Existence and L_∞ estimates of some mountain-pass type solutions

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Abstract

We prove the existence of a positive solution to an equation of the form $(\Phi(t)u'(t))' = f(t,u(t))$ with mixed Neuman and Dirichlet conditions. Our method combines variational and topological arguments providing an L_∞ estimate of the solution. Our results can be applied to certain type of elliptic problems in annular domains.

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1 Introduction

Early since its publication in 1973, the Mountain Pass Theorem of Ambrosetti and Rabinowitz (see [10]) has provided existence and multiplicity results in Differential Equations as well as a comprehensive perspective of variational methods. The characterization of Mountain Pass type solutions became itself a subject of interest. As examples one may cite works of del Pino and Felmer (see [5] and references therein) where the shape of the solutions to the Dirichlet problem

$$\epsilon^2 \Delta u - u + f(u) = 0 \quad \text{in } \Omega\,; \quad u > 0 \ \text{ in } \Omega \ \text{ and } u = 0 \ \text{ in } \Omega\,,$$

is established as ϵ tends to zero. In the same spirit, Bonheure, Habets and the author ([3]) have showed that for a superlinear elliptic problem with sign-changing non-linearity the major contribution of volume of mountain pass type solutions should concentrate in prescribed regions of the domain as a certain parameter $\mu \to \infty$. In the above examples the role played by a parameter as it approaches some limit is crucial. In [6] the author established existence and L_∞ estimates of positive Mountain Pass type solutions to a class of singular differential equations with an increasing friction term and Dirichlet boundary conditions. The bound is just the L_∞ norm of any regular function where the Euler-Lagrange functional J attains a negative value. Our method combined arguments in the Direct Calculus of Variations with phase plane techniques. In fact, pursuing the nature of the optimal min-max path connecting the origin to some function where J is negative, we were lead to consider a family of minimizers of truncated functionals containing, as a particular element, a classical solution to our b.v.p. In this work we approach with similar arguments a more general class of equations that include some elliptic problems in an annulus. More precisely we will be interested in positive solutions to

$$(\Phi(t)u'(t))' + f(t,u(t)) = 0 \tag{1}$$

$$u'(0) = u(1) = 0 \tag{2}$$

By positive solution we mean a C^2 function u verifying the above equalities and such that $u(t) > 0$ for all $t \in [0,1[$. Similar problems have been considered in [2], [8] and [9].

2 Variational setting and results

We begin by listing the assumptions on the terms of equation (1)-(2).

$\Phi \in C^1([0,1])$ is strictly positive and we choose $m, \overline{m} > 0$ such that, for all $t \in]0,1[$,

$$0 < m \leq \Phi(t) \leq \overline{m}\,. \tag{3}$$

We assume that

$$f(t,u)\Phi(t) \text{ is decreasing in } t \text{ for every } u \geq 0\,. \tag{4}$$

Also

$$f : [0,1] \times [0,+\infty[\mapsto \mathbb{R} \quad \text{is locally lipschitz in the variable } u\,, \tag{5}$$

and that, for some $\delta > 0$, $f(t,u)$ verifies

$$f(t,u) = 0 \ \ \forall (t,u) \in [0,1] \times [0,\delta] \text{ and } f(t,u) > 0 \ \text{ in } \ [0,1]\times]\delta,+\infty[\,. \tag{6}$$

The technical assumption (6) will be relaxed subsequently to a sub-linear growth near zero. Since we are looking for positive solutions we assume throughout the paper that f is extended by zero in $[0,1]\times]-\infty,0]$. The reader may easily verify that any non-trivial solution to (1)-(2) with this extension -which we will still denote by f- should be positive in $]0,1[$ therefore being a solution of the initial problem. We shall consider the Sobolev space $H \subset H^1_0(]0,1[)$ consisting in absolutely continuous functions u such that

$$\|u\|^2 := \int_0^1 {u'}^2(t)dt \ < \infty \ , \ u(1) = 0.$$

In the sequence we will also refer

$$\|u\|_\infty := \sup\{u(t) \ : \ t \in [0,1]\}\,,$$

the natural norm on the space of continuous functions $C([0,1])$. Note that Problem (1)-(2) may be viewed as the Euler-Lagrange equation of the functional $J : H \to \mathbb{R}$ defined by:

