NAVIER-STOKES EQUATIONS WITH SHEAR DEPENDENT VISCOSITY. REGULARITY UP TO THE BOUNDARY.

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Abstract. In the following we prove some sharp regularity results for the stationary and the evolution Navier-Stokes equations with shear dependent viscosity, see (1.1), under the non-slip boundary condition (1.4). We are interested in regularity results for the second order derivatives of the velocity and the first order derivatives of the pressure *up to the boundary*, in dimension $n \geq 3$. In reference [4] we consider the stationary problem in the half space $\mathbb{R}^n_+$ under slip and non-slip boundary conditions. Here, by working in a simpler context, we lay stress on the core of the proofs. We consider a cubic domain and impose our boundary condition (1.4) only on two opposite faces. On the other faces we assume periodicity, as a device to avoid effective boundary conditions. This choice is made so that we work in a bounded domain Ω and simultaneously with a flat boundary. In the last section we provide the extension of the results from the stationary to the evolution problem.

Mathematics Subject Classification (2000). Primary 35J25, 35Q30; Secondary 76D03, 76D05.

Keywords. Navier-Stokes equations, shear dependent viscosity, regularity up to the boundary.

1. Introduction

In the sequel u and π denote, respectively, the velocity and the pressure of a viscous incompressible fluid. We are mainly interested in studying and improving regularity results for solutions to the evolution Navier-Stokes equations for flows with shear dependent viscosity, namely

$$\begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)\, u - \nabla \cdot T(u, \pi) = f, \\ \nabla \cdot u = 0\,, \end{cases} \tag{1.1}$$

Partly supported by CMAF/UL and Fundação Calouste Gulbenkian (Lisbon).

under suitable boundary conditions, where T denotes the Cauchy stress tensor

$$T = -\pi\, I + \nu_T(u)\, \mathcal{D}\, u\,, \tag{1.2}$$

$\frac{1}{2}\,\mathcal{D}\, u$ denotes the symmetric gradient, i.e.,

$$\mathcal{D}\, u = \nabla\, u + \nabla\, u^T\,,$$

and

$$\nu_T(u) = \nu_0 + \nu_1 |\mathcal{D}\, u|^{p-2} \tag{1.3}$$

denotes the viscosity. Here ν_0 and ν_1 are strictly positive constants. In the following we consider the case $p \geq 2$.

The system of equations (1.1), for $p = 3$, was introduced by J.S. Smagorinsky, see [35], as a turbulence model. For arbitrary p the system was introduced and studied by O.A. Ladyzenskaya, already as a turbulence model, in references [14], [15], [16] and [17]. J.-L. Lions considered similar models, in which $\mathcal{D}\, u$ is replaced by ∇u. See [20] and [21], Chap.2, n.5. It is worth noting that (1.2) satisfies the Stokes Principle, see [37]. A clear and rigorous discussion on this subject is given by J. Serrin in reference [34], page 231, where the above physical principle is stated in a postulational form.

In order to avoid additional calculations we assume that $p \leq 3$. However this restriction is not at all necessary, in the sense that essentially the same argument gives similar results for $p > 3$. The case $2 \leq p \leq 3$ (specially $p = 3$) has been applied in the last forty years to model turbulence phenomena in fluid flows, a main problem in theoretical, applied and numerical Fluid Mechanics. See, for instance, [5], [7], [11], [12], [13], [19], [28], [35] and the references given by these authors. Nonlinear shear dependent viscosity also model of properties of materials. The cases $p > 2$ and $p < 2$ captures shear thickening and shear thinning phenomena, respectively. See, for instance, [29].

Higher order regularity results, up to the boundary, for solutions to problem (1.1) (and similar) in regular bounded open sets $\Omega \subset \mathbb{R}^3$, under the non-slip boundary condition

$$u_{|\Gamma} = 0\,, \tag{1.4}$$

are studied in depth in reference [23]. Nevertheless these results may be improved. In reference [4] particularly sharp regularity results in the half-space $\mathbb{R}^n_+$ (note the flat boundary) were obtained for the stationary problem

$$\begin{cases} -\nu_0\, \nabla \cdot\, \mathcal{D}\, u - \nu_1\, \nabla \cdot\, \Big(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u \Big) + \nabla\, \pi = f, \\ \\ \nabla \cdot u = 0 \end{cases} \tag{1.5}$$

under slip and non-slip boundary conditions. In the $\mathbb{R}^n_+$ case we do not have the inclusion $L^q \subset L^p$ if $q > p$. The lack of this property, which holds in a bounded domain Ω, implies some secondary but involved arguments which substantially upset the main stream of the proofs. In order to work once more with a flat boundary Γ and, simultaneously, in a functional framework in which the above functional inclusions hold, we are led to consider here a cubic domain Ω and to impose the boundary condition (1.4) just on two opposite faces. On the other pair of faces we assume periodicity conditions (in this way we avoid singularities due to the corner points). This enables us to emphasize the very basic ideas of our method.

We have already obtained the extension of our results to arbitrary regular open sets Ω. This is the subject of a paper in preparation.

Remark 1.1. **On the convective term.** In proving higher order regularity results for the classical Navier-Stokes equations the convective term plays a secondary rule, in spite of its responsibility for the lack of regularity of the solution. In fact, in proving these results, the central point in the known proofs is the higher order regularity for the Stokes linear equation (for instance, the classical works by Cattabriga and Solonnikov). The convective term is simply treated "as a right hand side" (for want of anything better!). We do not face a more favorable situation when $p \neq 2$. Hence we treat the stationary problem (1.5) without the convective term and show, just as a final byproduct, that the regularity results proved for the stationary generalized Stokes problem hold for the stationary generalized Navier-Stokes problem. The same holds in the evolution case provided that $p \geq 2 + \frac{2}{5}$.
Obviously, as in reference [4], no single term in the left hand side of (1.5) can be treated "as a right hand side" (as wrongly remarked somewhere).

Remark 1.2. **On the evolution problem.** Below we show that higher order regularity results for the evolution problem (1.1) can be obtained in quite a simple way as corollaries of the corresponding results for the stationary problem (1.5). Hence the **crucial point** is the study of the stationary problem (1.5).

Remark 1.3. **On the regularity up to the boundary.** When $p \neq 2$, there is an unusually increment of difficulty in passing from interior to boundary regularity for solutions to the system (1.5). A sign of this fact is the lower regularity obtained for the second order derivatives of the velocity (and for the first order derivatives of the pressure) in the normal direction in comparison to the other directions. One of the main reasons is the following one. In proving interior regularity by appealing to the classical translation method, translations are admissible in all the n independent directions. This allows suitable estimates for $\nabla \mathcal{D} u$. Note that the above full gradient ∇ is obtained here thanks to the possibility of appealing to translations in all the directions. Furthermore, it is easily shown that $c\,|\nabla \nabla u| \leq |\nabla \mathcal{D} u| \leq C\,|\nabla \nabla u|$. This two facts together lead to a not particularly distinct situation if we replace $\mathcal{D}u$ by ∇u in equation (1.5). However, in proving regularity up to the boundary, the two cases are completely distinct, as is well known to authors acquainted with these problems. In fact, solutions to the J.-L. Lions model belong to $W^{2,2}$ up to the boundary. It looks not accidental that there is a very extensive literature on interior regularity for the above problem but, as far as we know, few literature concerning regularity up to the boundary, at least in the $3-D$ case.

Remark 1.4. **On the slip boundary condition.** In [4] we also consider the more intricate slip boundary condition. For simplicity we take here into account only the non-slip boundary condition (1.4) and assume $n=3$. However, by following [4], we easily extend to the slip boundary condition all the results proved below.

Results. For the main results proved in the sequel see the Theorems 3.1, 3.2, 3.4 and 3.5(and also the Lemma 3.10) for the stationary problem, and the Theorems 10.3 and 10.4 for the evolution problem. This set of theorems improve the previous known results when applied under the same hypotheses.

Without any claim of completeness, and besides the articles quoted above, see [1],[4], [6], [8], [9], [10], [18], [22], [23], [24], [25], [30], [31], [32], [33], and the references given by these authors.

2. Notation, weak solutions and some auxiliary results.

In the sequel Ω denotes the 3-dimensional cube $\Omega = (]0,1[)^3$.
Further, we set

$$\Gamma_- = \{x : |x_1|,\, |x_2| < 1\,, x_3 = 0\}\,, \quad \Gamma_+ = \{x : |x_1|,\, |x_2| < 1\,, x_3 = 1\}\,.$$

The Dirichlet boundary condition (the condition in which we are interested here) will be imposed only on

$$\Gamma = \Gamma_- \cup \Gamma_+\,.$$

The problem will be assumed periodic, with period equal to 1, both in the x_1 and the x_2 directions. In the following the significant boundary is Γ. Actually $\Gamma = \partial\,\Omega$ provided that Ω and Γ are indefinitely reflected in the x_1 and x_2 directions. Sometimes we use the term "boundary" to denote Γ. For convenience we set

$$x' = (x_1, x_2)\,.$$

By x'-periodic we mean periodic of period 1 both in x_1 and x_2. A similar convention is assumed for expressions like x'-periodicity and so on.

