On a class of bounded trajectories for some non-autonomous systems

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Abstract. We prove, by variational arguments, the existence of a solution to the boundary value problem in the half line

$$\begin{cases} \ddot{x} + c\dot{x} = a(t)V'(x) \\ \\ x(0) = 0, \;\; x(+\infty) = 1. \end{cases} \tag{1}$$

where $c \geq 0$ and a belongs to a certain class of positive functions. The existence of such a solution in the case $c = 0$ means that the system (1) behaves in significantly different way from its autonomous counterpart.

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1 Introduction

Consider a smooth scalar potential $V(x),\ x \in I\!R,$ which is positive in $]0,1[$ and such that $V(0) = V(1) = 0$, a scalar function $a(t)$ which has a positive infimum in $I\!R$ and, finally, let $c \geq 0$. In this paper we are concerned with the existence of solutions to the following boundary value problem on the interval $]0, \infty[$:

$$\ddot{x} + c\dot{x} = a(t)V'(x) \tag{2}$$

$$x(0) = 0, \;\; x(+\infty) = 1. \tag{3}$$

More precise assumptions on the data will be given below. We would like first to make some comments about the problem.

Our interest in this problem has two motivations, according to whether $c = 0$ or $c > 0$. The first is that, in the case $c = 0$, if a and V are even functions, our problem is equivalent to that of finding a heteroclinic connection between non-consecutive equilibria ± 1 of a potential V having three minima at the same level; such a problem has no solution if a is a constant because energy is conserved. Hence it is meaningful to investigate conditions on the time dependence of the coefficient $a(t)$ under which the mentioned connection appears. Striking differences between autonomous and non-autonomous systems have been investigated by other authors, as an example see [3]. It follows from our results that with respect to (1) there are differences of this kind for $c = 0$ but not for $c > 0$. In fact, in the case $c > 0$ a solution of our boundary value problem exists in the autonomous case as well.

Several authors have considered the problem of finding trajectories between equilibria of non-autonomous equations: we refer the reader to the recent paper by Malaguti, Marcelli and Partsvania [6] and the references therein.

We shall present two existence theorems, under distinct sets of conditions on the data. It turns out that the way the (increasing) function $a(t)$ approaches its limit plays an important role in the sufficient conditions. The approach is variational in both cases. More precisely, in Theorem 1 problem (1) will be solved in a situation where weak regularity assumptions on V and its minima are assumed; while with respect to $a(t)$ it is required that $a(t)$ tends to its limit l in such a way that, if $l < \infty$, $l - a(t)$ is slower than $1/t$. On the other hand, in Theorem 2, dealing with the case $c = 0$ only, we prove the existence of solutions for a wider class of functions $a(t)$, while confining ourselves to the class of C^2 potentials V.

The description of our assumptions follows.

(H_1) $V \in C^1(I\!R)$ is a non negative function, $V(0) = V(1) = 0$ and $V > 0$ in $]0, 1[$.

(H_2) There exist $\delta > 0$ and A_1, $A_2 > 0$ such that $A_1 x^2 \leq V(x) \leq A_2 x^2$ for $|x| < \delta$.

(H_3) The function $a : [0, +\infty[\to]0, +\infty[$ is such that there exists $t_0 \geq 0$ with the property that a is increasing in $[t_0, +\infty[$.

Theorem 1 *Assume that (H_1) holds and $a : [0, +\infty[\to]0, +\infty[$ is a continuous function. If $c = 0$ assume that (H_2) and (H_3) hold as well and in addition that $\eta := \inf_{t \geq 0} a(t) > 0$ and $l := \lim_{t \to +\infty} a(t)$ has the property*

$$\lim_{t \to +\infty} t(l - a(t)) = +\infty \tag{4}$$

then the boundary value problem (2)-(3) has at least one solution that takes values in $[0, 1]$.