$$J(u) = \frac{1}{2}\int_0^1 \Phi(t){u'}^2(t)dt - \int_0^1 F(t,u(t))dt$$

where $F(t,u) = \int_0^u f(t,s)ds$. We will suppose that J satisfies the fundamental property:

$$\exists\, h \in H \ : J(h) < 0. \tag{7}$$

Remark 1 *Property (7) can be easily verified if, for some $\epsilon > 0$, $f(t,u) \geq \epsilon u^\alpha - C$ for all $u \geq 0$ and $t \in [0,1]$, where $\alpha > 1$ and $C > 0$. Note that we make no assumptions on the growth of f as $u \to \infty$ or require J to verify Palais-Smale condition.*

We denote $\overline{M} = \|h\|_\infty$. Since

$$\forall w \in H \ , \ \|w\|_\infty \leq \delta \ \Rightarrow \ J(w) \geq 0,$$

(where δ was defined in (6)) we have $\overline{M} > \delta$. For all $M \in \left[\delta, \overline{M}\right]$, we consider the following subset of H:

$$\mathfrak{C}_M = \{u \in H : \max u \geq M\} .$$

We also consider the truncated functional $J_M : H \to \mathbb{R}$,

$$J_M(u) = \frac{1}{2}\int_0^1 \Phi(t)u'^2(t)dt - \int_0^1 F_M(t, u(t))dt$$

where

$$F_M(u) = \begin{cases} F(t,u) \text{ if } u \leq M \\ \\ F(t,M) \text{ if } u > M \end{cases} .$$

Remark 2 *From the compact injection of $H_0^1(]0,1[)$ in $C([0,1])$ we conclude that $\mathfrak{C}_M$ is weakly sequentially closed and that J_M is coercive and weakly lower semi-continuous.*

We will be interested in the family of minimizers u_M of J_M in $\mathfrak{C}_M$. By Remark 2 we know that u_M exists for every $M \in \left[\delta, \overline{M}\right]$. We also know that:

Lemma 1 *Let u_M be a minimizer of J_M in $\mathfrak{C}_M$. Then*

$$\max_{[0,1]} u = M \quad \textit{and} \quad \min_{[0,1]} u = 0 \ .$$

.

Proof. Given $w \in \mathfrak{C}_M$ define

$$\overline{w}(t) = \max\{0, \min\{w(t), M\}\}.$$

Of course $\overline{w} \in H \cap \mathfrak{C}_M$. If $\overline{w} \neq w$ then,

$$\int_0^1 \Phi(t)\overline{w}'^2(t)\,dt < \int_0^1 \Phi(t)w'^2(t)\,dt$$

and

$$\int_0^1 F_M(t, \overline{w}(t))\,dt = \int_0^1 F_M(t, w(t))\,dt \ .$$

Then $J_M(\overline{w}) < J_M(w)$ which is absurd and the lemma follows. ■

Given $M \in \left[\delta, \overline{M}\right]$, we consider two types of minimizers of J_M in $\mathfrak{C}_M$:

Definition. Let u_M be a minimizer of J_M in $\mathfrak{C}_M$.

We say that u_M is a minimizer of type A if

$$u(0) = M\ , \quad u'_+(0) \leq 0 \text{ and } u(t) < M \text{ for all } t > 0.$$

We say that u_M is a minimizer of type B if, for some $\bar{t} \geq 0$, we have

$$u(t) = M \ \forall t \in [0, \bar{t}\,] \ , u'(\bar{t}) = 0 \text{ and } u(t) < M \text{ if } t > \bar{t}.$$

Remark 3 *If u_M is a minimizer of type A then u satisfies equation (1) in $]0,1[$. In fact, if $v \in C_0^1(]0,1[)$, then, for sufficiently small s, we have,*

$$u_M + sv \in \mathfrak{C}_M \text{ and } u_M(t) + sv(t) < M \ \forall t \in supp(v).$$

Since u_M is a minimizer, we conclude

$$\lim_{s\to 0} \frac{J_M(u_M + sv) - J_M(u_M)}{s} = \int_0^1 \Phi(t) u_M' v'(t) dt - \int_0^1 f(t, u_M(t)) v(t)\, dt = 0 \ ,$$

and the assertion follows. Similarly, if u_M is a minimizer of type B, it satisfies equation (1) in $]\bar{t}, 1[$. If u_M is simultaneously of type A and B, then u_M is a classical solution to problem (1)-(2).

