Existence and $L_\infty$ estimates of some mountain-pass type solutions

J. M. Gomes

Abstract

We prove the existence of a positive solution to an equation of the form

$$(\Phi(t)u'(t))' = f(t, u(t))$$

with mixed Neuman and Dirichlet conditions. Our method combines variational and topological arguments providing an $L_\infty$ estimate of the solution. Our results can be applied to certain type of elliptic problems in annular domains.

Keywords: second order singular differential equation, variational methods, Mountain Pass Theorem.

2000 Mathematics Subject Classification. 34B18 34C11

1 Introduction

Early since its publication in 1973, the Mountain Pass Theorem of Ambrosetti and Rabinowitz (see [10]) has provided existence and multiplicity results in Differential Equations as well as a comprehensive perspective of variational methods. The characterization of Mountain Pass type solutions became itself a subject of interest. As examples one may cite works of del Pino and Felmer (see [5] and references therein) where the shape of the solutions to the Dirichlet problem

$$\epsilon^2 \Delta u - u + f(u) = 0 \quad \text{in } \Omega; \quad u > 0 \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{in } \Omega,$$

is established as $\epsilon$ tends to zero. In the same spirit, Bonheure, Habets and the author ([3]) have showed that for a superlinear elliptic problem with sign-changing non-linearity the major contribution of volume of mountain pass type solutions should concentrate in prescribed regions of the domain as a certain parameter $\mu \to \infty$. In the above examples the role played by a parameter as it approaches some limit is crucial. In [6] the author established existence and $L_\infty$ estimates of positive Mountain Pass type solutions to a class of singular differential equations with an increasing friction term and Dirichlet boundary conditions. The bound is just the $L_\infty$ norm of any regular function where the Euler-Lagrange functional $J$ attains a negative value. Our method combined arguments in the Direct Calculus of Variations with phase plane techniques. In fact, pursuing the nature of the optimal min-max path connecting the origin to some function where $J$ is negative, we were lead to consider a family of minimizers of truncated functionals containing, as a particular element, a classical solution to our b.v.p. In this work we approach with similar arguments a more general class of equations that include some elliptic problems in an annulus. More precisely we will be interested in positive solutions to

$$(\Phi(t)u'(t))' + f(t, u(t)) = 0$$  \quad (1)
Existence and $L_\infty$ estimates of Mountain Pass solutions

By positive solution we mean a $C^2$ function $u$ verifying the above equalities and such that $u(t) > 0$ for all $t \in [0,1]$. Similar problems have been considered in [2], [8] and [9].

## 2 Variational setting and results

We begin by listing the assumptions on the terms of equation (1)-(2).

- $\Phi \in C^1([0,1])$ is strictly positive and we choose $m, \overline{m} > 0$ such that, for all $t \in [0,1]$,
  \[ 0 < m \leq \Phi(t) \leq \overline{m} \tag{3} \]

We assume that

\[ f(t, u)\Phi(t) \text{ is decreasing in } t \text{ for every } u \geq 0. \tag{4} \]

Also

\[ f : [0,1] \times [0, +\infty[ \mapsto \mathbb{R} \text{ is locally lipschitz in the variable } u, \tag{5} \]

and that, for some $\delta > 0$, $f(t, u)$ verifies

\[ f(t, u) = 0 \quad \forall (t, u) \in [0,1] \times [0, \delta] \text{ and } f(t, u) > 0 \text{ in } [0,1] \times \delta, +\infty[. \tag{6} \]

The technical assumption (6) will be relaxed subsequently to a sub-linear growth near zero. Since we are looking for positive solutions we assume throughout the paper that $f$ is extended by zero in $[0,1] \times ]-\infty, 0]$. The reader may easily verify that any non-trivial solution to (1)-(2) with this extension -which we will still denote by $f$- should be positive in $]0,1[$ therefore being a solution of the initial problem. We shall consider the Sobolev space $H \subset H^1_0([0,1])$ consisting in absolutely continuous functions $u$ such that

\[ \|u\|_2 := \int_0^1 u'^2(t) dt < \infty, \quad u(1) = 0. \]