Remark 1 *If $l \in I\!R^+$ it is easy to check that (4) holds if $a(t)$ is of the form $a(t) = l - \frac{\gamma}{(1+t)^\beta}$, $0 < \gamma < l$, $\beta < 1$.*

Remark 2 *The assumption (4) implies that for any $C, D, E > 0$ there exists $\sigma_0 > 0$ such that if $0 < \sigma < \sigma_0$*

$$a\left(\frac{C}{\sigma} + D\right) + E\sigma < l$$

Theorem 2 *Assume $c = 0$. Let $V \in C^2(I\!R)$ satisfy (H_1) and $V''(0) > 0$. Let $a : I\!R \to]0, +\infty[$ be a function of bounded variation with $\eta := \inf_{t \geq 0} a(t) > 0$ satisfying (H_3) and the property*

$$\lim_{t \to +\infty} (l - a(t)) e^{2\mu t} = +\infty, \tag{5}$$

where $l := \lim_{t \to +\infty} a(t)$ and $\mu = \sqrt{\eta V''(0)}$.

Then the boundary value problem (2)-(3) has at least one solution taking values in $[0, 1]$.

Corollary 1 *Let $c = 0$, a and V be even functions satisfying the assumptions of Theorem 1 or Theorem 2. Moreover assume that 1 is an isolated minimizer of V. Then equation (2) has a heteroclinic solution connecting the equilibria -1 and 1.*

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2 A comparison between autonomous and non autonomous problems

First, note that if $c = 0$ and $a(t)$ is constant, then problem (1), which corresponds now to an autonomous equation, has no solution. Then as a next step and still maintaining $c = 0$, the simplest non autonomous system one can discuss is one where $a(t)$ is a 'bang-bang' function with only one switch. More precisely, if $0 < a < b$ and $T > 0$, we define

$$a(t) := \begin{cases} a, & 0 \leq t \leq T \\ b & t > T. \end{cases} \tag{6}$$

Then, consider a C^1 potential $V(x)$ as above and which is locally bounded from above by Ax^2 and $A(x-1)^2$, $A > 0$, respectively around $x = 0$ and $x = 1$.

We note that problem (1) has a solution if and only if there exists a solution of $\ddot{x} = aV'(x)$ on $[0, T]$, with $x(0) = 0$ such that the corresponding solution curve $(x(t), \dot{x}(t))$ in the phase plane $(x, \dot{x})$ intersects at time T the heteroclinic orbit between $(0, 0)$ and $(0, 1)$ corresponding to the equation $\ddot{x} = bV'(x)$.

Set $\xi = x(T) \in]0, 1[$. By the conservation of energy, the heteroclinic solution of the second equation satisfies $\dot{x} = \sqrt{2bV(x)}$, whereas the solution of the first equation with $x(0) = 0$ satisfies $\dot{x} = \sqrt{2aV(x) + C}$ for some constant C. Then, imposing that the two solution curves intersect at time T in the phase plane, we get

$$C = 2(b-a)V(\xi) \tag{7}$$

and, if our problem admits a solution, then the following representation holds for $T = T(\xi)$:

$$T(\xi) = \int_0^\xi \frac{dx}{\sqrt{2aV(x) + 2(b-a)V(\xi)}} \tag{8}$$

By the quadratic growth of $V(x)$ in a neighbourhood of $x = 0$, there exists a constant $\underline{c}$ such that

$$\underline{c} \int_0^\xi \frac{dx}{\sqrt{2ax^2 + 2(b-a)\xi^2}} \leq T(\xi) \tag{9}$$

for any sufficiently small $\xi > 0$. Since

$$\int_0^\xi \frac{dx}{\sqrt{2ax^2 + 2(b-a)\xi^2}} = \frac{1}{\sqrt{2a}} \log\left(\frac{\sqrt{a}+\sqrt{b}}{\sqrt{b}}\right)$$

we infer that $T(\xi)$ is bounded away from zero in a right neighbourhood of $\xi = 0$. In a similar way it can be shown that $T(\xi) \to +\infty$ as $\xi \to 1^-$. Then, since $T(\xi)$ is a continuous function of ξ, we conclude that there exists $T_0 > 0$ such that $T(]0,1[) = [T_0, +\infty[$ or $T(]0,1[) =]T_0, +\infty[$ and therefore problem (1) has no solution if $T < T_0$ (and admits a solution for any $T > T_0$.)

