On the Nullity of Isometric Immersions from Kähler Manifolds

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Abstract

In this article we study isometric immersions from Kähler manifolds into space forms which generalize immersed Riemann surfaces with parallel mean curvature (cmc isometric immersions). In the Kähler setting the complexified second fundamental form α splits according to types. The $(1,1)$ part of the second fundamental splitting plays the role of the mean curvature for sufaces and will be called the plurimean curvature. Therefore isometric immersions with parallel plurimean curvature (*ppmc* isometric immersions) generalize in a natural way the cmc immersions. For isometric immersions of Riemann surfaces into space forms it is well known that the traceless part of the complexified second fundamental form constitutes a holomorphic quadratic differential. In the Kähler approach this differential corresponds to a operator Q which is the $(2,0)$ part of α. It turns out that for *ppmc* immersions Q is also a holomorphic vector bundle valued quadratic differential. We study *ppmc* immersions for which Q has big nullity index. For immersions with identically zero plurimean curvature, Dajczer and Rodrigues ([3]) have studied immersions with big relative index of nulity. We remark that in this situation, the nullity indez of α correponds to the nullity index of Q.

1 Introduction

In this work we study the geometry of Kähler submanifolds of space forms. The almost complex structure J of such a manifold M coupled with the second fundamental form α of the immersion give rise to two operators which are crucial ingredients in the study of the intrinsic and extrinsic geometry of M. Indeed the complexified α splits in a natural way, according to types, giving rise to

$$\alpha = \alpha^{(1,1)} + \alpha^{(2,0)} + \alpha^{(0,2)}$$

Maps with $\alpha^{(1,1)} = 0$ are called pluriharmonic maps. Holomorphic maps between Kähler manifolds are examples of pluriharmonic maps. Isometric immersions with $\alpha^{(1,1)} = 0$ are called $pluri\min imal$ immersions and have been

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extensively studied (see, by instance, [3] [4], [5], [6]). When M is a Riemann surface we have $\alpha^{(1,1)} =< .,. > H$, where $H = trace\ \alpha$ is the mean curvature vector field of the immersion. In this case the pluriminimal immersions are precisely the minimal ones. In general, the immersion is pluriminimal if and only if its restriction to each holomorphic curve of M is a minimal immersion. The operator $\alpha^{(1,1)}$ is naturally called the plurimean curvature of the immersion. When the ambient space is R^n, it is well known that $\alpha^{(1,1)} = 0$ if and only if $H = 0$ ([4]), so that the class of pluriminimal immersions and the class of minimal immersions coincide for Kähler manifolds. It is also well known ([5]) that the class of pluriminimal immersions into a space form Q_c, with sectional curvature $c \neq 0$, restricts to the class of minimal surfaces into Q_c.

In the present paper we are mainly interested in isometric immersions which have $\alpha^{(1,1)}$ parallel, otherwise named, isometric immersions with *parallel pluriminimal curvature* operator (*ppmc* isometric immersions). They constitute a natural generalization to higher dimensions of the isometric immersions from Riemann surfaces with parallel mean curvature. In fact *ppmc* isometric immersions into space-forms display some special features of the parallel mean curvature surfaces, namely the existence of a $1-$parameter deformation through a smooth family of *ppmc* isometric immersions which, up to a parallel isomorphism, have the same normal bundle ([2]). Just as in the case of immersions with parallel mean curvature, *ppmc* isometric immersions can also be characterized by the pluriharmonicity of its Gauss map ([7]).

Studying immersed surfaces in R^3, H. Hopf discovered in 1955 that for surfaces with constant mean curvature (*cmc* surfaces) the complexification of the traceless part of α is a holomorphic quadratic differential Q. This holomorphic quadratic differential has been an important ingredient in the investigation of geometric properties of *cmc* surfaces ([16], [12], [13]). The operator Q is nothing but the $(2,0)$ part of the complex bilinear extension of α. A straightforward computation shows that, for *ppmc* isometric immersions, Q is again a vector-bundle valued holomorphic quadratic differential. Isometric immersions with $\alpha^{(2,0)} = \alpha^{(0,2)} = 0$ are called $(2,0) - geodesic$ immersions. Curiously that is a strong condition. Indeed, it can be deduced from Codazzi equation that $(2,0) - geodesic$ immersions into space forms have parallel second fundamental form. Ferus ([10], [9]), Takeuchi ([15]) and Strübing ([14]) classified the $(2,0) - geodesic$ immersed immersions into space forms. It turns out that they are extrinsically symmetric.

In this article we consider *ppmc* immersions whith big nullity of $\alpha^{(2,0)}$. In ([3]) Dajczer and Rodrigues proved the following result:

Theorem 1 *Let M be a Kähler manifold with complex dimension m and $\varphi : M \to R^n$ be a pluriminimal immersion such that, for every $x \in M$, the index of relative nullity of α at x is greater or equal than $2m-2$. Then $M^m = R^{2n-2} \times M^1$ and $\varphi = id \times \varphi_2$, where M^1 is a Riemann surface.*

Notice that that for pluriminimal immersions, the index of relative nullity of α and the index of relative nullity of $\alpha^{(2,0)}$ coincide. From now on M^m will denote a connected complete Kähler manifold with complex dimension m, S^n_c $(c>0)$ the n-dimensional euclidean sphere with sectional curvature c and H^n_c the n-dimensional hyperbolic space with constant seccional curvature c $(c<0)$.

