$$Tu(t) = 1 + \int_0^t \frac{\lambda^2}{\tau^k} \int_0^\tau s^k u(s) \, ds \, d\tau,$$

defined in some functional space. Considering the space \(Z = \{u \in C[0,t_0] : u(0) = 1\}\), for some \(t_0\) small enough, \(T\) has a unique fixed point since it is a contraction. The singularity of equation (6) is at the point \(t = 0\), so it is obvious that this solution can be extended to the interval \([0,1]\). Let \(u_1\) be this solution, and consider the function \(v_1(t) = u_1(t) \int_0^1 \frac{ds}{s u_1(s)^r} \), which is the solution of (6) obtained by the standard method of reducing the order of an ordinary differential equation. The solutions \(u_1\) and \(v_1\) are linearly independent and their associated Wronskian is \(W(t) = u_1(t)v_1'(t) - u_1'(t)v_1(t) = -t^{-k}\). Furthermore, they satisfy the following properties, which we shall use in the next proposition: \(u_1'(t) \geq 0, v_1(1) = 0, v_1(t) \sim t^{-(k-1)}, \) and \(v_1'(t) \sim t^{-k}\) as \(t \to 0\) (we write \(f(t) \sim g(t)\) as \(t \to 0\) if and only if \(\lim_{t \to 0} \frac{f(t)}{g(t)} = L \neq 0\).

**Proposition 2.2.** Let \(h \in X \equiv \{h(t) \text{ measurable} : \exists c \in \mathbb{R}, h_0 \in L^2_{k+2}(0,1), h(t) = \frac{c}{t} + h_0(t)\}\). Then the boundary value problem

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \quad t \in [0,1], \quad u(1) = 0,$$

has a unique solution in \(C^2[0,1] \cap C^1[0,1]\), given by the integral expression

$$u(t) = -u_1(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)} \, ds - v_1(t) \int_0^t \frac{u_1(s)h(s)}{W(s)} \, ds.$$  

**Proof.** Let us first note that \(L^2_{k+2}(0,1) \subset X \subset L^2_{k+2}(0,1)\), so that equation (4) has a unique solution in \(H_k(0,1)\), that satisfies \(u(1) = 0\).

Suppose that \(h \in L^2_{k+1}(0,1)\), that is, \(c = 0\). Applying the method of undetermined coefficients, we see that the unique solution of

$$-u''(t) - \frac{k}{t}u'(t) + \lambda^2 u(t) = h(t), \quad t \in [0,1], \quad u'(0) = u(1) = 0,$$

is given by the well defined integral expression

$$u(t) = -u_1(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)} \, ds - v_1(t) \int_0^t \frac{u_1(s)h(s)}{W(s)} \, ds.$$  

If we differentiate this expression, we get

$$u'(t) = -u_1'(t) \int_t^1 \frac{v_1(s)h(s)}{W(s)} \, ds - v_1'(t) \int_0^t \frac{u_1(s)h(s)}{W(s)} \, ds,$$

from which, after some computation, we can confirm that \(u'(0) = 0\).
Suppose now that \( h(t) \notin L^2_1(0, 1) \), that is, \( h(t) = \xi + h_0(t) \) for some \( c \neq 0 \), \( h_0 \in L^2_1(0, 1) \). In this case, the integral expression (8) is still well defined, satisfies equation (4), and

\[
u'(0) = -\lim_{t \to 0} \frac{1}{t} \int_0^t u_1(s) h(s) \, ds = \lim_{t \to 0} c \, v_1(t) \int_0^t u_1(s) s^{k-1} \, ds = -\frac{c}{k}.
\]

Remark 2.3. Expression (8) can obviously be written in the form

\[
\int_0^1 H_\lambda(t, s) h(s) \, ds,
\]

which allows us to get the explicit form of the Green’s function associated to (9). From the expression of \( H_\lambda \), it is a simple matter to verify that it is continuous in \([0, 1] \times [0, 1]\) and positive in \([0, 1[\times[0, 1]\).

From the proof of the previous proposition, we infer that formula (11), where the Green’s function \( H_\lambda \) appears, provides us the unique solution of (4) for all the boundary conditions \( u'(0) = a \in \mathbb{R}, u(1) = 0 \), whenever \( h(t) + \frac{ka}{t} \in L^2_1(0, 1) \).

The boundary value problem

\[-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = h(t), \quad u(1) = b,
\]

with \( b \neq 0 \), has also a unique solution in \( C^2_0[0, 1] \cap C^1[0, 1] \) if we had two different solutions \( w_1, w_2 \), then \( w_1 - w_2 \) would be the unique solution of the homogeneous problem, which is identically zero, given by \( u_0(t) + \frac{b}{u_1''(t)} u_1(t) \), where \( u_0(t) \) is the unique solution of

\[-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = h(t),
\]

in \( H_k(0, 1) \). Note that for some functions \( h(t) \notin X \) we can still obtain a solution of equation (4) via the Green’s function, which possibly has infinite derivative at \( t = 0 \), or simply does not have derivative at \( t = 0 \), but we will not consider these cases.