If X is a Banach space, we denote by X' its strong dual space. We use the same notation for functional spaces and norms for both scalar and vector fields. The symbol $\|\cdot\|_p$ denotes the canonical norm in $L^p(\Omega)$, and $\|\cdot\|$ that in $L^2(\Omega)$. $W^{1,p}(\Omega)$ denotes the usual Sobolev space.

We set

$$V_p = \big\{ v \in W^{1,p}(\Omega) : \, (\nabla \cdot v)_{|\Omega} = 0\,;\; v_{|\Gamma} = 0\,;\, v \text{ is } \; x' - \text{periodic} \big\}\,. \tag{2.1}$$

Note that, by appealing to inequalities of Korn's type, one gets the following result.

Lemma 2.1. There is a positive constant c such that the estimate

$$\|\nabla v\|_p + \|v\|_p \leq c\,\|\mathcal{D}v\|_p \tag{2.2}$$

holds, for each $v \in V_p$. Hence the two above quantities are equivalent norms in V_p.

For the proof see, for instance, [28], Proposition 1.1.

Definition 2.1. Assume that

$$f \in (V_2)'\,. \tag{2.3}$$

We say that u is a *weak solution* to problem (1.5), (1.4) if $u \in V_p$ satisfies

$$\frac{1}{2}\int_\Omega \nu_T(u)\,\mathcal{D}u \cdot \mathcal{D}v\,dx = \int_\Omega f \cdot v\,dx \tag{2.4}$$

for all $v \in V_p$.

By defining $< A\,u,\, v >$, for each pair $u,\, v \in V_p$, as the left hand side of (2.4), the operator $A:\, V_p \to V_p'$ satisfies the assumptions in the Theorems 2.1 and 2.2, Chap.2, Sect.2, [21]. This shows existence and uniqueness of the weak solution.

By replacing v by u in equation (2.4) one gets

$$\nu_0\,\|\nabla u\|^2 + \nu_1\,\|\mathcal{D}u\|_p^p = < f, u >\,, \tag{2.5}$$

where the symbols $< \cdot, \cdot >$ denote a duality pairing. Note that the left hand side of equation (2.5) is just $< A\,u,\, u >$. This shows that the assumption (2.3) in Theorem 2.1 of reference [21] holds.

From (2.5) there readily follows the basic estimates

$$\begin{cases} \nu_0^2\,\|\nabla u\|^2 + 2\,\nu_0\,\nu_1\,\|\mathcal{D}u\|_p^p \leq c\,\|f\|^2\,, \\[2ex] \nu_0\,\nu_1^{\frac{1}{p-1}}\,\|\nabla u\|^2 + \nu_1^{p'}\,\|\mathcal{D}u\|_p^p \leq c\,\|f\|_{p'}^{p'}\,. \end{cases} \tag{2.6}$$

In particular

$$\begin{cases} \nu_0\,\|\nabla u\| \leq c\,\|f\|\,, \\[2ex] \nu_1\,\|\nabla u\|_p \leq c\,\|f\|_{p'}^{\frac{1}{p-1}}\,. \end{cases} \tag{2.7}$$

By restriction of (2.4) to divergence-free test-functions v with compact support in Ω, and by De Rham's theorem, there follows the existence of a distribution π (determined up to a constant) such that

$$\nabla\,\pi = -\,\nabla \cdot \left[\nu_0 \nabla\,u + \nu_1\,|\mathcal{D}u|^{p-2}\,\mathcal{D}u\right] + f \equiv \nabla \cdot (U_1 + U_2\,) + f\,, \tag{2.8}$$

Equation (2.8) shows that the first equation (1.5) holds in the distributions sense.

The following result is well known.

Lemma 2.2. If a distribution g is such that $\nabla\,g \in W^{-1,\alpha}(\Omega)$ then $g \in L^\alpha(\Omega)$ and

$$\|g\|_{L^\alpha_\#} \leq c\,\|\nabla\,g\|_{W^{-1,\alpha}}\,, \tag{2.9}$$

where $L^\alpha_\# = L^\alpha/\mathbb{R}$.

From (2.8) and (2.7) it readily follows that $\pi \in L^{p'}(\Omega)$ and that

$$\|\pi\|_{L^{p'}_{\#}} \leq c(\|f\| + \|f\|_{p'})\,.$$

We end this section by introducing some more notation.

We denote by $D^2\,u$ the set of all the second derivatives of u. The meaning of expressions like $\|D^2\,u\|$ is clear. The symbol $D^2_*\,u$ denotes any of the second order derivatives $\partial^2 u_j/\,\partial x_i\,\partial x_k$ except for the derivatives $\partial^2 u_j/\,\partial x_3^2$, if $j=1$ or $j=2$. Moreover,

$$|D^2_*\,u|^2 := \left|\frac{\partial^2\,u_3}{\partial\,x_3^2}\right|^2 + \sum_{\substack{i,j,k=1\\(i,k)\neq(3,3)}}^{3} \left|\frac{\partial^2\,u_j}{\partial x_i\,\partial x_k}\right|^2\,. \tag{2.10}$$

Similarly, ∇^* may denote any first order partial derivative, except for $\partial\ /\partial\,x_3$.

Some integrability exponents play a crucial role in our proofs and are, for the reader's convenience, introduced here.

In the sequel p denotes an exponent that lies in the interval

$$2 \leq\,p\,\leq 3 \tag{2.11}$$

and q an exponent that lies in the interval

$$p \leq\,q \leq\,6\,.$$

We denote by p' the dual exponent

$$p' = \frac{p}{p-1}\,. \tag{2.12}$$

In general, for $1 < r < 3$ we define the Sobolev embedding exponent r^* by the equation

$$\frac{1}{r^*} = \frac{1}{r} - \frac{1}{3}\,. \tag{2.13}$$

Given p and q as above we define $r=r(q)$ by

$$\frac{1}{r} = \frac{p-2}{2\,q} + \frac{1}{2} \tag{2.14}$$

and $\overline{q} = \overline{q}(q)$ by

$$\frac{1}{\overline{q}} = \frac{p-2}{r^*} + \frac{1}{2} = \frac{(p-2)^2}{2q} + \frac{p-2}{6} + \frac{1}{2} \tag{2.15}$$

and set

$$\widetilde{q} = \min\{\overline{q}, r\}\,. \tag{2.16}$$

The assumption $p \geq 2$ is essential in many points of our proofs. However the assumption $p \leq 3$ can be relaxed, or even dropped, in many statements (for instance, $2 \leq p$ is sufficient in Theorem 3.1 and $2 \leq p < 4$ in theorem 3.4). However, in order to avoid cumbersome distinctions, we assume the condition (2.11).

We denote by c a generic positive constant that may change from equation to equation. The positive constants c do not depend on the parameters p and q, in the usual sense (i.e., they are bounded from above for p and q varying in the ranges considered here). As a rule, we let the constants c depend on ν_0 and ν_1. It is easily seen that if $0 < \underline{\nu} \leq \nu_0,\ \nu_1 \leq \overline{\nu}$ the constants c depend only on $\underline{\nu}$ and $\overline{\nu}$. Nevertheless we may let the constants ν_0 and ν_1 appear when this provides a better understanding of some manipulation.

3. The stationary problem. Main results

In this section we state our main results concerning the stationary problem. We also include some explanation regarding the "architecture" of the proofs. We start with the following very basic result.

Theorem 3.1. Assume that

$$f \in L^2(\Omega) \tag{3.1}$$

and let u, π be the weak solution to problem (1.5) under the boundary condition (1.4) plus x'-periodicity (problem (2.4)).

Then the derivatives $D_*^2\, u$ belong to $L^2(\Omega)$, moreover

$$\nu_0\, \|D_*^2\, u\| + (\nu_0\, \nu_1)^{\frac{1}{2}} \, \left\| \, |\mathcal{D}u|^{\frac{p-2}{2}}\, \nabla^*\, \mathcal{D}u \right\| \leq c\, \|f\|\,. \tag{3.2}$$

Furthermore $D^2\, u$, $|\mathcal{D}u|^{p-2}\, \nabla^*\, \mathcal{D}u$ and $\nabla^*\, \pi$ belong to $L^{p'}(\Omega)$ and satisfy the estimate

$$\|\nabla^*\, \pi\|_{p'} + \|D^2\, u\|_{p'} + \|\, |\mathcal{D}u|^{p-2}\, \nabla^*\, \mathcal{D}u\|_{p'} \leq \mathcal{K}_p \tag{3.3}$$

where $\mathcal{K}_p$ has the form

$$\mathcal{K}_p = c\, \|f\| + c\, \|\mathcal{D}\, u\|_p^{\frac{p-2}{2}}\, \|f\|\,. \tag{3.4}$$

Finally,

$$\frac{\partial\, \pi}{\partial\, x_3} \in L^{p_0}(\Omega),$$

and

$$\|\, \nabla\, \pi\|_{p_0} \leq c\, \left[1 + \mathcal{K}_p^{p-2}\right]\, \|f\| + c\, \mathcal{K}_p\,, \tag{3.5}$$

where $p_0 = \min\{\overline{q}, p'\}$ and $\overline{q}$ is given here by setting $q = p$ in equation (2.15).