Lemma 2 *Let u be a minimizer of J_M in $\mathfrak{C}_M$. Then u is of type A or B (possibly both).*

Proof. Let us consider

$$\bar{t} := \sup\{t \in [0,1] \, : \, u(t) = M\} \, .$$

Since $H \subset C([0,1])$, we have $u(\bar{t}) = M$ and we may therefore consider $w \in H$

$$w := \begin{cases} M & \text{if } t \le \bar{t} \\ u(t) & \text{if } t > \bar{t} \end{cases} .$$

Moreover,

$$\int_0^1 F_M(t, w(t))\, dt \ge \int_0^1 F_M(t, u(t))\, dt \ ,$$

and

$$\int_0^1 \Phi(t) {w'}^2(t)\, dt \le \int_0^1 \Phi(t) {u'}^2(t)\, dt \ ,$$

the last inequality being strict if $w \ne u$ in $[0, \bar{t}]$. Since u is a minimum of J_M, we conclude $u \equiv w$. Note that by Lemma 1 and Remark 3, $u'_+(\bar{t})$ is well defined and non-positive. If $\bar{t} = 0$ then u is of type A. Suppose in view of a contradiction that $\bar{t} > 0$ and $u'_+(\bar{t}) < 0$. Choose $\theta, \epsilon > 0$ such that $u'(t) \le -\theta$ for every $t \in]\bar{t}, \bar{t} + \epsilon[$ and, assuming $\epsilon < \bar{t}/2$, define the "triangular" perturbation

$$v_\epsilon(t) = -(|t - \bar{t}| - \epsilon)_- \quad . \tag{8}$$

We assert that, for a small ϵ,

$$\lim_{s\to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} < 0. \tag{9}$$

If (9) holds, then, for sufficiently small $s^* > 0$, we have $u + s^* v_\epsilon \in \mathfrak{C}_M$ (since, by our choice of ϵ, $(u + s^* v_\epsilon)(0) = M$) and $J_M(u + s^* v_\epsilon) < J_M(u)$ thereby contradicting the assumption that u is a minimizer of J_M in $\mathfrak{C}_M$. In fact, Lemma 1 and (8) imply $u + s^* v_\epsilon \le M$. Therefore

$$\lim_{s\to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} = \lim_{s\to 0} \frac{J(u + sv_\epsilon) - J(u)}{s} =$$

$$= \int_0^1 \Phi(t)u'(t)v'_\epsilon(t)dt - \int_0^1 f(t,u(t))v_\epsilon(t)dt \leq$$
$$-\theta \int_{\bar t}^{\bar t+\epsilon} \Phi(t)\,dt - \int_{\bar t-\epsilon}^{\bar t+\epsilon} f(t,u(t))v_\epsilon(t)dt.$$

We observe that, by (3),

$$-\theta \int_{\bar t}^{\bar t+\epsilon} \Phi(t)\,dt \leq -m\theta\epsilon \tag{10}$$

and for some $C > 0$ depending only on f,

$$\int_{\bar t-\epsilon}^{\bar t+\epsilon} f(t,u(t))v_\epsilon(t)dt \leq C\epsilon^2. \tag{11}$$

Therefore, by (10) and (11), we have

$$\lim_{s\to 0} \frac{J_M(u+sv_\epsilon) - J_M(u)}{s} \leq -m\theta\epsilon + C\epsilon^2\,,$$

and the assertion follows for sufficiently small ϵ. ■

In the next lemma we prove a necessary ordering relation between type A and type B minimizers of J_M in $\mathfrak{C}_M$.

