Friction boundary conditions on thermal incompressible viscous flows*

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Abstract

This work adresses an unsteady heat flow problem involving friction and convective heat transfer behaviors on a part of the boundary. The problem is constituted by a variational motion inequality with energy dependent coefficients, and the energy equation in the framework of L^1-theory for the dissipative term. Using the duality theory of convex analysis, it also envolves the existence of Lagrange multipliers. Weak solutions of an approximate coupled system are proven by a fixed point argument for multivalued mappings and compactness methods. Then the existence result for the initial coupled system is proven by the passage to the limit.

1 Introduction

Isothermal fluids are well known and they have been exaustively studied ([18] and the references therein). At the recent years, a rigorous mathematical study of thermal flows is appearing ([8, 21] and the references therein), and also what type of boundary conditions should be taken into account. We refer, for instance, to [3, 23] and the references therein, that are devoted to several slip-type boundary conditions.

Here, a coupled system of parabolic inequations and equations derived from the motion and energy equations models the motion of non-Newtonian fluids undergoing the friction and convective heat transfer processes on a part of the boundary. The nonstandard boundary conditions as well as the energy dependent parameters are the main goal of the present work. Also the dissipative heating is not neglected leading analysis of the energy equation in the scope of L^1- theory.

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A somewhat related problems to the coupled system constituted by motion and energy equations, studying the thin film flow behavior in lubrification area and the roughness-induced effective boundary surface in hydrodynamic drag, are proposed in recent papers, e.g. [6, 15] and the literature quoted in these papers. These equations are also motivated by the turbulence modelling (see [19]), and other related problems [1]. For the study of the energy equation without any information on the fluid velocity vector $\mathbf{u}$, resulting from no consider the material in the framework of processes at high temperatures, see for instance [22] and references therein whose the operator is not coercive in $H^1(\Omega)$ due to the convective term, and the Joule effect is given as L^1 datum.

The present work extends the works [8] and [9], capturing the frictional phenomenum to the first work and the thermal phenomenum to the second one. We emphasize that at the present work we prove that the velocity gradient has strong convergence and, denoting by p and q the exponents given at the coercivity properties of the operators A and $\mathbf{a}$ representing the differential parts of the stress tensor and the heat flux, respectively, we present an existence result in which these values have lower critical values than the ones of the above mentioned works and have no dependence on each other if higher critical values are taken into account. Next Section is devoted to the statement of the problem. In Sect. 3, we set the weak variational coupled problem under general assumptions on the data and we state the existence results. In Sect. 4, we present the existence results for the velocity, the stress tensor and the internal energy problems when uncoupled parabolic equations and inequations are considered. In Sect. 5, we prove the existence of at least one weak solution to an approximate coupled system via a fixed point argument for multivalued mappings. Finally, exploiting the L^1-theory for partial differential equations, Sect. 6 leads to the main existence result due to the passage to the limit on the approximate problem.

2 Statement of the problem

Let Ω be a bounded open subset of $\mathbb{R}^n\,(n = 2,3)$ with Lipschitz continuous boundary $\partial\Omega$, $T > 0$ and $Q = \Omega\times]0,T[$. Incompressible viscous flows are governed by

$$\nabla\cdot\mathbf{u} = \sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i} = 0 \qquad \text{in } Q; \tag{1}$$

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\tau = -\nabla\pi + \mathbf{f} \qquad \text{in } Q \tag{2}$$

where $\mathbf{u}$ is the velocity vector, π denotes the pressure, $\mathbf{f}$ denotes the external forces, and $\tau = (\tau_{ij})$ denotes the viscous part of the Cauchy stress tensor $\sigma = (\sigma_{ij})$, that is, $\tau = \pi I + \sigma$, where I is the identity matrix. The density is constant and it is assumed equal to 1.

The class of non-Newtonian fluids considered in this paper is such that the viscous part τ belongs to the subdifferential of a continuous convex functional

$\mathcal{F}$ at the point given by $D\mathbf{u} = (D_{ij}) = (\partial_i u_j + \partial_j u_i)/2$, i, $j = 1, \cdots, n$: [7]

$$\tau - \mu(\cdot, e)A(D\mathbf{u}) \in \partial\mathcal{F}(D\mathbf{u}) = \partial\left[\eta(\cdot, e)F(|D\mathbf{u}|)\right], \tag{3}$$

where μ and η are the viscosities relative to the differential part (A) and the non-differential one (F), respectively, and e is a fixed internal energy. The following generalization is available

$$\mathcal{F}(D\mathbf{u}) = \sum_{l=1}^{L} \nu_l(\cdot, e)F_l(|D\mathbf{u}|),$$

where ν_l $(l = 1, ..., L)$ are different viscosities, and F_l $(l = 1, ..., L)$ are arbitrary convex functions. The well-known operator $A(\varkappa) = |\varkappa|^{p-2}\varkappa, p \geq 2$, describes the Newtonian $(p = 2)$ or the dilatant $(p > 2)$ behaviors, and the nondifferentiable part F includes, for instance, the Bingham behavior (F =id). The abstract mathematical data can be illustrated by the known Norton-Hoff law

$$\mu(\theta)D = \tau_{II}^{n-1}\tau$$

where θ represents the temperature and τ_{II} denotes the second invariant of the deviator tensor τ, $2\tau_{II} = \tau : \tau = |\tau|^2$, which includes the thermal Navier-Stokes fluid $(n = 1)$, and the Glen law for glaciology $(n = 3)$; and by the Arrhenius type law

$$\mu(\theta) = \mu_0 \exp\left[Q\left(1/\theta - 1/\theta_r\right)/R\right] \quad \text{for } \theta, \theta_r > 0,$$

where Q denotes the activation energy, R is the gas constant, and μ_0 and θ_r are constants and correspond to some convenient reference state where the viscosity and the temperature are known.

The internal energy e is a nonlinear invertible function on the temperature θ through the specific heat c_p

$$e = \int^{\theta} c_p(s)ds \Leftrightarrow \theta = \theta(e).$$

Thus the heat flux $\mathbf{q}$ given by the Fourier law is

$$\mathbf{q} = -k(\cdot, \theta)\nabla\theta = -\frac{k(\cdot, \theta)}{c_p(\theta)}\nabla e = -\chi(\cdot, e)\nabla e$$

where $\chi = (k/c_p) \circ \theta$ denotes the thermal diffusity. Then, considering a generalized Fourier law, the energy equation reads

$$\partial_t e + \mathbf{u} \cdot \nabla e - \nabla \cdot (\chi(\cdot, e)\mathbf{a}(\nabla e)) = \tau : D\mathbf{u} \text{ in } Q, \tag{4}$$

assuming that the external source is only constituted by the dissipative term. Notice that if only there is a weak solution to (1)-(3), i.e., $\nabla\mathbf{u} \in [L^p(Q)]^{n\times n}$ where the exponent p corresponds to the value given at the coercivity property of the operator A, then the Joule effect term $\tau : D\mathbf{u}$ belongs to $L^1(Q)$ and the existence of a solution of the energy equation (4) requires L^1-theory.

The boundary $\partial\Omega$ can be decomposed in two different ways as $\partial\Omega = \bar{\Gamma}_D \cup \bar{\Gamma}_F$ and $\partial\Omega = \bar{\Gamma}_T \cup \bar{\Gamma}_N$, where $\Gamma_\alpha,\ \alpha = D, F, T, N$, are open subsets of $\partial\Omega$ with smooth boundaries and such that $\Gamma_D \cap \Gamma_F = \emptyset, \Gamma_T \cap \Gamma_N = \emptyset$, and $\Gamma_D, \Gamma_T \neq \emptyset$. Hencefurther, let us assume that $\Gamma_D \equiv \Gamma_T$, where we consider the Dirichlet conditions

$$\mathbf{u} = \mathbf{0} \quad \text{and} \quad e = 0 \quad \text{on } \Gamma_D\times]0,T[; \tag{5}$$

and denote the friction and convective boundaries, Γ_F and Γ_N, simply by Γ, and $\Sigma = \Gamma\times]0,T[$.