In the simple example above the switch time T_0 for the function $a(t)$ actually depends on $V(x)$ through $\underline{c}$. This suggests that, generally speaking, despite the fact that the variables x and t are separate in the right-hand side of our equation, the conditions given on $a(t)$ to solve problem (1) when $c = 0$ may naturally involve the potential V. This is unlike the heteroclinic problem $x(-\infty) = 0,\ x(+\infty) = 1$ associated to the same equation for the class of potentials considered above. In fact, in this case a general result guarantees existence of solutions if $\lim_{|t|\to\infty} a(t) \to l \in I\!\!R$, and $a(t) \leq l$ with strict inequality holding on a set of positive measure. See [1].

We now turn to consider the features of the case $c > 0$. If we again take a to be a constant, the problem (1) may be related to a kind of boundary value problems that arise in the theory of travelling waves for reaction-diffusion (see [2]). Indeed, if one looks for strictly monotone solutions then (1) is easily transformed into a first order problem for the new unknown function $\psi = \Phi^2$ where Φ describes the graph of the curve $\dot{x} = \Phi(x)$ in the phase plane. See [1, 4, 5]. The new formulation may be written as

$$(10) \qquad \begin{cases} \psi' = 2(aV'(x) - c\sqrt{\psi}) \\ \\ \psi(1) = 0, \ \ \psi(x) > 0 \ \ \forall x \in [0, 1[. \end{cases}$$

It is not difficult to conclude, directly from phase-plane analysis or by studying (10), that for any $c > 0$ and $a > 0$ constant, the problem (1) has a solution. In fact if we consider, for $\epsilon > 0$, the Cauchy problem

$$(11) \qquad \begin{cases} \psi' = 2(aV'(x) - c\sqrt{\psi_+ + \epsilon}) \\ \\ \psi(1) = 0, \end{cases}$$

it turns out that it has a solution in $[0,1]$ that stays above $2aV(x)$. Then, by taking the limit as $\epsilon \to 0$, our claim follows. From our main results we shall see that the solution still exists when a depends on t.

3 Proof of the main results

Proof of theorem 1. We shall minimize the functional

$$\mathcal{F}(x) = \int_0^{+\infty} e^{ct} \left(\frac{\dot{x}(t)^2}{2} + a(t)V(x(t)) \right) dt \tag{12}$$

in the functional space

$$X := \{x \in C([0,+\infty[) \cap H^1_{loc}([0,+\infty[) \,:\, x(0) = 0, \;\; x(+\infty) = 1\}.$$

Let $\mathcal{I} = \inf_X \mathcal{F}$ and consider $x_n \in X$ such that $\mathcal{F}(x_n) \to \mathcal{I}$. For $t \geq 0$ we define $x_0(t) := \min\{t,1\} \in X$, and we have $\mathcal{I} \leq K := \mathcal{F}(x_0) = \int_0^1 e^{ct}(1 + a(t))V(t))\, dt$. It is clear that, for any $M > 0$, $(x_n)_n$ is bounded in $H^1(0,M)$ and $(e^{ct/2}\dot{x}_n)_n$ is bounded in $L^2(0,+\infty)$. Then we can take a subsequence, still denoted by $(x_n)_n$, which converges to some absolutely continuous function x uniformly on compact sets and in such a way that

$$e^{ct/2}\dot{x}_n \rightharpoonup e^{ct/2}\dot{x} \quad \text{in} \quad L^2(0,+\infty).$$

Moreover we may assume $0 \leq x_n(t) \leq 1$, since otherwise we could replace x_n with $\min(\max(x_n,0),1)$, still obtaining a minimizing sequence.