Lemma 3 *Suppose that for a certain $M \in \left]0, \overline{M}\right]$ there exist minimizers u and v of J_M in $\mathfrak{C}_M$ such that u is of type A and v is of type B. Then $u(t) < v(t)$ for all $t \in]0,1[$ or else $u \equiv v$.*

Proof. Assume $u \neq v$. Necessarily, we will have $u(t) < v(t)$ for all $t \in]0,\epsilon[$ provided ϵ is sufficiently small. Suppose that for some $t^* \in]0,1[$ we had

$$u(t^*) = v(t^*) \quad \text{and} \quad u'(t^*) > v'(t^*)\,,$$

(the case $u'(t^*) = v'(t^*)$ is excluded by (5) together with Existence and Uniqueness Theorem). Moreover, suppose that

$$\frac{1}{2}\int_{t^*}^1 \Phi u'^2 - \int_{t^*}^1 \Phi F_M(t,u) \leq \frac{1}{2}\int_{t^*}^1 \Phi v'^2 dt - \int_{t^*}^1 \Phi F_M(t,v), \tag{12}$$

and let

$$v^*(t) = \begin{cases} v(t) \text{ if } & 0 \leq t \leq t^* \\ u(t) \text{ if } & t^* < t \leq 1 \end{cases}.$$

Then $v^* \in H$ and

$$J_M(v^*) \leq J_M(v),$$

therefore v^* is also a minimizer in $\mathfrak{C}_M$. But this is absurd since v^* is not differentiable at t^* (see remark 3). In case where, instead of (12), we had the reversed inequality we would get the same contradiction by considering:

$$u^*(t) = \begin{cases} u(t) \text{ if } & 0 \leq t \leq t^* \\ v(t) \text{ if } & t^* < t \leq 1 \end{cases}.$$

■

In the next lemma we establish an important fact concerning the coexistence of type A and type B minimizers at a same truncating level.

Lemma 4 *Assume that conditions (4), (3) and (5) hold. Suppose that for a certain $M \in \left]0, \overline{M}\right]$ there exist minimizers u and v of J_M in $\mathfrak{C}_M$ such that u is of type A and v is of type B. Then the minimizer u is a classical solution to (1)-(2).*

Proof. We define an inverse function for $u(t)$. By Remark 3, we may write

$$u'(t) = \frac{1}{\phi(t)} \left(\phi(0) u'_+(0) - \int_0^t f(s, u(s))\, ds \right) .$$

Note that, if $u(0) \le \delta$, necessarily $u'_+(0) < 0$. In fact by Lemma 1 we have $u'(0) \le 0$. In case $u'(0) = 0$ condition (6) and Remark 3 imply $u \equiv \delta$ contradicting the assumption that $u \in H$. If $u(0) > \delta$ then our assumptions on f imply $u'(t) < 0$ for all $t \in]0, 1]$. In both cases we conclude that u is strictly decreasing in $[0, 1]$ and we may define the inverse function

$$[0, M] \to [0, 1] \ ,$$

$$u \mapsto t_A(u) \ .$$

Using similar arguments we may define an inverse function for v

$$[0, M[\to]\bar{t}, 1] \ ,$$

$$v \mapsto t_B(v) \ .$$

We consider the continuous extension of this function to $[0, M]$ (that we still denote by t_B) verifying $t_B(M) = \bar{t}$. By Lemma 3 we have

$$t_A(u) \le t_B(u) \quad \forall u \in [0, M] \ .$$

We consider the functions [1]

$$Z_A(u) = \Phi(t_A(u)) u'(t_A(u)) \ ,$$

and

$$Z_B(v) = \Phi(t_B(v)) v'(t_B(v)) \ .$$

Since the functions $u(t)$ and $v(t)$ verify (1)-(2) for $t \ge 0$ and $t \ge \bar{t}$ respectively, we write, for $u, v \in]0, M[$

$$-\frac{d\, Z_A}{du} \frac{du}{dt} = f(t_A(u), u) \quad \text{and} \quad -\frac{d\, Z_B}{dv} \frac{dv}{dt} = f(t_B(v), v)$$

or

$$\frac{d\, Z_A}{du} = -\frac{\Phi(t_A(u))}{Z_A} f(t_A(u), u) \quad \text{and} \quad \frac{d\, Z_B}{dv} = -\frac{\Phi(t_B(v))}{Z_B} f(t_B(v), v) \ . \qquad (13)$$

Suppose in view of a contradiction that $u(t)$ is not a type B minimizer, i.e.

$$Z_A(M) < 0 = Z_B(M) \, . \qquad (14)$$

[1] This change of variables was suggested by the reading of [7].