In the sequence we will also refer

\[ \|u\|_{\infty} := \sup \{ u(t) : t \in [0,1] \}, \]

the natural norm on the space of continuous functions $C([0,1])$. Note that Problem (1)-(2) may be viewed as the Euler-Lagrange equation of the functional $J : H \rightarrow \mathbb{R}$ defined by:

\[ J(u) = \frac{1}{2} \int_0^1 \Phi(t)u'^2(t) dt - \int_0^1 F(t, u(t)) dt \]

where $F(t, u) = \int_0^u f(t, s) ds$. We will suppose that $J$ satisfies the fundamental property:

\[ \exists h \in H : J(h) < 0. \tag{7} \]

**Remark 1**. Property (7) can be easily verified if, for some $\epsilon > 0$, $f(t, u) \geq \epsilon u^a - C$ for all $u \geq 0$ and $t \in [0,1]$, where $a > 1$ and $C > 0$. Note that we make no assumptions on the growth of $f$ as $u \rightarrow \infty$ or require $J$ to verify Palais-Smale condition.
We denote $M = \|h\|_{\infty}$. Since
\[
\forall w \in H, \quad \|w\|_{\infty} \leq \delta \Rightarrow J(w) \geq 0,
\]
(where $\delta$ was defined in (6)) we have $M > \delta$. For all $M \in [\delta, M]$, we consider the following subset of $H$:
\[
\mathcal{C}_M = \{ u \in H : \max u \geq M \}.
\]
We also consider the truncated functional $J_M : H \to \mathbb{R},$
\[
J_M(u) = \frac{1}{2} \int_0^1 \Phi(t)u'^2(t)dt - \int_0^1 F_M(t, u(t))dt
\]
where
\[
F_M(u) = \begin{cases} 
F(t, u) & \text{if } u \leq M \\
F(t, M) & \text{if } u > M
\end{cases}
\]

**Remark 2** From the compact injection of $H^1_0([0,1])$ in $C([0,1])$ we conclude that $\mathcal{C}_M$ is weakly sequentially closed and that $J_M$ is coercive and weakly lower semi-continuous.

We will be interested in the family of minimizers $u_M$ of $J_M$ in $\mathcal{C}_M$. By Remark 2 we know that $u_M$ exists for every $M \in [\delta, M]$. We also know that:

**Lemma 1** Let $u_M$ be a minimizer of $J_M$ in $\mathcal{C}_M$. Then
\[
\max_{[0,1]} u = M \quad \text{and} \quad \min_{[0,1]} u = 0.
\]

**Proof.** Given $w \in \mathcal{C}_M$ define
\[
\overline{w}(t) = \max\{0, \min\{w(t), M\}\}.
\]
Of course $\overline{w} \in H \cap \mathcal{C}_M$. If $\overline{w} \neq w$ then,
\[
\int_0^1 \Phi(t)\overline{w}'^2(t)dt < \int_0^1 \Phi(t)w'^2(t)dt
\]
and
\[
\int_0^1 F_M(t, \overline{w}(t))dt = \int_0^1 F_M(t, w(t))dt.
\]
Then $J_M(\overline{w}) < J_M(w)$ which is absurd and the lemma follows. \qed

Given $M \in [\delta, M]$, we consider two types of minimizers of $J_M$ in $\mathcal{C}_M$:

**Definition.** Let $u_M$ be a minimizer of $J_M$ in $\mathcal{C}_M$.

We say that $u_M$ is a minimizer of type A if
\[
u(0) = M, \quad u'(0) \leq 0 \quad \text{and} \quad u(t) < M \quad \text{for all } t > 0.
\]
We say that $u_M$ is a minimizer of type B if, for some $\tilde{t} \geq 0$, we have
\[
u(t) = M \quad \forall t \in [0, \tilde{t}], \quad u'(\tilde{t}) = 0 \quad \text{and} \quad u(t) < M \quad \text{if } t > \tilde{t}.
Remark 3 If $u_M$ is a minimizer of type A then $u$ satisfies equation (1) in $]0,1[$. In fact, if $v \in C_0^1([0,1])$, then, for sufficiently small $s$, we have,

$$u_M + sv \in \mathcal{E}_M \text{ and } u_M(t) + sv(t) < M \quad \forall t \in \text{supp}(v).$$

Since $u_M$ is a minimizer, we conclude

$$
\lim_{s \to 0} \frac{J_M(u_M + sv) - J_M(u_M)}{s} = \int_0^1 \Phi(t)u'_M v'(t) dt - \int_0^1 f(t,u_M(t))v(t) \, dt = 0,
$$

and the assertion follows. Similarly, if $u_M$ is a minimizer of type B, it satisfies equation (1) in $]1,\infty[$. If $u_M$ is simultaneously of type A and B, then $u_M$ is a classical solution to problem (1)-(2).

