Sufficient conditions for the convexity of the level sets of ground-state solutions

J. M. Gomes *

Abstract

Let Ω be a bounded convex domain in $\mathbb{R}^n$. We consider constrained minimization problems related to the Euler-Lagrange equation

$$-\Delta u + V(x)u = \lambda u^p \quad \text{in } \Omega \text{ , } u > 0 \, ,$$

over classes of functions $u \in H_0^1(\Omega)$ with convex super level sets. We then search for sufficient conditions insuring that the minimizer obtained is a classical solution to the above equation.

Keywords: variational methods, constrained minimizers, quasi-concave functions.

2000 Mathematics Subject Classification. 35J20 35J60

1 Introduction

The problem of characterizing level sets of solutions to the elliptic problem

$$\Delta u + f(u) = 0 \;\; \text{in } \Omega \, , \; u > 0 \qquad (1)$$
$$u|_{\partial\Omega} = 0$$

where f is a non-negative Lipschitz or continuous function has been a subject of great interest since the early eighty's. The fundamental work by Gidas, Ni and Nirenberg [3] inspired by the technique of the moving planes of Alexandroff, relying itself on the maximum principle, established the radial symmetry of the solutions of (1) when Ω is a ball. As a natural extension of this result, several authors (see [6] for a rich bibliography on the subject) studied the convexity of the super level sets when Ω is convex for elliptic free boundary and boundary value problems. In 1983, Korevaar (see [8]) obtained significant results proving, with a maximum principle, the convexity of a composed function $g(u(x))$ where u is a positive solution of the original problem. This method has been successfully exploited by several authors (see for instance [5] or [7]). In [2], the convexity of the set which is delimited by the free boundary to a quasilinear elliptic equation is proven for analytic solutions.

Although a vast range of problems has a variational origin, variational arguments seem to be less effective since it is hard to conceive a "convex rearrangement" of the level sets diminishing the energy $\int |\nabla u|^2$ of a constrained

*Supported by ESF activity "Global and geometrical aspects of nonlinear P.D.E.'s."

minimum. This opinion seems to be shared with B. Kawohl as one may deduce from [strategy 6, p.105,[6]]. We point out that Colesanti and Salani prove in [1] the coincidence of the solution with its quasi-concave envelop for a certain class of elliptic problems in convex ring-shaped domains using the maximum principle.

In this work we focus on a still unsolved problem, the convexity of the super level sets of a positive solution to

$$-\Delta u + V(x)u = \lambda u^p \quad \text{in } \Omega\,,\ u > 0 \tag{2}$$
$$u|_{\partial\Omega} = 0$$

with $1 < p < \frac{N+2}{N-2}$, Ω is a convex bounded domain of $\mathbb{R}^N$ and $V(x) \geq l > -\Lambda$ where Λ is the first eigenvalue of $L(u) = -\Delta u$ in $H_0^1(\Omega)$. We can trace back the conjecture to Kawohl-Sacks (see [5]). In case $n = 2$ and $V(x) \equiv 0$ the result has been established by Lin in [10], although its proof cannot be extended to higher dimensions. The proof in case $0 < p \leq 1$ and $V(x) \equiv 1$ can be found in [6].

Basically, the solution to (2) can be obtained as a constrained minimum of the energy functional $\int_\Omega |\nabla u|^2 + V(x)u^2$ on the manifold consisting on the functions with prescribed volume $\int_\Omega |u|^{p+1} = 1$; this solution, which we denote by $\underline{u}$, is known as a ground-state solution. In our approach we perform the same minimization with the additional constrain of convexity of the super level sets (quasi-concavity) obtaining a second minimizer $\bar{u}$. We then search for sufficient conditions insuring that $\bar{u} = \underline{u}$. In particular we prove in Section 3 that if $\bar{u}$ and $\underline{u}$ coincide "near" the boundary $\partial\Omega$ then they must be equal in all Ω. Also we define a H^1-distance minimizer to $\underline{u}$ over the set of quasi-concave functions and observe that local coincidence with $\underline{u}$ implies global coincidence. The author would like to express his thanks to P. Girão and B. Kawohl for a careful reading of the manuscript and suggestions.