Note that by (2.7) one has, in particular,

$$\mathcal{K}_p \leq c\|f\| + c\, \|f\|^{\frac{3p-4}{2p-2}}\,.$$

Moreover, if $p = 2$ we reobtain the classical result for the Stokes linear equation, namely, if f is square integrable so is $\nabla\, \pi$. It is curious enough that in the very important and significant case of the Smagorinsky exponent $p = 3$ it follows that $p_0 = \frac{6}{5}$. Hence $p_0^* = 2$, i.e. the pressure π is square integrable. The exponents p' and p_0 in the estimates (3.3) and (3.5) will be improved below. Nevertheless, for completness, we remark that $p_0 = p'$ if $2 \leq p \leq 2 + \frac{1}{4}$ and $p_0 = \overline{q}$ if $p \geq 2 + \frac{1}{4}$. For $p = 2 + \frac{1}{4}$ one has $p_0 = p' = \overline{p} = \frac{9}{5}$.

If we assume that (3.6) below holds for some $q > p$ then the Theorem 3.1 can be improved. Actually we will show that (3.6) holds provided that $p < 3$. However it is more convenient to start by establishing the result in the conditional form below. The assumption $3 \leq q \leq 6$ is essentially superfluous.

Theorem 3.2. Let f, u and π be as in Theorem 3.1 and assume, in addition, that

$$\mathcal{D}\, u \in L^q(\Omega) \tag{3.6}$$

for some $3 \leq q \leq 6$. Then, in addition to (3.2), one has

$$D^2\, u,\ |\mathcal{D}\, u|^{p-2}\, \nabla^*\, \mathcal{D}\, u\,,\ \nabla^*\, \pi \in L^r(\Omega)\,. \tag{3.7}$$

More precisely,

$$\|\nabla^*\, \pi\|_r + \|D^2\, u\|_r + \|\, |\mathcal{D}\, u|^{p-2}\, \nabla^*\, \mathcal{D}\, u\|_r \leq \mathcal{K}_q\,, \tag{3.8}$$

where $\mathcal{K}_q$ has the form

$$\mathcal{K}_q = c\, \|f\| + c\, \|\mathcal{D}\, u\|_q^{\frac{p-2}{2}}\, \|f\| \tag{3.9}$$

and r is given by (2.14).

Concerning the regularity of the derivative $\frac{\partial \pi}{\partial x_3}$ one has the following result.

Lemma 3.3. Under the assumptions of Theorem 3.2 one has

$$\left\| \frac{\partial \pi}{\partial x_3} \right\|_{\widetilde{q}} \leq c\, \left[1 + \mathcal{K}_q^{p-2}\right]\, \|f\| + c\, \mathcal{K}_q \tag{3.10}$$

where $\widetilde{q}$ is defined in (2.16). In particular, by (3.8),

$$\|\nabla \pi\|_{\widetilde{q}} \leq c\, \left[1 + \mathcal{K}_q^{p-2}\right]\, \|f\| + c\, \mathcal{K}_q\,. \tag{3.11}$$

Remark. We note that the above quantity $\mathcal{K}_q$ does not correspond to the quantity defined in reference [4] by the same symbol. In fact, the quantity $\mathcal{K}_q$ defined by (3.9) corresponds to the quantity defined in [4] equation (5.5) by the symbol $\mathcal{K}_r$, where r is related to q by (2.14).

Theorem 3.4. Let f, u and π be as in Theorem 3.1. Then, in addition to (3.2), one has

$$D^2\, u\,,\ |\mathcal{D} u|^{p-2}\, \nabla^*\, \mathcal{D} u\,,\ \nabla^*\, \pi \in L^l(\Omega),$$

where

$$l = 3\, \frac{4-p}{5-p}\,. \tag{3.12}$$

More precisely,

$$\|\nabla^*\, \pi\|_l + \|D^2\, u\|_l + \|\, |\mathcal{D} u|^{p-2}\, \nabla^*\, \mathcal{D} u\|_l \leq c\, \|f\| + c\, \|f\|^{\frac{2}{4-p}}\,. \tag{3.13}$$

Finally,

$$\frac{\partial\, \pi}{\partial\, x_3} \in L^m(\Omega), \tag{3.14}$$

where

$$m = \frac{6(4-p)}{8-p}\,. \tag{3.15}$$

In particular,

$$\nabla\,\pi \in L^m(\Omega)\,,$$

and

$$\|\nabla\,\pi\,\|_m \leq c\,\left(\|f\|^{\frac{2}{p}} + \|f\|^{\frac{p}{4-p}}\right)\,. \tag{3.16}$$

Remarks.

Note that (3.13) improves (3.5) since $p' < l$ if $2 < p < 3$. Moreover

$$u \in W^{1,l^*}(\Omega)\,,$$

where $l^* = 3\,(4-p)$. Clearly $l^* > p$ for $2 < p < 3$. In addition, $u \in C^{0,\alpha}(\Omega)$, where $\alpha = \frac{3-p}{4-p}$. Also note that $m > p'$ if $p < 2 + \frac{2}{5}$.

It is significant that, when $p = 2$, the statements and estimates established in Theorems 3.1 and 3.4 coincide with the classical results for the linear Stokes problem.

Theorem 3.5. All the regularity results stated in the Theorems 3.1, 3.2 and 3.4, and in the Lemma 3.10, hold for the generalized Navier-Stokes equations

$$\begin{cases} -\nu_0\,\nabla\cdot\,\mathcal{D}\,u - \nu_1\,\nabla\cdot\,\left(|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u\right) + (u\cdot\nabla)\,u + \nabla\,\pi = f, \\ \\ \nabla\cdot u = 0\,. \end{cases} \tag{3.17}$$

Moreover all the estimates shown in the above statements hold provided that we replace $\|f\|$ by $\|f\| + \|\nabla\,u\|_p^2$.

4. Main lines.

In order to help following the proofs we briefly illustrate the main lines. The starting point is the proof of (3.2), given in section 4. Then in section 5 we prove *under the assumption* (3.6), the estimate (3.8). In section 6 we prove the estimate (3.10). At this point the Theorem 3.1 is completely proved since, if $q = p$, weak solutions satisfy (3.6) and the estimates (3.3) and (3.5) coincide with (3.8) and (3.3) respectively. In particular $r = p'$.

Now we comment on Theorem 3.2, which is a main step in order to prove the Theorem 3.4. By Theorem 3.2 for $q = p$ (i.e. by Theorem 3.1) it follows that $u \in W^{2,p'}$. A Sobolev embedding theorem shows that $u \in W^{1,q_2}$, where $q_2 = (p')^* = \frac{3p}{2p-3}$. If $p < 3$ then q_2 is larger than p. This fact opens the way to a bootstrap argument by applying again the Theorem 3.2, now with $q = q_2$. The bootstrap argument works well and leads to a chain of "intermediate" W^{2,l_n} regularity results, by applying at each steep the Theorem 3.2 to the previous value of the parameter q. The Theorem 3.1 is just the first element of this chain. By the above argument we prove an infinite sequence of regularity results. A further, natural, problem is trying "to pass to the limit" in the above sequence of regularity results and proving in this way that $u \in W^{2,l}$, where l is the upper bound of the exponents l_n for which $u \in W^{2,l_n}$. We succeed in proving this last step. This leads to the Theorem 3.4. In this Theorem the exponent l turns out to be just the exponent for which the Theorem 3.2 with $q = l^*$ yields $u \in W^{2,l}$. Then, by a Sobolev embedding Theorem, $u \in W^{1,l^*}$. In other words, l^* is the fixed point of the map $q \to r \to r^*$. So, further regularity cannot be obtained by appealing to Theorem 3.2.

Finally, the reason that leads us to separate Lemma 3.3 from Theorem 3.2 is to emphasize that the regularity of $\frac{\partial \pi}{\partial x_3}$ is simply obtained as a final by product (in contrast with the main rule of the regularity of all the other derivatives of u and π in each steep of the bootstrap argument).