Lemma 2 Let $u$ be a minimizer of $J_M$ in $\mathcal{E}_M$. Then $u$ is of type A or B (possibly both).

Proof. Let us consider

$$\tilde{t} := \sup\{t \in [0,1] : u(t) = M\}.$$

Since $H \subset C([0,1])$, we have $u(\tilde{t}) = M$ and we may therefore consider $w \in H$

$$w := \begin{cases} M & \text{if } t \leq \tilde{t} \\ u(t) & \text{if } t > \tilde{t} \end{cases}.$$

Moreover,

$$\int_0^1 F_M(t, w(t)) \, dt \geq \int_0^1 F_M(t, u(t)) \, dt,$$

and

$$\int_0^1 \Phi(t)u''^2(t) \, dt \leq \int_0^1 \Phi(t)u''^2(t) \, dt,$$

the last inequality being strict if $w \neq u$ in $[0,\tilde{t}]$. Since $u$ is a minimum of $J_M$, we conclude $u \equiv w$. Note that by Lemma 1 and Remark 3, $u'_\epsilon(\tilde{t})$ is well defined and non-positive. If $t = 0$ then $u$ is of type A. Suppose in view of a contradiction that $\tilde{t} > 0$ and $u'_\epsilon(\tilde{t}) < 0$. Choose $\theta, \epsilon > 0$ such that $u'(t) \leq -\theta$ for every $t \in [\tilde{t},\tilde{t} + \epsilon]$ and, assuming $\epsilon < \tilde{t}/2$, define the “triangular” perturbation

$$v_\epsilon(t) = -[(t - \tilde{t}) - \epsilon]_+.$$  \quad (8)

We assert that, for a small $\epsilon$,

$$\lim_{s \to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} < 0. \quad (9)$$

If (9) holds, then, for sufficiently small $s^* > 0$, we have $u + s^*v_\epsilon \in \mathcal{E}_M$ (since, by our choice of $\epsilon$, $(u + s^*v_\epsilon)(0) = M$) and $J_M(u + s^*v_\epsilon) < J_M(u)$ thereby contradicting the assumption that $u$ is a minimizer of $J_M$ in $\mathcal{E}_M$. In fact, Lemma 1 and (8) imply $u + s^*v_\epsilon \leq M$. Therefore

$$\lim_{s \to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} = \lim_{s \to 0} \frac{J(u + sv_\epsilon) - J(u)}{s} =$$
Existence and $L_\infty$ estimates of Mountain Pass solutions

\[ \int_0^1 \Phi(t)u'(t)v_\epsilon'(t)dt - \int_0^1 f(t, u(t))v_\epsilon(t)dt \leq \int_{t_\epsilon}^{t+\epsilon} \Phi(t) dt - \int_{t-\epsilon}^{t+\epsilon} f(t, u(t))v_\epsilon(t)dt. \]

We observe that, by (3),

\[ -\theta \int_{\bar{t}+\epsilon}^{\bar{t}+\epsilon} \Phi(t) dt \leq -m\theta \epsilon \]

and for some $C > 0$ depending only on $f$,

\[ \int_{t-\epsilon}^{t+\epsilon} f(t, u(t))v_\epsilon(t)dt \leq C\epsilon^2. \]

Therefore, by (10) and (11), we have

\[ \lim_{s \to 0} \frac{J_M(u + sv_\epsilon) - J_M(u)}{s} \leq -m\theta \epsilon + C\epsilon^2, \]

and the assertion follows for sufficiently small $\epsilon$. 

In the next lemma we prove a necessary ordering relation between type A and type B minimizers of $J_M$ in $\mathcal{C}_M$.

**Lemma 3** Suppose that for a certain $M \in [0, M]$ there exist minimizers $u$ and $v$ of $J_M$ in $\mathcal{C}_M$ such that $u$ is of type A and $v$ is of type B. Then $u(t) < v(t)$ for all $t \in [0, 1]$ or else $u \equiv v$.