Consider now the equation for \( k = 1 \)

\[-u''(t) - \frac{1}{t} u'(t) + \lambda^2 u(t) = h(t), \quad t \in [0, 1],
\]

where \( \lambda > 0 \) and \( h \in L^q_1(0, 1) \equiv \left\{ h(t) \text{ measurable: } \int_0^1 th(t)^q \, dt < \infty \right\} \), for some \( 1 < q < 2 \). Consider also the functional

\[J(u) := \int_0^1 \frac{1}{2} \left( t u'(t)^2 + \lambda^2 t u^2(t) \right) + t h(t) u(t) \, dt,
\]

defined in \( H_1(0, 1) \). We have

\[
\int_0^1 t h(t) u(t) \, dt \leq \left( \int_0^1 t u(t)^p \, dt \right)^{1/p} \left( \int_0^1 t h(t)^q \, dt \right)^{1/q}.
\]

Following [3], since for any \( p > 2 \) we have \( u^p \leq Ce^{\delta u^{2-\eta}} \), for some \( C, \eta > 0 \), we know that \( \int_0^1 t u(t)^p \, dt \leq \infty \), and therefore the functional \( J(u) \) is well defined in \( H_1(0, 1) \).

We can state exactly the same results obtained above for \( k > 1 \) in the case \( k = 1 \), just noticing that in this case \( v_1(t) \sim \ln t \), and \( v_1'(t) \sim t^{-1} \). The fact that \( 1 < q < 2 \) allows us to conclude that with a function \( h(t) \sim \frac{1}{t} \), \( J(u) \) is well defined, and the associated solution is the one obtained via Green function, with non-zero derivative at \( t = 0 \).
3 Nonlocal Linear Problems

Let us consider the linear boundary value problem in the interval $[0, 1]$

$$-u''(t) + \lambda^2 u(t) = h(t), \quad u(0) = u(1) = 0,$$  \hspace{1cm} (13)

where $\lambda > 0$ and $h \in C[0, 1]$.

This problem has a well known Green’s function

$$G_\lambda(t, s) = \begin{cases} \sinh(\lambda) \cosh(\lambda t) \sinh(\lambda s) - \cosh(\lambda) \sinh(\lambda t) \sinh(\lambda s), & t \geq s \\ \sinh(\lambda) \cosh(\lambda s) \sinh(\lambda t) - \cosh(\lambda) \sinh(\lambda s) \sinh(\lambda t), & t \leq s. \end{cases}$$

and therefore we have

$$u(t) = \int_0^1 G_\lambda(t, s) h(s) \, ds.$$

**Proposition 3.1.** Let $w \in C[0, 1] \cap C^2[0, 1]$ be such that

$$-w''(t) + \lambda^2 w(t) + M \int_0^1 w(\tau) \, d\tau = 0, \quad w(0) = w(1) = 0,$$  \hspace{1cm} (14)

for some $\lambda > 0$, $M > 0$. Then we have $w(t) = 0$ for all $t \in [0, 1]$.

**Proof.** Assume towards a contradiction that there exists $w(t) \neq 0$ satisfying (14).

If $w(t) \geq 0$ (by $\geq$ we mean $\geq$ and $\neq$), then $w$ reaches a positive maximum for some $t_0 \in [0, 1]$, where we would have the contradiction

$$0 < -w''(t_0) + \lambda^2 w(t_0) + M \int_0^1 w(\tau) \, d\tau = 0.$$

If $w(t) \leq 0$, we get a contradiction with a similar argument. So $w(t)$ must have a positive maximum for some $t_1 \in [0, 1]$ and a negative minimum for some $t_2 \in [0, 1]$. With $t = t_1$ in (14) we get $\int_0^1 w(\tau) \, d\tau < 0$, and with $t = t_2$ in (14) we get $\int_0^1 w(\tau) \, d\tau > 0$. The conclusion now follows. $\square$

**Lemma 3.2.** Let $u \in C[0, 1] \cap C^2[0, 1]$ be such that

$$-u''(t) + \lambda^2 u(t) + M \int_0^1 u(\tau) \, d\tau = f(t) \geq 0, \quad u(0) = a \geq 0, \quad u(1) = b \geq 0,$$  \hspace{1cm} (15)

for some $\lambda > 0$, $M > 0$, and consider the $C^2[0, 1]$ functions $U, V$, where $U(t)$ is the unique solution of (13) with $h(t) = 1$ and $V(t)$ is the unique solution of $-V''(t) + \lambda^2 V(t) = 0$, with boundary conditions $V(0) = a$, $V(1) = b$ (note that $U$ and $V$ depend on $\lambda$).

