# Multiplicity of solutions of Dirichlet problems associated to second order equations in $\mathbb{R}^2$

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*Abstract.* We study existence of multiple solutions for a two-point boundary value problem associated to a planar system of second order ordinary differential equations by using a shooting technique. We consider asymptotically linear nonlinearities satisfying suitable sign conditions. Multiplicity is ensured by assumptions involving the Morse indexes of the linearizations at zero and at infinity.

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## 1 Introduction

In this paper we are interested on the existence of multiple solutions to the equation

$$x'' + A(t,x)x = 0, \tag{1.1}$$

$x \in \mathbb{R}^2$, $t \in [0,\pi]$, satisfying the Dirichlet boundary conditions $x(0) = x(\pi) = 0$. We will assume that $A : [0,\pi] \times \mathbb{R}^2 \to GL_s(\mathbb{R}^2)$,

$$A(t,x) = \begin{bmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{12}(t,x) & a_{22}(t,x) \end{bmatrix},$$

is a continuous function such that

$$\lim_{|x|\to 0} A(t,x) = A_0(t) \text{ uniformly in } t \in [0,\pi],$$
$$\lim_{|x|\to \infty} A(t,x) = A_\infty(t) \text{ uniformly in } t \in [0,\pi],$$

that is, we assume asymptotically linear conditions at the origin and at infinity.

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There exists an extensive literature concerning the existence of solutions of boundary value problems associated with asymptotically linear Hamiltonian systems. Regarding first order Hamiltonian systems and existence of periodic solutions, we can mention, for example, the papers [3], [4], [9], [11] and, among others, the more recent works [18], [24], [35] (see also to the references therein). In these papers, the existence of at least one (in some cases two) solution is guaranteed when the Maslov-type indexes [2] of the linearizations at zero and at infinity are different. When some additional conditions (like convexity, symmetry in the space variable, or for autonomous equations) are guaranteed, multiplicity results are achieved in [1], [14], [17], [20], [21], [28], [29]. For the particular case of $\mathbb{R}^2$, multiplicity results were obtained in [27] using no additional conditions through the Poincaré - Birkhoff theorem.
Existence of solutions of Hamiltonian systems satisfying Dirichlet and Bolza boundary conditions was studied in [7] and [16], respectively.

The next references we wish to quote deal with existence and multiplicity results for second order asymptotically linear systems.
Interesting contributions in the periodic setting can be found, among others, in the works [5], [25], [33] in which existence results are obtained and in [6] where multiplicity of solutions is proved in the autonomous case.
The literature is not so rich in contributions as far as Dirichlet problems associated to second order systems are concerned. In this direction, we refer to the papers [10], [24] and [32] containing existence results for PDEs systems. Multiplicity results have been recently attained under some extra assumptions in [8], [15] for ODEs and in [30], [34] for PDEs.
It is worth noticing that in these works it is shown that the bigger the gap between suitable indexes associated to the linearizations of the problem at the origin and at infinity, the larger the number of multiple solutions. In particular, in the paper [8] the authors consider (1.1) in $\mathbb{R}^n$ and obtain multiplicity of solutions satisfying the Dirichlet boundary conditions under asymptotically linear growth conditions. The results are achieved through a generalized shooting approach using the notion of moments of verticality and phase angles. However, the number of solutions obtained depends on the cardinality of a suitable set which sometimes can be empty. On the other hand, in [15] the author proved the existence of multiple solutions to the Dirichlet problem associated to the equation $x'' + V'(t,x) = 0$ (which is a particular case of (1.1), as it is shown in [8]) assuming asymptotically linear conditions and a symmetric condition on the potential $V$, that is $V(t,x) \equiv V(t,-x)$.

In our paper we reexamine this problem in the case of $\mathbb{R}^2$ and prove the existence of multiple solutions of (1.1) satisfying Dirichlet boundary conditions. The aim of this paper consisted in trying to generalize the results of [15] to a context where no symmetric assumptions were required. To reach this goal, we had to assume some sign conditions for the matrix $A(t,x)$ (see Theorem 2.3). Under these conditions and whenever there is a gap equal to $N$ between the indexes of the linearizations at the origin and at infinity ([15]), we are able to guarantee the existence of $2N$ solutions to (1.1) satisfying Dirichlet boundary conditions. Our proof is developed in the framework of the shooting methods. Multiplicity results follow by combining degree theory with some preliminary results about eigenvalues and eigenvectors of second order Dirichlet problems, proved in Proposition 2.4 and Proposition 2.6.

In the following we denote by $GL_s(\mathbb{R}^2)$ the group of $2 \times 2$ real symmetric matrices and by $I_2$ the identity matrix in that group. According to the notation of [15], for any $B_1, B_2 \in L^1([0,\pi]; GL_s(\mathbb{R}^2))$ we write $B_1 < B_2$ if $B_1(t) \le B_2(t)$ for a.e. $t \in (0,\pi)$ and $B_1(t) < B_2(t)$ on a subset of $(0,\pi)$ with positive measure. We set $\mathcal{Q}_1 := [0,+\infty[\times[0,+\infty[$ and $\mathcal{Q}_3 :=]-\infty,0]\times]-\infty,0]$, representing the first and the third quadrant, respectively. Finally, we denote by $\mathbb{R}^+$ the set of positive real numbers.