**Proof.** Assume $u \neq v$. Necessarily, we will have $u(t) < v(t)$ for all $t \in [0, \epsilon]$ provided $\epsilon$ is sufficiently small. Suppose that for some $t^* \in [0, 1]$ we had

\[ u(t^*) = v(t^*) \quad \text{and} \quad u'(t^*) > v'(t^*), \]

(the case $u'(t^*) = v'(t^*)$ is excluded by (5) together with Existence and Uniqueness Theorem). Moreover, suppose that

\[ \frac{1}{2} \int_{t^*}^1 \Phi u'^2 dt - \int_{t^*}^1 \Phi F_M(t, u) \leq \frac{1}{2} \int_{t^*}^1 \Phi v'^2 dt - \int_{t^*}^1 \Phi F_M(t, v), \]  

and let

\[ v^*(t) = \begin{cases} v(t) & 0 \leq t \leq t^* \\ u(t) & t^* < t \leq 1 \end{cases} \]

Then $v^* \in H$ and

\[ J_M(v^*) \leq J_M(v), \]

therefore $v^*$ is also a minimizer in $\mathcal{C}_M$. But this is absurd since $v^*$ is not differentiable at $t^*$ (see remark 3). In case where, instead of (12), we had the reversed inequality we would get the same contradiction by considering:

\[ u^*(t) = \begin{cases} u(t) & 0 \leq t \leq t^* \\ v(t) & t^* < t \leq 1 \end{cases} \]
In the next lemma we establish an important fact concerning the coexistence of type A and type B minimizers at a same truncating level.

**Lemma 4** Assume that conditions (4), (3) and (5) hold. Suppose that for a certain $M \in [0, \overline{M}]$ there exist minimizers $u$ and $v$ of $J_M$ in $C_M$ such that $u$ is of type A and $v$ is of type B. Then the minimizer $u$ is a classical solution to (1)-(2).

**Proof.** We define an inverse function for $u(t)$. By Remark 3, we may write

$$u'(t) = \frac{1}{\phi(t)} \left( \phi(0)u'_u(0) - \int_0^t f(s, u(s)) \, ds \right).$$

Note that, if $u(0) \leq \delta$, necessarily $u'_u(0) < 0$. In fact by Lemma 1 we have $u'(0) \leq 0$. In case $u'(0) = 0$ condition (6) and Remark 3 imply $u \equiv \delta$ contradicting the assumption that $u \in H$. If $u(0) > \delta$ then our assumptions on $f$ imply $u'(t) < 0$ for all $t \in [0,1]$. In both cases we conclude that $u$ is strictly decreasing in $[0,1]$ and we may define the inverse function

$$[0, M] \rightarrow [0,1] ,
\quad u \mapsto t_A(u) .$$

Using similar arguments we may define an inverse function for $v$

$$[0, M[ \rightarrow [\overline{t},1] ,
\quad v \mapsto t_B(v) .$$

We consider the continuous extension of this function to $[0, M]$ (that we still denote by $t_B$) verifying $t_B(M) = \overline{t}$. By Lemma 3 we have

$$t_A(u) \leq t_B(u) \ \forall u \in [0, M] .$$

We consider the functions

$$Z_A(u) = \Phi(t_A(u))u'(t_A(u)) ,$$

and

$$Z_B(v) = \Phi(t_B(v))v'(t_B(v)) .$$

Since the functions $u(t)$ and $v(t)$ verify (1)-(2) for $t \geq 0$ and $t \geq \overline{t}$ respectively, we write, for $u, v \in [0, M]$

$$-dZ_A \frac{du}{dt} = f(t_A(u), u) \quad \text{and} \quad -dZ_B \frac{dv}{dt} = f(t_B(v), v)$$

or

$$\frac{dZ_A}{du} = -\frac{\Phi(t_A(u))}{Z_A} f(t_A(u), u) \quad \text{and} \quad \frac{dZ_B}{dv} = -\frac{\Phi(t_B(v))}{Z_B} f(t_B(v), v) . \quad (13)$$

Suppose in view of a contradiction that $u(t)$ is not a type B minimizer, i.e.

$$Z_A(M) < 0 = Z_B(M) . \quad (14)$$

1This change of variables was suggested by the reading of [7].
In fact, this assumption implies that $Z_A(u) < Z_B(u)$ for all $u \in [0, M]$. We start by noting that if $Z_A(0) = Z_B(0)$, then $u'(1) = v'(1)$. Since $u(1) = v(1) = 0$, Existence Uniqueness Theorem implies $u(t) = v(t)$ for all $t \in [\bar{t}, 1]$ thereby contradicting (14). Admit that for some $u^* \in [0, M]$ we had $Z_A(u^*) = Z_B(u^*)$.

