A $(p-q)$ coupled system in elliptic nonlinear problems with nonstandard boundary conditions

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Abstract

We state an abstract variational formulation to a coupled system consisted by an inequality and an equality motivated by the motion and energy equations, and the constitutive laws for the stress tensor and the heat flux, respectively, when non-Newtonian fluids are taken care of. Here the existence of a weak solution is proven via a fixed point argument to multivalued mappings. The nonstandard boundary conditions correspond to friction wall laws and energy transfer condition considered on a part of the boundary, whereas there exists the presence of the frictional work due to the friction of the fluid motion. We conclude by formulating the corresponding stationary heat conducting viscous incompressible flow problem and we establish an existence result.

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1 Introduction

In the present work we deal with a model problem motivated by the solid and/or fluid thermomechanics. For Ω an open bounded set of $\mathbb{R}^n$ ($n > 1$) with a sufficiently smooth boundary $\partial\Omega$ constituted by two disjoint complementar open subsets Γ_0 and Γ, i.e., $\Omega \in C^1$, $\partial\Omega = \overline{\Gamma}_0 \cup \overline{\Gamma}$, $\Gamma_0 \cap \Gamma = \emptyset$, $\mathrm{meas}(\Gamma_0) > 0$, $\mathrm{meas}(\partial\Omega \setminus (\Gamma_0 \cup \Gamma)) = 0$, we study the elliptic boundary value problem: find $u, e : \bar{\Omega} \to \mathbb{R}$ and $\tau : \bar{\Omega} \to \mathbb{R}^n$ such that

$$
\begin{aligned}
-\nabla \cdot \tau &= f(e, u, \nabla u) \quad \text{in} \quad \Omega; && (1)\\
\tau &\in \partial\mathcal{F}(e, \nabla u) \quad \text{in} \quad \Omega; && (2)\\
-\tau \cdot \mathbf{n} &\in \partial\mathcal{G}(e, u) \quad \text{on} \quad \Gamma; && (3)\\
-\nabla \cdot A(e, \nabla e) &= g(u, e, \nabla e) + \tau \cdot \nabla u \quad \text{in} \quad \Omega; && (4)\\
A(e, \nabla e) \cdot \mathbf{n} + \gamma(e) &= -(\tau \cdot \mathbf{n})u \quad \text{on} \quad \Gamma; && (5)\\
u = e &= 0 \quad \text{on} \quad \Gamma_0 := \partial\Omega \setminus \bar{\Gamma}. && (6)
\end{aligned}
$$

Here $\mathbf{n}$ denotes the unit outward normal vector to Γ. For simplicity, we will concentrate on the principal parts neglecting the possible lower order terms that can be taken into account, i.e., assuming $f(e, u, \nabla u) \equiv f$ defined in Ω, and $g(u, e, \nabla e) \equiv 0$.

We say that (1)-(6) is a $(p - q)$ *coupled system* if

- for $p > 1$, $\mathcal{F}(e, \zeta) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a continuous function, strictly convex on the second variable, such that

$$
\begin{aligned}
\exists \alpha_{\#} > 0 : \mathcal{F}(e, \zeta) &\geq \alpha_{\#} |\zeta|^p, \quad \forall e \in \mathbb{R}, \zeta \in \mathbb{R}^n; && (7)\\
\exists \alpha^{\#} > 0 : \mathcal{F}(e, \zeta) &\leq \alpha^{\#} (1 + |\zeta|^p), \quad \forall e \in \mathbb{R}, \zeta \in \mathbb{R}^n; && (8)
\end{aligned}
$$

- for $q > 2 - 1/n$, $A(x, e, \zeta) : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory function, i.e., it is measurable in $x \in \Omega$ for all $e \in \mathbb{R}$ and $\zeta \in \mathbb{R}^n$, and it is continuous in $e \in \mathbb{R}$ and $\zeta \in \mathbb{R}^n$ for a.e. in $x \in \Omega$, such that

$$
\begin{aligned}
&\exists \nu_{\#} > 0 : \ A(\cdot, e, \zeta) \cdot \zeta \geq \nu_{\#} |\zeta|^q, \ \text{a.e. in } \Omega, \ \forall e \in \mathbb{R}, \zeta \in \mathbb{R}^n; && (9)\\
&\exists \nu^{\#} > 0 : \ |A(\cdot, e, \zeta)| \leq \nu^{\#} (1 + |\zeta|^{q-1}), \ \text{a.e. in } \Omega, \forall e \in \mathbb{R}, \zeta \in \mathbb{R}^n; && (10)\\
&(A(\cdot, e, \zeta) - A(\cdot, e, \eta)) \cdot (\zeta - \eta) > 0, \quad \forall \eta \neq \zeta. && (11)
\end{aligned}
$$

In recent years, there has been an increasing interest in the study of different applications of the mathematical modelling of non-Newtonian fluids to coupled systems of partial differential equations. For instance, the ability of electro-rheological fluids to drastically change the mechanical properties under the influence of an external electromagnetic field [21, 23]. The main feature of the above system is that the equation for the electromagnetic field is uncoupled thus its regularity is as good as the obtained by the classical regularity theory.

The differential coupled system containing the Joule effects leads us to prove the existence of a solution of an elliptic equation with L^1-data. The mathematical modelling of such fluids was investigated by different authors [1, 3, 22], under Dirichlet conditions. We emphasize that the uniqueness of an elliptic problem with L^1-data is known for a linear operator and for monotone operators under different definitions of the solution, entropy solutions [5] or renormalized solutions [13] introduced for the study of the Boltzmann equations. Here we use the notion of SOLA, solution obtained by limit approximation, introduced in [12] for a strongly monotone operator and we prove that the uniqueness remains true for a strictly monotone operator.

In [8], the problem is studied under the Fourier linear heat flux and a Fourier-type boundary condition for the scalar energy. In [9], the particular case of Navier-Stokes-Fourier problem includes the radiation behavior at the two-dimensional space. In [11], the Joule effect is neglected and Dirichlet boundary condition for the fluid velocity is taken into account, since we deal with the existence of a weak solution capturing the radiation behavior in the three-dimensional space and the shear thinning phenomenon exhibited by a broader class of non-Newtonian fluids. That is, we provide upper bounds to the exponent of the nonlinear convective-radiative term and lower bounds to the exponent relative to the principal nonlinear elliptic operator than the classical restriction, $p > 3n/(n + 2)$, due to the existence of the convective term.

Recently slip conditions appear as, for instance, the nonlocal Colomb friction law (see [10] and the references therein) and the linear Navier law (see [4] and the references therein). This work adresses friction wall laws in the subdifferential form in section 6.

The outline of this work is as follows. In section 2, the weak variational formulation and the main results are stated. Section 3 is devoted to the existence of weak solutions of auxiliary problems and their characterizations. The proofs of the main results are presented in sections 4 and 5. Finally we present the weak variational formulation and the existence result (section 7) and its proof (section 8) relative to the heat conducting flow problem introduced in section 6.

