REMARKS ON THE NAVIER-STOKES EVOLUTION EQUATIONS UNDER SLIP TYPE BOUNDARY CONDITIONS WITH LINEAR FRICTION.*

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Abstract

In general terms, we show that the regularity results proved in [2] (see also [7]) for strong solutions to the *linear stationary* Stokes system under slip type boundary conditions allow to extend to these boundary conditions many of the well known results for solutions to the *nonlinear evolution* Navier-Stokes equations under the classical non-slip boundary condition.

1 Introduction and aim of this paper

In the sequel Ω is a bounded,connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of class $C^{2,1}$ (Lipschitz continuous second derivatives). We denote by $\underline{n}$ the unit outward normal to Γ. We set $Q_T =]0,T] \times \Omega$ and $\Sigma_T =]0,T] \times \Gamma$.

In the sequel we consider the Stokes evolution equations

$$(1.1) \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta\, u + \nabla p = \, f(t,x), \\ \nabla \cdot u = \, 0 \quad \text{in } Q_T\,, \\ u(0,x) = \, u_0(x) \quad \text{in } \Omega\,, \end{cases}$$

as well as the Navier-Stokes evolution equations (3.1), under the slip boundary condition

$$(1.2) \qquad \begin{cases} (u \cdot n)_{|\Gamma} = \, 0, \\ \\ \beta\, u_\tau + \underline{\tau}(u)_{|\Gamma} = \, 0 \quad \text{in } \Sigma_T\,. \end{cases}$$

For some references on this type of boundary conditions see, for instance, [2].

Our main purpose is to apply some of the regularity results proved in reference [2] for stationary solutions under slip boundary conditions to the study of the corresponding evolution problems. See Theorems 3.2 and 3.3 below.

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These notes are organized as follows. In section 2 we introduce the main notation and present some results for stationary solutions proved in reference [2]. In section 3 we apply these results to the study of the Navier-Stokes evolution problem (3.1). The proofs are straightforward.

2 Some known results

In reference [2] we study the problem

$$
\begin{cases} -\nu\Delta\, u - \mu\,\nabla(\nabla\cdot u) + \nabla p = \, f(x), \\ \lambda p + \nabla \cdot u = \, g(x) \quad \text{in } \Omega\,, \end{cases} \tag{2.1}
$$

under the non homogeneous slip boundary conditions

$$
\begin{cases} (u\cdot n)_{|\Gamma} = a(x), \\ \\ \beta\, u_\tau + \underline{\tau}(u)_{|\Gamma} = b(x)\,, \end{cases} \tag{2.2}
$$

where $\nu > 0$ and $\beta \geq 0$ are given constants, and $a(x)$ and $b(x)$ are, respectively, a given scalar field and a given tangential vector field on Γ. We denote by $u_\tau = u - (u\cdot \underline{n})\,\underline{n}$ the tangential component of u. For the definition of $\underline{\tau}(u)$ see below. Let

$$
T = -p\, I + \nu(\nabla u + \nabla u^T)
$$

be the stress tensor, and set $\underline{t} = T\cdot \underline{n}$. Hence,

$$
T_{ik} = -\delta_{ik} p + \nu \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right), \tag{2.3}
$$

and

$$
t_i = \sum_{k=1}^{3} T_{ik} n_k . \tag{2.4}
$$

We define the linear operator $\underline{\tau}$ as

$$
\underline{\tau}(u) = \underline{t} - (\underline{t}\cdot\underline{n})\underline{n}. \tag{2.5}
$$

Hence

$$
\tau_i(u) = \nu \sum_{k=1}^{3} \left(\frac{\partial u_i}{\partial x_k} + \frac{\partial u_k}{\partial x_i} \right) n_k - 2\nu \left[\sum_{k,l=1}^{3} \frac{\partial u_l}{\partial x_k} n_k n_l \right] n_i . \tag{2.6}
$$

Note that $\underline{\tau}(u)$ is tangential to the boundary and independent of the pressure p.