We assume that $u^*$ is the maximum point satisfying the previous equality. Then

$$\frac{dZ_A}{du}(u^*) \leq \frac{dZ_B}{du}(u^*).$$

If there is equality of the derivatives then Existence Uniqueness Theorem implies $Z_A(0) = Z_B(0)$ which, as previously noticed, is absurd. Since $Z_A(u^*) = Z_B(u^*) < 0$, (4) implies that the right hand-sides of the equalities in (13) are decreasing functions of $t$. Recalling that $t_B(u^*) \geq t_A(u^*)$ we conclude

$$\frac{dZ_A}{du}(u^*) > \frac{dZ_B}{du}(u^*)$$

contradicting the maximal property of our choice of $u^*$. In particular we have proved that if $u$ is not a type B minimizer then $Z_A(0) < Z_B(0)$.

Finally we conclude $u'(1) < v'(1) < 0$ which in turn implies that $u(t^*) > v(t^*)$ for some $t^* < 1$, a contradiction with Lemma 3 and the proof is complete. □

We are now in a position to prove

**Proposition 5** Assume that conditions (3), (4), (5), (6) and (7) hold. Then there exists a positive solution $u$ to (1)-(2) such that

$$\|u\|_\infty \leq \|h\|_\infty$$

where $h$ was defined in (7).

**Proof.**

Recalling our notation $\overline{M} = \|h\|_\infty$, let $I = [\delta, \overline{M}]$ and consider the following subsets $I_A$ and $I_B$:

$$I_A(I_B) = \{M \in [\delta, \overline{M}] : J_M \text{ has a minimizer in } C_M \text{ of type A (B)}\}.$$

By Lemma 2 we have $I = I_A \cup I_B$. We assert that $I_A$ and $I_B$ are non-empty. In fact $\delta \in I_A$ since, as previously noticed, if $u_\delta$ is a minimizer of $J_\delta$ in $C_\delta$ and $u_\delta'(0) = 0$ the Existence and Uniqueness Theorem implies $u_\delta(t) = \delta$ for all $t \in [0, 1]$, which is absurd.

**Claim 1:** $I_B$ is non-empty.

Suppose that $\overline{M} \notin I_B$. In this case $u_{\overline{M}}$ is a type A minimizer. Let

$$\tilde{f}(t, x) := f(t, \min\{x, u_{\overline{M}}(t)\}).$$
Define, for $u \in H$

$$J(u) := \frac{1}{2} \int_0^1 \phi(t) u'^2(t) \, dt - \int_0^1 F(t, u(t)) \, dt,$$

where $F(t, x) = \int_0^x f(t, s) \, ds$. Trivially, by (7) we have $J(u_{\overline{w}}) < 0$. Also $J$ is coercive and lower semi-continuous in $H$ and therefore attains a minimum at some function $w \in H$ such that $J(w) < 0$. In fact

$$0 < w(t) < \pi_M(t) \quad \forall t \in [0, 1],$$

(0 and $\pi_M$ are a pair of well ordered lower and upper solutions respectively) and $w$ is a classical solution to (1)-(2) (see for instance [4], ch. 4 for details).

Claim 2: $I_A$ and $I_B$ are closed subsets of $I$.

Let $(M_n)$ be a sequence in $I_A$ such that $M_n \to M$. Let $u_n$ be a corresponding sequence of type A (B) minimizers of $J_{M_n}$ in $\mathcal{C}_{M_n}$. Since $(u_n)$ is trivially bounded we may extract a weakly convergent subsequence (still denoted by $u_n$) such that

$$u_n \rightharpoonup u \text{ in } H \quad \text{and} \quad u_n \to u \text{ in } C([0, 1]).$$