Suppose that

$$\frac{M}{1 + M \int_0^1 U(\tau) \, d\tau} \leq \inf_{0 < t < 1} \frac{G_\lambda(t, s)}{U(t) U(s)}, \quad \text{and} \quad \frac{M U(t)}{1 + M \int_0^1 U(\tau) \, d\tau} \leq \frac{V(t)}{\int_0^1 V(\tau) \, d\tau}. \hspace{1cm} (16)$$

Then we have $u(t) \geq 0$ for all $t \in [0, 1]$.

**Proof.** Let $v$ and $w$ be such that

$$-v''(t) + \lambda^2 v(t) = f(t), \quad v(0) = a, \quad v(1) = b,$$

$$-w''(t) + \lambda^2 w(t) = \frac{M \int_0^1 v(\tau) \, d\tau}{1 + M \int_0^1 U(\tau) \, d\tau}, \quad w(0) = w(1) = 0.$$
As \( w(t) = \frac{M \int_{0}^{t} v(\tau) \, d\tau}{1 + M \int_{0}^{1} U(\tau) \, d\tau} \) \( U(t) \), it can be easily verified that \( v - w \) satisfies (15). Proposition 3.1 allows us to conclude that \( u = v - w \), so we only need to prove that \( v \geq w \).

Using the Green’s function \( G_\lambda \) defined above and the fact that \( G_\lambda(t, s) = G_\lambda(s, t) \), we have

\[
v(t) = \int_{0}^{1} G_\lambda(t, s) f(s) \, ds + V(t), \quad \text{and}
\]

\[
w(t) = \frac{M}{1 + M \int_{0}^{1} U(\tau) \, d\tau} \int_{0}^{1} G_\lambda(t, \sigma) \, d\sigma \int_{0}^{1} \left[ \int_{0}^{1} G_\lambda(\tau, s) f(s) \, ds + V(\tau) \right] \, d\tau
\]

\[
= \frac{M}{1 + M \int_{0}^{1} U(\tau) \, d\tau} \left[ \int_{0}^{1} U(t) U(s) f(s) \, ds + U(t) \int_{0}^{1} V(\tau) \, d\tau \right],
\]

and therefore, if the conditions in (16) are verified, we have \( v \geq w \).

\[\square\]

**Remark 3.3.** The explicit form of \( U \) and \( V \) is:

\[
U(t) = -\frac{e^{-\lambda t} (-1 + e^{\lambda t}) (-e^\lambda + e^{\lambda t})}{(1 + e^\lambda) \lambda^2}
\]

\[
V(t) = \frac{e^{-\lambda t} (-a e^{2\lambda} - a e^{2\lambda} + b e^{2\lambda} + 2 e^{2\lambda})}{-1 + e^{2\lambda}}.
\]

Let us now consider the linear boundary value problem

\[ -u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) = h(t), \quad u(1) = b, \quad u \in C^2[0, 1] \cap C^1[0, 1], \quad (17) \]

where \( k \geq 1, \lambda > 0 \) and \( h \in X \).

As stated before, this problem has a unique solution given by

\[
u(t) = \int_{0}^{1} H_\lambda(t, s) h(s) \, ds + \frac{b}{u_1(1)} u_1(t),
\]

where, as before, \( u_1(t) \) is the solution of the homogeneous equation with \( u_1(0) = 1, u_1'(0) = 0 \).

**Lemma 3.4.** The Green’s function \( H_\lambda(t, s) \) satisfies the following symmetry property:

\[
t^k H_\lambda(t, s) = s^k H_\lambda(s, t).
\]

**Proof.** Let \( u_1, u_2 \) be such that

\[-u''_i(t) - \frac{k}{t} u'_i(t) + \lambda^2 u_i(t) = f_i(t), \quad u'_i(0) = u_i(1) = 0, \quad \text{i = 1, 2}\]

for some continuous functions \( f_1, f_2 \). The equations above are obviously equivalent to

\[- \left( t^k u'_i(t) \right)' + \lambda^2 t^k u_i(t) = t^k f_i(t)\]

Using this form of the equations, integrating by parts we obtain

\[
\int_{0}^{1} t^k f_1(t) u_2(t) \, dt = \int_{0}^{1} t^k f_2(t) u_1(t) \, dt,
\]

and therefore

\[
\int_{0}^{1} \int_{0}^{1} t^k f_1(t) H_\lambda(t, s) f_2(s) \, ds \, dt = \int_{0}^{1} \int_{0}^{1} t^k f_2(t) H_\lambda(t, s) f_1(s) \, ds \, dt.
\]

Given the arbitrariness of \( f_1 \) and \( f_2 \), the conclusion follows now easily. \[\square\]
Proposition 3.5. Let \( w \in C^2[0,1] \) be such that
\[
-w''(t) - \frac{k}{t} w'(t) + \lambda^2 w(t) + M \int_0^1 \tau^k w(\tau) \, d\tau = 0, \quad w'(0) = w(1) = 0,
\]
for some \( \lambda > 0, M > 0 \). Then we have \( w(t) = 0 \) for all \( t \in [0,1] \).