The existence of weak and strong solutions to problem (2.1), (2.2) was first proved by Solonnikov and Scadilov in the classical paper [7], in the case $\beta = \lambda = \mu = 0$ and $g = 0$. These authors also proved the existence of the strong solution under the additional hypothesis $a = 0$, $b = 0$.

In [1] we consider the problem (2.1), (2.2), with $\beta = 0$, in the half-space $\mathbb{R}^n$, for arbitrary n. In reference [2] we consider the stationary problem in the above open set Ω, and prove that this problem admits a strong solution $(u,\, p) \in \mathbb{H}^2 \times\, H^1$ provided that $\nu > 0, \lambda \geq\, 0, \beta \geq\, 0$ and

$$(2.7) \qquad \begin{cases} f \in \mathbb{L}^2(\Omega), \\ g \in \mathbb{H}^1(\Omega), \\ a \in H^{3/2}(\Gamma), \\ b \in \mathbb{H}^{1/2}(\Gamma). \end{cases}$$

If $\lambda =\, 0$ we must assume the following (necessary) compatibility condition

$$(2.8) \qquad \int_\Omega g\, dx = \int_\Gamma a\, d\Gamma\, .$$

Moreover, when $\beta \;=\; 0$ and Ω is axial symmetric, compatibility conditions between f and b occur. See [7] and [2]. However, this situation does not affect the regularity of the solutions. For convenience, we assume here that $\beta >\, 0$ and also that $\lambda =\, \mu =\, 0$. Moreover, we consider the homogeneous boundary value problem. Without loosing generality, we set $\nu =\, 1$.

Hence we start by considering the homogeneous problem

$$(2.9) \qquad \begin{cases} -\Delta\, u + \nabla p =\, f(x), \\ \nabla \cdot u = 0 \quad \text{in } \Omega; \\ (u \cdot n)_{|\Gamma} =\, 0\,, \\ \beta\, u_\tau +\, \underline{\tau}(u)_{|\Gamma} =\, 0, \end{cases}$$

where $\beta >\, 0$.

From the general estimates proved in reference [2] it follows that the (unique) solution (u, p) to problem (2.9) satisfies the estimate

$$(2.10) \qquad \|u\|_2^2 +\, \|p\|_1^2 \leq\, c\, \|f\|^2\, ,$$

where c is a positive constant.

Let us introduce some notation. We denote by c, $\bar{c}$, c_1, c_2, etc., positive constants that depend, at most, on Ω and on β. The same symbol c may denote different constants, even in the same equation.

Given a scalar function p we set

$$(2.11) \qquad \overline{p} =\, p -\, |\Omega|^{-1} \int_\Omega p\, dx\, .$$

The symbol $\|\,.\,\|$ denotes the canonical norm in $L^2(\Omega)$. The symbol $L^2_{\#}(\Omega)$ denotes the subspace of $L^2(\Omega)$ consisting of functions with mean value equal to zero. We denote by $H^k(\Omega), k$ a positive integer, the usual Sobolev space of order k, by $H^1_0(\Omega)$ the closure in $H^1(\Omega)$ of $C_0^\infty(\Omega)$ and by $H^{-1}(\Omega)$ the strong dual of $H^1_0(\Omega)$. The canonical norms in this spaces are denoted by $\|\,.\,\|_k$. We denote by $H^1_{\#}(\Omega)$ the subspace of $H^1(\Omega)$ consisting of functions with mean value equal to zero.

In notation concerning duality pairings and norms, we will not distinguish between scalar and vector fields. Very often we also omit from the notation the symbols indicating the domains Ω or Γ, provided that the meaning remains clear.

If X is a Banach space we denote by X' its (strong) dual space. The symbol $< \,.\,,\,.\, >$ denotes a generic duality pairing. In particular it may denote the scalar product in L^2.