2 Preliminary results

Let Ω be a regular convex set of $\mathbb{R}^N$, $1 \leq p < 2^* - 1 = \frac{n+2}{n-2}$. We recall that, given $u \in H_0^1(\Omega)$, a super level set $\mathcal{L}_t(u)$ is defined by:

$$\mathcal{L}_t(u) := \{x \in \Omega \,:\, u(x) > t\}.$$

We consider a particular subset $\mathfrak{C}$ of $H_0^1(\Omega)$ whose functions have convex super level sets. More precisely, we say that

$\underline{u \in \mathfrak{C}}$ iff $u \in H_0^1(\Omega)$ and

$$u(\lambda x + (1-\lambda)y) \geq \min\{u(x), u(y)\}\,,\ \text{a.e } (x, y, \lambda) \in \Omega \times \Omega \times [0, 1] \tag{3}$$

This characterization of quasi-concave functions is common in control theory.

For $1 < p < \frac{n+2}{n-2}$, let us consider the following minimization problem

$$\inf\{(\frac{1}{2}\int_\Omega |\nabla u|^2 + V(x)|u|^2)/(\frac{1}{p+1}\int_\Omega |u|^{p+1})^{\frac{2}{p+1}} \;:\; u \in \mathfrak{C}\,,\ u \geq 0\ ,\ u \neq 0\}. \tag{4}$$

Lemma 1 *The infimum defined in (4) is in fact a minimum and is attained at some function $\bar{u}$ whose super level sets are convex.*

Proof. By standard re-scaling arguments the assertion of the Lemma is equivalent to the existence of a minimizer to the following variational problem

$$\inf\{J(u) \ : \ u \in \mathfrak{V} \cap \mathfrak{C}\} \tag{5}$$

where

$$J(u) = \frac{1}{2}\Big(\int_\Omega |\nabla u|^2 + V(x)|u|^2\, dx\Big),$$

$$\mathfrak{V} := \{u \in H_0^1(\Omega) : \frac{1}{p+1}\int_\Omega {u_+}^{p+1} = 1\},$$

and $u_+ := \max\{u, 0\}$. Of course $\mathfrak{V} \cap \mathfrak{C}$ is a non-empty set. Let for instance $x_0 \in \Omega$ be such that $B(x_0, \epsilon) \subset \Omega$; then the function

$$u(x) := \frac{(\epsilon^2 - |x - x_0|^2)_+}{\left(\frac{1}{p+1}\int_\Omega (\epsilon^2 - |z - x_0|^2)_+{}^{p+1}\, dz\right)^{\frac{1}{p+1}}},$$

belongs to $\mathfrak{V} \cap \mathfrak{C}$.

Let (u_n) be a minimizing sequence. Since (u_n) is trivially bounded we may extract a weakly convergent subsequence still denoted by (u_n). In particular, by Sobolev's Embedding Theorem, we have a function $\bar{u}$ such that:

$$u_n \stackrel{H_1}{\rightharpoonup} \bar{u} \ \text{ and } \ u_n \stackrel{L^{p+1}}{\to} \bar{u}.$$

We may consider a null Lebesgue measure subset $N \subseteq \Omega$ such that

$$u_n(x) \to \bar{u}(x)\,,$$

for every $x \in \Omega \backslash N$. Considering the subset

$$L = (N \times N \times [0,1]) \cup \Big(\bigcup_{n=1}^{\infty} A_n\Big)$$

where

$$A_n := \{(x, y, \lambda) : u_n(\lambda x + (1 - \lambda)y) < \min\{u_n(x), u_n(y)\}$$

we have that $m(L) = 0$ where m is the Lebesgue measure in $\Omega \times \Omega \times [0, 1]$. Therefore

$$\bar{u}(\lambda x + (1 - \lambda)y) \geq \min\{\bar{u}(x), \bar{u}(y)\}$$

for every $(x, y, \lambda) \in (\Omega \times \Omega \times [0, 1])\backslash L$, that is $\bar{u} \in \mathfrak{C}$. ■

According to the previous lemma, we have

$$J(\bar{u}) := \min\{J(u) \ : \ u \in \mathfrak{V} \cap \mathfrak{C}\}. \tag{6}$$

As the reader may easily verify, the minimum of (4) is achieved at every $\lambda\bar{u}$, with $\lambda > 0$.