Proof. We obtain \( w(t) = 0 \) using similar arguments to those used in the proof of Proposition 3.1.

Lemma 3.6. Let \( u \in C^2[0,1] \) be such that
\[
-w''(t) - \frac{k}{t} w'(t) + \lambda^2 u(t) + M \int_0^1 \tau^k u(\tau) \, d\tau = f(t) \geq 0, \quad u'(0) = a \leq 0, \quad u(1) = b \geq 0,
\]
for some \( \lambda > 0, M > 0 \). Suppose that
\[
\frac{M}{1 + M \int_0^1 \tau^k \mu(\tau) \, d\tau} \leq \inf_{0 < t < 1} \frac{H_\lambda(t, s)}{U(t)U(s)k}, \quad \text{and} \quad \frac{M U(t)}{1 + M \int_0^1 \tau^k \mu(\tau) \, d\tau} \leq \frac{u_1(t)}{\int_0^1 \tau^k u_1(\tau) \, d\tau}
\]
where \( U(t) \) is the unique solution of (17) with \( h(t) = 1, a, b = 0 \). Then we have \( u(t) \geq 0 \) for all \( t \in [0,1] \).

Proof. Note that \( f \in X \). Let \( v \) and \( w \) be such that
\[
-v''(t) - \frac{k}{t} v'(t) + \lambda^2 v(t) = f(t), \quad v \in C^2[0,1] \cap C^1[0,1], \quad v(1) = b,
\]
\[
-w''(t) - \frac{k}{t} w'(t) + \lambda^2 w(t) = \frac{M \int_0^1 \tau^k v(\tau) \, d\tau}{1 + M \int_0^1 \tau^k \mu(\tau) \, d\tau}, \quad w'(0) = w(1) = 0.
\]
As \( w(t) = \frac{M \int_0^1 \tau^k v(\tau) \, d\tau}{1 + M \int_0^1 \tau^k \mu(\tau) \, d\tau} U(t) \), it can be easily verified that \( v - w \) satisfies (19). Proposition 3.5 allows us to conclude that \( u = v - w \), so we only need to prove that \( v \geq w \).

Using the Green’s function \( H_\lambda \) defined above and the previous lemma, we have
\[
v(t) = \int_0^1 H_\lambda(t, s) f(s) \, ds + \frac{b}{u_1(1)} u_1(t), \quad \text{and}
\]
\[
w(t) = \frac{M}{1 + M \int_0^1 \tau^k \mu(\tau) \, d\tau} \int_0^1 H_\lambda(t, \tau) \, ds \left( \int_0^1 H_\lambda(\tau, s) f(s) \, ds + \frac{b}{u_1(1)} u_1(\tau) \right) \, d\tau
\]
\[
= \frac{M}{1 + M \int_0^1 \tau^k \mu(\tau) \, d\tau} \left( \int_0^1 U(t)U(s)k f(s) \, ds + \frac{b U(t)}{u_1(1)} \int_0^1 \tau^k u_1(\tau) \, d\tau \right),
\]
and therefore, if the conditions in (20) are verified, we have \( v \geq w \).

Remark 3.7. In the two previous results we do not need to consider \( C^2[0,1] \) functions, the same conclusions are valid in \( C^1[0,1] \cap C^2[0,1] \).

4 Nonlocal Semi-Linear Problems

Consider the boundary value problem
\[
-u''(t) + \lambda^2 u(t) + M \int_0^1 |u(\tau)| \, d\tau = f(t), \quad u(0) = a \geq 0, \quad u(1) = b \geq 0.
\]
Proposition 4.1. If

\[ M < \min_{u \in H^1_0(0,1), u \neq 0} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau) \, d\tau}{\left( \int_0^1 |u(\tau)| \, d\tau \right)^2}, \]

then the problem (21) has a unique solution.

Proof. We shall consider two cases:

(i) If \( f(t) = 0 \), and \( a = b = 0 \), multiplying the equation in (21) by \( u \) and integrating by parts, we have

\[ \int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau) \, d\tau = -M \int_0^1 |u(\tau)| \, d\tau \int_0^1 u(\tau) \, d\tau \leq M \left( \int_0^1 |u(\tau)| \, d\tau \right)^2, \]

and the conclusion follows.