We set

$$\mathbb{L}^2 = [L^2(\Omega)]^3, \quad \mathbb{H}^s = [H^s(\Omega)]^3, \quad \mathbb{H}^s(\Gamma) = [H^s(\Gamma)]^3.$$

Moreover, we denote by $\mathcal{V}$ the set of the $C_0^\infty(\Omega)$ divergence free vector fields in Ω and by H its closure in the $\mathbb{L}^2$ norm. It is worth noting that

$$H = \{v \in \mathbb{L}^2 : \nabla \cdot v = 0 \quad \text{and} \quad (v \cdot n)_{|\Gamma} = 0\}.$$

Set

$$G = \{\nabla p \, : \, p \in H^1_{\#}(\Omega)\}\,.$$

It is well known that

$$\mathbb{L}^2 = H \oplus G\,,$$

and that H and G are orthogonal to each other. We denote by P the orthogonal projection of $\mathbb{L}^2$ onto H. Moreover, we define

$$\mathbb{H}^1_\tau = \left\{v \in \mathbb{H}^1 \, : \, (v \cdot \underline{n})_{|\Gamma} = 0\right\}.$$

and

$$V = \{v \in \mathbb{H}^1_\tau(\Omega) : \nabla \cdot v = 0\}\,. \tag{2.12}$$

Note that $\|\nabla v\|$ is a norm in $\mathbb{H}^1_\tau$, equivalent to the canonical $\mathbb{H}^1$-norm $\|v\|_1$.

Remark. Concerning the notation used in reference [2], and by tacking into account that here $Z = \{0\}$, one has $\mathbb{H}^1_z = \mathbb{H}^1$.

By identifying H with its dual space H', one has the following typical situation:

$$V \hookrightarrow H = H' \hookrightarrow V'\,, \tag{2.13}$$

with dense embeddings. In particular, if $f \in H$ and $v \in V$, one has

$$< f,\, v >_{V' \times V} = (f,\, v)\,.$$

Hence, in the sequel, we use only the symbol $<,>$.

The next result is well known. For a simple and complete proof see, for instance, the section 8 in reference [2]. Recall that $\overline{p}$ is defined by (2.11).

Proposition 2.1. *Let p be a scalar field in L^2. There is a constant c such that*

$$\|\overline{p}\| \le c\, \|\nabla p\|_{-1}. \tag{2.14}$$

In [2], equation (2.3), we define the bilinear form

$$B(u, \phi) := \int_\Omega \left[\frac{1}{2}(\nabla u + \nabla u^T) \cdot (\nabla \phi + \nabla \phi^T) - (\nabla \cdot u)(\nabla \cdot \phi)\right] dx. \tag{2.15}$$

By integrations by parts one easily shows that ([2], equation (2.5))

$$B(u,\phi) = -\int_\Omega \Delta u \cdot \phi \, dx + \int_\Gamma \underline{\tau}(u) \cdot \phi \, d\Gamma \,, \tag{2.16}$$

for each $\phi \in \mathbb{H}^1_\tau(\Omega)$.

In the particular case under consideration, the definition of weak solution given in reference [2] reads:

Definition. We say that a pair (u,p) is a <u>weak solution</u> to problem (2.9) if it belongs to $\mathbb{H}^1_\tau \times L^2_\#$, and if

$$B(u,\phi) - < p, \nabla\cdot\phi > + \beta < u,\phi >_\Gamma \; + \; < \nabla \cdot u, \psi > = < f, \phi > \,, \tag{2.17}$$

for each $(\phi,\psi) \in \mathbb{H}^1_\tau \times L^2_\#$.

Remarks. a) Since here $\lambda = 0$, the pressure p is defined up to a constant. Hence we may consider $p = \overline{p}$ and take test-functions ψ in $L^2_\#$.

b) In [2] the given function f is taken in the space $(\mathbb{H}^1_\tau)'$. Here we may assume that $f \in V'$ since V is a closed subspace of $\mathbb{H}^1_\tau$.