In fact, this assumption implies that $Z_A(u) < Z_B(u)$ for all $u \in [0, M]$. We start by noting that if $Z_A(0) = Z_B(0)$, then $u'(1) = v'(1)$. Since

$$u(1) = v(1) = 0\ ,$$

Existence Uniqueness Theorem implies $u(t) = v(t)$ for all $t \in [\bar{t}, 1]$ thereby contradicting (14). Admit that for some $u^* \in]0, M[$ we had

$$Z_A(u^*) = Z_B(u^*)\,.$$

We assume that u^* is the maximum point satisfying the previous equality. Then

$$\frac{d\,Z_A}{du}(u^*) \leq \frac{d\,Z_B}{du}(u^*)\,.$$

If there is equality of the derivatives then Existence Uniqueness Theorem implies $Z_A(0) = Z_B(0)$ which, as previously noticed, is absurd. Since $Z_A(u^*) = Z_B(u^*) < 0$, (4) implies that the right hand-sides of the equalities in (13) are decreasing functions of t. Recalling that $t_B(u^*) \geq t_A(u^*)$ we conclude

$$\frac{d\,Z_A}{du}(u^*) > \frac{d\,Z_B}{du}(u^*)$$

contradicting the maximal property of our choice of u^*. In particular we have proved that if u is not a type B minimizer then $Z_A(0) < Z_B(0)$.

Finally we conclude $u'(1) < v'(1) < 0$ which in turn implies that $u(t^*) > v(t^*)$ for some $t^* < 1$, a contradiction with Lemma 3 and the proof is complete.■

We are now in a position to prove

Proposition 5 *Assume that conditions (3), (4), (5), (6) and (7) hold. Then there exists a positive solution u to (1)-(2) such that*

$$\|u\|_\infty \leq \|h\|_\infty$$

where h was defined in (7).

Proof.

Recalling our notation $\overline{M} = \|h\|_\infty$, let $I = \left[\delta, \overline{M}\right]$ and consider the following subsets I_A and I_B:

$$I_A\,(I_B) = \{M \in \left[\delta, \overline{M}\right] : J_M \text{ has a minimizer in } \mathfrak{C}_M \text{ of type A (B)}\}.$$

By Lemma 2 we have $I = I_A \cup I_B$. We assert that I_A and I_B are non-empty. In fact $\delta \in I_A$ since, as previously noticed, if u_δ is a minimizer of J_δ in $\mathfrak{C}_\delta$ and $u'_\delta(0) = 0$ the Existence and Uniqueness Theorem implies $u_\delta(t) = \delta$ for all $t \in [0, 1]$, which is absurd.

Claim 1: I_B is non-empty.

Suppose that $\overline{M} \notin I_B$. In this case $u_{\overline{M}}$ is a type A minimizer. Let

$$\bar{f}(t, x) := f(t, \min\{x, u_{\overline{M}}(t)\})\,.$$

Define, for $u \in H$

$$\bar{J}(u) := \frac{1}{2}\int_0^1 \phi(t){u'}^2(t)\,dt - \int_0^1 \overline{F}(t,u(t))\,dt\,,$$

where $\overline{F}(t,x) = \int_0^x \bar{f}(t,s)ds$. Trivially, By (7) we have $\bar{J}(u_{\overline{M}}) < 0$. Also $\bar{J}$ is coercive and lower semi-continuous in H and therefore attains a minimum at some function $w \in H$ such that $\bar{J}(w) < 0$. In fact

$$0 < w(t) < \overline{u}_M(t) \quad \forall t \in [0,1[\,,$$

(0 and $\overline{u}_M$ are a pair of well ordered lower and upper solutions respectively) and w is a classical solution to (1)-(2)(see for instance [[4], ch. 4] for details).

Claim 2: I_A and I_B are closed subsets of I.