In the stationary case the above sequence of results obtained by the bootstrap argument are stronger for larger values of the "step number" n. Each of these single results gives rise to a regularity result for the evolution problem, as follows immediately from section 9. However, in the evolutionary case, as n increases the space-regularity exponents still increase but the time-regularity exponents *decrease*. See Theorems 10.3 and 10.4. The mathematical motivation for this situation is clear from the proofs given in section 9.

Remark. For convenience, in treating the evolutionary case, we state in explicit form only the two "extreme" cases of the above chain of possible results.

5. Regularity of the $D_*^2\, u$ derivatives. Proof of estimate (3.2).

in this section we prove Theorem 5.2 below concerning the Stokes stationary problem (1.5). The proof of the following auxiliary result is left to the reader.

Lemma 5.1. Let U, V be two arbitrary vectors in $\mathbb{R}^N, N \geq 1$ and $p \geq 2$. Then

$$\begin{gathered} \left(|U|^{p-2}U - |V|^{p-2}V\right) \cdot (U - V) \geq \frac{1}{2}\left(|U|^{p-2} + |V|^{p-2}\right) |U - V|^2\,, \\ \big|\, |U|^{p-2}U - |V|^{p-2}V \big| \leq \frac{p-1}{2}\left(|U|^{p-2} + |V|^{p-2}\right) |U - V|\,. \end{gathered} \tag{5.1}$$

Theorem 5.2. Assume that $2 < p$ and that f, u and π are as in Theorem 3.1. Then the derivatives $D_*^2\, u$ belong to $L^2(\Omega)$ and satisfy the estimate (3.2).

Proof of Theorem 5.2.

Proof. Let u be a weak solution, i.e. $u \in V_p$ is a solution to the problem

$$\frac{\nu_0}{2} \int \mathcal{D}\, u \cdot \mathcal{D}\, v \, dx + \frac{\nu_1}{2} \int |\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u \cdot \mathcal{D}\, v \, dx = \int f \cdot v \, dx \,, \tag{5.2}$$

for each $v \in V_p$. For arbitrary scalar or vector fields v we set

$$\tau_h\, v(x) = v(x_1, ..., x_{k-1}, x_k + h, x_{k+1}, ..., x_n)\,,$$

where $h \in \mathbb{R}$ and k, $k \neq n$, is assumed to be fixed. Here, $n = 3$. We also set

$$v^h = \tau_h\, v\,; \qquad \Delta_h\, v = \frac{v - v^{-h}}{h}\,,$$

Note that the above translations are done in the tangential directions.

By writing (5.2) with v replaced by v^h and by replacing, in the integrals on the left hand side, the variable x_k by $x_k - h$, one easily shows that

$$\frac{\nu_0}{2} \int \mathcal{D}\, u^{-h} \cdot \mathcal{D}\, v \, dx + \frac{\nu_1}{2} \int |\mathcal{D}\, u^{-h}|^{p-2}\, \mathcal{D}\, u^{-h} \cdot \mathcal{D}\, v \, dx = \int f \cdot v^h \, dx \,. \tag{5.3}$$

By taking the difference between equations (5.2) and (5.3), respecting the left and right sides, and by dividing by h one gets

$$\frac{\nu_0}{2} \int (\mathcal{D}\, \Delta_h\, u) \cdot \mathcal{D}\, v \, dx + \frac{\nu_1}{2\,h} \int \left(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u - |\mathcal{D}\, u^{-h}|^{p-2}\, \mathcal{D}\, u^{-h}\right) \cdot \mathcal{D}\, v \, dx = \frac{1}{h} \int f \cdot (v - v^h) \, dx \,. \tag{5.4}$$

By setting $v = \Delta_h\, u$ in equation (5.4) and by taking into account the estimate

$$\left| \frac{1}{h} \int f \cdot (v - v^h) \, dx \right| \leq \|f\| \left\| \frac{v - v^h}{h} \right\| \leq \|f\| \, \|\nabla v\| \,, \tag{5.5}$$

it follows that

$$
\begin{gathered}
\frac{\nu_0}{2} \int |\mathcal{D}\, \Delta_h\, u|^2\, dx + \\
\frac{\nu_1}{2\,h} \int \left(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u - \, |\mathcal{D}\, u^{-h}|^{p-2}\, \mathcal{D}\, u^{-h} \right) \cdot \, (\mathcal{D}\, \Delta_h\, u)\, dx \\
\leq \, c\, \|f\|\, \|\nabla\, (\Delta_h\, u)\|\, .
\end{gathered}
\tag{5.6}
$$

On the other hand, due to the divergence-free property, one has

$$
\int |\mathcal{D}\, \Delta_h\, u|^2 \, dx = 2 \int |\nabla\, \Delta_h\, u|^2 \, dx\, .
$$

Since the second term on the left hand side of (5.6) is nonnegative it follows that $D_*^2\, u \in L^2(\Omega)$, moreover,

$$
\nu_0\, \|D_*^2\, u\| \equiv \nu_0 \left(\|\frac{\partial^2\, u_3}{\partial\, x_3^2}\| + \sum_{\substack{i,j,k=1 \\ (i,k)\neq(3,3)}}^{3} \|\frac{\partial^2\, u_j}{\partial x_i\, \partial x_k}\| \right) \leq \, c\, \|f\|\, . \tag{5.7}
$$

The inclusion of the derivative $\partial^2 u_3/\partial x_3^2$ in the above estimate follows by differentiation with respect to x_n of the equation $\nabla \cdot u = 0$. This proves the first part of the estimate (3.2). Next we prove the second part of this estimate. Since

$$
\|\nabla\, \Delta_h\, u\| \leq \, \|D_*^2\, u\|
$$

it readily follows from (5.6) and (5.7) that

$$
\begin{gathered}
\frac{\nu_0}{2} \int |\mathcal{D}\, \Delta_h\, u|^2\, dx + \\
\frac{\nu_1}{2\,h} \int \left(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u - \, |\mathcal{D}\, u^{-h}|^{p-2}\, \mathcal{D}\, u^{-h} \right) \cdot \, \mathcal{D}\, \Delta_h\, u\, dx \\
\leq \, \frac{c}{\nu_0}\, \|f\|^2\, .
\end{gathered}
\tag{5.8}
$$

Setting $U = \mathcal{D}\, u$ and $V = \mathcal{D}\, u^{-h}$ in equation (5.1) it follows that

$$
\begin{aligned}
&\frac{1}{h} \, \left(|\mathcal{D}\, u|^{p-2} \mathcal{D}\, u \, - |\mathcal{D}\, u^{-h}|^{p-2} \mathcal{D}\, u^{-h} \right) \cdot \, \mathcal{D}\, \Delta_h\, u \\
&\geq \, \frac{1}{2} \, \left(|\mathcal{D}\, u|^{p-2} + \, |\mathcal{D}\, u^{-h}|^{p-2} \right) \, |\mathcal{D}\, \Delta_h\, u|^2
\end{aligned}
\tag{5.9}
$$

almost everywhere in Ω. From (5.8) and (5.9) it follows that

$$\nu_0 \int |\mathcal{D}\,\Delta_h\, u|^2\, dx$$

$$+\,\nu_1 \int \left\{ \left(|\mathcal{D}\, u|^{p-2} + \, |\mathcal{D}\, u^{-h}|^{p-2} \right) |\mathcal{D}\,\Delta_h\, u|^2 \right\}. \tag{5.10}$$

$$\leq\, c\,\nu_0^{-1}\, \|f\|^2 .$$

Next we pass to the limit in (5.10), as $h \to 0$. Clearly, $\mathcal{D}\, u^{-h} \to \mathcal{D}\, u$ almost everywhere in Ω. On the other hand, due to (5.7), we know that

$$\nabla\,\Delta_h\, u \to \nabla \frac{\partial u}{\partial x_k}\,,$$

almost everywhere in Ω. In particular, the same property holds by replacing ∇ by $\mathcal{D}$. The above considerations, together with the nonnegativity of the integrands that appear on the left hand side of inequality (5.10), allow us to pass to the limit by using Fatou's lemma. This yields

$$\nu_0 \int |\mathcal{D}\frac{\partial u}{\partial x_k}|^2\, dx \,+ \nu_1 \int |\mathcal{D}u|^{p-2}\, |\mathcal{D}\frac{\partial u}{\partial x_k}|^2\, dx \leq\, c\,\nu_0^{-1}\, \|f\|^2 , \tag{5.11}$$

for each $k \neq 3$. Hence,

$$\nu_0\, \|D_*^2\, u\|^2 + \,\nu_1 \sum_{k=1}^{2} \left\| \,|\mathcal{D}\, u|^{\frac{p-2}{2}}\, \mathcal{D}\, \frac{\partial u}{\partial x_k} \right\|^2 \leq\, c\,\nu_0^{-1}\, \|f\|^2 . \tag{5.12}$$

The proof of the estimate (3.2) is accomplished. $\square$

6. Proof of Theorem 3.2.

For convenience, from now on the positive constants c may depend on ν_0 and ν_1. It is easily seen, in particular, that if $0 < \underline{\nu} \leq \nu_0,\ \nu_1 \leq \overline{\nu}$ the constant c depends only on $\underline{\nu}$ and $\overline{\nu}$. Nevertheless, in some calculations we let the constants ν_0 and ν_1 explicitly appear for a better understanding of the manipulations.