2 Main results

Let us define our admissible spaces, for $p>1$ and $l\geq 1$,

$$\begin{array}{rcl} W_0^{1,p}(\Omega;\Gamma_0) & = & \{v\in W^{1,p}(\Omega):\ v=0 \text{ on } \Gamma_0\};\\ L^p_{\rm div} & = & \{\tau\in \mathbf{L}^p(\Omega):\ \nabla\cdot\tau\in L^p(\Omega)\};\\ X_{p,l} & = & \{v\in W_0^{1,p}(\Omega;\Gamma_0):\ v\in L^l(\Gamma)\}, \end{array}$$

and the norms

$$\|v\|_{1,p;\Omega}:=\|\nabla v\|_{p,\Omega}\ ,\ \|v\|_{X_{p,l}}:=\|v\|_{1,p;\Omega}+\|v\|_{l,\Gamma}.$$

We say that $(u,\tau,e)\in W_0^{1,p}(\Omega;\Gamma_0)\times L^{p'}_{\rm div}\times W_0^{1,r}(\Omega;\Gamma_0)$ is a *weak solution* to the $(p-q)$ coupled system (1)-(6) if it verifies (6) and satisfies

$$\int_\Omega\{\mathcal{F}(e,\nabla v)-\mathcal{F}(e,\nabla u)\}dx+\int_\Gamma\{\mathcal{G}(e,v)-\mathcal{G}(e,u)\}d\Gamma\geq\int_\Omega f(v-u)dx,\quad \forall v\in W_0^{1,p}(\Omega;\Gamma_0);\tag{12}$$

$$\int_\Omega \tau\cdot\nabla u dx=\int_\Omega\mathcal{F}(e,\nabla u)dx+\int_\Omega\mathcal{F}^*(e,\tau)dx;\tag{13}$$

$$-\int_\Gamma(\tau\cdot\mathbf{n})ud\Gamma=\int_\Gamma\mathcal{G}(e,u)d\Gamma+\int_\Gamma\mathcal{G}^*(e,-\tau\cdot\mathbf{n})d\Gamma;\tag{14}$$

$$\int_\Omega A(e,\nabla e)\cdot\nabla\phi dx+\int_\Gamma\gamma(e)\phi d\Gamma=\int_\Omega(\tau\cdot\nabla u)\phi dx-\int_\Gamma(\tau\cdot\mathbf{n})u\phi d\Gamma,\quad \forall\phi\in W_0^{1,r/(r-q+1)}(\Omega;\Gamma_0).\tag{15}$$

Here we define $\mathcal{F}^*$ and $\mathcal{G}^*$ as the conjugate functions to $\mathcal{F}$ and $\mathcal{G}$, respectively.

Theorem 1 *Assume that* $f\in L^{p'}(\Omega)$, $\mathcal{G}(x,e,u):\Gamma\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ *is a nonnegative Carathéodory function, convex on the third variable, and there exists* $1\leq s<p(n-1)/(n-p)$ *if* $p<n$ *or an arbitrary* $s\geq 1$ *otherwise such that*

$$\exists\varphi^\#>0:\ 0\leq\mathcal{G}(\cdot,e,u)\leq\varphi^\#(1+|u|^s),\ \ a.e.\ on\ \Gamma,\forall e,u\in\mathbb{R}.\tag{16}$$

Suppose that $\gamma : \mathbb{R} \to \mathbb{R}$ is a continuous function, $\gamma(0) = 0$,

$$\exists l \geq 1,\ \gamma^{\#} > 0: \qquad |\gamma(e)| \leq \gamma^{\#}(|e|^l + 1); \tag{17}$$

$$(\gamma(e) - \gamma(\xi))\mathrm{sign}(e - \xi) \geq 0,\ \forall e, \xi \in \mathbb{R}. \tag{18}$$

Then, if $1 \leq l < n(q-1)/(n-q)$ and $q < n$ or an arbitrary $l \geq 1$ otherwise, there exists a weak solution to the $(p-q)$ coupled system (1)-(6), for all $1 < r < (q-1)n/(n-1)$.

Theorem 2 *Under the assumptions of Theorem 1, and additionally*

$$\exists \gamma_{\#} > 0: \ \gamma(e)\mathrm{sign}(e) \geq \gamma_{\#}|e|^l, \quad \forall e \in \mathbb{R}; \tag{19}$$

for each $l \geq 1$ there exists a weak solution to the $(p-q)$ coupled system (1)-(6) such that $e \in X_{r,l}$, for all $1 < r < (q-1)n/(n-1)$.

The proofs are based on the application of the Tychonov-Kakutani-Glicksberg fixed point theorem [2, pages 218-220] for multivalued mappings, using three auxiliary existence results. The first one results from the classical elliptic theory for inequalities [24]. The second one envolves the duality theory on convex optimization and the existence of Lagrange multipliers [15]. The third one results from the elliptic equations theory with L^1-data [6, 12, 20]. Finally, compactness arguments are used to conclude the proofs. For reader's convenience, let us recall the fixed point theorem.

Theorem 3 (Tychonov-Kakutani-Glicksberg) *Let X be a locally convex Hausdorff topological vector space and K be a nonempty convex compact set in X. If $\mathcal{L} : K \to \mathcal{P}(K)$ is an upper semicontinuous mapping and $\mathcal{L}(z) \neq \emptyset$ is a convex and closed subset in K for every $z \in K$, then there exists at least one fixed point, $z \in \mathcal{L}(z)$.*

3 Auxiliary existence results

The first existence result is a consequence of the elliptic theory on inequalities [24, pp. 875].

Proposition 3.1 *For given $\xi \in W^{1,1}(\Omega)$, if the assumptions (7)-(8) and (16) are fulfilled then there exists $u \in W_0^{1,p}(\Omega;\Gamma_0)$ a unique weak solution to the problem (12), with ξ replacing e, satisfying the estimate*

$$\|u\|^p_{1,p;\Omega} \leq C\|f\|^{p'}_{p',\Omega}, \tag{20}$$

where C denotes a constant dependent on $\Omega, p, \alpha_{\#}$.

In the following, the same symbol C may denote different positive constants that dependent, at most, on the data. Let us state the following existence result due to the duality theory on the convex optimization (cf. [15, pp. 50-52]).