The following result is a corollary of Theorem 1.2 in reference [2].

Proposition 2.2. *To each $f \in V'$ it corresponds a (unique) weak solution $(u,p) \in \mathbb{H}^1_\tau \times L^2_\#$ of problem* (2.9)*, i.e., a solution to problem* (2.17)*. Moreover*

$$\|u\|_1^2 + \|p\|^2 \leq c\, \|f\|^2_{V'} \,, \tag{2.18}$$

where c is a positive constant.

Clearly (2.17) and (2.20) below are equivalent formulations of problem (2.9). In fact define the bilinear form

$$a(u,\,\phi) := \frac{1}{2} \int_\Omega (\nabla u + \nabla u^T) \cdot (\nabla \phi + \nabla \phi^T) \, dx \; + \; \beta < u,\phi >_\Gamma \; . \tag{2.19}$$

By setting in equation (2.17) $\phi = 0$ and $\psi = \nabla \cdot u$, it easily follows that $u \in V$. Hence, the solution u of (2.17) belongs to V and satisfies

$$a(u,\,\phi) = < f, \phi > \,, \quad \forall \quad \phi \in V \,. \tag{2.20}$$

On the other hand the solution to this last problem is unique since from Lemma 2.3 in reference [2] it follows that (coerciveness)

$$a(v,\,v) \geq c\, \|v\|_1^2 \,, \qquad \forall v \in V. \tag{2.21}$$

Consequently, one has the following result.

Proposition 2.3. *A pair (u,p) is a weak solution to problem* (2.9)*, in the sense of definition* (2.17)*, if and only if u belongs to V and satisfies* (2.20)*. In this last case $p = \overline{p}$ is determined, in the distributional sense, by the first equation* (2.9)*. Moreover,* (2.18) *holds.*

Note that the "abstract" formulation (2.20) and the functional framework (2.13) are, formally, typical formulations used to study Stokes and Navier-Stokes equations under the nonslip boundary condition $u = 0$ on Γ. In this last case one replaces $a(u, v)$ by

$$a_0(u, v) = \frac{1}{2} \int_\Omega \nabla u \cdot \nabla \phi \, dx$$

and V by

$$V_0 = \{u \in H^1_0(\Omega) : \nabla \cdot u = 0 \}.$$

Following a well known way we define the linear operator $A : V \to V'$ by the equation

$$< A\, u,\, v >= a(u, v), \quad \forall\, u, v \in V\, .$$

The coerciveness of the bilinear continuous form a shows that A is an homeomorphism from V onto V', hence $A^{-1} \in \mathcal{L}\big(V', V\big)$.

Next we consider the restriction of the operator A as an unbounded operator in H, i.e., with domain

$$D(A) = \{v \in V : \, Av \in H\}\, .$$

Moreover we set

$$\text{(2.22)} \quad \mathbb{D} = \{u \in H^2(\Omega) : \nabla\, u = 0\,, \quad (u \cdot n)_{|\Gamma} = 0 \quad \text{and}\; \beta\, u_\tau + \underline{\tau}(u)_{|\Gamma} = 0\}\, .$$

The Theorem 1.2 in reference [2] shows that $D(A) = \mathbb{D}$. Moreover, if $f \in \mathbb{L}^2$, the weak solution (u, p) belongs to $\mathbb{D} \times \mathbb{H}^1_\#$ and satisfies the estimate (2.10). In other words, the problem (2.9) admits a (unique) solution $(u, p) \in \mathbb{H}^2 \times \mathbb{H}^1_\#$, and (2.10) holds.