We start this section by recalling the following result.

Lemma 6.1. Let $g(x)$ be a scalar field in Ω such that

$$g = \,\nabla \cdot\, w_0\,,\ \text{and}\ \nabla\, g = \,\nabla \cdot\, W\,,$$

where $w_0 \in L^\beta(\Omega)$ and $W \in L^\alpha(\Omega)$, for some $\alpha \geq \beta > 1$. Then

$$\|g\|_{L^\alpha(\Omega)} \leq\, c\, \left(\|w_0\|_{L^\beta(\Omega)} + \,\|W\|_{L^\alpha(\Omega)} \right)\,. \tag{6.1}$$

The above result (for a bounded domain with a Lipschitz-continuous boundary) and $\beta = \alpha$ is proved in reference [26]. The above extension is easily proved by applying (2.9) to $g - \overline{g}$, together with simple devices. Here $\overline{g}$ denotes the mean value of g.

It is also worth noting that the constant c may be chosen independently of α and β, provided that $1 < \alpha_1 \leq \beta \leq \alpha \leq \alpha_2$, for some fixed exponents α_1 and α_2.

It is worth noting that if $2 \leq p \leq 3$ and $p \leq q \leq 6$ then $\frac{4}{3} \leq r \leq 2$. The lack of dependence of the constants c on p, q, r follows from this fact, since the constants that appear in the embedding theorems used in the sequel, as well as in (2.9), are uniformly bounded from above if the exponents in the Lebesgue spaces lie away from 1 and from ∞.

Proof of Theorem 3.2.

Lemma 6.2. Assume (3.6). For $k = 1, 2$, the terms $|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, \frac{\partial u}{\partial x_k}$ and the derivatives $\frac{\partial \pi}{\partial x_k}$ satisfy the estimate (3.8). In particular,

$$\left\| \frac{\partial \pi}{\partial x_k} \right\|_r \leq \mathcal{K}_q \,. \tag{6.2}$$

Proof. Straightforward calculations show that

$$\begin{gathered} \frac{\partial}{\partial\, x_k} \left(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u \right) = \\ |\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, \frac{\partial u}{\partial x_k} + (p-2)\, |\mathcal{D}\, u|^{p-4} \left(\mathcal{D}\, u \cdot \mathcal{D}\, \frac{\partial u}{\partial x_k} \right) \mathcal{D}\, u \,. \end{gathered} \tag{6.3}$$

On the other hand, by differentiation of equation (1.5) with respect to x_k, $k = 1, 2$, it follows that

$$\begin{aligned} \nabla \frac{\partial \pi}{\partial x_k} &= \nabla \cdot \left[-\nu_0\, \mathcal{D}\, \frac{\partial u}{\partial x_k} \right] + \nabla \cdot \left[-\nu_1\, \frac{\partial}{\partial x_k} \left(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u \right) \right] + \nabla \cdot G \\ &\equiv \nabla \cdot [U_3 + U_4 + G] \,, \end{aligned} \tag{6.4}$$

where, for uniformity of notation, we introduce $G_{ij} = \delta_{kj}\, f_i$. Hence $\nabla \cdot G = \frac{\partial f}{\partial x_k}$, moreover $\|G\| = \|f\|$.

Next we estimate suitable norms of the terms that appear inside square brackets on the right hand side of equation (6.4). By (5.7),

$$\|U_3\| \equiv \|\nu_0\, \mathcal{D}\, \frac{\partial u}{\partial x_k}\| \leq c\, \|f\| \,. \tag{6.5}$$

On the other hand, by using (6.3), one shows that

$$|\, \frac{\partial}{\partial x_k} \left(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u \right) | \leq c\, |\mathcal{D}\, u|^{p-2} \left| \mathcal{D}\, \frac{\partial\, u}{\partial x_k} \right| \,, \tag{6.6}$$

almost everywhere in Ω. Moreover, by Hőlder's inequality and assumption (3.6), one has

$$\left\| \, |\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, \frac{\partial\, u}{\partial x_k} \right\|_r \leq \|\mathcal{D}\, u\|_q^{\frac{p-2}{2}} \; \left\| \, |\mathcal{D}\, u|^{\frac{p-2}{2}}\, \mathcal{D}\, \frac{\partial\, u}{\partial x_k} \right\| . \tag{6.7}$$

Hence, by (5.12), it follows that

$$\left\| \, |\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, \frac{\partial\, u}{\partial x_k} \right\|_r \leq c\, \frac{1}{\nu_0}\, \|\mathcal{D}\, u\|_q^{\frac{p-2}{2}}\, \|f\| . \tag{6.8}$$

This proves the first statement in the Lemma. Furthermore,

$$\begin{aligned} \|U_4\|_r \equiv \|\, \nu_1\, \frac{\partial}{\partial x_k} \left(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u \right) \|_r \leq \\ c\, \|\mathcal{D}\, u\|_q^{\frac{p-2}{2}}\, \|f\| . \end{aligned} \tag{6.9}$$

By using (6.1), with $g = \frac{\partial \pi}{\partial x_k}$, $\alpha = r$ and $\beta = p'$, and by (2.8), (2.7) and (6.4), it follows that

$$\left\| \frac{\partial \pi}{\partial x_k} \right\|_r \leq c\, \left(\|f\| + \|f\|_{p'} + \|U_3\| + \|G\| + \|U_4\|_r \right) . \tag{6.10}$$

By (6.5) and (6.9) we get (6.2). □

Note that from equations (6.8) and (6.2) we get the estimate (3.8) for the first and the last term on the left hand side. The missing term is the subject of the following lemma.

Lemma 6.3. The derivatives $\frac{\partial^2 u_j}{\partial x_3^2}$, $j = 1, 2$ satisfy the estimate

$$\nu_0 \sum_{l=1}^{2} \left\| \frac{\partial^2\, u_l}{\partial x_3^2} \right\|_r \leq \mathcal{K}_q\, . \tag{6.11}$$

Proof. By using (6.3), the $j.^{th}$ equation (1.5) may be written in the form

$$\begin{aligned} -\nu_0 \sum_{k=1}^{3} \frac{\partial^2\, u_j}{\partial x_k^2} - \nu_1\, |\mathcal{D}\, u|^{p-2} \sum_{k=1}^{3} \left(\frac{\partial^2\, u_j}{\partial x_k^2} + \frac{\partial^2\, u_k}{\partial x_j\, \partial x_k} \right) \\ -(p-2)\, \nu_1\, |\mathcal{D}\, u|^{p-4} \sum_{l,m,k=1}^{3} \mathcal{D}_{lm}\, \mathcal{D}_{jk} \left(\frac{\partial^2\, u_l}{\partial x_m\, \partial x_k} + \frac{\partial^2\, u_m}{\partial x_l\, \partial x_k} \right) \\ + \frac{\partial \pi}{\partial x_j} = f_j\, , \end{aligned} \tag{6.12}$$

where $\mathcal{D}_{ij} = (\mathcal{D}\, u)_{ij} = \frac{\partial\, u_i}{\partial x_j} + \frac{\partial\, u_j}{\partial x_i}$ and $1 \leq j \leq 3$. Let us write the first two equations (6.12), $k = 1, 2$, as follows:

$$\nu_0 \, \frac{\partial^2 \, u_j}{\partial x_3^2} + \nu_1 \, |\mathcal{D}\, u|^{p-2} \, \frac{\partial^2 \, u_j}{\partial x_3^2}$$

$$+ \, 2\,(p-2)\, \nu_1 \, |\mathcal{D}\, u|^{p-4} \, \mathcal{D}_{j3} \sum_{l=1}^{2} \mathcal{D}_{l3} \, \frac{\partial^2 \, u_l}{\partial x_3^2} = \tag{6.13}$$

$$F_j(x) + \frac{\partial \pi}{\partial x_j} - f_j \,,$$

where the $F_j(x), j \neq 3$, are given by

$$F_j(x) := -\nu_0 \sum_{k=1}^{2} \frac{\partial^2 \, u_j}{\partial x_k^2} - \nu_1 \, |\mathcal{D}\, u|^{p-2} \sum_{k=1}^{2} \frac{\partial^2 \, u_j}{\partial x_k^2}$$

$$-\nu_1 \, |\mathcal{D}\, u|^{p-2} \sum_{k=1}^{3} \frac{\partial^2 \, u_k}{\partial x_j \, \partial x_k} \tag{6.14}$$

$$-2\,(p-2)\, \nu_1 \, |\mathcal{D}\, u|^{p-4} \left\{ \mathcal{D}_{33} \, \mathcal{D}_{j3} \, \frac{\partial^2 \, u_3}{\partial x_3^2} + \sum_{\substack{l,m,k=1 \\ (m,k) \neq (3,3)}}^{3} \mathcal{D}_{lm} \, \mathcal{D}_{jk} \, \frac{\partial^2 \, u_l}{\partial x_m \, \partial x_k} \right\} .$$

In the sequel, the equations (6.13), $j = 1, 2$, will be treated as a 2×2 linear system in the unknowns $\frac{\partial^2 \, u_j}{\partial x_3^2}, j \neq 3$. Note that, with an obviously simplified notation, the measurable functions F_j satisfy

$$|F_j(x)| \leq c \, \left(\nu_0 + (p-1)\, \nu_1 \, |\mathcal{D}\, u(x)|^{p-2} \right) \, |D_*^2 \, u(x)| \,, \tag{6.15}$$

a.e. in Ω.