Of course we can consider, instead of (5), the classical constrained minimization problem

$$J(\underline{u}) = \min\{J(u) \ : \ u \in \mathfrak{V}\}. \tag{7}$$

Denoting by $\underline{u}$ the minimizer of the constrained energy, it is well known that $\underline{u}$ satisfies

$$-\Delta\underline{u} + V(x)\underline{u} = \frac{2J(\underline{u})}{p+1}\underline{u}^p \ \ , \ \ \underline{u} \in \mathfrak{V} \ , \ (1 < p < 2^*)\,. \tag{8}$$

It is the so called "ground-state solution".

Rewriting equation (8) in its weak form, we have that the classical ground state $\underline{u}$ verifies

$$\int_\Omega \nabla\underline{u}\nabla v\,dx + \int_\Omega V(x)\underline{u}v\,dx - \frac{2J(\underline{u})}{p+1}\int_\Omega \underline{u}^p v\,dx = 0, \tag{9}$$

for every $v \in H_0^1(\Omega)$. In the next lemma we establish the derivative of the constrained functional J at $\bar{u}$ along an admissible variation v:

Lemma 2 *Let $\bar{u}$ be as defined in Lemma 1 and let $v \in H_0^1(\Omega)$ be such that for some $\epsilon > 0$ and all $s \in]-\epsilon, \epsilon[$*

$$\bar{u} + sv \in \mathfrak{C}. \tag{10}$$

Then,

$$\int_\Omega \nabla\bar{u}\nabla v + \int_\Omega V(x)\bar{u}v - \frac{2J(\bar{u})}{p+1}\int_\Omega \bar{u}^p v = 0\,. \tag{11}$$

Proof. Consider the real function $\phi(s) := f(s)/g(s)$, where

$$f(s) := J(\bar{u} + sv) \quad \text{and} \quad g(s) := \left(\frac{1}{p+1}\int_\Omega (\bar{u} + sv)^{p+1}\right)^{\frac{2}{p+1}}.$$

The reader may easily verify that $\phi(s)$ is regular for sufficiently small values of ϵ and evaluates J along the path in $\mathfrak{V} \cap \mathfrak{C}$ which is parameterized by s as

$$s \mapsto \left(\frac{1}{p+1}\int_\Omega (\bar{u} + sv)^{p+1}\right)^{-\frac{1}{p+1}} (\bar{u} + sv),$$

for $s \in]-\epsilon, \epsilon[$, where ϵ is small. By (10) and the definition of $\bar{u}$ we conclude $\phi'(0) = 0$ that is

$$\frac{f'(0)g(0) - f(0)g'(0)}{g(0)^2} = 0. \tag{12}$$

We have

$$f'(0) = \int_\Omega \nabla\bar{u}\nabla v\,dx + \int_\Omega V(x)\bar{u}v \ , \ f(0) = J(\bar{u})\,, \, g(0) = 1\,,$$

and, by Leibniz rule,

$$g'(0) = \frac{2}{p+1}(\frac{1}{p+1}\int_\Omega (\bar{u} + sv)^{p+1}\,dx)^{\frac{1-p}{1+p}} \times \int_\Omega (\bar{u} + sv)^p v\,dx\,|_{s=0} = \frac{2}{p+1}\int_\Omega \bar{u}^p v\,dx.$$

Replacing these quantities in (12) we conclude (11). ∎

In the next lemma we prove the existence of a class of admissible variations to every u in $\mathfrak{C}$.