## 2 Main result

Let us consider the two-point boundary value problem

$$\begin{cases} x'' + A(t,x)x = 0, \ x \in \mathbb{R}^2, \ t \in [0,\pi] \\ x(0) = x(\pi) = 0, \end{cases} \tag{2.1}$$

where $A : [0,\pi] \times \mathbb{R}^2 \to GL_s(\mathbb{R}^2)$,

$$A(t,x) = \begin{bmatrix} a_{11}(t,x) & a_{12}(t,x) \\ a_{12}(t,x) & a_{22}(t,x) \end{bmatrix},$$

is a continuous function such that uniqueness of solutions of Cauchy problems associated to system (2.1) is guaranteed. We will assume that

$$\lim_{|x|\to 0} A(t,x) = A_0(t) \text{ uniformly in } t \in [0,\pi], \tag{2.2}$$

$$\lim_{|x|\to \infty} A(t,x) = A_\infty(t) \text{ uniformly in } t \in [0,\pi]. \tag{2.3}$$

Under the condition (2.3) we conclude that $A$ is bounded and hence the continuability of the solutions of Cauchy problems associated to system (2.1) is guaranteed.
In order to state our main result, we recall the definitions of index and of nullity of a path of symmetric matrices (see [15]). To do this, first we reformulate the proposition proved in [15].

**Proposition 2.1** *Given $B \in L^\infty([0,\pi]; GL_s(\mathbb{R}^2))$ there exists a sequence of eigenvalues of $B$, $\lambda_1(B) \le \lambda_2(B) \le \ldots \le \lambda_j(B) \to +\infty$ as $j \to +\infty$ such that, for each $j$, there exists a space of dimension one of nontrivial solutions (eigenvectors of $B$) of the problem*

$$\begin{cases} x'' + (B(t) + \lambda_j(B)I_2)x \ = \ 0 \\ x(0) = x(\pi) = 0. \end{cases}$$

*Moreover* $H_0^1([0,\pi];\mathbb{R}^2) := \{x : [0,\pi] \to \mathbb{R}^2 \,|\, x(\cdot)$ *is continuous on* $[0,\pi]$*, satisfies* $x(0) = 0 = x(\pi)$*, and* $x' \in L^2([0,\pi];\mathbb{R}^2)\}$ *admits a basis of eigenvectors of* $B$*.*

**Definition 2.2** *Given* $B \in L^\infty([0,\pi]; GL_s(\mathbb{R}^2))$*, its index* $i(B)$ *is defined as the number of negative eigenvalues and its nullity* $\nu(B)$ *the number of zero eigenvalues.*

The index of $B \in L^\infty([0,\pi]; GL_s(\mathbb{R}^2))$ as we have just defined coincides with the Morse index of the boundary value problem $x'' + B(t)x = 0$, $x(0) = x(\pi) = 0$.

Note that in the sequence of the eigenvalues of a matrix $B$ we cannot have the same value repeated more than twice. In the case it is repeated twice we say that the corresponding eigenvalue $\lambda(B) = \lambda_j(B) = \lambda_{j+1}(B)$, for some $j$, has a space of eigenvectors of dimension two. Otherwise, we say that the space of eigenvectors has dimension one.

Now we are in position to state the main result.

**Theorem 2.3** *Assume that* $A(t,x)$ *satisfies (2.2) and (2.3). Suppose moreover that*

$$\begin{aligned} a_{11}(t,x) < 0\,, \qquad a_{22}(t,x) < 0 \quad \forall (t,x) \in [0,\pi]\times\mathbb{R}^2 \ \textit{and} \\ \textit{either} \ \ a_{12}(t,x) \geq 0 \quad \textit{or} \quad a_{12}(t,x) \leq 0, \quad \forall (t,x) \in [0,\pi]\times\mathbb{R}^2. \end{aligned} \tag{2.4}$$

*Then if* $i(A_0) > i(A_\infty)$ *and* $\nu(A_\infty) = 0$ *(or* $i(A_0) < i(A_\infty)$ *and* $\nu(A_0) = 0$*), the problem (2.1) has at least* $2\,|i(A_0) - i(A_\infty)|$ *nontrivial solutions.*

Before proving the theorem, we need to state some preliminary results. At first we present some results about eigenvalues and eigenvectors of a matrix $B$ which will be useful in the proof of Theorem 2.3. Analogous results for the case of a second order equation can be found in [22] and [23].

**Proposition 2.4** *For each* $j = 1, \dots, +\infty$*,* $B \to \lambda_j(B)$ *is continuous in* $\big\{B \in L^1([0,\pi]; GL_s(\mathbb{R}^2)) : \ B < 0\big\}$.

*Proof.* According to [13], each eigenvalue $\lambda_j(B)$ satisfies $\lambda_j(B) = \frac{1}{\mu_j(B)}$ where

$$\mu_j(B) = \sup_{F_j} \inf \left\{ \int_0^\pi |u|^2 : \ \|u\|_{a_B} = 1, \ u \in F_j \right\},$$

where $F_j$ varies over all $j$-dimensional subspaces of $H_0^1([0,\pi];\mathbb{R}^2)$ and $\|\cdot\|_{a_B}$ is the norm associated to the inner product