We assert that $u$ is a minimizer of $J_M$ in $\mathcal{C}_M$. In fact, since,

$$\lim_{n \to \infty} \int_0^1 \Phi(t) F_{M_n}(t, u_n(t)) \, dt = \int_0^1 \Phi(t) F_M(t, u(t)) \, dt$$

and

$$\int_0^1 \Phi(t) u'^2(t) \, dt \leq \liminf_{n \to \infty} \int_0^1 \Phi(t) u_n'^2(t) \, dt,$$

we conclude

$$J_M(u) \leq \liminf_{n \to \infty} J_{M_n}(u_n).$$

However, if we set $w_n = (M_n/M)u$, we have $w_n \rightharpoonup u$ in $H$ and $w_n \in \mathcal{C}_{M_n}$, for all $n \in \mathbb{N}$. Therefore

$$J_M(u) = \lim_{n \to \infty} J_{M_n}(w_n)$$

and

$$J_{M_n}(w_n) \geq J_{M_n}(u_n),$$

for all $n \in \mathbb{N}$. We conclude

$$J_M(u) \geq \limsup_{n \to \infty} J_{M_n}(u_n) \geq \liminf_{n \to \infty} J_{M_n}(u_n) \geq J_M(u),$$

or,

$$\lim_{n \to \infty} J_{M_n}(u_n) = J_M(u).$$

If, for some $u^*$ in $\mathcal{C}_M$, we had $J_M(u^*) < J_M(u)$ then for sufficiently large $n$, we would have

$$J_{M_n}(w_n^*) < J_{M_n}(u_n),$$

where $w_n^* = (M_n/M)u^*$, which is absurd. Note that so far we have just used the fact that $u_n$ is a sequence of minimizers. It remains to prove that the limit function $u$ is of type A (B). If $(u_n)$ is a type A sequence, since $u_n \rightharpoonup u$ in $L_\infty$ and (1) is verified for all $u_n$ in $[0, 1]$ implies that $u$ satisfies (1) in $[0, 1]$. This
trivially implies that \( u \) itself is of type A. In case of a type B sequence, then the same \( L_\infty \)-convergence insures that \( u \) must be constant in some interval \([0, \bar{t}]\)
where \( \bar{t} \) is a limit point of the sequence \((t_n)\) where
\[
t_n = \max \{ t : u_n(t) \text{ is constant in } [0, t] \},
\]
and satisfies (1) in \([\bar{t}, 1] \). Also, since the \( u_n \)'s are of type B and, for small \( \epsilon > 0 \),
\[
u_n \rightharpoonup u \quad \text{C1 in } [\bar{t} + \epsilon, 1]
\]
we have \( u'(\bar{t}) = 0 \) and the claim is proved.

We may therefore conclude, since \( I \) is connected, that \( I_A \cap I_B \neq \emptyset \). By Lemma 4 it implies the existence of a classical solution \( u \) such that \( \max \| h \| \leq \| h \|_{\infty} \).

**Remark 4** Note that instead of (2) we may consider the more general boundary conditions
\[
u'(a) = u(b) = 0 \quad (a < b).
\]

In the next result we relax condition (6) using an approximating standard technique.

**Theorem 6** Suppose that \( f(t, u) \) is locally Lipschitz in the variable \( u \), verifies (4) and
\[
0 \leq f(t, u) \leq Cu^p \quad \text{for } (t, u) \in [0, 1] \times [0, \rho]
\]
for some \( p > 1 \) and \( \rho > 0 \). Also assume (3) and that condition (7) is fulfilled by some non-negative \( h \in H \). Then there exists a positive solution \( u \) to (1)-(2) such that \( \max u \leq \| h \|_{\infty} \).

**Proof.** We may suppose that \( f \) is bounded above by \( \| h \|_{\infty} \). Consider the following translation of the nonlinearity:
\[
f_\delta(t, u) = f(t, (u - \delta)_+) \quad .
\]
Observe that, since \( (u - \delta)_+ \) is an increasing function of \( u \), assumption (4) is verified by \( f_\delta \) for all \( \delta > 0 \) as well as (15) for the same constant \( C \). Also (7) is fulfilled by the same function \( h \) for all the functionals
\[
J_\delta(u) := \frac{1}{2} \int_0^1 \Phi u'^2 - \int_0^1 F_\delta(t, u) \, dt,
\]
where \( F_\delta(t, u) = \int_0^u f_\delta(t, s) \, ds \), provided \( \delta \) is small. We may therefore apply Proposition 5 and conclude the existence of a solution \( u_\delta \) to the problem
\[
(\Phi(t) u_\delta')' = f_\delta(t, u_\delta) \quad , \quad u_\delta'(0) = u_\delta(1) = 0 .
\]
Since \( u_\delta \) is a critical point of \( J \), \( H \) is continuously injected in \( L^{p+1}(0, 1) \) with \( p > 1 \), we have, by (15) and classical estimates, for some \( K_1, K \) independent of \( \delta \),
Existence and $L_{\infty}$ estimates of Mountain Pass solutions