Lemma 3 *Let $u \in \mathfrak{C}$ and $f : \mathbb{R} \to \mathbb{R}$ a non-decreasing Lipschitz function such that $f(0) = 0$. Then*

$$u \pm sf(u) \in \mathfrak{C}$$

for all $s \in]-\frac{1}{k}, \frac{1}{k}[$ where $k > 0$ is a Lipschitz constant of f.

Proof.

It is known that $f(u) \in H_0^1(\Omega)$. Let $(x, y, \lambda) \in \Omega \times \Omega \times [0, 1]$ be such that $u(\lambda x + (1 - \lambda y)) \geq \min\{u(x), u(y)\}$. Without loss of generality we may assume $u(y) = \min\{u(x), u(y)\}$. Trivially, since f is non-decreasing, we have for $s \geq 0$,

$$(u + sf(u))(\lambda x + (1 - \lambda)y) \geq \min\{u(x) + sf(u(x)), u(y) + sf(u(y))\}\,.$$

Recalling that $(1 - sk) > 0$, we obtain,

$$(u - sf(u))(\lambda x + (1 - \lambda)y) \geq u(\lambda x + (1 - \lambda)y) - s(f(u(y)) + k(u(\lambda x + (1 - \lambda)y) - u(y))) \geq$$

$$(1 - sk)u(\lambda x + (1 - \lambda)y) + sku(y) - sf(u(y)) \geq u(y) - sf(u(y))\,.$$

Since this inequality is verified for almost all $(x, y, \lambda) \in \Omega \times \Omega \times [0, 1]$, the claim follows. ■

Remark. By the previous Lemma, one may consider as admissible variations for $\bar{u}$, the set of functions

$$\bar{u}_t = \min\{\bar{u}, t\}$$

for every $t \geq 0$. We conclude, by Lemma 2,

$$z(t) := \int_\Omega \nabla\bar{u}\nabla\bar{u}_t + \int_\Omega V(x)\bar{u}\bar{u}_t - \frac{2J(\bar{u})}{p+1}\int_\Omega \bar{u}^p\bar{u}_t = 0 \quad \forall t \geq 0\,.$$

Assuming that u is o class C^1 in some relatively open set containing the boundary, differentiating $z(t)$ in $t = 0$, we may derive the following equality that is reminiscent of Stokes' Theorem:

$$\int_{\partial\Omega} |\nabla\bar{u}| do(z) = \frac{2J(\bar{u})}{(p+1)}\int_\Omega \bar{u}^p(x)\,dx - \int_\Omega V(x)\bar{u}(x)\,dx.$$

3 Sufficient conditions for the coincidence of $\underline{u}$ with some quasi-concave minimizers

In this section we study conditions implying the coincidence of the classical ground-state $\underline{u}$ with some quasi-concave constrained minimizers. For simplicity we will consider the two dimensional case although our arguments remain valid in $\mathbb{R}^N$.

We say that a bounded smooth domain $\Omega \subset \mathbb{R}^2$ is strictly convex when $\partial\Omega$ is parameterized by a regular curve $r : [0, 1] \to \mathbb{R}^2$ with strictly positive curvature k at all $x \in \partial\Omega$. By regular we mean that $r'(s) \neq 0$ for all $s \in [0, 1[$. We recall that the curvature of a level set of a $C^2(\overline{\Omega})$ function can be evaluated at some point $x \in \overline{\Omega}$ such that $\nabla u(x) \neq 0$ by the formula

$$k_u(x) = -\frac{Du(x)_{\tau\tau}}{|\nabla u|(x)}. \tag{13}$$

Here $Du(x)_{\tau\tau}$ is the second derivative of u at x evaluated along the direction τ, the unitary tangent to the level line at x.