We denote by $\widetilde{F}_j$ the right hand sides

$$\widetilde{F}_j(x) := F_j(x) + \frac{\partial \pi}{\partial x_j} - f_j \,, \tag{6.16}$$

that appear in the above 2×2 system (6.13).

Let us show that the 2×2 system (6.13) can be solved for the unknowns $\frac{\partial^2 \, u_j}{\partial x_3^2}, j = 1, 2$, for almost all $x \in \Omega$.

The elements $a_{j\,l}$ of the matrix system A are given by

$$a_{j\,l} = (\nu_0 + \nu_1 \, |\mathcal{D}\, u|^{p-2}) \, \delta_{j\,l} + 2\,(p-2)\, \nu_1 \, |\mathcal{D}\, u|^{p-4} \, \mathcal{D}_{l3} \, \mathcal{D}_{j3} \,,$$

for $j, l \neq 3$. Note that $a_{j\,l} = a_{l\,j}$. One easily shows that

$$\sum_{j,l=1}^{2} a_{j\,l}\xi_j\xi_l = (\nu_0 + \nu_1\,|\mathcal{D}\,u|^{p-2})\,|\xi|^2 + 2\,(p-2)\,\nu_1\,|\mathcal{D}\,u|^{p-4}\,\,[(\mathcal{D}\,u)\cdot\xi]_3^2\,.$$

Hence the matrix A is symmetric and positive definite. Moreover, the above identity shows that all the eigenvalues are larger than or equal to $\nu_0 + \nu_1\,|\mathcal{D}\,u|^{p-2}$. Hence,

$$det\,A \geq (\nu_0 + \nu_1\,|\mathcal{D}u|^{p-2})^2\,.$$

Next, by setting $\xi_l = \frac{\partial^2 u_l}{\partial x_3^2}$, we get from (6.13), i.e. from

$$\sum_{l=1}^{2} a_{j\,l}\,\xi_l = \widetilde{F}_j\,, \tag{6.17}$$

that

$$\sum_{l,j=1}^{2} a_{j\,l}\,\xi_l\xi_j = \sum_{j=1}^{2} \widetilde{F}_j\,\xi_j\,. \tag{6.18}$$

Consequently $(\nu_0 + \nu_1\,|\mathcal{D}u|^{p-2})\,|\xi|^2 \leq |\widetilde{F}|\,|\xi|$, which shows that

$$(\nu_0 + \nu_1\,|\mathcal{D}u|^{p-2})\,\sum_{l=1}^{2}\left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \leq |\widetilde{F}| := \left(\sum_{j=1}^{2}|\widetilde{F}_j|^2\right)^{1/2}\,, \tag{6.19}$$

almost everywhere in Ω. By appealing to (6.15) and (6.16) one shows that

$$(\nu_0 + \nu_1\,|\mathcal{D}u|^{p-2})\,\sum_{l=1}^{2}\left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \tag{6.20}$$

$$\leq c(\,\nu_0 + \nu_1\,|\mathcal{D}u|^{p-2})\,|D_*^2\,u(x)| + c\,(|\nabla^*\,\pi| + |f|)$$

where the bounded quantity $p-1$ was incorporated in the constant c. In particular,

$$\sum_{l=1}^{2}\left|\frac{\partial^2 u_l}{\partial x_3^2}\right| \leq c\,|D_*^2\,u(x)| + c\,\nu_0^{-1}\,(|\nabla^*\,\pi| + |f|)\,, \tag{6.21}$$

almost everywhere in Ω. There readily follows, by appealing to (6.2) and(5.7), that (6.11) holds. The proof of Proposition 3.2 is accomplished. □

7. Proof of Lemma 3.3

Proof. We define r^* as the Sobolev embedding exponent

$$\frac{1}{r^*} = \frac{1}{r} - \frac{1}{3} = \frac{p-2}{2q} + \frac{1}{6} \tag{7.1}$$

and $\overline{q}$ by equation (2.15). By (6.11), (5.7) and a Sobolev embedding theorem,

$$\nu_0 \|\mathcal{D}u\|_{r*} \leq \mathcal{K}_q\,. \tag{7.2}$$

Hence, by Hőlder's inequality,

$$\|\,|\mathcal{D}u|^{p-2}\, D^2_* u\|_{\overline{q}} \leq \|\mathcal{D}u\|^{p-2}_{r^*}\, \|D^2_* u\|\,. \tag{7.3}$$

By (5.7) one gets

$$\|\,|\mathcal{D}u|^{p-2}\, D^2_* u\|_{\overline{q}} \leq c\, \|\mathcal{D}u\|^{p-2}_{r^*}\, \nu_0^{-1}\, \|f\|\,. \tag{7.4}$$

From equation (6.12) written for $j = 3$, we get an expression for $\frac{\partial \pi}{\partial x_3}$ in terms of functions already estimated. In particular,

$$\begin{gathered} \left|\frac{\partial \pi}{\partial x_3}\right| \leq c\, \big(\nu_0 + (p-1)\,\nu_1\, |\mathcal{D}\, u(x)|^{p-2}\big)\, |D^2_*\, u(x)| + \\ c\,(p-2)\,\nu_1\, |\mathcal{D}\, u(x)|^{p-2} \sum_{l=1}^{2} \left|\frac{\partial^2\, u_l}{\partial x_3^2}\right| + |f_3(x)|\,, \end{gathered} \tag{7.5}$$

almost everywhere in Ω.

By appealing to (6.19), (6.16) and (6.15) we prove that

$$\left|\frac{\partial \pi}{\partial x_3}\right| \leq c\, \big[\, \big(\nu_0 + \nu_1\, |\mathcal{D}\, u(x)|^{p-2}\big)\, |D^2_*\, u(x)| + |\nabla^*\, \pi| + |f|\,\big]\,, \tag{7.6}$$

where c is independent of p since p is bounded from above. Hence, by (7.4) and (5.7),

$$\left\|\frac{\partial \pi}{\partial x_3}\right\|_{\widetilde{q}} \leq c\,(1 + \frac{\nu_1}{\nu_0}\, \|\mathcal{D}\, u\|^{p-2}_{r^*})\, \|f\| + c\, \|\nabla^*\, \pi\|_r\,. \tag{7.7}$$

By appealing to (6.2) and (7.2) one proves (3.10). $\square$

8. Proof of Theorem 3.4.

In the sequel $\|\quad\|_{k,s}$ denotes the norm in the Sobolev space $W^{k,s}(\Omega)$.

We define $r = r(q)$ by (2.14), and the Sobolev embedding exponent r^* by (2.13). Hence $r^* = r^*(q)$ is defined by

$$r^*(q) = \frac{6\, q}{3(p-2) + q}\,, \tag{8.1}$$

for $p \leq q \leq 6$. In the following $r = r(q)$ and $r^* = r^*(q)$.

Theorem 3.2 shows that if $u \in W^{1,q}$ then $u \in W^{2,r}$. Moreover, by (3.8),

$$\|u\|_{2,r} \leq \mathcal{K}_q\,.$$

Hence, by a Sobolev embedding theorem, $u \in W^{1,r^*}$ and

$$\|u\|_{1,r^*} \leq c_0\, \|u\|_{2,r} \leq \mathcal{K}_q\,.$$

Since $1+\frac{2}{p-2} \leq r \leq 2$, the distinct values of the embedding constants c_0 are bounded from above by a constant independent of r. We incorporate this constant (once and for all) in $\mathcal{K}_q$.

This shows the following result.