Proposition 3.2 *Let $u = u(\xi)$ be the solution given at Proposition 3.1. Then there exists a Lagrange multiplyer $(\tau, \varsigma) \in \mathbf{L}^{p'}(\Omega) \times L^{s'}(\Gamma)$ such that*

$$\begin{aligned} -\Upsilon^*(0,\tau,\varsigma) &= \sup\{-\Upsilon^*(0,\zeta,\psi) : \ \zeta \in \mathbf{L}^{p'}(\Omega), \ \psi \in L^{s'}(\Gamma)\} \\ &= \inf\{\Upsilon(v,0,0) : \ v \in W_0^{1,p}(\Omega;\Gamma_0)\} \end{aligned}$$

where Υ^* is the conjugate function of $\Upsilon : W_0^{1,p}(\Omega;\Gamma_0) \times \mathbf{L}^p(\Omega) \times L^s(\Gamma) \to \mathbb{R}$ is defined by

$$\Upsilon(v,\zeta,\psi) = \int_\Omega \mathcal{F}(\xi, \nabla v + \zeta)dx + \int_\Gamma \mathcal{G}(\xi, v+\psi)d\Gamma - \int_\Omega fvdx.$$

Moreover, (13)-(14) are verified and the estimates hold

$$\|\tau\|_{p',\Omega}^{p'} \leq \frac{2^p}{p-1}(\alpha^\# p)^{p'}\|\nabla u\|_{p,\Omega}^p + \frac{2}{p-1}(\alpha^\# p)^{p'}\mathrm{meas}(\Omega); \tag{21}$$

$$\|\varsigma\|_{s',\Gamma}^{s'} \leq \frac{2^s}{s-1}(\varphi^\# s)^{s'}\|u\|_{s,\Gamma}^s + \frac{2}{s-1}(\varphi^\# s)^{s'}\mathrm{meas}(\Gamma). \tag{22}$$

Proof. The existence of a Lagrange multiplier $(\tau,\varsigma) \in \mathbf{L}^{p'}(\Omega) \times L^{s'}(\Gamma)$, verifying (13)-(14), is a consequence of the existence of the minimizer u:

$$\Upsilon(u,0,0) = \inf\{\Upsilon(v,0,0) : \ v \in W_0^{1,p}(\Omega;\Gamma_0)\};$$

and the convexity of $\mathcal{F}$ and $\mathcal{G}$. From (8) and (16) we obtain

$$\mathcal{F}^*(\xi,\tau) \geq \alpha^\# \left((p-1)(\alpha^\# p)^{-p'}|\tau|^{p'} - 1\right);$$

$$\mathcal{G}^*(\xi,\varsigma) \geq \varphi^\# \left((s-1)(\varphi^\# s)^{-s'}|\varsigma|^{s'} - 1\right).$$

Then, it follows

$$\begin{aligned} -\alpha^\#|\Omega| + (\alpha^\#)^{1-p'}p^{-p'}(p-1)\int_\Omega |\tau|^{p'}dx &\leq \int_\Omega \mathcal{F}(\xi,\nabla u)dx + \int_\Omega \mathcal{F}^*(\xi,\tau)dx \\ &= \langle \tau, \nabla u\rangle \leq \frac{1}{p'}\|\tau\|_{p',\Omega}^{p'} + \frac{1}{p}\|\nabla u\|_{p,\Omega}^p, \end{aligned}$$

analogously for ς, resulting the estimates (21) and (22). Since $\tau \in \mathbf{L}^{p'}(\Omega)$ and $\nabla \cdot \tau = -f \in L^{p'}(\Omega)$, that is, $\tau \in L^{p'}_{\text{div}}$, then there exists a linear continuous mapping $\mathcal{T}$ such that $\mathcal{T}\tau \in L^{p'}(\Gamma)$ and the Green formula holds

$$\langle \tau, \nabla v \rangle + \langle \nabla \cdot \tau, v \rangle = \langle \mathcal{T}\tau, v \rangle, \quad \forall v \in W^{1,p}_0(\Omega; \Gamma_0).$$

So it results (cf. [10])

$$\varsigma = -\mathcal{T}\tau \text{ on } L^{s'}(\Gamma) \cap L^{p'}(\Gamma),$$

or simply $\varsigma = -\tau \cdot \mathbf{n}$. □

REMARK 3.1 *We remark that* $u \in W^{1,p}_0(\Omega; \Gamma_0) \hookrightarrow\hookrightarrow L^t(\Gamma)$, *with* $\max\{s, p\} \leq t < p(n-1)/(n-p)$ *if* $p < n$ *or an arbitrary* $t \geq \max\{s, p\}$ *otherwise.*

The third existence result on L^1-theory is proven via solutions obtained by limit approximation (SOLA) and the uniqueness is shown for that kind of solutions (cf. [6, 12, 20]).

Proposition 3.3 *Let the assumptions of Theorem 1 be fulfilled,* $\xi \in L^1(\Omega)$, $g \in L^1(\Omega)$ *and* $h \in L^1(\Gamma)$. *Then there exists a unique SOLA,* $e \in W^{1,r}_0(\Omega; \Gamma_0)$, *for all* $1 < r < (q-1)n/(n-1)$, *to the problem*

$$\int_\Omega A(\cdot, \xi, \nabla e) \cdot \nabla \phi dx + \int_\Gamma \gamma(e) \phi d\Gamma = \int_\Omega g \phi dx + \int_\Gamma h \phi d\Gamma, \tag{23}$$

for all $\phi \in W^{1,r/(r-q+1)}_0(\Omega; \Gamma_0)$. *Moreover, the estimate holds*

$$\|e\|_{1,r;\Omega} \leq C(\|g\|_{1,\Omega} + \|h\|_{1,\Gamma})^\lambda, \tag{24}$$

with $\lambda = \lambda(n, r, q)$ *and* C *a constant dependent on the structural constant data.*

PROOF. This proof is divided in four steps: (1) for each $M \in \mathbb{N}$, there exists a weak solution e_M of an approximate problem $(23)_M$; (2) $\gamma(e_M) \rightharpoonup \gamma(e)$ in $L^1(\Gamma)$; (3) $A(\cdot, \xi, \nabla e_M) \rightharpoonup A(\cdot, \xi, \nabla e)$ in $L^{r/(q-1)}(\Omega)$; (4) the uniqueness of SOLA.

STEP 1. Let M be a fixed natural number. Since $W^{1,q}_0(\Omega; \Gamma_0)$ is a reflexive Hausdorff Banach space ($q > 1$) the existence of a weak solution to the problem (23), with regular data $g = g_M$ and $h = h_M$ such that $g_M \in L^{q'}(\Omega)$

and $h_M \in L^{q'}(\Gamma)$, is consequence of the coercive, continuity, boundedness, and strictly monotone properties of the left hand side of the equality (23). Indeed

$$\int_\Omega A(\cdot,\xi,\nabla e)\cdot\nabla e dx + \int_\Gamma \gamma(e) e d\Gamma \geq \nu_\# \|\nabla e\|^q_{q,\Omega}; \quad (25)$$

$$|\int_\Omega A(\cdot,\xi,\nabla e)\cdot\nabla\phi dx + \int_\Gamma \gamma(e)\phi d\Gamma| \leq \quad (26)$$

$$\leq \|A(\cdot,\xi,\nabla e)\|_{q',\Omega}\|\nabla\phi\|_{q,\Omega} + \|\gamma(e)\|_{q(n-1)/(nq-n),\Gamma}\|\phi\|_{q(n-1)/(n-q),\Gamma} \leq$$

$$\leq C(1+\|\nabla e\|^{q-1}_{q,\Omega} + \|e\|^l_{lq(n-1)/(nq-n),\Gamma})\|\nabla\phi\|_{q,\Omega},$$

remarking that $W^{1,q}(\Omega) \hookrightarrow L^{lq(n-1)/(nq-n)}(\Gamma)$ for $l \leq n(q-1)/(n-q)$ and $q < n$. The estimate (24) is derived taking a suitable test function in the approximate problem $(23)_M$ (for details see, for instance, [6]).