More precisely,

Proposition 2.4. *To each $f \in H$ it corresponds a (unique) $u \in \mathbb{D}$ such that* (2.9) *holds for some $p \in \mathbb{H}^1_\#$. This p is unique, and* (2.10) *holds. Moreover, u is the (unique) solution to problem* (2.20). *If $f \in V'$, the solution u of* (2.10) *belongs to $\mathbb{D}$ if and only if $f \in H$.*

3 The evolution problem

Define

$$b(u, v, w) = \int_\Omega (u \cdot \nabla) v \, \cdot w \, dx\, .$$

It readily follows that the crucial property

$$b(u, v, w) = -b(u, w, v)\,, \quad \text{for each } u \in V$$

and for each pair $v,\, w \in \mathbb{H}^1(\Omega)$, still holds if V is given by (2.12). In particular, $b(u, v, v) = 0$ if $u \in V$.

The "abstract" definition of weak solution to the evolution Navier-Stokes equations

$$\text{(3.1)} \qquad \begin{cases} \frac{\partial u}{\partial t} - \Delta\, u + (u \cdot \nabla)\, u + \nabla p = f(t, x), \\ \nabla \cdot u = 0 \quad \text{in } Q_T\,, \\ u(0, x) = u_0(x) \quad \text{in } \Omega\,, \end{cases}$$

with the boundary conditions (1.2) can be stated as follows. Let $f \in L^2(0,T;V')$. The vector field u is a weak solution if $u(t) \in V$ for almost all $t \in (0,T)$, if $u(0) = u_0$ and if

$$\frac{d}{dt}(u(t),\, v) + a(u(t),\, v) + b(u(t), u(t), v) = < f(t), v >\,, \tag{3.2}$$

for all $v \in V$. As immediately seen, this corresponds to the formal definition used on dealing with the classical non-slip boundary condition $u = 0$ on Γ, in which a and V are replaced by a_0 and V_0 respectively. Since the fundamental abstract properties of the bilinear forms and of the functional spaces used in both cases coincide, most of the main results for the non-slip boundary condition can be easily extended to slip boundary conditions by following *exactly* the same proofs. Let us illustrate this claim by considering just a couple of possible situations.

In the following we consider strong solutions. We assume that $f \in H$ and that (u,p) is the solution referred in Proposition 2.4.

Since $-\Delta\, u = f \oplus \nabla(-p)$ one has

$$f = P(-\Delta u)\,, \quad \nabla p = -(I-P)(-\Delta u)\,.$$

From (2.20) it follows that

$$A\, u = -P\,\Delta u\,, \qquad \forall u \in \mathbb{D}\,.$$

Clearly, the domain $\mathbb{D}$ of A is dense in H.

Proposition 2.4 shows that the linear operator A is an homeomorphism from $\mathbb{D}$ onto H. In particular

$$c_1\, \|A\, u\| \leq \|u\|_2 \leq c_2\, \|A\, u\|\,, \quad \forall u \in \mathbb{D}\,. \tag{3.3}$$

We recall that a linear operator $A : \mathbb{D} \to H$ is said monotone if $(Av,\, v) \geq 0$ for each $v \in \mathbb{D}$. It is said **maximal monotone** if, moreover, the image of $I + A$ is the whole H. See [3], chapter VII. From (2.21) it follows that A is monotone. Recall that $\|v\| \leq c\,\|v\|_1$. Since the immage of A is the whole H, it readily follows that A is maximal monotone. On the other hand, by restriction of equation (2.16) to $\mathbb{D} \times \mathbb{D}$, it follows that $B(u,v) = (Au,v)$. Hence A is symmetric. These two last properties imply that A is necessarily selfadjoint. See [3], Proposition VII.6.

Hence, one has the following result (see [3] Section VII.2, and references):

Theorem 3.1. *The operator A is m.m. and self-adjoint in H. In particular, A is the generator of an analytical semigroup of contractions in H. The semigroup is also compact.*

The compactness follows from the analyticity together to the compactness of the embedding $\mathbb{D} \hookrightarrow H$.