In the sequel the liquid-solid interface constitutive law for the tangential deviator tensor τ_T has the general form on Σ [12]

$$\begin{cases} u_N = 0, \\ -\tau_T \in \partial[\varphi(\cdot, e)G(|\mathbf{u}_T|)], \qquad \forall e \in \mathbb{R}, \end{cases} \tag{6}$$

where $u_N, \mathbf{u}_T$ are the normal and the tangential components of the velocity vector, respectively, $\tau_T = \tau \cdot \mathbf{n} - \tau_N \mathbf{n}$ is the tangential component of τ, which coincides with the tangential stress tensor σ_T, and φ denotes the friction yield coefficient. Here $\mathbf{n} = (n_i)$ denotes the unit outward normal to $\partial\Omega$. For the power-law $G(|\mathbf{u}_T|) = |\mathbf{u}_T|^s, s \geq 1$, we have $\tau_T = -s\varphi(e)|\mathbf{u}_T|^{s-2}\mathbf{u}_T$. This class includes the linear Navier law ($s = 2$) for Newtonian fluids, the Coulomb law ($s = 1$), or the Chezy-Manning law ($s = 3$) between other related laws (see [13] and the references therein). The problem can be generalized for $\Gamma_i (i = 1, \cdots, n)$.

The Newton law of cooling on the boundary Γ should involve the frictional work [14]

$$\chi(\cdot, e)\mathbf{a}(\nabla e) \cdot \mathbf{n} + \gamma(\cdot, e)e = \tau_T \cdot \mathbf{u}_T, \tag{7}$$

where the energy dependent function γ represents the convective heat transfer coefficient.

3 Assumptions and main result

In the framework of Sobolev and Lebesgue functional spaces, we introduce the following Banach spaces, for $p, q, r > 1$,

$$\begin{aligned}
\mathcal{V} &= \{\mathbf{v} \in \mathbf{C}^\infty(\Omega) : \ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega\} \\
V_p &= \{\mathbf{v} \in \mathbf{W}^{1,p}(\Omega) : \ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D, \quad v_N = 0 \text{ on } \Gamma\}; \\
H_p &= \{\mathbf{v} \in \mathbf{L}^p(\Omega) : \ \nabla \cdot \mathbf{v} = 0 \text{ in } \Omega, \quad v_N = 0 \text{ on } \partial\Omega\}; \\
V_T &= \{\mathbf{v}|_\Gamma : \ \mathbf{v} \in V_p\}; \\
L^p_{\text{sym}} &= \{\tau = (\tau_{ij}) : \ \tau_{ij} = \tau_{ji} \in L^p(Q)\}; \\
W_q &= \{e \in W^{1,q}(\Omega) : \quad e = 0 \text{ on } \Gamma_D\}; \\
Z_{r,i} &= \{e \in L^r(0,T;W_r) : \ \partial_t e \in L^{(q-i)/(q-1)}(0,T;(W_{r/(r-q+1)})')\} \qquad (i = 0, 1)
\end{aligned}$$

endowed with the standard norms

$$\|\mathbf{u}\|_{V_p} = \|D\mathbf{u}\|_{p,\Omega}, \quad \|e\|_{W_q} = \|\nabla e\|_{q,\Omega}, \quad \|\mathbf{u}\|_{V_T} = \inf_{\substack{\mathbf{v}\in V_p \\ \mathbf{u}=\mathbf{v}|_\Gamma}} \|\mathbf{v}\|_{V_p}$$

$$\|\tau\|_{L^p_{\rm sym}} = \|(\tau:\tau)^{1/2}\|_{p,Q}, \quad \|e\|_{Z_{r,i}} = \|e\|_{r,W_r} + \|\partial_t e\|_{(q-i)/(q-1),(W_{r/(r-q+1)})'}$$

where $\|\cdot\|_{p,Q}$ is the canonical norm in $L^p(Q)$.

Definition. We say that $(\mathbf{u},\tau,e)$ is a weak solution to the problem (1)-(7) if $\mathbf{u}\in L^p(0,T;V_p)\cap L^\infty(0,T;H_2)$, $\partial_t\mathbf{u}\in L^{p'}(0,T;(V_p)')$, $\tau\in L^{p'}_{\rm sym}$, $e\in Z_{r,1}\cap L^\infty(0,T;L^1(\Omega))$:

$$\langle\partial_t\mathbf{u},\mathbf{v}-\mathbf{u}\rangle + \int_Q D\mathbf{u}:\mathbf{u}\otimes\mathbf{v} + \int_Q \mu(\cdot,e)A(D\mathbf{u}):D(\mathbf{v}-\mathbf{u}) +$$

$$+\int_Q \eta(\cdot,e)\{F(|D\mathbf{v}|)-F(|D\mathbf{u}|)\} + \int_\Sigma \varphi(\cdot,e)\{G(|\mathbf{v}_T|)-G(|\mathbf{u}_T|)\}\geq$$

$$\geq \langle\mathbf{f},\mathbf{v}-\mathbf{u}\rangle, \quad \forall\mathbf{v}\in L^p(0,T;V_p); \tag{8}$$

$$\tau-\mu(\cdot,e)A(D\mathbf{u})\in\partial[\eta(\cdot,e)F(D\mathbf{u})]; \tag{9}$$

$$\langle\partial_t e,\phi\rangle - \int_Q e\mathbf{u}\cdot\nabla\phi + \int_Q \chi(\cdot,e)\mathbf{a}(\nabla e)\cdot\nabla\phi + \int_\Sigma \gamma(\cdot,e)e\phi =$$

$$= \langle\tau:D\mathbf{u},\phi\rangle - \langle\tau_T\cdot\mathbf{u}_T,\phi\rangle, \quad \forall\phi\in L^\infty(0,T;W_{r/(r-q+1)}); \tag{10}$$

$$\mathbf{u}(\cdot,0)=\mathbf{u}_0, \quad e(\cdot,0)=e_0, \tag{11}$$

where $p'=p/(p-1)$ is the conjugate exponent to p, and the symbol $\langle\cdot,\cdot\rangle$ denotes a generic duality pairing, not distinguished between scalar and vector fields.

Remark 3.1 *The antisymmetry property of the convective terms* $\int_Q(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}$ *and* $\int_Q \mathbf{u}\cdot\nabla e\phi$ *are valid, considering the incompressibility property (1) and the boundary condition* $u_N=0$ *on* $\partial\Omega$ *given at (5)-(6).*

We assume that

$$e_0\in L^1(\Omega); \tag{12}$$

$$\mathbf{f}\in\mathbf{L}^{p'}(Q), \qquad \mathbf{u}_0\in H_2; \tag{13}$$

- $\mu,\eta,\chi: Q\times\mathbb{R}\to\mathbb{R}^+$ are Carathéodory functions, that is, measurable with respect to $(x,t)\in Q$ for every $e\in\mathbb{R}$, and continuous with respect to $e\in\mathbb{R}$ almost every $(x,t)\in Q$, and they satisfy, respectively,

$$\exists\mu_*,\mu^*>0:\ \mu_*\leq\mu(\cdot,e)\leq\mu^*, \quad \forall e\in\mathbb{R},\ \text{a.e. in } Q; \tag{14}$$

$$\exists\eta^*>0:\ 0\leq\eta(\cdot,e)\leq\eta^*, \quad \forall e\in\mathbb{R},\ \text{a.e. in } Q; \tag{15}$$

$$\exists\chi_*,\chi^*>0:\ \chi_*\leq\chi(\cdot,e)\leq\chi^*, \quad \forall e\in\mathbb{R},\ \text{a.e. in } Q; \tag{16}$$

- $A:\mathbb{M}_{n\times n}\to\mathbb{M}_{n\times n}$ is a continuous function such that $A(0)=0$, and satisfies

$$\exists p>\frac{3n+2}{n+2}, \qquad \exists\alpha_*>0:\ A(\varkappa):\varkappa\geq\alpha_*|\varkappa|^p, \tag{17}$$

$$\exists\alpha^*>0:\ |A(\varkappa)|\leq\alpha^*(|\varkappa|^{p-1}+1), \tag{18}$$

$$\exists\alpha>0: \qquad (A(\varkappa)-A(\zeta)):(\varkappa-\zeta)\geq\alpha|\varkappa-\zeta|^p,\ \forall\varkappa,\zeta\in\mathbb{M}_{n\times n}; \tag{19}$$

where $\mathbb{M}_{n\times n}$ denotes the set of real symmetric matrices of the type $n\times n$;

- $F : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is a continuous and convex function such that $F(0) = 0$, and

$$\exists \beta > 0 : \ 0 \leq F(d) \leq \beta(d^p + 1), \ \forall d \geq 0; \tag{20}$$