(ii) If \( f(t) \neq 0 \), let \( u_1, u_2 \) be such that

\[ -u_i''(t) + \lambda^2 u_i(t) + M \int_0^1 |u_i(\tau)| \, d\tau = f(t), \quad u_i(0) = a, \quad u_i(1) = b, \quad i = 1, 2. \]

Setting \( w = u_1 - u_2 \), we have

\[ -w''(t) + \lambda^2 w(t) + M \int_0^1 \theta(\tau) w(\tau) \, d\tau = 0, \quad w(0) = w(1) = 0, \]

where \( \theta(\tau) = \frac{|u_1(\tau)| - |u_2(\tau)|}{u_1(\tau) - u_2(\tau)} \). Since \( |\theta(\tau)| \leq 1 \), using an argument similar to the one in (i), we get \( w(t) = 0 \), and therefore there is a unique solution to (21).

Proposition 4.2. We have

\[ \min_{u \in H^1_0(0,1), u \neq 0} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau) \, d\tau}{\left( \int_0^1 |u(\tau)| \, d\tau \right)^2} = \min_{u \in H^1_0(0,1), u \neq 0} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau) \, d\tau}{\left( \int_0^1 u(\tau) \, d\tau \right)^2}. \]

Proof. If a function \( u_0 \) minimizes the left-hand side, then, since \( |u_0| \in H^1_0(0,1) \), the right-hand side has the same value.

Let

\[ l_1 = \min_{u \in H^1_0(0,1), u \neq 0} \frac{\int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau) \, d\tau}{\left( \int_0^1 u(\tau) \, d\tau \right)^2} = \min_{u \in H^1_0(0,1)} \int_0^1 u'^2(\tau) + \lambda^2 u^2(\tau) \, d\tau. \]

To find \( l_1 \), we need to solve a constrained extrema problem, which we can do using Lagrange Multipliers (the proposition above allows us to use a differentiable restriction). Our minimizer \( u_0 \) satisfies

\[ -u_0''(t) + \lambda^2 u_0(t) = m, \quad u_0(0) = u_0(1) = 0, \]
where \( m \) is the Lagrange Multiplier, so \( u_0(t) = mU(t) \). Since \( \int_0^1 u_0(\tau) \, d\tau = 1 \), we get \( m = \left( \int_0^1 U(\tau) \, d\tau \right)^{-1} \), and consequently

\[
l_1 = \int_0^1 u_0'(\tau)^2 + \lambda^2 u_0^2(\tau) \, d\tau = \frac{1}{\int_0^1 U(\tau) \, d\tau}.
\]

**Theorem 4.3** (Maximum Principle 1). Let \( \lambda, M \) be positive constants, \( G_\lambda \) the Green’s function associated to (13), \( U(t) = \int_0^1 G_\lambda(t, s) \, ds \), and \( V(t) \) the unique solution of \(-V''(t) + \lambda^2 V(t) = 0\), with boundary conditions \( V(0) = a \geq 0 \), \( V(1) = b \geq 0 \). Suppose that

\[
\frac{M}{1 + M \int_0^1 U(\tau) \, d\tau} \leq \inf_{0 < t, s < 1} \frac{G_\lambda(t, s)}{U(t) U(s)} \quad \text{and} \quad \frac{V(t)}{\int_0^1 V(\tau) \, d\tau} \geq \frac{M U(t)}{1 + M \int_0^1 U(\tau) \, d\tau},
\]

and

\[
M < \frac{1}{\int_0^1 U(\tau) \, d\tau}.
\]

Then, if \( u \in C[0, 1] \cap C^2[0, 1] \) satisfies

\[-u''(t) + \lambda^2 u(t) + M \int_0^1 |u(\tau)| \, d\tau \geq 0, \quad u(0) = a \geq 0, \quad u(1) = b \geq 0, \quad (22)\]

we have \( u(t) \geq 0 \).

**Proof.** Let \( f(t) = -u''(t) + \lambda^2 u(t) + M \int_0^1 |u(\tau)| \, d\tau \). By Lemma (3.2), we know that the linear problem (15) has a nonnegative solution, and therefore, this nonnegative solution has to be the only solution of (21). \( \square \)

Using *Mathematica*, we have the following estimates relative to the first pair of conditions:

- \( \lambda = 0.2 \quad \text{M}_{max} \approx 5.98 \)
- \( \lambda = 0.5 \quad \text{M}_{max} \approx 5.92 \)
- \( \lambda = 1 \quad \text{M}_{max} \approx 5.71 \)
- \( \lambda = 2 \quad \text{M}_{max} \approx 4.89 \)
- \( \lambda = 4 \quad \text{M}_{max} \approx 2.74 \)
- \( \lambda = 7 \quad \text{M}_{max} \approx 0.62 \)
- \( \lambda = 10 \quad \text{M}_{max} \approx 0.09 \)

The last condition is less restrictive, as it is shown by the following graph:

![Figure 1: \( l_1(\lambda) = \frac{1}{\int_0^1 U(\tau) \, d\tau} \)]
Using the same technique, we can reach similar results for the boundary value problem
\[-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) + M \int_0^1 \tau^k |u(\tau)| \, d\tau = f(t), \quad u'(0) = a \leq 0, \quad u(1) = b \geq 0. \quad (23)\]

Let us consider the Hilbert Space
\[H_k(0, 1) = \left\{ u \in AC[0, 1]: \int_0^1 \tau^k u^2(\tau) \, d\tau < \infty, \quad u(1) = 0 \right\},\]
with the norm \[\|u\| = \left( \int_0^1 \tau^k u^2(\tau) \, d\tau \right)^{1/2}.\] Following [1], for any \(u \in H_k(0, 1)\) with \(k > 1\), we have \(\int_0^1 \tau^k u^2 \leq C \|u\|^2\), for some \(C > 0\).