Proposition 1 *Let $\Omega \subset \mathbb{R}^2$ be a smooth strictly convex bounded domain and let $\underline{u}$ and $\bar{u}$ be the functions defined in (7) and Lemma 1 respectively. Suppose that for some open set ω we have that $\overline{\omega} \cap \partial\Omega \neq \emptyset$ and $\bar{u} = \underline{u}$ in $\omega \cap \Omega$. Then*

$$\bar{u}(x) \;=\; \underline{u}(x) \quad \forall x \in \Omega\,,$$

(so that the ground-state solution to (8) has convex level sets).

Proof. By a classical result in regularity theory (see for instance [4], [Theorem 8.13. pp 177]) we have that $\underline{u} \in C^{3,\alpha}(\overline{\Omega})$ $(0 < \alpha < 1)$. By the strong maximum principle we have $\nabla u \neq 0$ in $\partial\Omega$. By (13), the hypotheses $k(x) > 0$ for all $x \in \partial\Omega$ and the regularity of $\underline{u}$, there is an open subset $\omega' \subseteq \omega \cap \Omega$ such that $k(x) \geq \delta > 0$ for all $x \in \omega'$. Let $v \in C^\infty(\Omega)$ be such that $v \geq 0, v \neq 0$ and

$$supp\,(v) \subset \omega'\,.$$

Since, by our assumptions, $\bar{u} = \underline{u}$ in ω' we have that $\bar{u} + sv \in \mathfrak{C}$ for $s \in]-\epsilon, \epsilon[$ provided that ϵ is sufficiently small (just notice $k_{\bar{u}+sv}$ remains well defined by (13) and positive if the C^2 norm of sv is small). Then, by Lemma 2 and (8), we have

$$J(\bar{u}) = \frac{(p+1)\int_{\omega'} \nabla\bar{u}\nabla v + \int_{\omega'} V(x)\bar{u}v}{2\int_{\omega'} \bar{u}^p v} = \frac{(p+1)\int_{\omega'} \nabla\underline{u}\nabla v + \int_{\omega'} V(x)\underline{u}v}{2\int_{\omega'} \underline{u}^p v} = J(\underline{u})\,.$$

We infer from the previous equality that $\bar{u}$ is also a classical solution to (8). By a standard application of the Unique Continuation Principle (see [11] or [12]) we may conclude that $\bar{u} = \underline{u}$ in Ω. ■

Remark 1 *The previous and next result remain true if $\Omega \subset \mathbb{R}^N$ is a C^∞ strict convex bounded domain in the sense that $\partial\Omega$ is a regular $(N-1)$ manifold such that for every $x \in \partial\Omega$*

$$k_i(x) > 0.$$

Here the k_i's stand for the principal curvatures along the inward normal direction.

In the next result we establish an alternative minimization problem where the local coincidence of the minimizer implies the convexity of the level sets of $\underline{u}$. We will consider the more general set

$$\overline{\mathfrak{V}} := \{u \in H_0^1(\Omega) \,:\, \frac{1}{p+1}\int_\Omega (u_+)^{p+1} \geq 1\}.$$

Proposition 2 *Under the conditions of Proposition 1, let* $u^* \in \overline{\mathfrak{V}} \cap \mathfrak{C}$ *be such that*

$$J(u^* - \underline{u}) = \min\{J(u - \underline{u}) \,:\, u \in \overline{\mathfrak{V}} \cap \mathfrak{C}\}\,, \tag{14}$$

and suppose that for some open set ω *such that* $\overline{\omega} \cap \partial\Omega \neq \emptyset$ *we have*

$$u^*(x) = \underline{u}(x) \;\; \forall x \in \omega \cap \Omega.$$

Then $u^* \equiv \bar{u} \equiv \underline{u}$ *in* Ω*. In particular the classical ground-state* $\underline{u}$ *has convex level sets.*

Proof. We begin by noting that the existence of u^* satisfying (14) can be established using already familiar arguments. We minimize in $\overline{\mathfrak{V}} \cap \mathfrak{C}$ the "distance" functional

$$D(u) := \frac{1}{2}\int_\Omega |\nabla(\underline{u} - u)|^2\, dx \, + \frac{1}{2}\int_\Omega V(x)(u - \underline{u})^2\, dx,$$

Eliminating constant terms and using the fact that $\underline{u}$ satisfies (9), we conclude that u^* minimizes in $\overline{\mathfrak{V}} \cap \mathfrak{C}$ the functional

$$G(u) := J(u) - \frac{2J(\underline{u})}{p+1}\int_\Omega \underline{u}^p u\,.$$

Claim : $u^* \in \mathfrak{V}$.