- $G : \mathbb{R}_0^+ \longrightarrow \mathbb{R}_0^+$ is a continuous and convex function such that $G(0) = 0$, and

$$\exists 1 \leq s < p(n-1)/(n-p), \ \exists \nu > 0 : \quad 0 \leq G(d) \leq \nu(d^s + 1), \ \forall d \geq 0; \tag{21}$$

- $\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n$ is a nonlinear continuous function obeying

$$\exists q \geq 2, \qquad \exists \upsilon_* > 0 : \ \mathbf{a}(\zeta) \cdot \zeta \geq \upsilon_* |\zeta|^q; \tag{22}$$

$$\exists \upsilon^* > 0 : \ |\mathbf{a}(\zeta)| \leq \upsilon^*(|\zeta|^{q-1} + 1); \tag{23}$$

$$\exists \upsilon > 0 : \qquad (\mathbf{a}(\zeta) - \mathbf{a}(\varkappa)) \cdot (\zeta - \varkappa) \geq \upsilon |\zeta - \varkappa|^q, \quad \forall \zeta, \varkappa \in \mathbb{R}^n; \tag{24}$$

- $\varphi, \gamma : \Sigma \times \mathbb{R} \to \mathbb{R}_0^+$ are Carathéodory functions such that

$$\exists \varphi^* > 0 : \ 0 \leq \varphi(\cdot, e) \leq \varphi^*, \quad \forall e \in \mathbb{R}, \ \text{a.e. in } \Sigma; \tag{25}$$

$$\exists \gamma^* > 0 : \ 0 \leq \gamma(\cdot, e) \leq \gamma^*, \quad \forall e \in \mathbb{R}, \ \text{a.e. in } \Sigma; \tag{26}$$

$$(\gamma(\cdot, e)e - \gamma(\cdot, \xi)\xi)\text{sign}(e - \xi) \geq 0, \ \forall e, \xi \in \mathbb{R}, \ \text{a.e. in } \Sigma. \tag{27}$$

In what follows the letter C denotes constants which may vary from line to line and may depend on any data constant $\Omega, T, p, q, s, \mu_*, \alpha_*, \upsilon_*, \chi_*$, etc, if not specified. Hereafter in order to simplify the exposition we will neglect the dependence on space and time of the coefficients μ, η, χ, φ, and γ, unless specially required. We emphasize that this dependence is crucial to the impossibility of the use of the Kirchoff transformation.

REMARK 3.2 *All terms on (10) have meanful. From*

$$q \geq 2 > 2n/(n+1) \Longrightarrow 1 < r < q - n/(n+1) < n(q-1)/(n-1),$$

we have $W_{r/(r-q+1)} \hookrightarrow C(\bar{\Omega})$ *because* $r/(r-q+1) > n$. *On the other hand,* $r/(r-q+1)$ *is the conjugate exponent to* $r/(q-1)$ *and* $\mathbf{a}(\nabla e) \in \mathbf{L}^{r/(q-1)}(Q)$ *for* $\nabla e \in \mathbf{L}^r(Q)$. *Finally, the convective term has sense,* $e\mathbf{u} \in L^1(0,T; L^{r/(q-1)}(\Omega))$. *Indeed, by interpolation technique we have*

$$\mathbf{u} \in L^p(0,T;V_p) \cap L^\infty(0,T;H_2) \hookrightarrow \begin{cases} \mathbf{L}^{p(n+2)/n}(Q), & p < n \\ \mathbf{L}^{p+2}(Q), & p \geq n; \end{cases}$$

and $e \in L^r(0,T;W_r) \cap L^\infty(0,T;L^1(\Omega)) \hookrightarrow L^{r(n+1)/n}(Q)$. *If* $p \geq n$ *and* $q \geq 2$, *then*

$$(q-1)/r > 1/(p+2) + n/[r(n+1)] \quad \forall r < q - n/(n+1) \ (\forall n \in \mathbb{N}).$$

When $n > 2$, $n > p$ *and* $r < q - n/(n+1)$, *we have*

$$\frac{n}{p(n+2)} + \frac{n}{r(n+1)} < \frac{q-1}{r} \quad \text{if } q \geq \frac{(2n+1)(n+2)p - n^2}{(n+1)[(n+2)p-n]}.$$

Notice that $q \to 2$ if and only if $p \to n$ and $q \geq (35p-9)/(20p-12)$ if $n=3$ (see Fig. 1). In order to capture the case $q=2$ for lower critical values of p, we loose on the range of the exponent r:

$$1 < r \leq \frac{p(n+2)}{n}\left(q - \frac{2n+1}{n+1}\right) < q - \frac{n}{n+1}$$
$$\textit{if} \quad q > \frac{(2n+1)(n+2)p + n(n+1)}{(n+1)(n+2)p},$$

that means, $q \geq 2$ for $12/5 < p < 3$ if $n=3$ (see Fig. 1).

Remark 3.3 *The exponent s, given by (21), is such that the Rellich-Sobolev imbedding, namely $V_p \hookrightarrow\hookrightarrow \mathbf{L}^s(\Gamma)$, is valid.*

Figure 1: The (p,q) relations at the three-dimensional space ($n=3,\ p>11/5$): $q = \max(2, (35p+12)/(20p))$ (solid line) and $q(20p-12) = 35p-9$ (dotted line).

The main result of this work is the following theorem.

Theorem 1 *Under the assumptions (12)-(27), there exists at least a solution $(\mathbf{u}, \tau, e)$ to the problem (8)-(11), for all*

$$1 < r < q - n/(n+1) \qquad \textit{if} \quad \begin{cases} q \geq 2, & p \geq n \\ q \geq \frac{(2n+1)(n+2)p - n^2}{(n+1)[(n+2)p-n]}, & \frac{3n+2}{n+2} < p < n \end{cases}$$

$$1 < r \leq \frac{5p}{3}\left(q - \frac{7}{4}\right) \qquad \textit{if} \quad \frac{35p+12}{20p} < q < \frac{35p-9}{20p-12} \ (n=3).$$

The existence of a solution is based in the following approximate result.

Theorem 2 *Let the assumptions (13)-(26) be fulfilled. For each $M \in \mathbb{N}$, if $e_0^M \in L^2(\Omega)$, then there exists $(\mathbf{u}_M, \tau_M, e_M)$ in $L^p(0,T;V_p) \cap C([0,T];H_2) \times L^{p'}_{\rm sym} \times L^q(0,T;W_q) \cap L^\infty(0,T;L^2(\Omega))$ satisfying, a.e. $t \in]0,T[$,*

$$\langle \partial_t \mathbf{u}_M, \mathbf{v} - \mathbf{u}_M \rangle + \int_\Omega D\mathbf{u}_M : \frac{M\mathbf{u}_M}{M + |\mathbf{u}_M|} \otimes \mathbf{v} +$$

$$+ \int_\Omega \mu(e_M) A(D\mathbf{u}_M) : D(\mathbf{v} - \mathbf{u}_M) + \int_\Omega \eta(e_M)\{F(|D\mathbf{v}|) - F(|D\mathbf{u}_M|)\} +$$

$$+ \int_\Gamma \varphi(e_M)\{G(|\mathbf{v}_T|) - G(|\mathbf{u}_{MT}|)\} \geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}_M), \ \forall \mathbf{v} \in V_p; \tag{28}$$

$$\tau_M - \mu(e_M) A(D\mathbf{u}_M) \in \partial\left[\eta(e_M) F(|D\mathbf{u}_M|)\right]; \tag{29}$$

$$\langle \partial_t e_M, \phi \rangle + \int_\Omega \frac{M\mathbf{u}_M}{M + |\mathbf{u}_M|} \cdot \nabla e_M \phi + \int_\Omega \chi(e_M) \mathbf{a}(\nabla e_M) \cdot \nabla \phi +$$

$$+ \int_\Gamma \gamma(e_M) e_M \phi = \int_\Omega \overline{\tau_M : D\mathbf{u}_M} \phi + \int_\Gamma \overline{-\tau_{MT} \cdot \mathbf{u}_{MT}} \phi, \ \forall \phi \in W_q; \tag{30}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0, \quad e(\cdot, 0) = e_0^M,$$

where $\tau_{MT} \in \mathbf{L}^{s'}(\Sigma)$ means the Lagrange multiplyer that satisfies (6), and $\bar{\cdot}$ denotes the cut function

$$\bar{z} := \begin{cases} z, & \text{if } |z| \leq M \\ M \operatorname{sign}(z), & \text{otherwise.} \end{cases}$$

4 Preliminary results

In this section we deal with some auxiliary results that will be essential in the proof of existence result for the coupled system (28)-(30).