$$(u,v)_{a_B} = \int_0^\pi [u'(t)\cdot v'(t) - B(t)u(t)\cdot v(t)]\, dt.$$

The result follows from the fact that given $\varepsilon > 0$, $B \in L^1([0,\pi];GL_s(\mathbb{R}^2))$, $B < 0$ and $j \in \mathbb{N}$, there exists a positive constant $\delta = \delta(\varepsilon, B, j)$ such that for each $B_1 \in L^1([0,\pi];GL_s(\mathbb{R}^2))$ with $B_1 < 0$ and $\|B - B_1\|_{L^1} < \delta$, for each $j$-dimensional subspace $F_j$ of $H_0^1([0,\pi];\mathbb{R}^2)$ and for each $u \in F_j$ with $\|u\|_{a_B} = 1$ (and $\|u\|_{a_{B_1}} = 1$) there exists $v \in F_j$ with $\|v\|_{a_{B_1}} = 1$ (resp. $\|v\|_{a_B} = 1$) such that $\left|\int_0^\pi |u|^2 - \int_0^\pi |v|^2\right| < \varepsilon$. To prove this, for every $F_j$ we can choose an orthonormal basis with respect to the new inner product $(\cdot,\cdot)_{a_B}$, $\phi_i$, $i = 1, \cdots, j$. Taking into account the equivalence between $\|\cdot\|_{a_B}$ and the usual norm of the Hilbert space $H_0^1([0,\pi];\mathbb{R}^2)$ (cf. [15] and the references therein), it is easy to see that $|(\phi_i,\phi_k)_{a_{B_1}} - 1|$ and $|(\phi_i,\phi_k)_{a_{B_1}}|$ are small if $||B - B_1||_{L^1}$ is small, whenever $i, k \in \{1, \cdots, j\}$, $i \neq k$. Thus, for each $u = \sum_{i=1}^j c_i \phi_i$ we can choose $v = \sum_{i=1, i\neq k}^j c_i\phi_i + (c_k + \eta)\phi_k$, for an adequate $k$ and a sufficiently small $\eta$. ■

**Corollary 2.5** *Fixed $M > 0$, for each $j = 1, \ldots, +\infty$, $B \to \lambda_j(B)$ is continuous in $\{B \in L^1([0,\pi];GL_s(\mathbb{R}^2)) : \|B(t)\| < M$ for a.e. $t \in (0,\pi)\}$.*

*Proof.* Consider $B \in L^1([0,\pi];GL_s(\mathbb{R}^2))$ satisfying $\|B(t)\| < M$ for a.e. $t \in (0,\pi)$. It immediately follows that $|B(t)\,x \cdot x| \leq |B(t)\,x|\,|x| < M\,|x|^2$ for every $x \in \mathbb{R}^2$ and for a.e. $t \in (0,\pi)$. This implies that $B < MI_2$. According to [15] and [13], each eigenvalue can be expressed by the relation $\lambda_j(B) = \frac{1}{\tilde{\mu}_j(B)} - M$ where

$$\tilde{\mu}_j(B) = \mu_j(B^*) = \sup_{F_j} \inf \left\{ \int_0^\pi |u|^2 : \ \|u\|_{a_{B^*}} = 1, \ u \in F_j \right\},$$

with $B^*(t) = B(t) - MI_2$. By combining the continuous dependence of $\mu_j(B^*)$ with respect to $B^*$ ensured by the previous proposition with the continuity of the map $B \to B^*$ from $L^1([0,\pi];GL_s(\mathbb{R}^2))$ into itself, we achieve the thesis. ■

The next result concerns the possibility of considering continuous branches of eigenvectors when the equation depends continuously on a parameter. We state the result in the case of the zero eigenvalue but the result is still valid if we consider eigenvalues depending continuously on a parameter. A similar result can be found in [22].

**Proposition 2.6** *Let $\mathcal{C}$ be a continuum of $\mathbb{R}^2$ and assume that $B : [0,\pi] \times \mathcal{C} \to GL_s(\mathbb{R}^2)$ is continuous. Suppose that for each $\alpha \in \mathcal{C}$, zero is an eigenvalue of $B(\cdot,\alpha)$ and that there exists $(a,b) \in S^1$ such that for each $\alpha \in \mathcal{C}$ the solution of $x'' + B(t,\alpha)x = 0$ which satisfies $x(0) = 0$ and $x'(0) = (a,b)$ does not vanish at $t = \pi$. Then, we can choose a continuous function from $\mathcal{C}$ to $(C^1([0,\pi],\mathbb{R}^2))^2$, $\alpha \to (v_\alpha(\cdot), v'_\alpha(\cdot))$, such that for each $\alpha$, $v_\alpha$ is an eigenvector of $B(\cdot,\alpha)$ associated to the zero eigenvalue.*

*Proof.* Consider the solutions $U^i(\cdot,\alpha) : [0,\pi] \times \mathcal{C} \to \mathbb{R}^4$, $i = 1, 2$, of