STEP 2. Arguing as in [20], choose $\phi = T_k(e_M)$ as a test function in (23), where $T_k(e) = \min\{k, \max\{e,-k\}\}$, $(k > 0)$, considering g_M and h_M as regular data such that $\|g_M\|_{q',\Omega} \leq \|g\|_{1,\Omega}$ and $\|h_M\|_{q',\Gamma} \leq \|h\|_{1,\Gamma}$. Since the convective term vanishes, from the assumption (18) we obtain

$$k\int_{[|e_M|>k]} |\gamma(e_M)| d\Gamma \leq \int_{[|e_M|\leq k]} \gamma(e_M)e_M d\Gamma + k\int_{[|e_M|>k]} \gamma(e_M)\mathrm{sign}(e_M) d\Gamma =$$

$$= \int_\Gamma \gamma(e_M)T_k(e_M) d\Gamma \leq k(\|g\|_{1,\Omega} + \|h\|_{1,\Gamma}).$$

Passing to the limit, when k tends to zero, it results

$$\|\gamma(e_M)\|_{1,\Gamma} \leq \|g\|_{1,\Omega} + \|h\|_{1,\Gamma}. \quad (27)$$

From the compact embedding $W^{1,r}(\Omega) \hookrightarrow\hookrightarrow L^t(\Gamma)$, for $1 < t < r(n-1)/(n-r)$, we can extract a subsequence of e_M, still denoted by e_M, such that $e_M \to e$ a.e. in Γ. Thus $\gamma(e_M) \to \gamma(e)$ a.e. in Γ, and consequently the desired weak convergence arises.

STEP 3. To prove the convergence of ∇e_M to ∇e a.e. in Ω, it is sufficient to prove the convergence in measure:

$$\forall\delta, \forall\varepsilon, \exists m_0 : \ M \geq m_0 \Rightarrow \mathrm{meas}\{x\in\Omega : \ |(\nabla e_M - \nabla e)(x)| \geq \delta\} \leq \varepsilon.$$

Arguing as in [20], for $k, \eta > 0$ we have

$$\{x\in\Omega : \ |(\nabla e_M - \nabla e)(x)| \geq \delta\} \subseteq \cup_{i=1}^6 A_i$$

where

$$\begin{array}{rcll}
A_1 & = & \{x \in \Omega : \ |e_M(x)| \geq k\} & \Rightarrow \text{meas}(A_1) \leq \|e_M\|_{1,\Omega}/k; \\
A_2 & = & \{x \in \Omega : \ |e(x)| \geq k\} & \Rightarrow \text{meas}(A_2) \leq \|e\|_{1,\Omega}/k; \\
A_3 & = & \{x \in \Omega : \ |\nabla e_M(x)| \geq k\} & \Rightarrow \text{meas}(A_3) \leq \|\nabla e_M\|_{1,\Omega}/k; \\
A_4 & = & \{x \in \Omega : \ |\nabla e(x)| \geq k\} & \Rightarrow \text{meas}(A_4) \leq \|\nabla e\|_{1,\Omega}/k; \\
A_5 & = & \{x \in \Omega : \ |(e_M - e)(x)| \geq \eta\} & \Rightarrow \text{meas}(A_5) \leq \|e_M - e\|_{1,\Omega}/\eta,
\end{array}$$

$$\begin{array}{l}
A_6 = \{x \in \Omega : \ |(\nabla e_M - \nabla e)(x)| \geq \delta, \\
|e_M(x)| \leq k, |e(x)| \leq k, |\nabla e_M(x)| \leq k, |\nabla e(x)| \leq k, |(e_M - e)(x)| \leq \eta\}.
\end{array}$$

Since $e_M \to e$ in $L^1(\Omega)$, for all $\eta > 0$, there exists m_1 such that $M \geq m_1$, $\text{meas}(A_5) \leq \varepsilon$, and choosing k sufficiently large we have $\text{meas}(A_i) \leq \varepsilon, i = 1, 2, 3, 4$. To estimate $\text{meas}(A_6)$, choose $\phi = T_\eta(e_M - T_k(e))$ as a test function in $(23)_M$. Thus it results

$$\begin{array}{r}
\displaystyle\int_\Omega [A(\cdot, \xi, \nabla e_M) - A(\cdot, \xi, \nabla T_k(e))].\nabla T_\eta(e_M - T_k(e)) + \\
\displaystyle + \int_\Gamma [\gamma(e_M) - \gamma(T_k(e))] T_\eta(e_M - T_k(e)) = \\
\displaystyle = \int_\Omega g_M T_\eta(e_M - T_k(e)) + \int_\Gamma h_M T_\eta(e_M - T_k(e)) - \\
\displaystyle - \int_\Omega A(\cdot, \xi, \nabla T_k(e)) \cdot \nabla T_\eta(e_M - T_k(e)) - \\
\displaystyle - \int_\Gamma \gamma(T_k(e)) T_\eta(e_M - T_k(e)).
\end{array}$$

From the assumption (18) we have

$$[\gamma(e_M) - \gamma(T_k(e))] T_\eta(e_M - T_k(e)) \geq 0.$$

From the assumption (17) we get

$$\begin{array}{rcl}
\displaystyle\left| \int_\Gamma \gamma(T_k(e)) T_\eta(e_M - T_k(e)) \right| & \leq & \displaystyle\eta \gamma^{\#} \int_\Gamma (|T_k(e)|^l + 1) \\
& \leq & \eta \gamma^{\#} (k^l + 1) \text{meas}(\Gamma)
\end{array}$$

where k is already chosen sufficiently large. Thus η is chosen sufficiently small such that $\eta\gamma^{\#}(k^l+1)\text{meas}(\Gamma) < \varepsilon/C$, with C a constant independent on k, η and ε. Then we can proceed as in [20] to conclude that $\text{meas}(A_6) < \varepsilon/6$.

Then we conclude that $A(\cdot,\xi,\nabla e_M) \to A(\cdot,\xi,\nabla e)$ a.e. in Ω. From the assumption (10), $A(\cdot,\xi,\nabla e_M)$ belongs to a bounded subset of $L^{r/(q-1)}(\Omega)$. Therefore we get the desired weak convergence.