Proposition 4.1 *Under the assumptions (13)-(15), (17)-(21) and (25), for every $M \in \mathbb{N}$ and $\xi \in L^1(0,T;W^{1,1}(\Omega))$ there exists $\mathbf{u} = \mathbf{u}_M(\xi)$ a unique solution to the problem*

$$\mathbf{u} \in L^p(0,T;V_p) \cap C([0,T];H_2), \ \partial_t \mathbf{u} \in L^{p'}(0,T;(V_p)');$$

$$\langle \partial_t \mathbf{u}, \mathbf{v} - \mathbf{u} \rangle + \int_\Omega D\mathbf{u} : \frac{M\mathbf{u}}{M + |\mathbf{u}|} \otimes \mathbf{v} + \int_\Omega \mu(\xi) A(D\mathbf{u}) : D(\mathbf{v} - \mathbf{u}) +$$

$$+ \int_\Omega \eta(\xi)\{F(|D\mathbf{v}|) - F(|D\mathbf{u}|)\} + \int_\Gamma \varphi(\xi)\{G(|\mathbf{v}_T|) - G(|\mathbf{u}_T|)\} \geq$$

$$\geq \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}), \quad a.e.\ t \in]0,T[\quad \forall \mathbf{v} \in V_p; \tag{31}$$

$$\mathbf{u}(\cdot, 0) = \mathbf{u}_0. \tag{32}$$

Proof. We follow the argument already used in [8] and [9]. For reader's convenience, we sketch the principal idea of the proof. For every $\varepsilon > 0$, we consider the approximate system of equations

$$\langle \partial_t \mathbf{u}_\varepsilon + (\frac{M\mathbf{u}_\varepsilon}{M+|\mathbf{u}_\varepsilon|}\cdot\nabla)\mathbf{u}_\varepsilon, \mathbf{v}\rangle + \int_Q \{\mu(\xi)A(D\mathbf{u}_\varepsilon) + \eta(\xi)\nabla F_\varepsilon(\mathbf{u}_\varepsilon)\} : D(\mathbf{v}) +$$

$$+ \int_\Sigma \varphi(\xi)\nabla G_\varepsilon(\mathbf{u}_\varepsilon)\cdot\mathbf{v} = \langle \mathbf{f}, \mathbf{v}\rangle \quad \forall \mathbf{v}\in L^p(0,T;V_p); \tag{33}$$

$$\mathbf{u}_\varepsilon(0) = \mathbf{u}_0, \tag{34}$$

where $\nabla F_\varepsilon : L^p(0,T;V_p) \to L^{p'}(0,T;(V_p)')$ is defined by

$$\nabla F_\varepsilon(\mathbf{u}_\varepsilon) = F'_\varepsilon(D_{II}(\mathbf{u}_\varepsilon))D\mathbf{u}_\varepsilon$$

where D_{II} denotes the second invariant of the symmetrized gradient D, and $\nabla G_\varepsilon : L^p(0,T;V_T) \to L^{p'}(0,T;(V_T)')$ is defined by

$$\nabla G_\varepsilon(\mathbf{u}_\varepsilon) = G'_\varepsilon(|\mathbf{u}_{\varepsilon T}|^2/2)\mathbf{u}_{\varepsilon T},$$

i.e., they are the Gateaux derivatives of the Moreau approximations which coincide with the Yosida approximations of the operators ∂F and ∂G (cf. [5]), respectively. Indeed, for a given $\varepsilon > 0$ the mappings $\nabla F_\varepsilon(\mathbf{u}) \in \partial F(|D(J^p_\varepsilon \mathbf{u})|)$ and $\nabla G_\varepsilon(\mathbf{u}) \in \partial G(|(J^T_\varepsilon \mathbf{u})_T|)$ are monotone and Lipschitzian, where $J^i_\varepsilon \mathbf{u}$ $(i = p, T)$ are the minimum attached in the Moreau-Yosida approximation to the convex functionals F and G, respectively.

The solution $\mathbf{u}_\varepsilon$ to the approximate problem (33)-(34) is obtained as the limit of solutions $\mathbf{u}_m$ in finite dimensional spaces computed by the Faedo-Galerkin method (for details see [8] and [9]).

For $p \geq (3n+2)/(n+2)$, the estimates are valid (cf. [7, 17, 20])

$$\|\mathbf{u}_\varepsilon(t)\|^2_{2,\Omega} + \frac{2\mu_*}{p'}\int_0^t \|D\mathbf{u}_\varepsilon\|^p_{p,\Omega} \leq \|\mathbf{u}_0\|^2_{2,\Omega} + C\int_0^t \|\mathbf{f}\|^{p'}_{p',\Omega} := R \tag{35}$$

$$\|A(D\mathbf{u}_\varepsilon)\|^{p'}_{p',Q} \leq C(\|D\mathbf{u}_\varepsilon\|^p_{p,Q} + 1) \tag{36}$$

$$\int_0^T \|\nabla F_\varepsilon(\mathbf{u}_\varepsilon)\|^{p'}_{(V_p)'} + \|\nabla G_\varepsilon(\mathbf{u}_\varepsilon)\|^{p'}_{(V_T)'}dt \leq C(R)$$

$$\int_0^T \|\partial_t \mathbf{u}_\varepsilon\|^{p'}_{(V_p)'}dt \leq C\left(\|\mathbf{u}_\varepsilon\|^{p'}_{\infty,H_2} + 1\right)\|\mathbf{u}_\varepsilon\|^p_{p,V_p} + C\|\mathbf{f}\|^{p'}_{p',Q}.$$

From Rellich-Sobolev imbeddings, $V_p \hookrightarrow\hookrightarrow H_2 \hookrightarrow (V_p)'$ and $V_p \hookrightarrow\hookrightarrow \mathbf{L}^s(\Gamma) \hookrightarrow (V_p)'$ for $s < p(n-1)/(n-p)$ and in according to the classical compactness and continuity results, we get

$$\begin{aligned}\{\mathbf{v}\in L^p(0,T;V_p) : \partial_t\mathbf{v} \in L^{p'}(0,T;(V_p)')\} \quad &\hookrightarrow\hookrightarrow L^p(0,T;H_2);\\ &\hookrightarrow\hookrightarrow L^p(0,T;\mathbf{L}^s(\Gamma));\\ \text{and} \quad &\hookrightarrow C([0,T];H_2).\end{aligned}$$

In order to pass to the limit (33) when $\varepsilon \to 0^+$, we consider

$$\begin{aligned}&\langle \nabla F_\varepsilon(\mathbf{u}_\varepsilon), D(\mathbf{v}-\mathbf{u}_\varepsilon)\rangle =\\ &= \langle \nabla F_\varepsilon(\mathbf{u}_\varepsilon), D(\mathbf{v}-J^p_\varepsilon\mathbf{u}_\varepsilon)\rangle + \langle \nabla F_\varepsilon(\mathbf{u}_\varepsilon), D(J^p_\varepsilon\mathbf{u}_\varepsilon - \mathbf{u}_\varepsilon)\rangle \leq\\ &\leq F(|D\mathbf{v}|) - F(|D(J^p_\varepsilon\mathbf{u}_\varepsilon)|), \quad \forall \mathbf{v}\in L^p(0,T;V_p)\end{aligned}$$

analogously

$$\langle \nabla G_\varepsilon(\mathbf{u}_\varepsilon), \mathbf{v}-\mathbf{u}_\varepsilon\rangle \le G\left(|\mathbf{v}_T|\right) - G\left(|(J_\varepsilon^T \mathbf{u}_\varepsilon)_T|\right), \quad \forall \mathbf{v}\in L^p(0,T;V_T)$$

to rewrite (33) as

$$\langle \partial_t \mathbf{u}_\varepsilon + (\frac{M\mathbf{u}_\varepsilon}{M+|\mathbf{u}_\varepsilon|}\cdot\nabla)\mathbf{u}_\varepsilon, \mathbf{v}\rangle + \int_Q \mu(\xi)A(D\mathbf{u}_\varepsilon):D\mathbf{v} + \int_Q \eta(\xi)F(|D\mathbf{v}|) + \\ + \int_\Sigma \varphi(\xi)G\left(|\mathbf{v}_T|\right) \ge \langle \partial_t\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon\rangle + \int_Q \mu(\xi)A(D\mathbf{u}_\varepsilon):D\mathbf{u}_\varepsilon + \\ + \int_Q \eta(\xi)F\left(|D(J_\varepsilon^p\mathbf{u}_\varepsilon)|\right) + \int_\Sigma \varphi(\xi)G\left(|(J_\varepsilon^T\mathbf{u}_\varepsilon)_T|\right) + \int_Q \mathbf{f}\cdot(\mathbf{v}-\mathbf{u}_\varepsilon).$$