$$\begin{cases} x' = y \\ y' = -B(t,\alpha)x \end{cases} \tag{2.5}$$

satisfying $U^1(0,\alpha) = (0,0,a,b)$ and $U^2(0,\alpha) = (0,0,-b,a)$.
We want to construct a continuous function $\alpha \to (v_\alpha(\cdot), v'_\alpha(\cdot))$ such that, for each $\alpha$, $(v_\alpha(\cdot), v'_\alpha(\cdot))$ satisfies (2.5), $v_\alpha(\cdot)$ is not identically zero and $v_\alpha(0) = v_\alpha(\pi) = 0$. That is, for each $\alpha$, $(v_\alpha(\cdot), v'_\alpha(\cdot))$ will be a nonzero linear combination of $U^i(\cdot,\alpha)$, $i = 1,2$, satisfying $v_\alpha(\pi) = 0$.
Let us recall that, by assumption, $(U^1_3(0,\alpha), U^1_4(0,\alpha)) = (a,b)$ for each $\alpha \in \mathcal{C}$. This implies that $(U^1_1(\pi,\alpha), U^1_2(\pi,\alpha)) \neq (0,0)$ for each $\alpha \in \mathcal{C}$.
As a consequence of the theorems on continuous dependence of the parameters (cf., for instance, [19]), the functions $U^i$ are continuous on $\alpha$.
We now choose $c_1(\alpha) := -\frac{U^1_1\,U^2_1 + U^1_2\,U^2_2}{(U^1_1)^2+(U^1_2)^2}(\pi,\alpha)$ and $c_2(\alpha) := 1$. According to the remark above, $c_1$ and $c_2$ are well defined and are continuous on $\alpha$.
Finally, let us set $(v_\alpha(t), v'_\alpha(t)) = c_1(\alpha)\,U^1(t,\alpha) + c_2(\alpha)\,U^2(t,\alpha)$.
Note that the continuity of $\alpha \to (v_\alpha(\cdot), v'_\alpha(\cdot))$ is guaranteed. Since, by assumption, zero is an eigenvalue of $B(\cdot,\alpha)$ for every $\alpha \in \mathcal{C}$, it follows that $(U^1_1 U^2_2)(\pi,\alpha) = (U^2_1 U^1_2)(\pi,\alpha)$ implying $v_\alpha(\pi) = 0$ for every $\alpha \in \mathcal{C}$. To complete the proof, it remains to show that $v_\alpha(\cdot)$ is not identically zero for each $\alpha \in \mathcal{C}$ or, equivalently, that $v'_\alpha(0)$ never vanishes. This is a consequence of the fact that $v'_\alpha(0) = c_1(\alpha)\,(a,b) + \,(-b,a)$ and that $(a,b)$ and $(-b,a)$ are linearly independent. ■

Now we state two preliminary lemmas which will be important for the proof of the main result.

**Lemma 2.7** *Consider the problem*

$$\begin{cases} x'' + B(t)x = 0, \ t \in [0,\pi] \\ x(0) = x(\pi) = 0, \end{cases} \tag{2.6}$$

*where $B \in L^\infty([0,\pi]; GL_s(\mathbb{R}^2))$, and*

$$B(t) = \begin{bmatrix} b_{11}(t) & b_{12}(t) \\ b_{12}(t) & b_{22}(t) \end{bmatrix}.$$

*Assume that $b_{11}(t) < 0$ and $b_{22}(t) < 0$ for every $t \in [0,\pi]$. Then we have that if $b_{12}(t) \leq 0$ (or $b_{12}(t) \geq 0$) for every $t \in [0,\pi]$, there are no nontrivial solutions of the Dirichlet problem (2.6) such that $x'(0)$ lies in the first or the third (resp. second or fourth) quadrant.*

*Proof.* Assume that $b_{12}(t) \leq 0$ for every $t \in [0,\pi]$.
We are first interested in proving the strict monotonicity of each component of the solutions $x = (x_1, x_2)$ to the problem

$$\begin{cases} x_1'' = -b_{11}(t)x_1 - b_{12}(t)x_2 \\ x_2'' = -b_{12}(t)x_1 - b_{22}(t)x_2 \\ x(0) = 0, \end{cases} \tag{2.7}$$

whenever $x_1'(0)\, x_2'(0) > 0$.
Suppose that $x_i'(0) > 0$ for each $i \in \{1,2\}$. This implies that there exists $\delta > 0$ such that $x_i(t) > 0$ for each $t \in (0,\delta]$ and for each $i \in \{1,2\}$. According to the sign assumption, it is immediate to note that $x_i''(t)$ is positive and, consequently, $x_i'(t) > x_i'(0) > 0$ for every $t \in (0,\delta]$, $i \in \{1,2\}$. In particular, as long as $x_1''$ and $x_2''$ remain positive, $x_1$ and $x_2$ keep on increasing. This allows us to conclude that each component of $x''$ never vanishes in $\mathbb{R}^+$, whenever $x = (x_1, x_2)$ is a solution of (2.7) satisfying $x_i'(0) > 0$ for each $i \in \{1,2\}$. Thus, it turns out that $x_1$ and $x_2$ are strictly increasing in $\mathbb{R}^+$.
Consider now a solution $x = (x_1, x_2)$ of (2.7) with $x_i'(0) < 0$ for each $i \in \{1,2\}$. Being the problem (2.7) linear, also $-x$ solves it. By the previous step, it follows that $-x_i' > 0$ in $\mathbb{R}^+$ for each $i \in \{1,2\}$ and, consequently, $x_1$ and $x_2$ are strictly decreasing in $\mathbb{R}^+$.
We have finally proved that the problem (2.6) does not admit any solution $x = (x_1, x_2)$ satisfying $x_1'(0)\, x_2'(0) > 0$.