Case 1. $c > 0$. Let $T > 0$: from the Cauchy-Schwarz inequality we obtain, for any $t \geq T$,

$$|x_n(t) - 1| \leq \int_T^{+\infty} |\dot{x}_n(s)|\, ds \leq \frac{e^{-cT/2}}{\sqrt{c}} (\int_T^{+\infty} e^{cs}\dot{x}_n(s)^2\, ds)^{1/2} \leq \frac{e^{-cT/2}}{\sqrt{c}}(2K)^{1/2}.$$

Hence, for any $\delta > 0$, we can choose $T > 0$ such that

$$|x_n(t) - 1| \leq \delta \quad \forall t \geq T, \; \forall n.$$

Then the limit function x satisfies the same inequality, and since δ is arbitrary, $x \in X$. Moreover, by the weak lower semicontinuity of the L^2-norm and Fatou's Lemma, we have

$$\mathcal{F}(x) \leq \liminf_n \mathcal{F}(x_n) = \mathcal{I},$$

so that x actually minimizes $\mathcal{F}$ in X. Then, by standard arguments, x is a solution of (2)-(3).

Case 2. $c = 0$. Let us define, for fixed $\alpha \in]0, \min(\delta, 1/4)[$,

$$t_1(n) = \max\{t \geq 0 |\, x_n(t) \leq \alpha\}, \quad t_2(n) = \min\{t \geq t_1(n) |\, x_n(t) \geq 1 - \alpha\},$$

which will be simply denoted by t_1 and t_2. Then from the Schwarz inequality we obtain

$$1 - 2\alpha \leq (\int_{t_1}^{t_2} dt)^{1/2} (\int_{t_1}^{t_2} \dot{x}_n(t)^2\, dt)^{1/2} \leq (2K)^{1/2} \sqrt{t_2 - t_1},$$

so that

$$t_2 - t_1 \geq \frac{(1-2\alpha)^2}{2K}.$$

Now we can find t_3 and t_4 (depending on n) such that $x_n(t_3) = 1/4$, $x_n(t_4) = 3/4$ $t_1 < t_3 < t_4 < t_2$ and $1/4 \leq x_n(t) \leq 3/4$ for $t_3 \leq t \leq t_4$. With the previous computations (with $\alpha = 1/4$) we get $t_4 - t_3 \geq 1/8K$. Hence, we can consider the constant $\Delta := \dfrac{\min_{[1/4,3/4]} V}{8K}$ which is independent from α and has the property that $0 < \Delta \leq \int_{t_3}^{t_4} V(x_n(s))\, ds$. More generally, if we put $\underline{V}(\alpha) := \min_{[\alpha, 1-\alpha]} V(x)$, we have

$$\int_{t_1}^{t_2} a(t) V(x(t))\, dt \geq \eta (t_2 - t_1) \underline{V}(\alpha)$$

so that,

$$t_2 - t_1 \leq \frac{K}{\eta \underline{V}(\alpha)} \leq \frac{K}{\eta A_1 \alpha^2}, \tag{13}$$

where the last inequality follows from (H_2).

Moreover, by Remark 2 it follows that there exists $T \geq t_0$ and $\alpha < \min(1/4, \delta)$ for which

$$a(T) > a\left(\frac{K}{\eta A_1 \alpha^2} + t_0\right) + \frac{1}{\Delta}\left(\frac{1}{2t_0} + \frac{A_2 M_0 t_0}{3}\right)\alpha^2, \tag{14}$$

where $M_0 = \sup_{0 \leq t \leq t_0} a(t)$.

We will show that if $t_1 > T$, then we may replace the function x_n of the minimizing sequence with a function v_n for which $t_1 \leq T$. For consider the function:

$$v_n(t) := \begin{cases} \alpha t / t_0, & 0 \leq t \leq t_0 \\ \\ x_n(t + t_1 - t_0) & t \geq t_0. \end{cases} \tag{15}$$

We have

$$\begin{aligned}
\mathcal{F}(v_n) &= \frac{\alpha^2}{2t_0} + \int_{t_0}^{+\infty} \frac{\dot{x}(t+t_1-t_0)^2}{2}\,dt + \int_0^{t_0} a(t)V(\alpha t/t_0)\,dt + \int_{t_0}^{+\infty} a(t)V(x_n(t+t_1-t_0))\,dt \\
&= \frac{\alpha^2}{2t_0} + \int_{t_1}^{+\infty} \frac{\dot{x}_n(t)^2}{2}\,dt + \int_0^{t_0} a(t)V(\alpha t/t_0)\,dt + \int_{t_1}^{+\infty} a(t-t_1+t_0)V(x_n(t))\,dt.
\end{aligned}$$