Then the passage to the limit is due to the convergences

$$\begin{aligned}
\mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{in} \quad L^p(0,T;V_p); && (37)\\
\partial_t\mathbf{u}_\varepsilon &\rightharpoonup \partial_t\mathbf{u} && \text{in} \quad L^{p'}(0,T;(V_p)'); && (38)\\
\mathbf{u}_\varepsilon &\to \mathbf{u} && \text{in} \quad L^p(0,T;H_2) \text{ and a.e. in } Q; && (39)\\
\mathbf{u}_\varepsilon &\to \mathbf{u} && \text{in} \quad L^p(0,T;\mathbf{L}^s(\Gamma)) \text{ and a.e. in } \Sigma; && (40)\\
A(D\mathbf{u}_\varepsilon) &\rightharpoonup \varkappa && \text{in} \quad L^{p'}(0,T;(V_p)'); && (41)\\
J_\varepsilon^i\mathbf{u}_\varepsilon &\rightharpoonup \mathbf{u} && \text{in} \quad L^p(0,T;V_i); \quad i=p,T;
\end{aligned}$$

and taking into account the lower semicontinuity property of the convex and continuous functionals F and G, and the monotone property (19) of A (cf. [17] or [24]).

The functional equation (31) follows by standard arguments (cf. [24]). The uniqueness of a solution is a consequence of the assumptions (14), (19) and applying the Gronwall lemma to the following inequality

$$\|\mathbf{u}_1-\mathbf{u}_2\|_{2,\Omega}^2 + \mu_*\alpha\int_0^t \|D(\mathbf{u}_1-\mathbf{u}_2)\|_{p,\Omega}^p \le \\ \le \int_0^t D\mathbf{u}_2 : \left(\frac{M\mathbf{u}_1}{M+|\mathbf{u}_1|} - \frac{M\mathbf{u}_2}{M+|\mathbf{u}_2|}\right)\otimes(\mathbf{u}_1-\mathbf{u}_2) \\ \le 2M\int_0^t \|D\mathbf{u}_2\|_{p,\Omega}\|\mathbf{u}_1-\mathbf{u}_2\|_{2,\Omega}. \ \square$$

The existence of a deviator tensor is stated in the following proposition which detailed proof can be found in [8].

Proposition 4.2 *Let* $\mathbf{u}$ *be the solution given at Proposition 4.1. Then there exists a (in general nonunique) Lagrange multiplyer* $(\zeta,\varsigma)\in L_{\text{sym}}^{p'}\times\mathbf{L}^{s'}(\Sigma)$ *such that*

$$-\zeta \in \partial\left[\eta(\xi)F(|D\mathbf{u}|)\right]; \quad -\varsigma\in\partial\left[\varphi(\xi)G(|\mathbf{u}_T|)\right]; \qquad (42)$$

and the following estimates hold

$$\|\zeta\|_{p',Q}^{p'} \leq (\beta p\eta^*)^{p'}(p-1)^{-1}\left(2^p\|D\mathbf{u}\|_{p,Q}^p + 2|Q|\right) \tag{43}$$

$$\|\varsigma\|_{s',\Sigma}^{s'} \leq (\nu s\varphi^*)^{s'}(s-1)^{-1}\left(2^s\|\mathbf{u}\|_{s,\Sigma}^s + 2|\Sigma|\right). \tag{44}$$

Moreover, defining the viscous tensor $\tau \in L^{p'}_{\rm sym}$ *by* $\tau = \mu(\xi)A(D\mathbf{u}) - \zeta$*, we have* $\tau_T = \varsigma$ *and*

$$\|\tau\|_{p',Q}^{p'} \leq C(\|\mathbf{u}\|_{p,V_p}^p + 1). \tag{45}$$

Proof. Applying the duality theory of convex optimization [11, pp. 50-52], we define the operators

$$\begin{aligned}
\Upsilon: \quad & L^p(0,T;V_p) \to \mathbb{R} \\
& \Upsilon(\mathbf{v}) = \langle \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \mathbf{f}, \mathbf{v}\rangle + \langle \mu(\xi)A(D\mathbf{u}), D\mathbf{v}\rangle \\
\Lambda: \quad & L^p(0,T;V_p) \to L^p_{\rm sym} \times \mathbf{L}^s(\Sigma) \\
& \Lambda\mathbf{v} = (\Lambda_1\mathbf{u}, \Lambda_2\mathbf{v}) = (D\mathbf{v}, \mathbf{v}_T) \\
\mathcal{J}: \quad & L^p_{\rm sym} \times \mathbf{L}^s(\Sigma) \to \mathbb{R} \\
& \mathcal{J}(\zeta,\varsigma) = \int_Q \eta(\xi)F(|\zeta|) + \int_\Sigma \varphi(\xi)G(|\varsigma|),
\end{aligned}$$

where $\mathbf{u}$ is the solution given at Proposition 4.1. Then there exists a Lagrange multiplyer $(\zeta,\varsigma) \in L^{p'}_{\rm sym} \times \mathbf{L}^{s'}(\Sigma)$ solution of the dual problem defined by

$$\sup_{(\zeta,\varsigma)\in L^{p'}_{\rm sym}\times\mathbf{L}^{s'}(\Sigma)} [-\Upsilon^*(\Lambda^*(\zeta,\varsigma)) - \mathcal{J}^*(-(\zeta,\varsigma))].$$

For details, see [8] and [9]. Indeed, the extremality relations (42) can be denoted, respectively, in the forms

$$\int_Q \eta(\xi)F(|D\mathbf{u}|) + \int_Q \eta(\xi)F^*\left(\left|\frac{\zeta}{\eta(\xi)}\right|\right) = \langle -\zeta, D\mathbf{u}\rangle \tag{46}$$

$$\int_\Sigma \varphi(\xi)G(|\mathbf{u}_T|) + \int_\Sigma \varphi(\xi)G^*\left(\left|\frac{\varsigma}{\varphi(\xi)}\right|\right) = \langle -\varsigma, \mathbf{u}_T\rangle \tag{47}$$

where F^* and G^* are the conjugate functions of F and G, respectively, and $\zeta \in L^{p'}_{\rm sym}$ satisfies

$$\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot(\mu(\xi)A(D\mathbf{u})) + \nabla\cdot\zeta = \mathbf{f} \text{ in } L^{p'}(0,T;(V_p)'). \tag{48}$$

From Rham Theorem (see, for instance, [17, pp. 67-68]), there exists a pressure $\pi \in L^{p'}_0(Q)$ such that belongs to a bounded set of $L^{p'}(Q)$ independently of $\xi, \mathbf{u}$ and (τ,ς), and the following relation holds

$$\int_\Omega \{\partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot(\mu(\xi)A(D\mathbf{u})) + \nabla\cdot\zeta\}\cdot\mathbf{v} = \int_\Omega \{\mathbf{f} - \nabla\pi\}\cdot\mathbf{v}$$
$$\text{a.e. in }]0,T[; \quad \forall \mathbf{v} \in \mathbf{W}^{1,p}(\Omega): \ \mathbf{v}|_{\Gamma_D} = 0, \ v_N|_\Gamma = 0.$$

There exists uniquely determined linear continuous mapping ϖ_T defined on $\{\sigma \in L^{p'}_{\rm sym}(\Omega) : \nabla\cdot\sigma \in (V_p)'\}$ into $(V_T)'$ such that $\varpi_T(\sigma) = \sigma|_\Gamma\cdot\mathbf{n} - [(\sigma|_\Gamma\cdot\mathbf{n})\cdot\mathbf{n}]\mathbf{n}$ if $\sigma \in [C^1(\bar\Omega)]^{n\times n}$, and the Green formula is satisfied (see, for instance, [16, page 90])

$$\langle \sigma, D\mathbf{v}\rangle + \langle \nabla\cdot\sigma, \mathbf{v}\rangle = \langle \varpi_T\sigma, \mathbf{v}_T\rangle \quad \forall \sigma \in L^{p'}_{\rm sym}(\Omega),\ \forall \mathbf{v} \in V_p,$$

denoting by $\langle\cdot,\cdot\rangle$ every duality pairing. Therefore it follows

$$\zeta\cdot\mathbf{n} + \varsigma = \mu(\xi)A(D\mathbf{u})\cdot\mathbf{n} \text{ in } L^{p'}(0,T;(V_T)'). \tag{49}$$