STEP 4. The existence of a solution to (23) is a consequence of the passage to the limit at step 1, taking into account steps 2 and 3. To prove SOLA uniqueness, let $\{(g_M, h_M)\}$ and $\{(\tilde{g}_M, \tilde{h}_M)\}$ two different sequences such that

$$g_M, \tilde{g}_M \rightharpoonup g \text{ in } L^1(\Omega);$$
$$h_M, \tilde{h}_M \rightharpoonup h \text{ in } L^1(\Gamma).$$

Thus taking $e_M = e(\xi, g_M, h_M)$ and $\tilde{e}_M = e(\xi, \tilde{g}_M, \tilde{h}_M)$ the corresponding solutions to $(23)_M$, we have $e_M \to e = e(\xi,g,h)$ and $\tilde{e}_M \to \tilde{e} = e(\xi,g,h)$ in $W_0^{1,r}(\Omega;\Gamma_0)$. To prove that $e = \tilde{e}$ it remains to prove that $e_M - \tilde{e}_M \to 0$ in $W_0^{1,r}(\Omega;\Gamma_0)$. Indeed, arguing as in step 3 and choosing $\phi = T_\eta(e_M - T_k(\tilde{e}_M))$ as a test function in $(23)_M$ we obtain

$$\int_\Omega [A(\cdot,\xi,\nabla e_M) - A(\cdot,\xi,\nabla T_k(\tilde{e}_M))].\nabla T_\eta(e_M - T_k(\tilde{e}_M)) \leq$$
$$-\int_\Omega A(\cdot,\xi,\nabla T_k(\tilde{e}_M)) \cdot \nabla T_\eta(e_M - T_k(\tilde{e}_M)) +$$
$$+\eta\{\|g\|_{1,\Omega} + \|h\|_{1,\Gamma} + \gamma^{\#}(k^l+1)\text{meas}(\Gamma)\}$$

Then we can proceed as before concluding that e is the unique solution obtained as limit approximation (SOLA). □

Proposition 3.4 (Continuous dependence) *Let $\{\xi_m\}, \{g_m\}$ and $\{h_m\}$ be sequences such that $\xi_m \to \xi$ in $L^1(\Omega)$, $g_m \rightharpoonup g$ in $L^1(\Omega)$, $h_m \rightharpoonup h$ in $L^1(\Gamma)$ and there exist R_2, $R_3 > 0$ such that $\|g_m\|_{1,\Omega} \leq R_2$ and $\|h_m\|_{1,\Gamma} \leq R_3$. Then the solutions $e_m = e(\xi_m, g_m, h_m)$ to (23) are such that $e_m \to e$ in $W_0^{1,r}(\Omega;\Gamma_0)$, for all $1 < r < (q-1)n/(n-1)$, where $e = e(\xi,g,h)$ is the solution to (23).*

Proof. Let $\{\xi_m\}, \{g_m\}$ and $\{h_m\}$ be sequences in the conditions of the proposition. Take regular approximations g_{mM} and h_{mM}, $M \in \mathbb{N}$, to each g_m and h_m, respectively. By the diagonalization argument, we find g_{mm} and h_{mm} weakly convergent to g and h in $L^1(\Omega)$ and $L^1(\Gamma)$, respectively.

Proposition 3.3 (step 1) guarantees the existence of regular corresponding solutions $e_{mm} = e(\xi_m, g_{mm}, h_{mm})$ that verify (24) and (27). The argument used in Proposition 3.3 (step 3) remains valid replacing ξ by ξ_m (cf. [20]). Under these circumstances, we have $e_{mm} \to e = e(\xi, g, h)$, where e is the unique SOLA.

4 Proof of Theorem 1

In order to apply Theorem 3, let us consider the closed convex ball

$$K := \{(\xi, g, h) \in W_0^{1,1}(\Omega; \Gamma_0) \times L^1(\Omega) \times L^1(\Gamma) : \ \|\xi\|_{1,1;\Omega} \leq R_1, \\ \|g\|_{1,\Omega} \leq R_2, \ \|h\|_{1,\Gamma} \leq R_3\}$$

for some R_1, R_2 and $R_3 > 0$ conveniently chosen by (24), (20)-(21) and (20)-(22), respectively. The ball K is compact when the topological vector space is provided by the weak topology.

Let us built the operator $\mathcal{L} : K \subseteq W_0^{1,1}(\Omega; \Gamma_0) \times L^1(\Omega) \times L^1(\Gamma) \to \mathcal{P}(K)$ as follows

$$\mathcal{L}(\xi, g, h) = \{(e, \tau \cdot \nabla u, -(\tau \cdot \mathbf{n})u)\}$$

where e is the SOLA solution given at Proposition 3.3, u is the minimizer given at Proposition 3.1 and τ is a correspondent Lagrange multiplyer given at Proposition 3.2. The functional operator $\mathcal{L}$ is well defined, in the sense of Theorem 3, since $\mathcal{L}(\xi, g, h)$ is a convex set due to the uniqueness of the minimizer u, the convex property of the set of Lagrange multipliers, and the uniqueness of a solution obtained by limit approximation (cf. Proposition 3.3). To conclude the proof it remains to prove the closeness in $K \times K$ of the graph set:

$$G_{KK}(\mathcal{L}) := \{(y, z) \in K \times K : \ z \in \mathcal{L}(y)\}.$$

Take the sequences $(\xi_m, g_m, h_m) \in K$ and $(e_m, \tau_m \cdot \nabla u_m, -(\tau_m \cdot \mathbf{n})u_m) \in \mathcal{L}(\xi_m, g_m, h_m)$ satisfying

$$\begin{array}{rll} \xi_m \rightharpoonup \xi, \ e_m \rightharpoonup e & \text{in} & W_0^{1,1}(\Omega; \Gamma_0); \\ g_m \rightharpoonup g, \ \tau_m \cdot \nabla u_m \rightharpoonup \varkappa_1 & \text{in} & L^1(\Omega); \\ h_m \rightharpoonup h, \ (\tau_m \cdot \mathbf{n})u_m \rightharpoonup \varkappa_2 & \text{in} & L^1(\Gamma). \end{array}$$

The limit solution e is proven at Proposition 3.4. The proof of $u_m \rightharpoonup u$ in $W_0^{1,p}(\Omega; \Gamma_0)$ is classical on elliptic inequalities [19] and the convergence

$\tau_m \cdot \nabla u_m \rightharpoonup \tau \cdot \nabla u$ in $L^1(\Omega)$ follows by similar arguments already used in [8]. Finally, the convergence, $(\tau_m \cdot \mathbf{n})u_m \rightharpoonup (\tau \cdot \mathbf{n})u$ in $L^1(\Gamma)$, is a consequence of the strong convergence $u_m \to u$ in $L^p(\Gamma)$ and $\tau_m \rightharpoonup \tau$ in $L^{p'}(\Gamma)$. Therefore Theorem 3 can be applied and the existence of the required solution holds.