Then, if $t_1 > T$, taking into account that $a(t)$ is increasing in $[t_0, +\infty[$ and (14), it follows that

$$\begin{aligned}
\mathcal{F}(v_n) - \mathcal{F}(x_n) &\le \frac{\alpha^2}{2t_0} + \int_0^{t_0} a(t)V(\alpha t/t_0)\,dt + \int_{t_1}^{+\infty} \left(a(t-t_1+t_0) - a(t)\right) V(x_n(t))\,dt \\
&\le \frac{\alpha^2}{2t_0} + \frac{M_0 A_2 \alpha^2 t_0}{3} - \int_{t_1}^{t_2} \left(a(t) - a(t-t_1+t_0)\right) V(x_n(t))\,dt \\
&\le \left(\frac{1}{2t_0} + \frac{M_0 A_2 t_0}{3}\right)\alpha^2 - \Delta\left(a(t_1) - a(t_2 - t_1 + t_0)\right) \\
&\le \left(\frac{1}{2t_0} + \frac{M_0 A_2 t_0}{3}\right)\alpha^2 - \Delta\left(a(T) - a\Big(\frac{K}{\eta A_1 \alpha^2} + t_0\Big)\right) \le 0.
\end{aligned}$$

Then, for any x_n we may assume $t_1 \le T$: hence $x_n(t) \ge \alpha$ for any n and any $t \ge T$, and by pointwise convergence we get also $x(t) \ge \alpha$ for any $t \ge T$. Now, as in case 1), by the weak lower semicontinuity we obtain

$$\mathcal{F}(x) \le \liminf_n \mathcal{F}(x_n) = \mathcal{I}.$$

Since $\mathcal{F}(x) < +\infty$ and $V > 0$ in $]0,1[$, it is not difficult to see, using the arguments of Rabinowitz [7], Prop. 3.11, in this simpler situation, that $\lim_{t\to+\infty} x(t)$ exists and is equal to 1. Then $x \in X$, so that it minimizes $\mathcal{F}$ in X. The fact that x takes values in $[0,1]$ is a straightforward consequence of the construction of the minimizing sequence.

■

Proof of theorem 2. For $I \subset [0, +\infty[$ let

$$\mathcal{F}(x, I) := \int_I \left(\frac{\dot{x}(t)^2}{2} + a(t)V(x(t))\right) dt.$$

Consider the equation

(16) $$\ddot{x} = a(t)V'(x)$$

with the boundary condition (3).

We put:

$$X(\xi) = \{x \in X \,|\, x(0) = \xi, \; x(+\infty) = 1\}, \quad \xi \in \mathbb{R}. \tag{17}$$

X will be endowed with the norm $x \mapsto (|x(0)|^2 + \|\dot{x}\|_2^2)^{1/2}$.

We shall make use of the following three Lemmas.

Lemma 1 *$(l - a(t))\gamma(t)^2 \to +\infty$ as $t \to +\infty$, where γ is the solution of the Cauchy problem*

$$\begin{cases} \gamma''(t) = a(t)V''(0)\gamma(t) \\ \gamma(t_0) = 0, \; \gamma'(t_0) = 1 \end{cases} \tag{18}$$

Proof From (5) we obviously get, as $t \to +\infty$,

$$(l - a(t))\rho(t)^2 \to +\infty, \tag{19}$$

where $\rho(t) = (e^{\mu(t-t_0)} - e^{-\mu(t-t_0)})/4\mu$. On the other hand, since $\rho''(t) = \mu^2\rho(t)$, $\rho(t_0) = 0$, $\rho'(t_0) = 1/2$, it is easy to see that $\gamma(t) > \rho(t)$ for $t > t_0$. Indeed, from the initial conditions we obtain the assertion in a right neighbourhood of $t = t_0$. By contradiction, let τ the first point after t_0 at which $\gamma(\tau) = \rho(\tau)$. Then we must have $\gamma''(\sigma) < r''(\sigma)$ at some point $\sigma \in]t_0, \tau[$, while, on this interval,

$$\gamma''(t) \geq \eta V''(0)\gamma(t) = \mu^2\gamma(t) \geq \mu^2\rho(t) = \rho''(t)$$