From (20) and (46), and applying Hölder and Young inequalities we get

$$\begin{aligned}\frac{B}{p'}\|\zeta\|^{p'}_{p',Q} - \beta\eta^*|Q| \le \int_Q \eta(\xi)F^*\left(\left|\frac{\zeta}{\eta(\xi)}\right|\right) \le \langle -\zeta, D\mathbf{u}\rangle \\ \le \frac{B}{2p'}\|\zeta\|^{p'}_{p',Q} + \frac{2^{p/p'}}{pB^{p/p'}}\|D\mathbf{u}\|^p_{p,Q}\end{aligned}$$

where $B = (\beta p\eta^*)^{1-p'}$. Thus the estimate (43) is proven. Analogously, from (21) and (47), the estimate (44) arises.

We conclude the proof denoting $\tau_T = \varpi_T\tau$, and remarking that the estimate (45) can be proved with the help of the estimate (36). □

Remark 4.1 *Conversely, if* $\mathbf{u} \in L^p(0,T;V_p)$ *and* $(\zeta,\varsigma) \in L^{p'}_{\rm sym} \times \mathbf{L}^{s'}(\Sigma)$ *satisfy (42) and (48)-(49), then* $\mathbf{u}$ *is the solution to (31).*

Remark 4.2 *For* $F = G = id$ *in Proposition 4.2, the Lagrange multiplyer* $(\zeta,\varsigma) \in L^{p'}_{\rm sym} \times \mathbf{L}^{s'}(\Sigma)$ *satisfies*

$$\begin{aligned}|\zeta| \le \eta(\xi), \quad \zeta : D\mathbf{u} = \eta(\xi)|D\mathbf{u}| \text{ in } Q\\ |\varsigma| \le \varphi(\xi), \quad \varsigma\cdot\mathbf{u}_T = \varphi(\xi)|\mathbf{u}_T| \text{ in } \Sigma.\end{aligned}$$

Proposition 4.3 *Let* $\mathbf{u}$ *be the solution given by Proposition 4.1 and let the assumptions (16), (22)-(24) and (26) be fulfilled. For* $M \in \mathbb{N}$, $e_0^M \in L^2(\Omega)$, $\xi \in L^1(0,T;W^{1,1}(\Omega))$, $g \in L^{q'}(Q)$ *and* $h \in L^{q'}(\Sigma)$, *then there exists* $e \in Z_{q,0} \cap L^\infty(0,T;L^2(\Omega))$ *a unique solution satisfying*

$$\begin{aligned}\langle \partial_t e + \frac{M\mathbf{u}}{M+|\mathbf{u}|}\cdot\nabla e, \phi\rangle + \int_Q \chi(\xi)\mathbf{a}(\nabla e)\cdot\nabla\phi + \int_\Sigma \gamma(\xi)e\phi = \\ = \langle g,\phi\rangle + \langle h,\phi\rangle, \quad \forall\phi\in W_q \quad a.e.\ t \in]0,T[; \quad (50)\\ e(\cdot,0) = e_0^M. \quad (51)\end{aligned}$$

Moreover, the estimate holds

$$\|e\|_{Z_{q,0}} \le C(M)\left(\|g\|^{q'}_{q',Q} + \|h\|^{q'}_{q',\Sigma} + \|e_0^M\|^2_{2,\Omega}\right). \tag{52}$$

PROOF. The existence and uniqueness result is a consequence of the monotone first-order evolution problems [24, pp. 770], considering, for each $t \in]0,T[$, the operator $\mathcal{A}(t) : W_q \to (W_q)'$ defined by

$$\langle \mathcal{A}e, \phi\rangle = \int_\Omega M(M+|\mathbf{u}|)^{-1}\mathbf{u}\cdot\nabla e\phi + \int_\Omega \chi(\xi)\mathbf{a}(\nabla e)\cdot\nabla\phi + \int_\Gamma \gamma(\xi)e\phi$$

and the linear form $g+h \in L^{q'}(0,T;(W_q)')$. From the assumptions (16), (22)-(24) and (26), the function $t \mapsto \mathcal{A}(t)$ is weakly measurable, and for each $t \in]0,T[$, the functional $\mathcal{A}(t)$ is monotone, hemicontinuous, coercive and satisfies the growth condition, for $q \geq 2$,

$$\|\mathcal{A}(t)e\|_{(W_q)'} \leq \chi^* \upsilon^* \left(\|e\|_{W_q}^{q-1} + 1\right) + (M+\gamma^*)\|e\|_{W_q}.$$

Then the initial value problem (50)-(51) has a unique solution $e \in Z_{q,0}$ satisfying (52).□

5 Proof of Theorem 2 (M fixed)

We use the Tychonov-Kakutani-Glicksberg fixed point theorem [2, pages 218-220] to the multivalued functional $\mathcal{L}$, defined in a nonempty convex compact ball of the locally convex Hausdorff topological space $L^{q'}(Q)\times L^{q'}(\Sigma)\times Z_{q,0}$, and constructed as follows. Let $\xi \in Z_{q,0}$ and denote by $\mathbf{u}=\mathbf{u}(\xi)$ the unique solution from Proposition 4.1. Next denote by τ the deviator stress tensor given at Proposition 4.2. Let $(g,h)\in L^{q'}(Q)\times L^{q'}(\Sigma)$ and denote by $e=e(g,h,\xi,\mathbf{u})$ the unique solution from Proposition 4.3. Then $\mathcal{L}$ is the well defined mapping:

$$\mathcal{L}(g,h,\xi) = \left\{\left(\overline{\tau : D\mathbf{u}}, \overline{-\tau_T\cdot\mathbf{u}_T}, e\right)\right\}$$

and $\mathcal{L}(g,h,\xi)$ is a convex set since the set of Lagrange multipliers is a convex set and from propositions 4.1 and 4.3 we have the uniqueness of $\mathbf{u}$ and of e, respectively. The ball's radius can be conveniently chosen taking into account the estimates (35), (43)-(45) and (52).

Let us prove the upper semicontinuity for $\mathcal{L}$, and therefore it results the closeness of the set $\mathcal{L}(g,h,\xi)$ as a particular case. Let us consider the sequences $\left(\overline{\tau_m : D\mathbf{u}_m}, \overline{-\tau_{mT}\cdot\mathbf{u}_{mT}}, e_m\right) \in \mathcal{L}(g_m,h_m,\xi_m)$ with $\xi_m \rightharpoonup \xi$ in $Z_{q,0}$,

$$g_m \rightharpoonup g \text{ in } L^{q'}(Q), \qquad h_m \rightharpoonup h \text{ in } L^{q'}(\Sigma); \tag{53}$$

where $(\mathbf{u}_m,\tau_m,e_m)$ are the respective solutions to the problems (31)-(32), (42) and (50)-(51). Since $Z_{q,0} \hookrightarrow\hookrightarrow L^1(Q)$ and $Z_{q,0}\hookrightarrow\hookrightarrow L^1(\Sigma)$, then the sequence $\{\xi_m\}\subset Z_{q,0}$ is such that $\xi_m\to\xi$ in $L^1(Q)$ and in $L^1(\Sigma)$, thus $\xi_m\to\xi$ a.e. in Q and on Σ. If $\mathbf{u}_m$ are the corresponding solutions given at Proposition 4.1, as in step 2 of Proposition 4.1, we can extract a subsequence, still denoted by $\mathbf{u}_m$, verifying (37)-(41). From the assumptions (14)-(15) and (25), the properties of Niemytski operator garantee the continuity of the coefficient mappings

μ, η, and φ, then there exists $\mathbf{u} = \mathbf{u}(\xi)$ the corresponding solution given at Proposition 4.1. For details, see [8] and [9].