Our next aim consists in showing that there are no nontrivial solutions $x$ of the Dirichlet problem (2.6) with $x_h'(0) = 0$, $h$ fixed in $\{1,2\}$. Let $\tilde{x}$ be the solution to (2.7) verifying $\tilde{x}_h'(0) = 0$, $\tilde{x}_k'(0) \neq 0$, with $h \neq k,\ h,k \in \{1,2\}$. We want to prove that $\tilde{x}(\pi) \neq 0$.
By the linearity of the problem, it is not restrictive to assume $\tilde{x}_k'(0) > 0$. Moreover, for every $\varepsilon > 0$, let us consider the solution $x_\varepsilon = (x_{\varepsilon,1}, x_{\varepsilon,2})$ to the Cauchy problem (2.7) with $x_{\varepsilon,h}'(0) = \varepsilon > 0$ and $x_{\varepsilon,k}'(0) = \tilde{x}_k'(0) > 0$. By the theorem of continuous dependence of the solutions to Cauchy problems with respect to the initial data, we can deduce that $(x_\varepsilon, x_\varepsilon')$ tends uniformly to $(\tilde{x}, \tilde{x}')$ on the interval $[0,\pi]$ as $\varepsilon$ tends to 0. Being, by the previous step, each component of $x_\varepsilon'$ positive in $[0,\pi]$, we deduce that $\tilde{x}_i'(t) \geq 0$ for every $t \in [0,\pi]$, $i \in \{1,2\}$. From the fact that $\tilde{x}_k'(0) > 0$, it follows that $\tilde{x}(\pi) \neq (0,0)$.
This completes the proof under the assumption $b_{12}(\cdot) \leq 0$ on $[0,\pi]$.

The case involving opposite inequalities can be treated in an analogous way. ∎

In order to state the other preliminary lemma, we consider the Cauchy problem

$$\begin{aligned} x'' + A(t,x)\,x &= 0 \\ x(0) &= 0 \\ x'(0) &= \alpha \end{aligned}$$

associated to the system in (2.1). For each $\alpha \in \mathbb{R}^2$, we denote by $x_\alpha$ its unique solution. We now concentrate on the linear, parameter dependent equation

$$x'' + A(t, x_\alpha(t))\, x(t) = 0 \tag{2.8}$$

with $\alpha \in \mathbb{R}^2 \setminus \{(0,0)\}$.

In [8], where equation (2.8) was considered, it is established a relation between the initial data $\alpha$ of the Cauchy problem and the behaviour of the parameter depending matrix introduced above, whenever the asymptotically linear assumptions (2.2) and (2.3) are verified. In particular, the following lemma holds

**Lemma 2.8 (cf. [8])** *Suppose that the continuous function* $A : [0, \pi] \times \mathbb{R}^2 \to GL_s(\mathbb{R}^2)$ *satisfies assumptions (2.2) and (2.3), then*

$$A(t, x_\alpha(t)) \to A_\infty(t) \quad in \quad L^1([0, \pi]) \quad if \quad |\alpha| \to +\infty,$$

$$A(t, x_\alpha(t)) \to A_0(t) \quad in \quad L^1([0, \pi]) \quad if \quad |\alpha| \to 0.$$

We remark that the previous Lemma has been used in [8] in order to achieve multiplicity of solutions to asymptotically linear vectorial problems.

**Proof of Theorem 2.3** Let us assume that $i(A_0) > i(A_\infty)$, the other case can be treated similarly. By the definition of index there are exactly $i(A_0)$ negative eigenvalues $\lambda_l(A_0)$, $l \in \{1, \ldots, i(A_0)\}$. Also there are exactly $i(A_\infty)$ negative eigenvalues $\lambda_j(A_\infty)$, $j \in \{1, \ldots, i(A_\infty)\}$. Moreover, from the further assumption $\nu(A_\infty) = 0$ we get that $\lambda_j(A_\infty)$ is positive for every $j \in \mathbb{N}$ with $j \geq i(A_\infty) + 1$.
Consider now $h \in \mathbb{N}$ satisfying $i(A_0) \geq h \geq i(A_\infty)+1$. From the monotonicity properties of the sequence of eigenvalues, we immediately deduce that

$$\lambda_h(A_0) \, < \, 0 \, < \lambda_h(A_\infty). \tag{2.9}$$

We now concentrate on the study of the parameter dependent problem

$$\begin{aligned} &x'' \, + \, A(t, x_\alpha(t))\, x(t) \, = \, 0 \\ &x(0) \, = \, x(\pi) \, = \, 0. \end{aligned} \tag{2.10}$$

Assume that $a_{12}(t, x) \geq 0$ for every $(t, x) \in [0, \pi] \times \mathbb{R}^2$. Lemma 2.7 ensures that there are no solutions of the Dirichlet problem (2.10) such that $x'(0)$ lies in the second or the fourth quadrant.

Our next aim consists in proving the existence of $\alpha_{i,h} \in \mathcal{Q}_i \setminus \{(0, 0)\}$ such that $\lambda_h(A(\cdot, x_{\alpha_{i,h}}(\cdot))) = 0$ and $x_{\alpha_{i,h}}(\pi) = 0$ for $i \in \{1, 3\}$. We will focus on the search of $\alpha_{1,h} \in \mathcal{Q}_1 \setminus \{(0, 0)\}$, we observe that the case $\alpha_{3,h} \in \mathcal{Q}_3 \setminus \{(0, 0)\}$ can be treated analogously.
By combining Lemma 2.8 and Corollary 2.5 with the inequalities (2.9), we obtain

$$\lim_{|\alpha| \to 0} \lambda_h(A(\cdot, x_\alpha(\cdot))) \, < \, 0 \, < \lim_{|\alpha| \to +\infty} \lambda_h(A(\cdot, x_\alpha(\cdot))). \tag{2.11}$$