5 Proof of Theorem 2

The proof follows each step of the proof of Theorem 1. The difference comes from the choice of the functional space $X_{q,l+1}$ in step 1 of Proposition 3.3. Indeed, $e_M \in X_{q,l+1}$ satisfies (23) for all $\phi \in X_{q,l+1}$, taking into account that the estimates (25) and (26) read

$$\int_\Omega A(\cdot,\xi,\nabla e)\cdot\nabla e dx + \int_\Gamma \gamma(e) e d\Gamma \geq \min\{\nu_\#, \gamma_\#\}\|e\|_{X_{q,l+1}};$$

$$|\int_\Omega A(\cdot,\xi,\nabla e)\cdot\nabla\phi dx + \int_\Gamma \gamma(e)\phi d\Gamma| \leq$$
$$\leq \|A(\cdot,\xi,\nabla e)\|_{q',\Omega}\|\nabla\phi\|_{q,\Omega} + \|\gamma(e)\|_{(l+1)/l,\Gamma}\|\phi\|_{l+1,\Gamma} \leq$$
$$\leq C(1+\|\nabla e\|_{q,\Omega}^{q-1} + \|e\|_{l+1,\Gamma}^{l})\|\phi\|_{X_{q,l+1}}.$$

Applying (19), the estimate (27) implies

$$\|e\|_{l,\Gamma}^{l} \leq (\|g\|_{1,\Omega} + \|h\|_{1,\Gamma})/\gamma_\#. \tag{28}$$

Thus we can proceed as before.

6 A heat conducting flow problem

We consider the existence, at least on a part of the boundary of the heat conducting viscous incompressible fluid, of slip friction condition constituted by the non-penetration condition plus a subdifferential relation between the tangential velocity $\mathbf{u}_T$ and the tangential viscous stress τ_T [16]

$$\mathbf{u}_N = 0 \text{ and } -\tau_T \in \partial[\varphi(\cdot,e)|\mathbf{u}_T|^s], \quad s \geq 1. \tag{29}$$

This formulation includes the Coulomb friction law ($s=1$), the linear Navier law ($s=2$), as well as the Chezy-Manning law ($s=3$) when non-Newtonian fluids are taken into account. Here the yield coefficient φ depends on the specific internal energy e. The viscous part τ of the Cauchy stress tensor $\sigma = (\sigma_{ij})$ belongs to the subdifferential of a functional $\mathcal{F}$ at the point given by $D\mathbf{u} = (\nabla \mathbf{u} + (\nabla \mathbf{u})^T)/2$, for a fixed internal energy [7]

$$\tau = \pi I + \sigma \in \partial \mathcal{F}(e, D(\mathbf{u})) \tag{30}$$

where π denotes the pressure and I is the identity matrix. For a differentiable $\mathcal{F}$, this constitutive law includes the asymptotically Newtonian class of fluids, such as for instance the Prandtl-Eyring, Cross, Williamson and Carreau models, and the so well-known Navier-Stokes fluid. The governing equations for steady-state heat conducting viscous incompressible fluids under study are the energy, motion and incompressibility equations in the following form

$$\mathbf{u} \cdot \nabla e - \nabla \cdot (\chi(\cdot, e)\mathbf{a}(\nabla e)) = \tau : D\mathbf{u} \qquad \text{in } \Omega; \tag{31}$$

$$(\mathbf{u} \cdot \nabla)\mathbf{u} - \nabla \cdot \tau = -\nabla \pi + \mathbf{f} \qquad \text{in } \Omega; \tag{32}$$

$$\nabla \cdot \mathbf{u} = \sum_{i=1}^{n} \frac{\partial u_i}{\partial x_i} = 0 \qquad \text{in } \Omega, \tag{33}$$

where the density is assumed constant and equal to one, for simplicity, and χ denotes the thermal diffusity.

The heat transfer due to conduction, radiation, and convection on a non-black surface Γ is

$$\chi(\cdot, e)\mathbf{a}(\nabla e) \cdot \mathbf{n} + \gamma(e) = -\tau_T \cdot \mathbf{u}_T. \tag{34}$$

The continuous boundary operator γ characterizes the radiation emitted, absorbed, reflected and/or transmitted on the given surface and it includes the Newton law $\gamma = h_c I$ with h_c denoting the convective heat transfer coefficient, and the blackbody or gray radiation $\gamma(e) = \varepsilon e^4$ with ε denoting the Stefan-Boltzmann constant or the gray percentage coefficient [17, pp. 232-235]. The right hand side represents the frictional work, also known as frictional heat, because it denotes the heat generated by the boundary friction at (29). Notice that the condition (34) can even be generalized to

$$\chi(\cdot, e)\mathbf{a}(\nabla e) \cdot \mathbf{n} + \tau_T \cdot \mathbf{u}_T \in \partial\gamma(e) \quad \text{on } \Gamma;$$

with γ a convex function [14]. We assume Dirichlet conditions on the remaining part Γ_0 of the boundary $\partial\Omega$:

$$\mathbf{u} = \mathbf{0} \qquad \text{and} \qquad e = 0. \tag{35}$$

7 A $(p-q)$ coupled fluid system

Let us fix the functional setting, for $p, q, s > 1$,

$$\begin{aligned} H_p &= \{\mathbf{u} \in \mathbf{L}^p(\Omega) : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \quad \mathbf{u} \cdot \mathbf{n} = 0 \text{ on } \partial\Omega\}; \\ V_p &= \{\mathbf{u} \in \mathbf{W}^{1,p}(\Omega) : \ \nabla \cdot \mathbf{u} = 0 \text{ in } \Omega, \ \mathbf{u} = \mathbf{0} \text{ on } \Gamma_0, \ u_N = 0 \text{ on } \Gamma\}; \\ L^p_{\text{sym}} &= \{\tau = (\tau_{ij}) : \tau_{ij} = \tau_{ji} \in L^p(\Omega)\}, \end{aligned}$$

endowed with their canonical Lebesgue and Sobolev norms, assuming always that meas $(\Gamma_0) > 0$ such that the Poincaré inequality holds in Ω.

We say that (29)-(35) is a $(p-q)$ *coupled fluid system* if

$$\mathcal{F}(e, \varkappa) = \mu(\cdot, e) F_1(|\varkappa|) + \eta(\cdot, e) F_2(|\varkappa|), \qquad \varkappa \in \mathbb{M}_{n\times n}$$

where $\mathbb{M}_{n\times n}$ is the set of symmetric matrices of the type $n\times n$. The viscosities $\mu, \eta : \Omega \times \mathbb{R} \to \mathbb{R}^+_0$ are Carathéodory functions, $F_1, F_2 : \mathbb{R}^+_0 \to \mathbb{R}^+_0$ are convex functions, $F_1(0) = F_2(0) = 0$, furthermore suppose that F_1 is strictly convex, and

$$\begin{aligned} \exists \mu_\#, \mu^\# > 0 : &\quad \mu_\# \leq \mu(x, e) \leq \mu^\#; && (36) \\ \exists \eta_\#, \eta^\# > 0 : &\quad \eta_\# \leq \eta(x, e) \leq \eta^\# && (37) \\ \exists p > 1, \ \alpha_\# > 0 : &\quad F_1(d) \geq \alpha_\# d^p; && (38) \\ \exists \alpha^\# > 0 : &\quad F_1(d) \leq \alpha^\# (d^p + 1); && (39) \\ \exists 1 \leq p_2 \leq p, \ \beta > 0 : &\quad 0 \leq F_2(d) \leq \beta(d^{p_2} + 1), && (40) \end{aligned}$$