Let (M_n) be a sequence in I_A such that $M_n \to M$. Let u_n be a corresponding sequence of type A (B) minimizers of J_{M_n} in $\mathfrak{C}_{M_n}$. Since (u_n) is trivially bounded we may extract a weakly convergent subsequence (still denoted by u_n) such that

$$u_n \rightharpoonup u \text{ in } H \quad \text{and} \quad u_n \to u \text{ in } C([0,1])\ .$$

We assert that u is a minimizer of J_M in $\mathfrak{C}_M$. In fact, since,

$$\lim_{n\to\infty}\int_0^1 \Phi(t)F_{M_n}(t,u_n(t))\,dt = \int_0^1 \Phi(t)F_M(t,u(t))\,dt$$

and

$$\int_0^1 \Phi(t){u'}^2(t)\,dt \le \liminf_{n\to\infty}\int_0^1 \Phi(t){u_n'}^2(t)\,dt\,,$$

we conclude

$$J_M(u) \le \liminf J_{M_n}(u_n).$$

However, if we set $w_n = (M_n/M)u$, we have $w_n \to u$ in H and $w_n \in \mathfrak{C}_{M_n}$, for all $n \in \mathbb{N}$. Therefore

$$J_M(u) = \lim_{n\to\infty} J_{M_n}(w_n)$$

and

$$J_{M_n}(w_n) \ge J_{M_n}(u_n),$$

for all $n \in \mathbb{N}$. We conclude

$$J_M(u) \ge \limsup_{n\to\infty} J_{M_n}(u_n) \ge \liminf_{n\to\infty} J_{M_n}(u_n) \ge J_M(u),$$

or,

$$\lim_{n\to\infty} J_{M_n}(u_n) = J_M(u).$$

If, for some u^* in $\mathfrak{C}_M$, we had $J_M(u^*) < J_M(u)$ then for sufficiently large n, we would have

$$J_{M_n}(w_n^*) < J_{M_n}(u_n),$$

where $w_n^* = (M_n/M)u^*$, which is absurd. Note that so far we have just used the fact that u_n is a sequence of minimizers. It remains to prove that the limit function u is of type A (B). If (u_n) is a type A sequence, since $u_n \to u$ in L_∞ and (1) is verified for all u_n in $]0,1[$ implies that u satisfies (1) in $]0,1[$. This trivially implies that u itself is of type A. In case of a type B sequence, then the same L_∞-convergence insures that u must be constant in some interval $[0, \bar{t}]$ where $\bar{t}$ is a limit point of the sequence (t_n) where

$$t_n = \max\{t : u_n(t) \text{ is constant in} [0, t]\}\,,$$

and satisfies (1) in $]\bar{t}, 1[$. Also, since the u_n's are of type B and, for small $\epsilon > 0$,

$$u_n \underset{C^1}{\longrightarrow} u \;\text{ in }\; [\bar{t} + \epsilon, 1]$$

we have $u'(\bar{t}) = 0$ and the claim is proved.

We may therefore conclude, since I is connected, that $I_A \cap I_B \neq \varnothing$. By Lemma 4 it implies the existence of a classical solution u such that $\max u \in I_A \cap I_B$. ∎

Remark 4 *Note that instead of* (2) *we may conside the more general boundary conditions*

$$u'(a) = u(b) = 0 \quad (a < b)\,.$$

In the next result we relax condition (6) using an approximating standard technique.

Theorem 6 *Suppose that $f(t, u)$ is locally Lipschitz in the variable u, verifies (4) and*

$$0 \leq f(t, u) \leq Cu^p \quad \text{for } (t, u) \in [0, 1] \times [0, \rho] \tag{15}$$

for some $p > 1$ and $\rho > 0$. Also assume (3) and that condition (7) is fulfilled by some non-negative $h \in H$. Then there exists a positive solution u to (1)-(2) such that $\max u \leq \|h\|_\infty$.

Proof.