Suppose, in view of a contradiction, that $\int_\Omega (u^*_+)^{p+1} > 1$. Then, since the path $\lambda \mapsto (1+\lambda)u^*$ remains in $\overline{\mathfrak{V}} \cap \mathfrak{C}$ for $\lambda \in]-\epsilon, \epsilon[$ when $\epsilon > 0$ is sufficiently small, we obtain, by our definition of u^*,

$$\frac{d\,G((1+\lambda)u^*)}{d\lambda}|_{\lambda=0} = 0\;,$$

or

$$\int_\Omega |\nabla u^*|^2 + \int_\Omega V(x)(u^*)^2 - \frac{2J(\underline{u})}{p+1}\int_\Omega \underline{u}^p u^* \;=\; 0. \tag{15}$$

Since $\underline{u}$ satisfies (9) we conclude

$$\int_\Omega |\nabla u^*|^2 + \int_\Omega V(x)(u^*)^2 = \int_\Omega \nabla\underline{u}\nabla u^* + \int_\Omega V(x)\underline{u}u^*\,,$$

which in turn implies, by Schwarz inequality,

$$J(u^*) \le J(\underline{u})\,.$$

But this is absurd by our definition of $\underline{u}$ and our assumption $\int_\Omega (u^*_+)^{p+1} > 1$ (just note $\mu u^* \in \mathfrak{V}$ for some $0 < \mu < 1$ and $J(\mu u^*) < J(\underline{u})$).

Suppose that u and u^* coincide in ω as stated in the proposition. Take $v \in C^\infty(\Omega)$, $supp(v) \subset \omega'$, $v \ge 0$, $v \neq 0$ where ω' is chosen as in Proposition 1 and consider

$$\phi^*(s) := G\left(\frac{u^* + sv}{(\frac{1}{p+1}\int_\Omega (u^* + sv)^{p+1})^{\frac{1}{p+1}}}\right)\,,$$

or

$$\phi^*(s) = \frac{J(u^* + sv)}{[\frac{1}{p+1}\int_\Omega (u^* + sv)^{p+1}]^{2/(p+1)}} - \frac{2J(\underline{u})\int_\Omega \underline{u}^p(u^* + sv)}{(p+1)[\frac{1}{p+1}\int_\Omega (u^* + sv)^{p+1}]^{1/(p+1)}}.$$

Note that for small values of s, $\phi^*(s)$ is regular and evaluates the functional G along a path contained in $\mathfrak{C} \cap \mathfrak{V}$. Since $\phi^*(0)$ is a local minimum of ϕ we conclude that

$$\frac{d\phi^*}{ds}|_{s=0} = 0\,.$$

Computing this derivative, recalling that, by our assumptions, $u^* = \underline{u}$ in ω', $u^* \in \mathfrak{V}$ and $supp(v) \subset \omega'$, we obtain the following equality :

$$\int_\Omega \nabla\underline{u}\nabla v + \int_\Omega V(x)\underline{u}v - \frac{2J(u^*)}{p+1}\int_\Omega \underline{u}^p v - \frac{2J(\underline{u})}{p+1}\int_\Omega \underline{u}^p v(1 - (p+1)^{-1}\int_\Omega \underline{u}^p u^*) = 0\,. \tag{16}$$

Now observe the following facts:

1. Since $v \geq 0$ is not identically zero, $\underline{u}$ satisfies (9), $J(\underline{u}) \leq J(u^*)$, we have

$$\int_\Omega \nabla\underline{u}\nabla v + \int_\Omega V(x)\underline{u}v - \frac{2J(u^*)}{p+1}\int_\Omega \underline{u}^p v \leq 0.$$

2. By Holder's inequality and the fact that $\underline{u}$ and u^* are in $\mathfrak{V}$,

$$|\int_\Omega \underline{u}^p u^*| \leq (\int_\Omega \underline{u}^{p+1})^{\frac{p}{p+1}}(\int_\Omega (u^*)^{p+1})^{\frac{1}{p+1}} = (p+1)\,,$$

therefore

$$\frac{2J(\underline{u})}{p+1}\int_\Omega \underline{u}^p v(1 - (p+1)^{-1}\int_\Omega \underline{u}^p u^*) \geq 0\,.$$

Fact 1, Fact 2 and (16) imply that the last inequality is in fact an equality, i.e.

$$(1 - (p+1)^{-1}\int_\Omega \underline{u}^p u^*) = 0$$

or

$$\int_\Omega \underline{u}^p u^* = \|\underline{u}\|_{p+1}^p \|u^*\|_{p+1}.$$

This equality is achieved iff $\underline{u} = \lambda u^*$ a.e. (see [9]). Since $\underline{u} \in \mathfrak{V}$ we conclude $\lambda = 1$, and we may state

$$u^* \equiv \underline{u} \equiv \bar{u} \ \text{ in } \Omega\,.$$

■

Remark 2 *According to Proposition 2 we may expect the coincidence of $\bar{u}$ and $\underline{u}$ since it seems reasonable that the nearest function to $\underline{u}$ (in the norm of $H_0^1(\Omega)$) in $\mathfrak{V}$ with convex level sets should coincide with $\underline{u}$ near the boundary, where $\underline{u}$ has in fact convex level sets. The proof of this claim will establish the conjecture of Kawohl-Sacks.*

References

[1] Colesanti A., Salani P., Quasi concave envelope of a function and convexity of level sets of solutions to elliptic equations, Math. Nachr. 258, 3-15 (2003).

[2] Dolbeault J., Monneau R, Convexity estimates for nonlinear elliptic equations and applications to free boundary problems, Ann. I. H. Poincaré-AN 19, 903-926 (2002).

[3] Gidas B., Ni W. and Nirenberg L., Symmetry and related properties via the maximum principle, Commun. Math. Phys. 68, 209-243 (1979)

[4] Gilbarg D. and Trudinger N., Elliptic Partial Differential Equations of Second Order, Springer-Verlag, 1977.

[5] Kawohl B., When are solutions to nonlinear elliptic boundary value problems convex?, Comm. in Partial Differential Equations, 10, 1213-1225 (1985)

[6] Kawohl B., Rearrengements and Convexity of Level Sets in PDE, Lecture Notes in Mathematics, Springer-Verlag 1985.

[7] Kennington A., An improved concavity maximum principle and some applications. Thesis, Adelaide, 1984.

[8] Korevaar N., Convex solutions to nonlinear elliptic and parabolic boundary value problems, Indiana Univ. Math. J. 32, 603-614 (1983).

[9] Lieb E. and Loss M., Analysis, Graduate Studies in Mathematics, AMS 1996.

[10] Lin C.,Uniqueness of least energy solutions to a semilinear elliptic equation in $\mathbb{R}^2$, manuscripta math. 84, 13-19 (1994).

[11] Lopes O., Radial symmetry of minimizers for some translation and rotation invariant functionals, J. Differential Equations 124, 378-388 (1996).

[12] Willem M., Minimax Theorems, P.N.L.D.E 24, Birkhäuser, 1996.

José Maria Gomes

CMAF-Centro de Matemática e Aplicações Fundamentais

Avenida Professor Gama Pinto, 2, 1649-003 Lisboa

E-mail address: `zemaria@cii.fc.ul.pt`