Moreover, there is a positive real λ_0 such that

$$\|e^{-tA}\, u_0\| \leq e^{-\lambda_0\, t}\, \|u_0\|\,, \quad \forall\, u_0 \in H\,.$$

From the above facts many interesting applications follow. Let us consider a simple and typical one (which shows how the pressure can be explicitly reintroduced into the problem).

Let $u_0 \in \mathbb{D}$ and $f \in C^1([0,T];H)$. It is well known (see [3] Theorems VII.7 and VII.10) that there is a unique solution $u \in C([0,T];\mathbb{D}) \cap C^1([0,T];H)$ of the problem

$$
\begin{cases} \frac{d\,u}{dt} + A\,u = f(t)\,, \quad \text{in } H\,, \\ \\ u(0) = u_0. \end{cases} \tag{3.4}
$$

Equation (3.4) means that

$$
\begin{cases} \frac{\partial u}{\partial t} - P\Delta\,u = f(t), \\ \nabla \cdot u = 0 \quad \text{in } Q_T\,; \\ (u \cdot n)_{|\Gamma} = 0, \\ \beta\,u_\tau + \underline{\tau}(u) = 0 \quad \text{in } \Sigma_T\,; \\ u(0,x) = u_0(x) \quad \text{in } \Omega\,. \end{cases} \tag{3.5}
$$

Hence, for each $t \in [0,T]$, one has $-P\,\Delta\,u(t) = f(t) - \partial_t\,u$. Since

$$
-\Delta\,u(t) = -P\;\Delta\,u(t) \oplus (I-P)\,(-\Delta\,u(t)),
$$

it readily follows that there is a (unique) $p(t) \in H^1_{\#}(\Omega)$ such that the first equation (1.1) holds. Since $\nabla\,p = (I-P)\,(\Delta\,u(t))$, it follows that the H^1 norm of $p(t)$ is bounded in terms of the H^2 norm of u. The boundary conditions follow from the fact that $u(t) \in \mathbb{D}$, for each t. Hence the following result holds.

Theorem 3.2. *Let $u_0 \in \mathbb{D}$ and $f \in C^1([0,T];L^2)$. Then, there is a (unique) solution (u,p) of problem* (3.5)*. Moreover, $u \in C([0,T];\mathbb{H}^2) \cap C^1([0,T];L^2)$ and $p \in C([0,T];H^1_{\#})$.*

It is now clear that we can easily apply to the slip boundary condition a large part of the classical proofs established for strong solutions (for instance, via the variational method as well as via the *analytic semigroups theory*) to solutions to non-slip boundary conditions. See [3], Chapter X.

Concerning the formal parallelism between the two problems, one may add the interpolation result

$$
[\mathbb{D}, H]_{1/2} = V\,,
$$

and, in particular, the embedding

$$
L^2(0,T;\,\mathbb{D}) \cap W^{1,2}(0,T;H) \subset C(0,T;V)\,.
$$

The great part of the proofs that apply for strong solutions to the non slip boundary condition, via the variational method, apply as well to our slip boundary condition. See, for instance, [4],[5], [6], [8]. We just replace V_0, $a_0(u,v)$ and $A_0 := -P\Delta\,u$ with domain

$$
D(A_0) := \{v \in V_0 : A_0\,v \in H\} = \mathbb{H}^2 \cap V_0
$$

by the new counterparts V, $a(u,v)$ and A. For instance, one may show in this way that

Theorem 3.3. *Let* $f \in L^2(0, +\infty; L^2(\Omega))$ *and* $u_0 \in V$. *Then there is a positive* T *such that the problem* (3.1) *with the boundary conditions* (1.2), *has a unique solution* (u, p) *where*

$$u \in L^2(0, T; \mathbb{H}^2) \cap W^{1,2}(0, T; \mathbb{L}^2)$$

and

$$p \in L^2(0, T; H^1_{\#}).$$

Moreover, if the norms of the data are sufficiently small, the above strong solution is global in time.

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