Lemma 8.1. If a solution u belongs to $W^{1,q}$ then u belongs to W^{1,r^*}, where $r^*(q)$ is given by (8.1), moreover

$$\|u\|_{1,r^*} \leq c\,\|f\| + c\,\|u\|_{1,q}^{\frac{p-2}{2}}\,\|f\|\,. \tag{8.2}$$

Since $p \geq 2$ the function $r^*(q)$ is increasing and bounded from above (for instance, by 6). Next we define the increasing sequence

$$\begin{cases} q_1 = p\,, \\ \\ q_{n+1} = r^*(q_n)\,. \end{cases} \tag{8.3}$$

Clearly

$$q_\infty = 3\,(4-p) \tag{8.4}$$

is a fixed point of r^*, $r^*(q_\infty) = q_\infty$, moreover

$$\lim_{n\to\infty} q_n = q_\infty\,. \tag{8.5}$$

From (8.2) it follows that

$$\|u\|_{1,q_{n+1}} \leq c\,\|f\| + c\,\|f\|\,\|u\|_{1,q_n}^{\frac{p-2}{2}}\,. \tag{8.6}$$

Next we appeal to an induction argument. Note that for $n=1$ one has

$$\|u\|_{1,q_1} = \|u\|_{1,p}\,.$$

If we are able to show that the quantities $a_n = \|u\|_{1,q_n}$, at least for large values of n, are uniformly bounded by a finite number L then well know results in Functional Analysis, together with (8.5), yield

$$\|u\|_{1,q_\infty} \leq L\,. \tag{8.7}$$

For convenience set $b = \|f\|$ and $\alpha = \frac{p-2}{2}$. Note that $0 \leq \alpha < 1$ provided that $2 \leq p < 4$. Denote by λ the (unique) solution of the equation $\lambda = c\,b + c\,b\,\lambda^\alpha$. By (8.6) one has $a_{n+1} \leq c\,b + c\,b\,a_n^\alpha$. Set $b_1 = a_1$ and $b_{n+1} = c\,b + c\,b\,b_n^\alpha$. Clearly $a_n \leq b_n$ for each n. It is easily seen that if $b_1 < \lambda$ then the sequence b_n is strictly increasing an converges to the fixed point λ. If $b_1 > \lambda$ then the sequence decreases to the value λ. Hence the sequence b_n converges to λ, so $a_n < 2\,\lambda$ for large values of n. On the other hand one easily shows that

$$\lambda \leq 2cb + (2\,c\,b)^{\frac{1}{1-\alpha}}\,.$$

Hence, under the hypothesis of Theorem 3.4, one has

$$\|u\|_{1,q_\infty} \leq c\|f\| + c\,\|f\|^{\frac{2}{4-p}}\,. \tag{8.8}$$

The Theorem 3.4 follows now by applying once more the Theorem 3.2, now with $q = q_\infty$ given by (8.4). In this case the equation (2.14) shows that $r = r(q_\infty) = l$, with l given by (3.12). Hence, from (3.8), it follows that

$$\|\nabla^* \pi\|_l + \|D^2 u\|_l + \| |\mathcal{D} u|^{p-2} \nabla^* \mathcal{D} u\|_l \leq \mathcal{K}_{q_\infty} \leq c\,\|f\| + c\,\|\mathcal{D} u\|_{q_\infty}^{\frac{p-2}{2}} \|f\| . \quad (8.9)$$

Finally, by appealing to (8.8) we get (3.13).

Regularity and estimates for $\frac{\partial \pi}{x_3}$ follows immediately from the Lemma 3.3. Actually,

$$\left\| \frac{\partial \pi}{\partial x_3} \right\|_m \leq c\,(\|\mathcal{D} u\|_p + \|\mathcal{D} u\|_p^{\frac{p(p-1)}{2}} + \|f\| + \|f\|^{\frac{p}{4-p}}\,) . \quad (8.10)$$

The estimate (3.16) follows by appealing to (2.6). Concerning the exponent m, from (2.15) with $q = q_\infty$ it follows that

$$\overline{q}_\infty = m .$$

Since $m \leq l$ and $r = l$, it follows from (2.16) that

$$\widetilde{q}_\infty = \min\{\overline{q}_\infty, l\} = m .$$

9. Proof of Theorem 3.5.

Since

$$\int_\Omega (u \cdot \nabla) u \cdot u \, dx = 0 ,$$

it readily follows that all the estimates stated in section 2 for weak solutions hold for solutions u to the problem (3.17), i.e. to the problem

$$\begin{cases} -\nu_0 \nabla \cdot \mathcal{D} u - \nu_1 \nabla \cdot \left(|\mathcal{D} u|^{p-2} \mathcal{D} u \right) + \nabla \pi = F, \\ \\ \nabla \cdot u = 0 . \end{cases} \quad (9.1)$$

where $F = f - (u \cdot \nabla) u$. In particular, by (2.7),

$$\|u\|_{W^{1,p}} \leq c\,\|f\|_{p'}^{\frac{1}{p-1}} . \quad (9.2)$$

On the other hand, by Hőlder's inequality,

$$\|(u \cdot \nabla) u\| \leq \|u\|_{p^*} \|\nabla u\|_s$$

where $s = \frac{6p}{5p-6}$. By well know embedding theorems it follows that

$$\|(u \cdot \nabla) u\| \leq c\,\|u\|_{W^{1,p}} \|u\|_{W^{\frac{3}{2},p'}} . \quad (9.3)$$

By appealing, in particular, to the compact embedding of $W^{2,p'}$ into $W^{\frac{3}{2},p'}$ one shows that to each positive real ϵ it corresponds a positive C_ϵ such that

$$\|v\|_{W^{\frac{3}{2},p'}} \leq C_\epsilon \|v\|_{W^{1,p}} + \epsilon\,\|v\|_{W^{2,p'}} .$$

Consequently,

$$\|F\| \leq \|f\| + c\,\|u\|_{W^{1,p}}\,(C_\epsilon\,\|u\|_{W^{1,p}} + \epsilon\,\|u\|_{W^{2,p'}})\,. \tag{9.4}$$

On the other hand, from (3.3),

$$\|u\|_{W^{2,p'}} \leq c\,(1 + \|\mathcal{D}\,u\|_p^{\frac{p-2}{2}})\,\|F\|\,. \tag{9.5}$$

Hence

$$\begin{aligned} \|u\|_{W^{2,p'}} \leq\; & c\,(1 + \|u\|_{1,p}^{\frac{p-2}{2}})\,(\|f\| + C_\epsilon\,\|u\|^2_{W^{1,p}}) + \\ & c_0\,\epsilon\,(1 + \|u\|_{1,p}^{\frac{p-2}{2}})\,\|u\|_{1,p}\,\|u\|_{2,p'}\,. \end{aligned} \tag{9.6}$$

By choosing a sufficiently small ϵ, say ϵ such that

$$c_0\,\epsilon\,(\|f\|_{p'}^{\frac{1}{p-1}} + \|f\|_{p'}^{\frac{p}{2(p-1)}}) \leq \frac{1}{2}\,,$$

we get the desired a priori estimate for $\|u\|_{W^{2,p'}}$ in terms of $\|f\|$.

10. The evolution Navier-Stokes equation

Let us write (1.1) in the more explicit form

$$\begin{cases} \frac{\partial u}{\partial t} + (u\,\cdot\,\nabla)\,u - \\ \nu_0\,\nabla\cdot\,\mathcal{D}\,u - \nu_1\,\nabla\cdot\big(|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u\big) + \nabla\,\pi = f, \\ \nabla\cdot u = 0\,, \\ u(0) = u_0(x)\,. \end{cases} \tag{10.1}$$

In the sequel we merely prove the a priori estimates that lead to our results. Complete proofs are done by applying the estimates to the approximate solutions obtained by the Faedo-Galerkin method. By now this is a well known device. See, for instance, [28] section 2 where this method is followed for the evolution Ladyzhenskaya model.

Multiplication by u, integration in Ω followed by suitable integrations by parts show that

$$\frac{1}{2}\,\frac{d}{d\,t}\,\|u(t)\|^2 + \frac{\nu_0}{2}\,\|\mathcal{D}\,u\|^2 + \frac{\nu_1}{2}\,\|\mathcal{D}\,u\|_p^p = \int_\Omega f\,u\,dx\,. \tag{10.2}$$

By integration of (10.2) with respect to time, one gets the following result:

Lemma 10.1. Let u be a weak solution to problem (10.1) under the boundary condition (1.4) plus x'-periodicity. Then u satisfies the estimate

$$\begin{aligned}&\|u(t)\|^2_{L^\infty(0,T;L^2)} + \nu_0\,\|u\|^2_{L^2(0,T;H^1)} + \nu_1\,\|u\|^p_{L^p(0,T;W^{1,p})} + \\ &\le c\,\Big(\|u(0)\|^2 + \tfrac{1}{\nu_0}\,\|f\|^2_{L^2(0,T;H^{-1})}\Big)\,.\end{aligned} \tag{10.3}$$

Next we prove a stronger estimate "in time". See (10.5). A complete proof of this estimate is done by passing through the solutions of a suitable family of approximate problems. This can be done by appealing to a Faedo-Galerkin procedure as, for instance, in Theorem 2.2 in reference [28].