**Remark 4.4.** Note that if \(u \in H_k(0, 1)\), then \(|u| \in H_k(0, 1)\).

**Proposition 4.5.** If
\[M < \min_{\substack{u \in H_k(0, 1) \\ u \neq 0}} \frac{\int_0^1 \tau^k \left( u''^2(\tau) + \lambda^2 u^2(\tau) \right) \, d\tau}{\left( \int_0^1 \int_0^1 \tau^{2k} |u(\tau)| \, d\tau \right)^2},\]
then the problem (23) has a unique solution.

**Proof.** As stated before we can write equation (23) in the form
\[-\left( t^k u'(t) \right)' + \lambda^2 t^k u(t) + M t^k \int_0^1 \tau^k |u(\tau)| \, d\tau = t^k f(t)\]
We shall consider two cases:

(i) If \(f(t) = 0, a, b = 0\), multiplying the equation in (23) by \(u\) and integrating by parts, we have
\[\int_0^1 \tau^k \left( u''^2(\tau) + \lambda^2 u^2(\tau) \right) \, d\tau = -M \int_0^1 \tau^k |u(\tau)| \, d\tau \int_0^1 \tau^k u(\tau) \, d\tau \leq M \left( \int_0^1 \tau^k |u(\tau)| \, d\tau \right)^2,\]
and the conclusion follows.

(ii) If \(f(t) \neq 0\), let \(u_1, u_2\) be such that
\[-u''_i(t) - \frac{k}{t} u'_i(t) + \lambda^2 u_i(t) + M \int_0^1 \tau^k |u_i(\tau)| \, d\tau = f(t), \quad u'_i(0) = a, \quad u_i(1) = b,\]
Setting \(w = u_1 - u_2\), we have
\[-\left( t^k w'(t) \right)' + \lambda^2 t^k w(t) + M t^k \int_0^1 |\theta(\tau)| \tau^k |w(\tau)| \, d\tau = 0, \quad w'(0) = w(1) = 0,\]
where \(\theta(\tau) = \frac{|u_1(\tau)| - |u_2(\tau)|}{u_1(\tau) - u_2(\tau)}.\) Since \(\theta(\tau) \leq 1\), using an argument similar to the one in (i), we get \(w(t) = 0\), and therefore there is a unique solution to (21).
Proposition 4.6. We have
\[
\min_{u \in H_k(0,1), u \neq 0} \int_0^1 \tau^k \left( u'^2 + \lambda^2 u^2 \right) d\tau = \min_{u \in H_k(0,1), u \neq 0} \int_0^1 \tau^k \left( u'^2 + \lambda^2 u^2 \right) d\tau.
\]

Proof. If a function \( u_0 \) minimizes the left-hand side, then, since \(|u_0| \in H_k(0,1)\), the right-hand side has the same value.

Let
\[
l_2 = \min_{u \in H_k(0,1), u \neq 0} \int_0^1 \tau^k \left( u'^2 + \lambda^2 u^2 \right) d\tau = \min_{u \in H_k(0,1), u \neq 0} \int_0^1 \tau^k \left( u'^2 + \lambda^2 u^2 \right) d\tau.
\]

So, to find \( l_2 \), we need to solve another constrained extrema problem. Our minimizer \( u_0 \) satisfies
\[-u_0''(t) - \frac{k}{t} u_0'(t) + \lambda^2 u_0(t) = m, \quad u_0'(0) = u_0(1) = 0,
\]
where \( m \) is the Lagrange Multiplier, so \( u_0(t) = m U(t) \). Since \( \int_0^1 \tau^k u_0(t) d\tau = 1 \), we get
\[m = \left( \int_0^1 \tau^k U(t) d\tau \right)^{-1} \]
and consequently
\[l_2 = \int_0^1 \tau^k \left( u_0'^2 + \lambda^2 u_0^2 \right) d\tau = \frac{1}{\int_0^1 \tau^k U(t) d\tau}.
\]

Theorem 4.7 (Maximum Principle 2). Let \( \lambda, M \) be positive constants, \( H_\lambda \) the Green’s function associated to (17), and \( U = \int_0^1 H_\lambda(t,s) ds \). Suppose that
\[
\frac{M}{1 + M \int_0^1 \tau^k U(t) d\tau} \leq \inf_{0 < t < 1} \frac{H_\lambda(t,s)}{U(t) U(s)} \leq \frac{M U(t)}{1 + M \int_0^1 U(t) d\tau} \leq \frac{u_1(t)}{\int_0^1 \tau^k u_1(t) d\tau},
\]
and
\[M < \frac{1}{\int_0^1 \tau^k U(t) d\tau}.
\]
Then, if for \( 0 < t \leq 1, u \in C^2[0,1] \) satisfies
\[-u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) + M \int_0^1 \tau^k |u(t)| ds \geq 0, \quad u'(0) = a \leq 0, u(1) = b \geq 0,
\]
we have \( u(t) \geq 0 \).