Then $\gamma > \rho > 0$ on $]t_0, +\infty[$, so that $\gamma^2 > \rho^2$, and our claim follows from (19). ■

Lemma 2 *For any $\xi \in]0,1]$ $\mathcal{F}$ attains its minimum on the class $X(\xi)$.*

Proof Let $(y_k)_k$ be a minimizing sequence for $\mathcal{F}$ on $X(\xi)$, and let $t_1 > t_0$ be fixed. For any $k \in \mathbb{Z}^+$ the following properties may be assumed to hold:

(a) $0 \leq y_k(t) \leq 1$,

(b) y_k solves (16) on $J := [0, t_1]$

Indeed, if these conditions are not satisfied it is enough to replace y_k, respectively:

(a) by $\min(\max(y_k, 0), 1)$

(b) on J by a function which minimizes $\mathcal{F}(\cdot, J)$ on $y_k + H_0^1(J)$;

It is easy to check that $\mathcal{F}$ does not increase after these procedures. Furthermore, $(y_k)_k$ is bounded in $H^1(J)$: then we can suppose, up to a subsequence, that $(y_k)_k$ converges uniformly on J to some absolutely continuous function y and that $\dot{y}_k \to \dot{y}$ weakly in $L^2(J)$. Then, on the interval J where y_k solves (16), $\ddot{y}_k$ is uniformly bounded: since the sequence $(\dot{y}_k)_k$ is bounded in L^2, we conclude that it is actually bounded in $H^1(J)$ and, even more so, in $L^\infty(J)$. Then we may suppose that $(\dot{y}_k(0))_k$ converges, so that the continuous dependence on initial data of the solutions of a differential equation ensures that the limit function y solves (16) on J (and also the C^1- convergence on that interval). Furthermore, $0 \leq y(t) \leq 1$. On the other hand, if y vanishes at t_0, we should also get $\dot{y}(t_0) = 0$, since t_0 is in the interior of J and y cannot take negative values. But the conditions $y(t_0) = \dot{y}(t_0) = 0$, together with (16), would imply $y(t) \equiv 0$ on J, in contrast with $y(0) = \xi > 0$. Hence $y(t_0) > 0$, and we can find $\delta > 0$ such that $y_k(t_0) \geq \delta$ for large k's. We then redefine y_k by replacing its restriction to $S := [t_0, +\infty[$ by the shifted function $t \mapsto y_k(t + \tau_k)$, where $\tau_k \geq t_0$ is the last point such that $y_k(\tau) = y_k(t_0)$. Since a is increasing in S, this operation does not increase the value of $\mathcal{F}$. If we still denote by y_k the modified functions, we can actually suppose that $y_k(t) \geq y_k(t_0)$ for any $t \in S := [t_0, +\infty[$. Now we can apply the arguments of theorem 1 and take a subsequence, still denoted by $(y_k)_k$, which converges to some absolutely continuous function x uniformly on compact sets and in such a way that $\dot{y}_k \rightharpoonup \dot{x}$ in $L^2(0, +\infty)$. Of course, $x \equiv y$ in J. By pointwise convergence,

$$x(t) \geq \delta \quad \text{for all} \quad t \in S. \tag{20}$$

Furthermore, $\mathcal{F}(x) < +\infty$: again by the arguments at the end of the proof of Theorem 1, this entails that $x(+\infty) \in \{0, 1\}$. But (20) excludes the case $x(+\infty) = 0$, and actually $x \in X(\xi)$. Now, thanks again to the weak lower semicontinuity of $\mathcal{F}$, x minimizes $\mathcal{F}$ on $X(\xi)$. Of course, x takes values in $[0, 1]$.