and $\chi : \Omega \times \mathbb{R} \to \mathbb{R}$ is Carathéodory function, $\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous function and $\gamma, \varphi : \Gamma \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions, $\gamma(\cdot, 0) = 0$, satisfying (17)-(19), and

$$\begin{aligned} \exists \chi_\#, \chi^\# > 0 : &\quad \chi_\# \leq \chi(x, e) \leq \chi^\#; && (41) \\ \exists q > 2 - 1/n, \ \upsilon_\# > 0 : &\quad \mathbf{a}(\varkappa) \cdot \varkappa \geq \upsilon_\# |\varkappa|^q; && (42) \\ \exists \upsilon^\# > 0 : &\quad |\mathbf{a}(\varkappa)| \leq \upsilon^\# (|\varkappa|^{q-1} + 1); && (43) \\ &\quad (\mathbf{a}(\varkappa) - \mathbf{a}(\zeta)) \cdot (\varkappa - \zeta) > 0; && (44) \\ \exists \varphi^\# > 0 : &\quad 0 < \varphi(s, e) \leq \varphi^\#; && (45) \end{aligned}$$

almost everywhere $x \in \Omega$ and $s \in \Gamma$, for every $e \in \mathbb{R}$, $d \geq 0$ and $\varkappa, \zeta \in \mathbb{R}^n$.

We assume that

$$\mathbf{f} \in \mathbf{L}^{p'}(\Omega). \qquad (46)$$

DEFINITION. We say that $(\mathbf{u},\tau,e)$ is a *weak solution* to the (p-q) coupled fluid system (29)-(35) if $(\mathbf{u},\tau,\tau_T,e)\in V_p\times L^{p'}_{\rm sym}\times \mathbf{L}^{p'}(\Gamma)\times X_{r,l}$ satisfies (29)-(30) and

$$\int_\Omega D\mathbf{u}:\mathbf{u}\otimes\mathbf{v}dx+\int_\Omega\{\mathcal{F}(e,D\mathbf{v})-\mathcal{F}(e,D\mathbf{u})\}dx+$$
$$+\int_\Gamma\varphi(e)\{|\mathbf{v}_T|^s-|\mathbf{u}_T|^s\}d\Gamma\geq\int_\Omega\mathbf{f}\cdot(\mathbf{v}-\mathbf{u})dx,\quad\forall\mathbf{v}\in V_p; \tag{47}$$
$$\int_\Omega\mathbf{u}\cdot\nabla e\phi dx+\int_\Omega\chi(e)\mathbf{a}(\nabla e)\cdot\nabla\phi dx+\int_\Gamma\gamma(e)\phi d\Gamma=$$
$$=\int_\Omega\tau:D\mathbf{u}\phi dx-\int_\Gamma\tau_T\cdot\mathbf{u}_T\phi d\Gamma,\qquad\forall\phi\in W^{1,r/(r-q+1)}_0(\Omega;\Gamma_0). \tag{48}$$

Theorem 4 *Under the above assumptions, for $l\geq 1$,*

$$1\leq s<\frac{p(n-1)}{n-p}\ ,\ q>\frac{n(2p-1)}{p(n+1)-n}\ \textit{and}\ \ \frac{3n}{n+2}<p<n \tag{49}$$

or $s\geq 1$, $q>2-1/n$ and $p\geq n$, there exists a weak solution to the (p-q) coupled fluid system (29)-(35), for all $1<r<(q-1)n/(n-1)$.

The convective term $\int_\Omega D\mathbf{u}:\mathbf{w}\otimes\mathbf{v}dx$ in (47) has meanful for $\mathbf{w}\in H_t,\mathbf{u},\mathbf{v}\in V_p$ if $t\geq pn/(np+p-2n)$ and $p<n$, or $t\geq p'$ and $p\geq n$. The antisymmetry property is valid, and the compact embedding $V_p\hookrightarrow\hookrightarrow H_t$ occurs when $p>3n/(n+2)$ [18, 19]. For $1<r<n(q-1)/(n-1)$ and $q>2-1/n$, we have $r/(r-q+1)>n$ and then it is valid the Sobolev embedding $W^{r/(r-q+1)}(\Omega)\hookrightarrow L^\infty(\overline{\Omega})$. Then the convective term in (48) has meanful for $\mathbf{w}\in H_t,\ e\in W^{1,r}(\Omega)$ and $\phi\in W^{1,r/(r-q+1)}(\Omega)$ if $t\geq r'$. Thus the requirement $\max(pn/(p(n+1)-2n),r')\leq t<pn/(n-p)$ leads to the restriction (49). For $1\leq s<p(n-1)/(n-p)$, the compact embedding $W^{1,p}(\Omega)\hookrightarrow\hookrightarrow L^s(\Gamma)$ is valid.

8 Proof of Theorem 4

The proof follows the argument described at Section 2. Let us consider the space $X:=V_p\times L^1(\Omega)\times L^1(\Gamma)\times X_{1,l}$ endowed with the product of weak topologies. Thus X becomes a locally convex Hausdorff topological vector space, and the ball

$$K=\{(\mathbf{w},g,h,\xi)\in X:\|\mathbf{w}\|_{V_p}\leq R_1,\|g\|_{1,\Omega}\leq R_2,\|h\|_{1,\Gamma}\leq R_3,\|\xi\|_{X_{1,l}}\leq R_4\}$$

is a nonempty convex compact set in X, considering R_i $(i = 1, \cdots, 4)$ chosen later. Let us set a multivalued mapping $\mathcal{L}$ defined by

$$\mathcal{L}(\mathbf{w}, g, h, \xi) = \{(\mathbf{u}, \tau : D\mathbf{u}, -\tau_T \cdot \mathbf{u}_T, e)\}$$

where $\mathbf{u}$ is the unique solution given by the known result for the stationary fluid problem with prescribed coefficients as stated in the following proposition.

Proposition 8.1 *Let the assumptions (36)-(40) and (45)-(46) be fulfilled. For all* $p > 1, s \geq 1$, $\mathbf{w} \in H_t$ *and* $\xi \in W^{1,1}(\Omega)$, *there exists a unique solution* $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi) \in V_p$ *satisfying*

$$\int_\Omega D\mathbf{u} : \mathbf{w} \otimes \mathbf{v} dx + \int_\Omega \{\mathcal{F}(\xi, D\mathbf{v}) - \mathcal{F}(\xi, D\mathbf{u})\} dx + \\ + \int_\Gamma \varphi(\xi)\{|\mathbf{v}_T|^s - |\mathbf{u}_T|^s\} d\Gamma \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) dx, \quad \mathbf{v} \in V_p. \tag{50}$$

Moreover, the following estimate holds

$$\|\mathbf{u}\|_{V_p} \leq C \|\mathbf{f}\|_{p',\Omega}^{1/(p-1)} := R_1.$$

The existence of the Lagrange multipliers τ and τ_T is consequence of Proposition 3.2 and the existence result has the following form.