Proof. Let \( f(t) = -u''(t) - \frac{k}{t} u'(t) + \lambda^2 u(t) + M \int_0^1 \tau^k |u(t)| ds \). By Lemma (3.6), we know that the linear problem (19) has a nonnegative solution, and therefore, this nonnegative solution has to be the only solution of (23).

We have the following estimates relative to the cases \( k = 1, 2, 3 \):

(i) \( k = 1 \):

\[
\begin{align*}
\lambda = 0.25 & \quad M_{max} \approx 15.95 \\
\lambda = 1 & \quad M_{max} \approx 15.30 \\
\lambda = 5 & \quad M_{max} \approx 5.71
\end{align*}
\]
(ii) $k=2$:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$M_{\text{max}} \approx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>29.9</td>
</tr>
<tr>
<td>1</td>
<td>28.9</td>
</tr>
<tr>
<td>5</td>
<td>12.2</td>
</tr>
</tbody>
</table>

(iii) $k=3$:

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$M_{\text{max}} \approx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.25</td>
<td>47.9</td>
</tr>
<tr>
<td>0.5</td>
<td>47.5</td>
</tr>
<tr>
<td>1</td>
<td>46.5</td>
</tr>
<tr>
<td>3</td>
<td>36.0</td>
</tr>
<tr>
<td>5</td>
<td>21.5</td>
</tr>
<tr>
<td>10</td>
<td>2.2</td>
</tr>
</tbody>
</table>

The last condition is also less restrictive. We present here the graph of $l_2(\lambda)$ in the case $k=3$:

![Graph](image-url)

Figure 2: $l_2(\lambda) = \frac{1}{\int_0^1 r^3 U(r) \, dr}$

5 Lower and Upper solutions and the monotone method

Consider the boundary value problem

$$-u''(t) - \frac{n-1}{t} u'(t) = f \left( u(t), \omega_n \int_0^1 s^{n-1} g(u(s)) \, ds \right) \text{ for } 0 < t \leq 1,$$

and

$$u'(0) = 0 = u(1),$$

where $n \in \mathbb{N}$, $f$, $g$ are continuous functions, and $\omega_n$ is the superficial measure of the unit sphere in $\mathbb{R}^n$. The solutions of this problem are radial solutions of

$$-\Delta u = f \left( u, \int_U g(u) \right)$$

$$u|_{\partial U} = 0,$$

where $U = B(0,1)$ is the unit sphere in $\mathbb{R}^n$ (see [4]).
We say that $\alpha(t)$ is a lower solution of (25)--(26) if
\[ -\alpha''(t) - \frac{n-1}{t} \alpha'(t) \leq f(\alpha(t), \omega_n \int_0^1 s^{n-1} g(\alpha(s)) \, ds), \quad \text{for } 0 < t \leq 1, \]
\[ \alpha'(0) \geq 0 \text{ and } \alpha(1) \leq 0. \]

A function $\beta$ satisfying the reversed inequalities is called an upper solution.

For a given function $u(t) \in C[0,1]$, consider the boundary value problem
\[ -u''(t) - \frac{n-1}{t} u'(t) + \lambda^2 u(t) = f(u(t), \omega_n \int_0^1 s^{n-1} g(v(s)) \, ds) + \lambda^2 u(t), \]
with $v'(0) = 0 = v(1)$. Using the operator $L u = -u'' - \frac{n-1}{t} u' + \lambda^2 u$, in the space $C^+ = \{ u \in C^2[0,1]: u'(0) = u(1) = 0 \}$ this equation is equivalent to the fixed point equation
\[ v = L^{-1} \left( f(u, \omega_n \int_0^1 s^{n-1} g(v(s)) \, ds) + \lambda^2 u \right) \equiv \Phi_u v. \quad (29) \]

It turns out that it is advantageous to look at $\Phi_u$ as an operator from $L^2_{n-1}(0,1)$ into itself. Noticing that $L^{-1}$ is a compact self-adjoint operator in this space with norm $\|L^{-1}\| = (\xi_n^2 + \lambda^2)^{-1}$ where $\xi_n$ is the first positive zero of the Bessel function $J_{\frac{n}{2}}$, it is easy to see that if $f(u,v)$ is $k_1$-Lipschitz in $v$, $g$ is $k_2$-Lipschitz, then $\Phi_u$ is Lipschitz with constant $\frac{\omega_n k_1 k_2}{(\xi_n^2 + \lambda^2)n}$. In particular, when the condition
\[ \frac{\omega_n k_1 k_2}{(\xi_n^2 + \lambda^2)n} < 1 \quad (30) \]
is satisfied, $\Phi_u$ is a contraction mapping, and therefore has a unique fixed point.