From an application of the theorems on continuous dependence on initial data, we can deduce the continuity of the map $\gamma : \mathbb{R}^2 \to C([0, \pi], GL_s(\mathbb{R}^2))$, defined by $\gamma(\alpha) := A(\cdot, x_\alpha(\cdot))$.
According to this continuity result and taking into account (2.11) and Corollary 2.5 we can choose $0 < R_1 < R_2$ such that $\lambda_h(A(\cdot, x_\alpha(\cdot))) < 0$ for every $\alpha \in \mathcal{Q}_1$ with $|\alpha| = R_1$ and $\lambda_h(A(\cdot, x_\alpha(\cdot))) > 0$ for every $\alpha \in \mathcal{Q}_1$ with $|\alpha| = R_2$.
Consider now $g : [R_1, R_2] \times \left[0, \frac{\pi}{2}\right] \to \mathbb{R}$ defined by $g(r, \theta) = \lambda_h(A(\cdot, x_{(r\cos(\theta), r\sin(\theta))}(\cdot)))$. As for each $\theta$, $g(R_1, \theta) < 0 < g(R_2, \theta)$, we have that $\deg(g(\cdot, 0), ]R_1, R_2[, 0) \neq 0$, where

we denote by deg the Brower degree, and also that $g(r,\theta) \neq 0$ if $|r| = R_1$ or $|r| = R_2$. Hence using the Leray-Schauder continuation theorem [26, Théorème Fondamental] we infer the existence of a closed connected set $\mathcal{C}^* \subset \left\{(r,\theta) \in ]R_1, R_2[\times \left[0, \frac{\pi}{2}\right] \,:\, g(r,\theta) = 0\right\}$ such that $\mathcal{C}^* \cap ([R_1, R_2] \times \{0\}) \neq \emptyset$ and $\mathcal{C}^* \cap \left([R_1, R_2] \times \left\{\frac{\pi}{2}\right\}\right) \neq \emptyset$. Thus we conclude the existence of a closed connected set $\mathcal{C} \subset \mathcal{Q}_1 \setminus \{(0,0)\}$ such that $\mathcal{C} \cap (\{0\} \times \mathbb{R}^+) \neq \emptyset$, $\mathcal{C} \cap (\mathbb{R}^+ \times \{0\}) \neq \emptyset$ and

$$\lambda_h(A(\cdot, x_\alpha(\cdot))) = 0 \qquad \forall \alpha \in \mathcal{C}.$$

Let us now prove that all the assumptions of Proposition 2.6, considering $B : [0,\pi] \times \mathcal{C} \to GL_s(\mathbb{R}^2)$ defined by $B(t,\alpha) = A(t, x_\alpha(t))$, are satisfied. From the continuity of $\gamma$, it easily turns out that the map $B(t,\alpha)$ is continuous too. Finally, by combining Lemma 2.7 with assumptions (2.4) we deduce that for every $\alpha \in \mathbb{R}^2$ it does not exist any solution $\phi_\alpha = (\phi_{1,\alpha}, \phi_{2,\alpha})$ to the Dirichlet problem (2.10) with $\phi'_{1,\alpha}(0) = 0$. As a remark we point out that this implies that all the solutions of $x'' + A(t, x_\alpha(t))\, x(t) = 0$ satisfying $x(0) = 0$ and $x'(0) = (0,1)$ do not vanish at $t = \pi$ (and hence the space of eigenvectors associated to a zero eigenvalue has dimension one). We can now apply Proposition 2.6 and conclude the existence of a continuous function defined on $\mathcal{C}$, $\alpha \to (v_\alpha(\cdot), v'_\alpha(\cdot))$, such that, for each $\alpha$, $v_\alpha$ is an eigenvector of $A(\cdot, x_\alpha(\cdot))$ associated to the zero eigenvalue. For each $\alpha \in \mathcal{C}$ we can set $\beta(\alpha) := v'_\alpha(0) \in \mathbb{R}^2 \setminus \{(0,0)\}$, hence $v_\alpha$ is a nontrivial solution of the system

$$\begin{array}{l} x'' + A(t, x_\alpha(t))\, x = 0 \\ x(0) = x(\pi) = 0 \end{array} \tag{2.12}$$

satisfying $x'(0) = \beta(\alpha)$.

Taking into account Lemma 2.7 and the fact that $a_{12}(t,x) \geq 0$ for every $(t,x) \in [0,\pi] \times \mathbb{R}^2$, we note that $\beta(\alpha) = (\beta_1(\alpha), \beta_2(\alpha)) \in \mathcal{Q}_1 \cup \mathcal{Q}_3$ and $\beta_1(\alpha)\beta_2(\alpha) \neq 0$. Being the problem (2.12) linear, we can restrict ourselves to the case $\beta(\alpha) \in \mathcal{Q}_1 \setminus \{0,0\}$.

Now we prove that for some $\bar{\alpha} \in \mathcal{C}$ there exists $C > 0$ such that $\beta(\bar{\alpha}) = C\,\bar{\alpha}$, from which we obtain $x_{\bar{\alpha}} = \frac{x_{\beta(\bar{\alpha})}}{C}$ and, consequently, $x_{\bar{\alpha}}(\pi) = 0$. In particular, we can choose $\alpha_{1,h} = \bar{\alpha}$.