Proposition 8.2 *For each solution* $\mathbf{u}$ *given at the Proposition 8.1, there exists a stress tensor* $\sigma \in L^{p'}_{\text{sym}}$ *such that*

$$\sigma = -\pi I + \tau \ , \ \tau = -(\zeta_1 + \zeta_2) \quad \textit{and} \quad \tau_T = \varsigma; \tag{51}$$

where $\pi \in L_0^{p'}(\Omega)$, *the subspace of* $L^{p'}(\Omega)$ *consisting of functions with mean value equal to* 0, $\zeta_1, \zeta_2 \in L^{p'}_{\text{sym}}, \varsigma \in \mathbf{L}^{p'}(\Gamma)$ *satisfy*

$$s = 1: \qquad \int_\Gamma \varphi(\xi)|\mathbf{u}_T| d\Gamma = -\int_\Gamma \varsigma \cdot \mathbf{u}_T d\Gamma \ \textit{and} \ |\varsigma| \leq \varphi(\xi) \ \textit{on} \ \Gamma; \tag{52}$$

$$s > 1: \qquad \int_\Gamma \varphi(\xi)|\mathbf{u}_T|^s d\Gamma + C(s) \int_\Gamma \varphi(\xi) \left| \frac{\varsigma}{\varphi(\xi)} \right|^{s'} d\Gamma = -\int_\Gamma \varsigma \cdot \mathbf{u}_T d\Gamma;$$

$$\int_\Omega \mu(\xi) F_1(|D\mathbf{u}|) dx + \int_\Omega \mu(\xi) F_1^* \left(\left| \frac{\zeta_1}{\mu(\xi)} \right| \right) dx = -\int_\Omega \zeta_1 : D\mathbf{u} dx;$$

$$\int_\Omega \eta(\xi) F_2(|D\mathbf{u}|) dx + \int_\Omega \eta(\xi) F_2^* \left(\left| \frac{\zeta_2}{\eta(\xi)} \right| \right) dx = -\int_\Omega \zeta_2 : D\mathbf{u} dx;$$

$$(\mathbf{w} \cdot \nabla)\mathbf{u} + \nabla \cdot (\zeta_1 + \zeta_2) = \mathbf{f} - \nabla \pi \ \textit{in} \ \Omega. \tag{53}$$

Moreover, the estimates hold

$$\|\tau : D\mathbf{u}\|_{1,\Omega} \leq C(\|D\mathbf{u}\|^p_{p,\Omega} + 1) \leq C(R_1^p + 1) := R_2;$$
$$\|\tau_T \cdot \mathbf{u}_T\|_{1,\Gamma} \leq C\|\mathbf{u}\|_{V_p} \leq CR_1 := R_3.$$

Conversely, if $\mathbf{u} \in V_p$, $\sigma \in L^{p'}_{\rm sym}$ *and* $\tau_T \in \mathbf{L}^{p'}(\Gamma)$ *satisfy (51)-(53), then* $\mathbf{u}$ *is the solution to (50).*

The existence of a unique SOLA solution in accordance to L^1-theory (cf. Proposition 3.3 and the estimate (28)) is given at the following proposition.

Proposition 8.3 *Let the assumptions (17)-(19) and (41)-(44) be fulfilled. For each* $\mathbf{w} \in H_t$, $\xi \in L^1(\Omega)$, $g \in L^1(\Omega)$ *and* $h \in L^1(\Gamma)$, *there exists a SOLA solution* $e \in X_{r,l}$, *for all* $1 < r < (q-1)n/(n-1)$, *satisfying*

$$\int_\Omega \mathbf{w} \cdot \nabla e \phi dx + \int_\Omega \chi(\xi)\mathbf{a}(\nabla e) \cdot \nabla \phi dx + \int_\Gamma \gamma(e)\phi d\Gamma =$$
$$= \int_\Omega g\phi dx + \int_\Gamma h\phi d\Gamma, \quad \forall \phi \in W_0^{1,r/(r-q+1)}(\Omega; \Gamma_0). \tag{54}$$

Moreover, the estimate holds, independently on $\mathbf{w}$ *and* ξ,

$$\|e\|_{X_{r,l}} \leq C(\|g\|_{1,\Omega} + \|h\|_{1,\Gamma})^\lambda \leq C(R_2 + R_3)^\lambda := R_4,$$

with $\lambda = \lambda(n, r, q)$.

The set $\mathcal{L}(\mathbf{w}, g, h, \xi)$ is convex due to convex property of the set of Lagrange multipliers and the uniqueness of the solutions $\mathbf{u}$ and e. The upper semicontinuity of $\mathcal{L}$, and in particular the closeness of $\mathcal{L}(\mathbf{w}, g, h, \xi)$, comes from the closed graph property. Take the sequences $(\mathbf{w}_m, g_m, h_m, \xi_m) \in K$ and $(\mathbf{u}_m, \tau_m : D\mathbf{u}_m, -\varsigma_m \cdot \mathbf{u}_{mT}, e_m) \in \mathcal{L}(\mathbf{w}_m, g_m, h_m, \xi_m)$ satisfying

$$\begin{aligned}
\mathbf{w}_m \rightharpoonup \mathbf{w},\ \mathbf{u}_m \rightharpoonup \mathbf{u} \quad &\text{in} \quad V_p \hookrightarrow\hookrightarrow H_t \cap \mathbf{L}^p(\Gamma);\\
g_m \rightharpoonup g,\ \tau_m : D\mathbf{u}_m \rightharpoonup \varkappa_1 \quad &\text{in} \quad L^1(\Omega);\\
h_m \rightharpoonup h,\ \varsigma_m \cdot \mathbf{u}_{mT} \rightharpoonup \varkappa_2 \quad &\text{in} \quad L^1(\Gamma);\\
\xi_m \to \xi,\ e_m \to e \quad &\text{in} \quad X_{1,l} \hookrightarrow\hookrightarrow L^1(\Omega) \cap L^1(\Gamma),
\end{aligned}$$

where it is implicit that the symbol $\cap$ represents the function and its trace. The weak convergence yields $(\mathbf{u}, \varkappa_1, \varkappa_2, e) \in K$. The continuity property of the Niemytski operators μ, η and φ implies the strong convergence of the coefficients. From Remark 3.1, it follows the convergence $|\mathbf{u}_{mT}|^s \to |\mathbf{u}_T|^s$ in $\mathbf{L}^1(\Gamma)$. Thus, we can pass to the limit in $(50)_m$, when m tends to infinity, obtaining the solution $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi)$ to (50). Hence, we conclude that $\varkappa_2 = \varsigma \cdot \mathbf{u}_T$. The proof of $\varkappa_1 = \tau : D\mathbf{u}$ can be found in [8]. Arguing as in Theorem 2 we recognise that the weak limit e is the SOLA solution to (54). Indeed recalling the convergence of ξ_m to ξ a.e. in Ω and a.e. on Γ, the continuity of the Niemytsky operator χ allows the passage to the limit on the thermal diffusity coefficient. Then Theorem 4 is finished using Theorem 3.

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