$$m\|u_\delta\|^2 \leq \int_0^1 \Phi(t)u_\delta(t)^2 \, dt = \int_0^1 f_s(t, u_\delta(t))u_\delta(t) \, dt \leq K_1 \int_0^1 |u|^p \, dt \leq K\|u_\delta\|^p.$$  

We conclude, for $k^* = (m/K)^{1/p-1}$,

$$\|u_\delta\| \geq k^* > 0,$$

for all sufficiently small $\delta$. Consider a sequence $\delta_n \to 0$ and the corresponding sequence $u_n$ of solutions to (16). Noting that the sequence $(\|u_n\|)$ is trivially bounded by the variational characterization of the $u_n$'s, we may consider $u \in H$ and a subsequence (still denoted by $(u_n)$) such that

$$u_n \rightharpoonup u \quad \text{in} \quad H \quad \text{and} \quad u_n \to u \quad \text{in} \quad C([0,1]).$$

We may conclude

$$\int_0^1 f(t, u) \, dt = \lim_{\delta_n \to 0} \int_0^1 f_0(t, u_n) \, dt = \lim_{\delta_n \to 0} \int_0^1 \Phi(t)u_n'(t)^2 \, dt \geq mk^*, $$

i.e. $u$ is non-trivial. Standard arguments now insure that $u$ is a classical solution to (1)-(2) with $\|u\|_\infty \leq \|h\|_\infty$.

**Remark 5** Some type of sublinear condition like (15) is necessary, as one may deduce from the following example. Consider the existence of a positive solution to the boundary value problem:

$$u'' + \lambda u = 0 \quad u'(\frac{\pi}{2}) = u(\pi) = 0.$$  

As the reader may easily verify, all conditions of Proposition 6 are fulfilled except (15), provided $\lambda$ is sufficiently large. If $\lambda \in \mathbb{N}$ there is an infinity of solutions all multiples of $\sin((2\lambda + 1)t)$ functions. If $\lambda \notin \mathbb{N}$ the previous B.V.P. has no solution.

Finally we apply our results to an elliptic problem in an annulus.

**Corollary 7** Consider the annular domain $\Omega := B_R \setminus B_r \subset \mathbb{R}^N$ (where $B_L$ is the $N$-dimensional euclidean ball of center 0 and radius $L$) and the B.V.P.

$$-\Delta u = f(||x||, u) \quad \text{for all} \quad x \in \Omega, \quad (17)$$

$$u = 0 \quad \text{in} \quad \partial B_R \quad \text{and} \quad \frac{\partial u}{\partial n} = 0 \quad \text{in} \quad \partial B_r. \quad (18)$$

Suppose that $f(t, u)$ satisfies (4)-(7) for $\Phi(t) = t^{N-1}$ and (15). Then there exists a radial symmetric positive solution $u$ to (17)-(18) with $L_{\infty}$ norm bounded from above by $\|h\|_{\infty}$, where $h$ is defined by (7).

**Proof.** Just observe that a positive radial symmetric solution to (17)-(18) can be obtained as a solution to

$$(t^{N-1}u'(t))' + t^{N-1}f(t, u(t)) = 0, \quad u'(r) = u(R) = 0,$$

and consider Remark 4.

\[\blacksquare\]
Remark 6 We may apply our results to the existence of a positive radial solution to:

\[-\Delta u = \exp(-L\|x\|)u^\alpha \text{ for all } x \in \Omega ,
\]

\[u = 0 \text{ in } \partial B_R \text{ and } \frac{\partial u}{\partial n} = 0 \text{ in } \partial B_r ,\]

provided \(L\) is large.

References


José Maria Gomes
CMAF-Centro de Matemática e Aplicações Fundamentais,
Avenida Professor Gama Pinto, 2, 1649-003 Lisboa
E-mail address: zemaria@cii.fc.ul.pt