Consider the polar coordinates $(\vartheta, \rho)$ of $\gamma$ in the plane, given by $\gamma_1 = \rho\cos\vartheta$, $\gamma_2 = \rho\sin\vartheta$. Since the function $\alpha \mapsto \beta(\alpha)$ is continuous from $\mathcal{C} \subset \mathcal{Q}_1 \setminus \{0,0\}$ to $\mathcal{Q}_1 \setminus \{0,0\}$, the function $\alpha \mapsto \vartheta(\beta(\alpha)) - \vartheta(\alpha)$ from $\mathcal{C}$ to $]-\frac{\pi}{2}, \frac{\pi}{2}[$ is continuous as well.

There exist $\tilde{\alpha} = (0, \tilde{\alpha}_2)$, $\hat{\alpha} = (\hat{\alpha}_1, 0) \in \mathcal{C}$. Observe that $\vartheta(\beta(\tilde{\alpha})) - \vartheta(\tilde{\alpha}) < 0$ and $\vartheta(\beta(\hat{\alpha})) - \vartheta(\hat{\alpha}) > 0$. Hence, recalling that $\mathcal{C}$ is a connected set, we infer the existence of $\bar{\alpha} \in \mathcal{C}$ such that $\vartheta(\beta(\bar{\alpha})) = \vartheta(\bar{\alpha})$.

Arguing as above in the third quadrant, at the end we find $\alpha_{i,h} \in \mathcal{Q}_i \setminus \{(0,0)\}$ such that $\lambda_h(A(\cdot, x_{\alpha_{i,h}}(\cdot))) = 0$ and $x_{\alpha_{i,h}}(\pi) = 0$ for every $i \in \{1,3\}$. In particular, for each $i \in \{1,3\}$, $x_{\alpha_{i,h}}$ is a nontrivial solution of the Dirichlet problem (2.1), satisfying $\lambda_h(A(\cdot, x_{\alpha_{i,h}}(\cdot))) = 0$, where $h$ is an arbitrary natural number with $i(A_0) \geq h \geq i(A_\infty) + 1$.

To complete the proof of the case $a_{12}(\cdot) \geq 0$ on $[0, \pi] \times \mathbb{R}^2$, it remains to show that all the values $\alpha_{i,h}$ that we have found above are mutually different, or, equivalently, that all the solutions of the form $x_{\alpha_{i,h}}$ are mutually different.
Assume, by contradiction, that there exist two natural numbers $h$, $k \in [i(A_\infty)+1, i(A_0)]$ with $h \neq k$ such that $\alpha_{i,h} = \alpha_{i,k}$. Let us set $\tilde{\alpha} := \alpha_{i,h} = \alpha_{i,k}$. In this case $\lambda_h(A(\cdot, x_{\tilde{\alpha}}(\cdot)) = \lambda_k(A(\cdot, x_{\tilde{\alpha}}(\cdot)) = 0$ and this contradicts the fact that under our assumptions the space of eigenvectors associated to the zero eigenvalue has dimension one.

Being the case $a_{12}(t, x) \leq 0$ for every $(t, x) \in [0, \pi] \times \mathbb{R}^2$ similar to the above one, we omit the corresponding proof. ■

**Remark 2.9** *By using arguments analogous to the ones adopted in this paper, existence of multiple solutions can be attained also for the scalar, Dirichlet problem*

$$x''(t) + A\,(t, x(t))\,x(t) = 0, \quad x(0) = 0 = x(\pi),$$

*where $A : [0, \pi] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous, satisfies the asymptotically linear conditions at the origin (2.2) and at infinity (2.3) and is such that uniqueness of solutions of Cauchy problems associated to the above equation is guaranteed. The multiplicity results which we are able to obtain in this scalar setting coincides with well-known results concerning asymptotically linear Dirichlet scalar problems (cf., among others, [12], [31] and references therein). We point out that in the literature more general nonlinearities have been studied and multiplicity of solutions has been achieved also without uniqueness assumptions on the solutions of the initial value problems.*

The following remarks are devoted to present some possible extensions of Theorem 2.3 to more general contexts. Both the generalizations stated below can be easily proved by following procedures analogous to the one adopted to achieve our main result.

**Remark 2.10** *In Theorem 2.3, instead of condition (2.4) we could have asked other kind of conditions which guarantee the result of Lemma 2.7.*
*In particular, the same conclusion of Theorem 2.3 holds if we replace the condition (2.4) with*

$$a_{11}(t, x) \leq 0, \quad a_{22}(t, x) \leq 0 \quad \textit{and} \quad a_{12}(t, x) \neq 0 \ \forall (t, x) \in [0, \pi] \times \mathbb{R}^2.$$

**Remark 2.11** *Note that by removing the assumption $\nu(A_\infty) = 0$ (or $\nu(A_0) = 0$) in the statement of Theorem 2.3, we can prove the existence at least $2\,|(i(A_0) - i(A_\infty)| - 4$ nontrivial solutions to problem (2.1), provided that we assume the positivity of the value $|(i(A_0) - i(A_\infty)| - 2$.*

**Remark 2.12** *Assume that there exists $(a, b) \in S^1$ such that for every continuous funcion $g : [0, \pi] \to \mathbb{R}^2$ there are no solutions of the Dirichlet problem*

$$\begin{cases} x'' + A(t, g(t))x = 0, \ t \in [0, \pi] \\ x(0) = x(\pi) = 0, \end{cases}$$