Next let us prove the strong convergence

$$\mathbf{u}_m \to \mathbf{u} \text{ in } L^p(0,T;V_p). \tag{54}$$

Choosing $\mathbf{v} = \mathbf{u}_m$ and $\mathbf{v} = \mathbf{u}$ as test functions in (31) for the solutions $\mathbf{u}$ and $\mathbf{u}_m$, respectively, applying the assumptions (14) and (19), after routine calculations we arrive to the inequality

$$\begin{aligned}\alpha\mu_* \|D(\mathbf{u}_m - \mathbf{u})\|_{p,Q}^p \leq \int_Q \{\mu(\xi_m) - \mu(\xi)\} A(D\mathbf{u}) : D(\mathbf{u} - \mathbf{u}_m) + \\ + \int_Q \{\eta(\xi) - \eta(\xi_m)\} F(|D\mathbf{u}_m|) + \int_Q \{\eta(\xi_m) - \eta(\xi)\} F(|D\mathbf{u}|) + \\ + \int_\Sigma \{\varphi(\xi) - \varphi(\xi_m)\} G(|\mathbf{u}_{mT}|) + \int_\Sigma \{\varphi(\xi_m) - \varphi(\xi)\} G(|\mathbf{u}_T|) + \\ + \int_Q D\mathbf{u} : (\mathbf{u}_m - \mathbf{u}) \otimes (\mathbf{u}_m - \mathbf{u}).\end{aligned} \tag{55}$$

The last term converges to zero, when m tends to infinity, in accordance to the convergence $\mathbf{u}_m \otimes \mathbf{u}_m \rightharpoonup \mathbf{u} \otimes \mathbf{u}$ in $L^{p'}(0,T;(V_p)')$ for $p > (3n+2)/(n+2)$ [17, pp. 76-77].

Let us state the crucial result in the following lemma that will lead the strong convergence to $D\mathbf{u} \in L^p_{\mathrm{sym}}$.

Lemma 5.1 *Let $\{b_m\} \subset L^\infty(Q)$ and $\{U_m\} \subset L^1(Q)$ be two sequences such that*

$$0 \leq b_m \leq b^* : \quad b_m \to b \quad \text{a.e. in } Q; \quad \text{and} \quad U_m \geq 0 : \quad \|U_m\|_{1,Q} \leq C.$$

Then

$$\int_Q (b_m - b) U_m \underset{m\to+\infty}{\longrightarrow} 0.$$

PROOF. Let $\delta > 0$ be arbitrary and choose $\varepsilon = \delta/(8b^*) > 0$. Applying the Egorov theorem, there exists a subset $E \subset Q$ such that $\mathrm{meas}(Q \setminus E) < \varepsilon$ and

$$\|b_m - b\|_{\infty,E} \underset{m\to+\infty}{\longrightarrow} 0.$$

Decomposing the integral in the following form

$$\begin{aligned}\int_Q (b_m - b)U_m &= \int_E (b_m - b)U_m + \int_{Q\setminus E} (b_m - b)U_m \\ &\leq \|b_m - b\|_{\infty,E} \|U_m\|_{1,E} + 2b^* \int_{Q\setminus E} U_m.\end{aligned}$$

We can apply the Tchebychev inequality

$$\int_{Q\setminus E} U_m \leq \mathrm{meas}\left(\{(x,t) \in Q \setminus E : \; U_m \geq 1\}\right).$$

Thus, from the uniform convergence we have $N \in \mathbb{N}$ such that

$$m \geq N \Longrightarrow \|b_m - b\|_{\infty,E} \leq \delta/(2C).$$

Then it results

$$\left| \int_Q (b_m - b) U_m \right| \leq \delta/2 + 4b^* \mathrm{meas}(Q \setminus E).$$

From the choice of ε, we conclude the desired result. □

Taking in lemma 5.1, $b_m = \eta(\xi_m)$ and $U_m = F(|D\mathbf{u}_m|)$ in Q, and $b_m = \varphi(\xi_m)$ and $U_m = G(|\mathbf{u}_{mT}|)$ in Σ, and using into (55) so we get (54) and consequently $D\mathbf{u}_m \to D\mathbf{u}$ a.e. in Q. Then, using the Lebesgue's dominated convergence theorem we obtain

$$\mu(\xi_m)A(D\mathbf{u}_m) \to \mu(\xi)A(D\mathbf{u}) \quad \text{in} \quad L^{p'}_{\text{sym}}; \tag{56}$$

$$\eta(\xi_m)F(|D\mathbf{u}_m|) \to \eta(\xi)F(|D\mathbf{u}|) \quad \text{in} \quad L^1(Q); \tag{57}$$

$$\varphi(\xi_m)G(|\mathbf{u}_{mT}|) \to \varphi(\xi)G(|\mathbf{u}_T|) \quad \text{in} \quad L^1(\Sigma). \tag{58}$$

If τ_m are Lagrange multipliers in the conditions of Proposition 4.2, from the estimates (43)-(44), we can extract subsequences such that

$$\zeta_m \rightharpoonup \zeta \quad \text{in} \quad L^{p'}_{\text{sym}}; \tag{59}$$

$$\varsigma_m \rightharpoonup \varsigma \quad \text{in} \quad \mathbf{L}^{s'}(\Sigma). \tag{60}$$

Arguing as in [17], gathering (46)-(47) and (48)-(49) we get

$$\begin{aligned} &\int_Q \eta(\xi_m)F(|D\mathbf{u}_m|) + \int_Q \eta(\xi_m)F^*(|\zeta_m/\eta(\xi_m)|) + \\ &+ \int_\Sigma \varphi(\xi_m)G(|\mathbf{u}_{mT}|) + \int_\Sigma \varphi(\xi_m)G^*(|\varsigma_m/\varphi(\xi_m)|) = \\ &= \langle \mu(\xi_m)A(D\mathbf{u}_m, D(\mathbf{v} - \mathbf{u}_m)\rangle + \\ &+ \langle \partial_t \mathbf{u}_m + (\mathbf{u}_m \cdot \nabla)\mathbf{u}_m - \mathbf{f}, \mathbf{v} - \mathbf{u}_m \rangle + \langle -\zeta_m, D\mathbf{v}\rangle + \langle -\varsigma_m, \mathbf{v}_T \rangle. \end{aligned}$$

Applying the convergences (56)-(58) and (59)-(60) we can pass to the limit the above expression, when m tends to infinity, and after taking $\mathbf{v} = \mathbf{u}$ it results

$$\begin{aligned} &\limsup_{m\to+\infty} \left\{ \int_Q \eta(\xi_m)F^*(|\zeta_m/\eta(\xi_m)|) + \int_\Sigma \varphi(\xi_m)G^*(|\varsigma_m/\varphi(\xi_m)|) \right\} \leq \\ &\leq - \int_Q \eta(\xi)F(|D\mathbf{u}|) - \int_\Sigma \varphi(\xi)G(|\mathbf{u}_T|) + \langle -\zeta, D\mathbf{u}\rangle + \langle -\varsigma, \mathbf{u}_T \rangle = \\ &= \int_Q \eta(\xi)F^*(|\zeta/\eta(\xi)|) + \int_\Sigma \varphi(\xi)G^*(|\varsigma/\varphi(\xi)|). \end{aligned}$$

Using the lower semicontinuity of the functionals F^* and G^*, we can conclude that

$$\begin{aligned} &\int_Q \eta(\xi_m)F^*(|\zeta_m/\eta(\xi_m)|) + \int_\Sigma \varphi(\xi_m)G^*(|\varsigma_m/\varphi(\xi_m)|) \to \\ &\to \int_Q \eta(\xi)F^*(|\zeta/\eta(\xi)|) + \int_\Sigma \varphi(\xi)G^*(|\varsigma/\varphi(\xi)|) \end{aligned}$$

and consequently

$$\lim_{m\to+\infty}\{\langle-\zeta_m, D\mathbf{u}_m\rangle + \langle-\varsigma_m, \mathbf{u}_{mT}\rangle\} = \langle-\zeta, D\mathbf{u}\rangle + \langle-\varsigma, \mathbf{u}_T\rangle =$$
$$= \|-\zeta : D\mathbf{u}\|_{1,Q} + \|-\varsigma\cdot\mathbf{u}_T\|_{1,\Sigma}.$$

From (59)-(60) and (54), we have $(\zeta_m : D\mathbf{u}_m, \varsigma_m\cdot\mathbf{u}_{m_T}) \rightharpoonup (\zeta : D\mathbf{u}, \varsigma\cdot\mathbf{u}_T)$ in $L^1(Q)\times L^1(\Sigma)$, and since it converges in norm $\|\cdot\| = \int_Q|\cdot| + \int_\Sigma|\cdot|$ on the Banach space $L^1(Q)\times L^1(\Sigma)$, then the strong convergence yields in $L^1(Q)\times L^1(\Sigma)$. Consequently we can select a subsequence such that $\zeta_m : D\mathbf{u}_m \to \zeta : D\mathbf{u}$ a.e. in Q and $\varsigma_m\cdot\mathbf{u}_{m_T} \to \varsigma\cdot\mathbf{u}_T$ a.e. in Σ. Thus we have the strong convergence

$$\tau_m : D\mathbf{u}_m \to \tau : D\mathbf{u} \text{ in } L^1(Q). \tag{61}$$

Furthermore, we obtain $\tau_T = \varsigma$.