■

Now, let $\xi_i \to 0^+$ as $i \to +\infty$, and apply the previous Lemma on the class $X(\xi_i)$ for any $i \in \mathbb{Z}^+$, so as to get functions x_i such that

$$\mathcal{F}(x_i) \leq \mathcal{F}(y) \quad \text{for any} \quad y \in X(\xi_i)$$

As before we can suppose, up to a subsequence, that $(x_i)_i$ converges uniformly on compact sets to some function $x \in X$. Furthermore, the same arguments as in the proof of the previous Lemma allow to suppose that

$$\dot{x}_i(t_0) \to \dot{x}(t_0). \tag{21}$$

Then x solves (16), like x_i, and the following properties hold: $0 \leq x(t) \leq 1$, $x(t) \geq x(t_0)$ on S, $x(0) = 0$, $x(+\infty) \in \{0, 1\}$.

Lemma 3 $x(t_0) > 0$

Proof Let us suppose, by contradiction, $x(t_0) = 0$: since $x \geq 0$, we have $\dot{x}(t_0) = 0$ as well, and from (16) we actually get $x(t) \equiv 0$. In particular, $x_i(t_0) < r$ for large $i's$, where $r > 0$ is such that $V'' > 0$ in $[0, r]$. Furthermore $t_i \to +\infty$ as $i \to +\infty$, where t_i is the first time at which x_i reaches the value r. We put $\rho_i = x_i(t_0)$, $\eta_i = \dot{x}_i(t_0)$ and recall that each x_i solves (16). Then, for any $\tau \geq t_0$:

$$\begin{aligned}\frac{1}{2}\dot{x}_i(\tau)^2 - \frac{1}{2}\dot{x}_i(t_0)^2 &= \textstyle\int_{t_0}^{\tau} \dot{x}_i(s)\ddot{x}_i(s)\, ds = \int_{t_0}^{\tau} a(s)V(x_i(s))\dot{x}_i(s)\, ds = \\ &= [a(s)V(x_i(s))]_{t_0}^{\tau} - \textstyle\int_{t_0}^{\tau} V(x_i(s))\, da(s).\end{aligned}$$

Since $\dot{x}_i(\tau) \to 0$ and $V(x_i(\tau)) \to V(1) = 0$ as $\tau \to +\infty$, we get

$$\frac{1}{2}\eta_i^2 = a(t_0)V(x_i(t_0)) + \int_{t_0}^{+\infty} V(x_i(s))\, da(s) \geq \int_{t}^{t_i} V(x_i(s))\, da(s),$$

for any $t \in [t_0, t_i]$. Now, let us denote by $t \mapsto \phi(t; \xi, \eta)$ the solution of (16) which fulfils the conditions $x(t_0) = \xi$, $\dot{x}(t_0) = \eta$, so as to write $x_i(t) = \phi(t; \rho_i, \eta_i)$. For $t \geq t_0$, and as long as $\phi(t; \xi, \eta) \leq r$, it is easy to check that the function $V(\phi(t; \xi, \eta))$ is increasing with respect to all its arguments, so that $V(x_i(s)) = V(\phi(s; \rho_i, \eta_i)) \geq W(t, \eta_i)$ for any $s \in [t, t_i]$, where we put $W(s, \eta) = V(\phi(s; 0, \eta))$. Then

$$\frac{1}{2} \geq \frac{W(t, \eta_i)}{\eta_i^2}(a(t_i) - a(t)), \tag{22}$$

and we can let $i \to +\infty$. Since, by virtue of (21), $\eta_i \to \dot{x}(t_0) = 0$, we look for the behaviour of $W(t, \eta)/\eta^2$ as $\eta \to 0^+$, which depends on the partial derivatives of W (hence of ϕ) with respect to η. To this end we apply well-known results on differentiability with respect to initial data of the solution of a differential equation, which hold because the differential of the map