*satisfying $x'(0) = (a, b)$. Then, the same conclusion of Theorem 2.3 holds.*

Theorem 2.3 holds true if we generalize the asymptotically linear conditions (2.2) and (2.3) by assuming the existence of $A_1$, $A_2$, $B_1$, $B_2 \in C^0([0,\pi]; GL_s(\mathbb{R}^2))$ such that

$$B_1(t)z \cdot z \le \liminf_{|x|\to 0} A(t,x)z \cdot z \le \limsup_{|x|\to 0} A(t,x)z \cdot z \le B_2(t)z \cdot z, \quad (2.13)$$

$$A_1(t)z \cdot z \le \liminf_{|x|\to \infty} A(t,x)z \cdot z \le \limsup_{|x|\to \infty} A(t,x)z \cdot z \le A_2(t)z \cdot z. \quad (2.14)$$

uniformly in $t \in [0,\pi]$ and $z \in \mathbb{R}^2$. In particular, the statement of Theorem 2.3 can be extended into the following

**Corollary 2.13** *Assume that $A(t,x)$ satisfies (2.4), (2.13) and (2.14). Then if $i(B_1) > i(A_2)$ and $\nu(A_2) = 0$ (or $i(A_1) > i(B_2)$ and $\nu(B_2) = 0$), the problem (2.1) has at least $2\,|i(B_1) - i(A_2)|$ (or $2\,|i(B_2) - i(A_1)|$) nontrivial solutions.*

*Sketch of the proof.* The first step of the proof consists in generalizing Lemma 2.8 by adopting arguments analogous to the one followed in the proof of Proposition 4.4 in [8] and by taking into account that, under the assumptions of Theorem 2.13, $\|A\|$ is bounded. More precisely, we prove that for every sequence $\alpha_n \in \mathbb{R}^2$ satisfying $\lim_{n\to+\infty} |\alpha_n| = +\infty$ and for a.e. $t \in [0,\pi]$ the following inequalities hold

$$A_1(t)z \cdot z \le \liminf_{n\to+\infty} A(t, x_{\alpha_n}(t))z \cdot z \le \limsup_{n\to+\infty} A(t, x_{\alpha_n}(t))z \cdot z \le A_2(t)z \cdot z, \quad (2.15)$$

uniformly in $z \in \mathbb{R}^2$. By using the Fatou lemma, the Lebesgue's dominated convergence theorem and the boundedness of the matrix $A(t,x)$, one can pass from (2.15) to integral inequalities. More precisely, for every sequence $z_n \in L^\infty([0,\pi]; \mathbb{R}^2)$ with $\lim_{n\to+\infty} z_n = z_0$ in $\|\cdot\|_\infty$, we get $\int_0^\pi A_1(t)z_0(t) \cdot z_0(t)\,dt \le \liminf_{n\to+\infty} \int_0^\pi A(t, x_{\alpha_n}(t))z_n(t) \cdot z_n(t)\,dt$ and $\limsup_{n\to+\infty} \int_0^\pi A(t, x_{\alpha_n}(t))z_n(t) \cdot z_n(t)\,dt \le \int_0^\pi A_2(t)z_0(t) \cdot z_0(t)\,dt$.
By using the same procedure, from (2.13) it is possible to deduce integral inequalities analogous to the one exhibited above, in which $\alpha_n$ is replaced by $\beta_n \to 0$ as $n \to +\infty$ and where $A_i$ is replaced by $B_i$ for each $i \in \{1,2\}$.
The final steps of the proof are based on some generalized Sturm comparison result contained in [15] and, in particular, on its Proposition 2.6, where it is proved that $i(B) \le i(C)$ if $B(t) \le C(t)$ for a.e. $t \in (0,\pi)$ and $i(B) + \nu(B) \le i(C)$ if $B < C$, whenever $B$, $C \in L^\infty([0,\pi]; GL_s(\mathbb{R}^2))$.
By combining this result with the continuity of the eigenvalues proved in Corollary 2.5, it is easy to show that

$$\exists\, \varepsilon_0 > 0 : \ \forall \varepsilon \in (0, \varepsilon_0] \quad i(B - \varepsilon I_2) = i(B) \ \text{ and } \ i(B + \varepsilon I_2) = i(B) + \nu(B). \quad (2.16)$$

Taking into account the techniques used in [15] to prove the above quoted Proposition 2.6, the integral inequalities exhibited before and (2.16), we prove that

$$\begin{aligned} &\exists\, R > 0 : \ \forall \alpha \in \mathbb{R}^2,\ |\alpha| > R \quad i(A_1) \le i(A(\cdot, x_\alpha(\cdot))) \le i(A_2), \\ &\exists\, \delta > 0 : \ \forall \alpha \in \mathbb{R}^2,\ |\alpha| < \delta \quad i(B_1) \le i(A(\cdot, x_\alpha(\cdot))) \le i(B_2). \end{aligned} \quad (2.17)$$

Let us now concentrate on the case in which $i(B_1) > i(A_2)$ and $\nu(A_2) = 0$. Consider $h \in \mathbb{N}$ satisfying $i(B_1) \geq h \geq i(A_2) + 1$. According to (2.17), it turns out that

$$\forall \alpha \in \mathbb{R}^2,\ |\alpha| > R: \quad \lambda_h(A(\cdot, x_\alpha(\cdot))) > 0 \quad \text{and} \quad \forall \alpha \in \mathbb{R}^2,\ |\alpha| < \delta: \quad \lambda_h(A(\cdot, x_\alpha(\cdot))) < 0.$$

This relation recalls the relation (2.11), on which it is based the proof of Theorem 2.3. The thesis follows by proceeding as in the proof of our main theorem. ■

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