We define $\mathcal{M}$ by the equation

$$\begin{aligned}&\mathcal{M}^2 = 2\,exp\Big\{\tfrac{c}{\nu_1}\int_0^T\|\mathcal{D}\,u\|_p^{4-p}\,dt\Big\}\cdot \\ &\Big\{\nu_0\,\|\mathcal{D}\,u_0\|^2 + \nu_1\,\|\mathcal{D}\,u_0\|_p^p + c\int_0^T\|f(t)\|^2\,dt\Big\}\,.\end{aligned} \tag{10.4}$$

Note that, by (10.3), the first integral in the right hand side of (10.4) can be estimated in terms of the data since $4-p \le p$.

One has the following result:

Lemma 10.2. Let u be as in Lemma 10.1 and assume that $u_0 \in V_p$, (10.9) holds and $f \in L^2(0,T;L^2)$. Then

$$\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;L^2)} + \nu_0\,\|\nabla u\|^2_{L^\infty(0,T;L^2)} + \nu_1\,\|\nabla u\|^p_{L^\infty(0,T;L^p)} \le c\,\mathcal{M}^2\,. \tag{10.5}$$

Proof. By suitable integrations by parts, it follows that

$$\begin{aligned}&-\int_\Omega\big[\nabla\cdot\big(\nu_0\,\nabla u + \nu_1\,|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u\big) + \nabla\pi\big]\cdot\tfrac{\partial u}{\partial t}\,dx = \\ &\tfrac{\nu_0}{2}\,\tfrac{d}{dt}\|\mathcal{D}\,u\|^2 + \tfrac{\nu_1}{2p}\,\tfrac{d}{dt}\|\mathcal{D}\,u\|_p^p\,.\end{aligned} \tag{10.6}$$

On the other hand

$$\int_\Omega |(u\cdot\nabla)\,u|^2\,dx \le c\,\|u\|^2_{\frac{2p}{p-2}}\,\|\nabla u\|_p^2\,. \tag{10.7}$$

Furthermore

$$\|u\|_{\frac{2p}{p-2}} \le c\,\|u\|_{p^*} \tag{10.8}$$

provided that

$$p \ge 2 + \frac{2}{5}\,. \tag{10.9}$$

Remark. The assumption (10.9) is superfluous if we drop the term $(u\cdot\nabla)\,u$ from equation (1.1).

By appealing to a Sobolev embedding theorem together with (2.2), one shows that

$$\|(u\cdot\nabla)\,u\| \leq c\,\|\mathcal{D}\,u\|_p^2\,. \tag{10.10}$$

Hence, from (10.1) and (10.6), one gets

$$\begin{aligned} &\|\tfrac{\partial u}{\partial t}\|^2 + \nu_0\,\tfrac{d}{dt}\|\nabla u\|^2 + \nu_1\,\tfrac{d}{dt}\|\mathcal{D}\,u\|_p^p \leq \\ &c\,\left(\|f\|^2 + \|\mathcal{D}\,u\|_p^{4-p}\,\|\mathcal{D}\,u\|_p^p\right)\,. \end{aligned} \tag{10.11}$$

From (10.11) straightforward, well known, manipulations show that

$$\left\|\frac{\partial u}{\partial t}\right\|^2_{L^2(0,T;L^2)} + \nu_0\,\|\mathcal{D}\,u\|^2_{L^\infty(0,T;L^2)} + \nu_1\,\|\mathcal{D}\,u\|^p_{L^\infty(0,T;L^p)} \leq \mathcal{M}^2\,. \tag{10.12}$$

Finally, by (2.2), (10.5) follows for some constants c. □

One has the following results.

Theorem 10.3. Let u be a weak solution to problem (10.1) under the boundary condition (1.4) plus x'-periodicity, where $u_0 \in V_p$ and $f \in L^2(0,T;L^2)$. Assume that p satisfies (10.9). Then

$$\begin{cases} u \in L^2(0,T;W^{2,p'}) \cap L^\infty(0,T;W^{1,p})\,, \\ \nabla\,\pi \in L^2(0,T;L^{p_0})\,, \\ \frac{\partial u}{\partial t} \in L^2(0,T;L^2)\,. \end{cases} \tag{10.13}$$

In particular (10.5), (10.16) and (10.17) hold, where $\mathcal{M}$ is given by the equation (10.4).

Theorem 10.4. Under the assumptions of Theorem 10.3

$$\begin{cases} u \in L^{4-p}(0,T;W^{2,\,l}) \cap L^\infty(0,T;W^{1,p})\,, \\ \nabla\,\pi \in L^{\frac{2(4-p)}{p}}(0,T;L^m)\,, \\ \frac{\partial u}{\partial t} \in L^2(0,T;L^2)\,. \end{cases} \tag{10.14}$$

Moreover the estimates (10.5), (10.19) and (10.20) hold.

Proof of theorem 10.3.

One has, almost everywhere in $]0,\mathrm{T}[$,

$$-\nu_0\Delta\,u - \nu_1\,\nabla\cdot(|\mathcal{D}u|^{p-2}\,\mathcal{D}u) + \nabla\,\pi = f(x) - (u\cdot\nabla)\,u - \frac{\partial u}{\partial t}\,.$$

Hence, by taking into account (3.3), one shows that

$$
\begin{gathered}
\|u\|_{2,p'} \leq c\,(\|f\| + \|\mathcal{D}\,u\|_p^{\frac{p-2}{2}}\,\|f\|)+ \\
c\,(\|\mathcal{D}\,u\|_p^2 + \|\mathcal{D}\,u\|_p^{\frac{p+2}{2}}) + c\left(\left\|\frac{\partial\,u}{\partial\,t}\right\| + \|\mathcal{D}\,u\|_p^{\frac{p-2}{2}}\left\|\frac{\partial\,u}{\partial\,t}\right\|\right).
\end{gathered}
\tag{10.15}
$$

By appealing to (10.5), straightforward calculations show that

$$
\begin{gathered}
\|u\|_{L^2(0,\,T;\,W^{2,p'})} \leq \\
c\,(\mathcal{M} + T^{\frac{1}{2}}\,\mathcal{M}^{\frac{4}{p}} + T^{\frac{1}{2}}\,\mathcal{M}^{\frac{p+2}{p}} + \mathcal{M}^{\frac{2\,(p-1)}{p}})\,,
\end{gathered}
\tag{10.16}
$$

in $]0,\,T[$. Note that we may easily obtain more stringent estimates.

Similarly, by appealing to (3.5), one easily proves that

$$
\|\nabla\,\pi\|_{L^2(0,\,T;\,L_{\overline{p}})} \leq \mathcal{F}(T,\,\mathcal{M})\,. \tag{10.17}
$$

An explicit expression for $\mathcal{F}$ is left to the reader.

In particular, (10.5), (10.16) and (10.17) show that (10.13) holds.

Proof of theorem 10.4.

Next we combine (10.5) with (3.13). Now p' is replaced by l. The main difference is that now there is the additional term $\|f\|^{\frac{2}{4-\,p}}$. Instead of (10.15) one gets

$$
\begin{gathered}
\|u\|_{2,l} \leq c\,(\|f\| + \|f\|^{\frac{2}{4-\,p}})+ \\
c\,(\|\mathcal{D}\,u\|_p^2 + \|\mathcal{D}\,u\|^{\frac{4}{4-\,p}}) + c\left(\left\|\frac{\partial\,u}{\partial\,t}\right\| + \left\|\frac{\partial\,u}{\partial\,t}\right\|^{\frac{2}{4-\,p}}\right),
\end{gathered}
\tag{10.18}
$$

a.e. in $]0,\,T[$. Hence, by taking the $(4\,-\,p).^{th}$ power of both sides of (10.18) and by integrating in Ω, one shows that

$$
\|u\|_{L^{4-\,p}(0,\,T;\,W^{2,l})} \leq \mathcal{F}_0(T,\,\mathcal{M})\,, \tag{10.19}
$$

where an expression for $\mathcal{F}_0(T,\,\mathcal{M})$ is easily obtained from (10.18) and (10.5). Finally, by appealing to (3.16), similar devices show that

$$
\|\nabla\,\pi\|_{L^{\frac{2\,(4-p)}{p}}(0,\,T;\,L^m)} \leq \mathcal{F}_1(T,\,\mathcal{M})\,. \tag{10.20}
$$

Remark. Note that stronger estimates for the terms $\nabla^*\,\pi$, $D^2\,u$ and $|\mathcal{D}\,u|^{p-\,2}\,\nabla^*\,\mathcal{D}\,u$ can be easily obtained.

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