$(x,y) \mapsto f(t,x,y) = (y, a(t)V'(x))$ is bounded uniformly with respect to t. Since $\phi(t;0,0) \equiv 0$, the evolution of $\gamma(t) = \phi'_\eta(t;0,0)$ is ruled by (18). Hence

$$\frac{\partial W}{\partial \eta}(t,0) = V'(0)\gamma(t) = 0, \quad \frac{\partial^2 W}{\partial \eta^2}(t,0) = V''(0)\gamma(t)^2,$$

so that

$$\lim_{\eta \to 0^+} \frac{W(t,\eta)}{\eta^2} = \frac{1}{2} V''(0)\gamma(t)^2$$

Now (22) entails $1 \geq V''(0)\gamma(t)^2(l - a(t))$, in contrast with Lemma 1. Then $x(t_0) > 0$, as claimed. ∎

Conclusion of the proof of Theorem 2. Since $x(t) \geq x(t_0) > 0$ for $t \geq t_0$, the previous arguments show that $x(+\infty) = 1$, so that $x \in X$. Now, let $y \in X$, $i \in \mathbb{Z}^+$: we can modify y by putting $y_i = y + u_i$, where $u_i(0) = \xi_i$, $u \equiv 0$ in $[1, +\infty[$, $\dot{u} \equiv -\xi_i$ in $[0,1]$, so that $y_i \in X(\xi_i)$ and $\varepsilon_i := |\mathcal{F}(y_i) - \mathcal{F}(y)| \to 0$ as $i \to +\infty$. Then $\mathcal{F}(x_i) \leq \mathcal{F}(y_i) \leq \mathcal{F}(y) + \varepsilon_i$, and the lower limit as $i \to +\infty$ yields $\mathcal{F}(x) \leq \mathcal{F}(y)$. Hence x minimizes $\mathcal{F}$ on X. Of course, x takes values in $[0,1]$.

Proof of Corollary 1. Under the assumptions of Corollary 1, (2)-(3) with $c = 0$ has a solution $x(t)$ taking values in $[0,1]$. Then the function

$$w(t) = \begin{cases} x(t) & t \geq 0 \\ \\ -x(-t) & t < 0. \end{cases}$$

is a solution of (2) such that

$$\lim_{t \to \pm\infty} w(t) = \pm 1.$$

This is indeed a heteroclinic solution because $\lim_{t \to \pm\infty} \dot{w}(t) = 0$. In fact, integrating (2) between 0 and $t > 0$ we have

$$\dot{x}(t) - \dot{x}(0) = \int_0^t a(s)V'(x(s))\, ds.$$

Since there exists a sequence $t_n \to +\infty$ such that $\dot{x}(t_n) \to 0$ and the integrand in the right-hand side does not change sign in a neighbourhood of $+\infty$, we conclude that $\int_0^{+\infty} a(s)V'(x(s))\, ds$ converges. Therefore $\lim_{t \to +\infty} \dot{x}(t) = 0$.

Final Remarks. 1) As is shown by example (6), condition (5) is not necessary for the existence of solutions to problem (2)-(3). We can also generalize this example, by simply requiring that $a(t) \equiv b$ for $t \geq T$. Then suitable computations show that $\mathcal{F}$ attains its minimum on X, provided that the solution of the linear Cauchy problem

$$\begin{cases} \gamma''(t) = a(t)V''(0)\gamma(t) \\ \gamma(0) = 0,\ \gamma'(0) = 1 \end{cases}$$

satisfies the inequality $\dot{\gamma}(T) < \sqrt{bV''(0)}\gamma(T)$.

2) Suppose that V has only one critical point in $]0,1[$. Then we assert that the solution of (2)-(3) found in Theorem 1 or in Theorem 2 is monotone increasing. In order to see this, we argue by contradiction. If $x(t)$ is not monotone, we can find $0 \leq s_1 < s_2 < s_3 < s_4$ such that $x(s_2) = \max_{[s_1,s_4]} x > \min_{[s_1,s_4]} x = x(s_3)$, $x(s_4) = x(s_2)$ and $x(s_1) = x(s_3)$. Then, replacing $x|_{[s_1,s_3]}$ or $x|_{[s_2,s_4]}$ respectively with the constants $x(s_3)$ or $x(s_2)$ we would obtain a smaller value of the functional $\mathcal{F}$.

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