We may suppose that f is bounded above by $\|h\|_\infty$. Consider the following translation of the nonlinearity:

$$f_\delta(t, u) = f(t, (u - \delta)_+)\;.$$

Observe that, since $(u - \delta)_+$ is an increasing function of u, assumption (4) is verified by f_δ for all $\delta > 0$ as well as (15) for the same constant C. Also (7) is fulfilled by the same function h for all the functionals

$$J_\delta(u) := \frac{1}{2} \int_0^1 \Phi {u'}^2 - \int_0^1 F_\delta(t, u)\, dt,$$

where $F_\delta(t, u) = \int_0^u f_\delta(t, s) ds$, provided δ is small . We may therefore apply Proposition 5 and conclude the existence of a solution u_δ to the problem

$$(\Phi(t) {u_\delta}')' = f_\delta(t, u_\delta)\;,\; u_\delta'(0) = u_\delta(1) = 0\,. \tag{16}$$

Since u_δ is a critical point of J, H is continuously injected in $L^{p+1}(0, 1)$ with $p > 1$, we have, by (15) and classical estimates, for some K_1, K independent of δ,

$$m\|u_\delta\|^2 \le \int_0^1 \Phi(t)u_\delta'(t)^2\,dt = \int_0^1 f_\delta(t,u_\delta(t))u_\delta(t)\,dt \le K_1\int_0^1 |u|^{p+1} \le K\|u_\delta\|^{p+1} .$$

We conclude, for $k^* = (m/K)^{\frac{1}{p-1}}$,

$$\|u_\delta\| \ge k^* > 0 ,$$

for all sufficiently small δ. Consider a sequence $\delta_n \to 0$ and the corresponding sequence u_n of solutions to (16). Noting that the sequence ($\|u_n\|$) is trivially bounded by the variational characterization of the u_n's, we may consider $u \in H$ and a subsequence (still denoted by (u_n)) such that

$$u_n \rightharpoonup u \text{ in } H \quad \text{and} \quad u_n \to u \text{ in } C([0,1]) .$$

We may conclude

$$\int_0^1 f(t,u)u\,dt = \lim_{\delta_n\to 0}\int_0^1 f_\delta(t,u_n)u_n\,dt = \lim_{\delta_n\to 0}\int_0^1 \Phi(t)u_n'(t)^2\,dt \ge mk^* ,$$

i.e. u is non-trivial. Standard arguments now insure that u is a classical solution to (1)-(2) with $\|u\|_\infty \le \|h\|_\infty$. ■

Remark 5 *Some type of sublinear condition like (15) is necessary, as one may deduce from the following example. Consider the existence of a positive solution to the boundary value problem:*

$$u'' + \lambda u = 0 \quad u'(\frac{\pi}{2}) = u(\pi) = 0 .$$

As the reader may easily verify, all conditions of Proposition 6 are fulfilled except (15), provided λ is sufficiently large. If $\lambda \in \mathbb{N}$ there is an infinity of solutions all multiples of $\sin((2\lambda+1)t)$ *functions. If $\lambda \notin \mathbb{N}$ the previous B.V.P. has no solution.*

Finally we apply our results to an elliptic problem in an annullus.

Corollary 7 *Consider the annular domain $\Omega := B_R\backslash B_r \subset \mathbb{R}^N$ (where B_L is the N-dimensional euclidean ball of center* 0 *and radius L) and the B.V.P.*

$$-\Delta u = f(\|x\|,u) \quad \text{for all } x \in \Omega , \tag{17}$$

$$u = 0 \quad \text{in } \partial B_R \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{in } \partial B_r . \tag{18}$$

Suppose that $f(t,u)$ satisfies (4)-(7) for $\Phi(t) = t^{N-1}$ and (15). Then there exists a radial symmetric positive solution u to (17)-(18) with L_∞ norm bounded from above by $\|h\|_\infty$, where h is defined by (7).

Proof. Just observe that a positive radial symmetric solution to (17)-(18) can be obtained as a solution to

$$(t^{N-1}u'(t))' + t^{N-1}f(t,u(t)) = 0 \quad , \quad u'(r) = u(R) = 0 ,$$

and consider Remark 4. ■

Remark 6 *We may apply our results to the existence of a positive radial solution to:*

$$-\Delta u = \exp(-L\|x\|)u^\alpha \quad \textit{for all } x \in \Omega\,,$$

$$u = 0 \quad \textit{in} \quad \partial B_R \quad \textit{and} \quad \frac{\partial u}{\partial n} = 0 \quad \textit{in} \quad \partial B_r\,,$$

provided L is large.

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