Using maximum principle 4.7, we get the following improved version of Theorem 4.10 in [2]:

**Theorem 5.1.** Suppose that $f(u,v)$ is $k_1$-Lipschitz in $v$, $g$ is $k_2$-Lipschitz. Suppose that $M \equiv k_1 k_2 \omega_n$ and $\lambda$ are in the conditions of the Maximum Principle 4.7, (30) holds and
\[ f(u_2,v) - f(u_1,v) \geq -\lambda^2(u_2 - u_1), \]
for all $v \in \mathbb{R}$, and $u_1 \leq u_2$. Let $\alpha_0$ and $\beta_0$ be a lower and an upper solution of (25)--(26) respectively, with $\alpha_0 \leq \beta_0$ in $[0,1]$. If we take $(\alpha_n)_{n \in \mathbb{N}_0}$ and $(\beta_n)_{n \in \mathbb{N}_0}$ such that,
\[ \alpha_{n+1} = \Phi_{\alpha_n} \alpha_{n+1} \quad \text{and} \quad \beta_{n+1} = \Phi_{\beta_n} \beta_{n+1}, \quad \text{for all } n \in \mathbb{N}_0, \]
we obtain
\[ \alpha_0 \leq \alpha_1 \leq \cdots \leq \alpha_n \leq \cdots \leq \beta_n \leq \cdots \leq \beta_1 \leq \beta_0. \]
The monotone bounded sequences $(\alpha_n)_{n \in \mathbb{N}_0}, (\beta_n)_{n \in \mathbb{N}_0}$ defined above are convergent in $C[0,1]$ to solutions of (25)--(26).

The main step of the proof consists in verifying that indeed a monotone sequence is obtained. This is a consequence of the following fact: Let $u_1(r) \leq u_2(r)$ be two given functions defined in $[0,1]$ and $v_1(r), v_2(r)$ the two respective solutions of (29). Then $v_1(r) \leq v_2(r)$.
Let us recall the argument (see [2]):

\[-(v_2 - v_1)'' - \frac{n-1}{r}(v_2 - v_1)' + \lambda^2(v_2 - v_1) =
\]

\[= \lambda^2(u_2 - u_1) + f \left( u_2, \omega_n \int_0^1 s^{n-1} g(v_2) \, ds \right) - f \left( u_1, \omega_n \int_0^1 s^{n-1} g(v_2) \, ds \right) +
\]

\[+ f \left( u_1, \omega_n \int_0^1 s^{n-1} g(v_1) \, ds \right) - f \left( u_1, \omega_n \int_0^1 s^{n-1} g(v_1) \, ds \right) \geq
\]

\[\geq -k_1 k_2 \omega_n \int_0^1 s^{n-1} |v_2 - v_1| \, ds.
\]

It suffices then to invoke the maximum principle 4.7 to obtain the conclusion.

Example 5.2. Let us consider the non-local differential equation

\[-u''(t) - \frac{2}{t} u'(t) = f \left( u, 4\pi \int_0^1 s^2 \left( \frac{u(s)^2 + 1}{3} \right) \, ds \right) \quad (31)
\]

where

\[f(u, v) = \begin{cases} \left( \sqrt{u} + 1 \right) \left( \sin v + 1 \right) + 4, & u \leq 1 \\ \left( \frac{1}{u} + 1 \right) \left( \sin v + 1 \right) + 4, & u \geq 1, \end{cases}
\]

with boundary conditions \( u'(0) = u(1) = 0 \).

Consider \( \alpha_0 = 1 - t^2 \) and \( \beta_0 = \frac{4}{3} \left( 1 - t^2 \right) \). After some computation, we can verify that \( \alpha_0, \beta_0 \) are respectively a lower and an upper solution of (31), both satisfying the considered boundary conditions. Since \( 0 \leq \alpha_0(t) \leq \beta_0(t) \leq \frac{4}{3} \), for all \( t \in [0, 1] \), we can consider \( k_1 = 2, k_2 = \frac{8}{3} \), and \( \lambda = 1 \). Moreover \( \xi_3 = \pi \). Setting \( M = \frac{64\pi}{9} \), the conditions of theorem 5.1 are satisfied, and therefore, using the described iterative method, we can approximate a solution \( u(t) \) of (31) satisfying \( u'(0) = u(1) = 0 \) and \( 1 - t^2 \leq u(t) \leq \frac{4}{3} \left( 1 - t^2 \right) \).

References