From the estimate (52), the solutions $e_m = e(g_m, h_m, \xi_m, \mathbf{u}_m)$ are such that, for a subsequence, $e_m \rightharpoonup e$ in $Z_{q,0}$, and consequently $e_m \to e$ a.e. in Q and in Σ. Then by the assumptions (16) and (26), and from the properties of Niemytskii operators for χ and γ, to pass to the limit in (50) in order to obtain the desired solution $e = e(g, h, \xi, \mathbf{u})$ it remains to check the nonlinear term $\mathbf{a}(\nabla e_m)$.

To prove the strong convergence of e_m to e in $L^q(0,T;W_q)$, we choose $\phi = e_m - e_n$ as a test function in (50) for e_m and e_n, and subtrating the obtained equations it follows

$$\chi_* v\|\nabla(e_m - e_n)\|^q_{q,Q} \leq \int_Q (g_m - g_n)(e_m - e_n) + \int_\Sigma (h_m - h_n)(e_m - e_n)$$

and from (53) we have the required strong convergence in $L^q(0,T;W_q)$.

Hence, we pass to the limit in (50). So the Tychonov-Kakutani-Glicksberg fixed point theorem garantees the existence of $(g, h, \xi) \in \mathcal{L}(g, h, \xi)$, concluding the proof of Theorem 2.□

6 Proof of Theorem 1

Let $(\mathbf{u}_M, \tau_M, e_M)$ be a solution given at Theorem 2 and let $M \to +\infty$. By similar arguments of the proof of Proposition 4.1, we have (37)-(41). Then as in the proof of Theorem 2 it results (54) and

$$g_M = \overline{\tau_M : D\mathbf{u}_M} \to \tau : D\mathbf{u} \quad \text{in} \quad L^1(Q); \tag{62}$$
$$h_M = \overline{-\tau_{MT}\cdot\mathbf{u}_{MT}} \to \tau\cdot\mathbf{u}_T \quad \text{in} \quad L^1(\Sigma). \tag{63}$$

Arguing as in [8] the following estimates are valid, for every $M, N \in \mathbb{N}$,

$$\begin{aligned}\|e_M - e_N\|_{\infty,L^1(\Omega)} &\leq \|g_M - g_N\|_{1,Q} + \|h_M - h_N\|_{1,\Sigma} + T\|e_0^M - e_0^N\|_{1,\Omega} + \\ &\quad +|Q|/2;\\ \|\nabla(e_M - e_N)\|^r_{r,Q} &\leq C\{\|g_M - g_N\|_{1,Q} + \|h_M - h_N\|_{1,\Sigma} + \|e_0^M - e_0^N\|_{1,\Omega}\}\times\\ &\quad \times\|e_M - e_N\|^{r(q-r)/(qn)}_{\infty,L^1(\Omega)}\end{aligned}$$

for every exponent $1 < r < q - n/(n+1)$ (cf. [4] or [10]), according to (27) and $\int_Q M(M+|\mathbf{u}|)^{-1}\mathbf{u}\cdot\nabla\phi(e) = 0$.

For $e_0^M \in L^2(\Omega)$ such that $e_0^M \to e_0$ in $L^1(\Omega)$ then it follows e_M is a Cauchy sequence in $L^r(0,T;W_r)$, and consequently there exists $e \in L^r(0,T;W_r)$ such that $e_M \to e$ in $L^r(0,T;W_r)$.

In order to prove that $\partial_t e_M$ is a Cauchy sequence, let us subtract the equations (50) for e_M and e_N obtaining

$$\begin{aligned}\langle\partial_t(e_M - e_N), \phi\rangle &= \langle\chi(e_N)\mathbf{a}(\nabla e_N) - \chi(e_M)\mathbf{a}(\nabla e_M), \nabla\phi\rangle + \\ &+\langle\gamma(e_N)e_N - \gamma(e_M)e_M, \phi\rangle + \langle g_M - g_N, \phi\rangle + \langle h_M - h_N, \phi\rangle + \\ &+\langle N(N+|\mathbf{u}_N|)^{-1}\mathbf{u}_N\cdot\nabla e_N - M(M+|\mathbf{u}_M|)^{-1}\mathbf{u}_M\cdot\nabla e_M, \phi\rangle.\end{aligned}$$

If $p < n$, we find (cf. Remark 3.2)

$$\begin{aligned}\left\| e_M\frac{M\mathbf{u}}{M+|\mathbf{u}|} - e_N\frac{N\mathbf{u}}{N+|\mathbf{u}|}\right\|_{r/(q-1),\Omega} &\le \|e_M - e_N\|_{n/[r(n+1)],\Omega}\|\mathbf{u}\|_{n/[p(n+2)],\Omega} \\ &+\|e_N\|_{n/[r(n+1)],\Omega}\left\|M(M+|\mathbf{u}|)^{-1}\mathbf{u} - N(N+|\mathbf{u}|)^{-1}\mathbf{u}\right\|_{n/[p(n+2)],\Omega}.\end{aligned}$$

Hence, it implies

$$\begin{aligned}\int_0^T\left\| e_M\frac{M\mathbf{u}}{M+|\mathbf{u}|} - e_N\frac{N\mathbf{u}}{N+|\mathbf{u}|}\right\|_{r/(q-1),\Omega} &\le C_1\|e_M - e_N\|_{r,W_r} + \\ +C_2\left\|M(M+|\mathbf{u}|)^{-1}\mathbf{u} - N(N+|\mathbf{u}|)^{-1}\mathbf{u}\right\|_{n/[p(n+2)],Q} &:= I_{M,N};\end{aligned}$$

where C_1 and C_2 are dependent on $\|e_M - e_N\|_{\infty,L^1(\Omega)}$, $\|\mathbf{u}\|_{n/[p(n+2)],Q}$ and $\|e_N\|_{n/[r(n+1)],Q}$, respectively. If $p \ge n$, the result yields by a similar argument.

By definition, we have

$$\begin{aligned}\|\partial_t(e_M - e_N)\|_{1,(W_{r/(r-q+1)})'} &= \int_0^T \sup_{\|\nabla\phi\|_{r/(r-q+1),\Omega}\le 1} |\langle\partial_t(e_M - e_N), \phi\rangle| dt \le \\ &\le \int_0^T \|\chi(e_N)\mathbf{a}(\nabla e_N) - \chi(e_M)\mathbf{a}(\nabla e_M)\|_{r/(q-1),\Omega} + \\ +\|\gamma(e_N)e_N - \gamma(e_M)e_M\|_{1,\Sigma} &+ \|g_N - g_M\|_{1,Q} + \|h_M - h_N\|_{1,\Sigma} + I_{M,N}.\end{aligned}$$

Applying the Lebesgue's dominated convergence theorem due to the convergences (54), (62)-(63), $e_M \to e$ in $L^r(0,T;W_r)$, $e_M \to e$ in $L^1(\Sigma)$, $\chi(e_M) \to \chi(e)$ a.e. in Q, and $\gamma(e_M) \to \gamma(e)$ a.e. in Σ, then $\partial_t e_M$ is a Cauchy convergent in $L^1(0,T;(W_{r/(r-q+1)})')$. Then $e_M \to e$ in $Z_{r,1}$. By a classical compactness result $Z_{r,1} \hookrightarrow C([0,T];(W_{r/(r-q+1)})')$. Moreover, the solution e is the unique solution obtained as limit of approximations, also named SOLA (cf. [10]). Therefore, the proof of Theorem 1 is completed. □

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