For *ppmc* immersions we get the following generalization:

Theorem 2 *Let $\varphi : M^m \to R^n$ be a ppmc immersion. If the index of nullity of $\alpha^{(2,0)}$ is everywhere greater or equal than $2m-2$, one of the following conditions holds:*

1. *φ is extrinsically symmetric*
2. *$M^m = M^{m-1} \times M^1$ and $\varphi = \varphi_1 \times \varphi_2 : M^{m-1} \times M^1 \to R^{n_1} \times R^{n_2}$, where φ_2 has parallel mean curvature and $\varphi_1 : M^{m-1} \to R^{n_1}$ is extrinsically symmetric.*

Corollary 3 *Let $\varphi : M^m \to S^n$ be a ppmc immersion such that, for every $x \in M^m$, the index of nullity of $\alpha^{(2,0)}$ at x is greater or equal than $2m-2$. Then one of the following conditions hold:*

1. *f is extrinsically symmetric*
2. *$M^m = M^{m-1} \times M^1$ and $\varphi = \varphi_1 \times \varphi_2 : M^{m-1} \times M^1 \to S^{n_1}_a \times S^{n_2}_b$ $(a^{-1/2} + b^{-1/2} = 1)$, where φ_2 $M^1 \to S^{n_2}_b$ has parallel mean curvature and $\varphi_1 : M^{m-1} \to S^{n_1}_a$ is extrinsically symmetric.*

Corollary 4 *Let $\varphi : M^m \to H^n$ be a ppmc immersion such that, for every $x \in M^m$, the index of nullity of $\alpha^{(2,0)}$ at x is greater or equal than $2m-2$. Then one of the following conditions hold:*

1. *φ is extrinsically symmetric*
2. *$M^m = M^{m-1} \times M^1$ and $\varphi = \varphi_1 \times \varphi_2 : M^{m-1} \times M^1 \to S^{n_1}_a \times H^{n_2}_b$ $(a^{-1} + b^{-1} = -1)$, where φ_2 $M^1 \to H^{n_2}_b$ has parallel mean curvature and $\varphi_1 : M^{m-1} \to S^{n_1}_a$ is extrinsically symmetric.*
3. *$M^m = M^{m-1} \times M^1$ and $\varphi = \varphi_1 \times f_2 : M^{m-1} \times M^1 \to H^{n_1}_a \times S^{n_2}_b$ $(a^{-1} + b^{-1} = -1)$, where φ_2 $M^1 \to S^{n_2}_b$ has parallel mean curvature and $\varphi_1 : M^{m-1} \to H^{n_1}_a$ is extrinsically symmetric.*

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2 On the nullity distribution

Let J denote the almost complex structure of M^m and $\varphi : M^m \to Q_c$ be an isometric immersion into a space form with seccional curvature c.. We let $C(TM)$ denote the space of smooth sections of TM. We use the notation TM

and $T^{\perp}M$ for the the tangent and normal bundles of φ. The complexification of TM, denoted by $T^C M$, decomposes as

$$T^C M = T'M + T''M$$

where $T'M$ and $T''M$ are the eigenbundles of J corresponding respectively to the eigenvalues i and $-i$ of J. The orthogonal projections of $T^C M$ onto $T'M$ and $T''M$ will be represented respectively by π' and π''. Of course, for any section X of TM, we have $X = \pi'(X) + \pi''(X)$.

The complex bilinear extension of the second fundamental form α splits in a natural way according to types, giving rise to

$$\alpha = \alpha^{(1,1)} + \alpha^{(2,0)} + \alpha^{(0,2)}$$

We have

$$\alpha^{(1,1)}(X, Y) = \alpha(X', Y'') + \alpha(X'', Y')$$

where $Z' = \pi'(Z)$ and $Z'' = \pi''(Z)$ for every $Z \in C(T^C M)$. We can also write $\alpha^{(1,1)}(X, Y) = C(X, Y)$, where $C(X, Y) = \frac{1}{2}\{\alpha(X, Y) + \alpha(JX, JY)\}$

Similarly

$$\alpha^{(2,0)}(X, Y) = \alpha(X', Y') = \frac{1}{2}Q(X, Y) - i\frac{1}{2}Q(X, JY)$$

where $Q(X, Y) = \frac{1}{2}\{\alpha(X, Y) - \alpha(JX, JY)\}$

We will use the same symbol ∇ to represent either the Levi-Civita connection of TM or the induced connection on $\varphi^{-1}TN$ and on $T^*M \otimes \varphi^{-1}TN$. The symbol $\overset{\perp}{\nabla}$ will be used to represent either the induced connection on $\overset{\perp}{T}M$ or on $T^*M \otimes \overset{\perp}{T}M$.

Let $\Delta_x = \{X \in T_xM : Q(X, Y) = 0 \quad \forall\ Y \in T_xM\}$ and $\Delta_x^{\perp}$ be its orthogonal complement in T_xM.

Δ_x and $\Delta_x^{\perp}$ are J_x invariant.since $Q(X, JY) = Q(JX, Y)$, for any $X, Y \in T_xM$

Proposition 5 *On an open set where the dimension of Δ is constant, Δ is a smooth integrable distribution whose leaves are totally geodesic in M.*

Proof. Let U be an open set where $\dim \Delta$ is constant. Notice that for $x \in U$, $X \in \Delta_x$ if and only if $\alpha(X', Y') = 0$ for every $Y \in T_xM$. It is enough to prove that if $X, Z \in \Delta_x$, $\alpha(\nabla_Z X', Y') = 0$, for all $Y \in T_xM$. Indeed using Cozazzi equation and the fact that φ is *ppmc*, $\alpha(\nabla_Z X', Y') = \alpha(\nabla_{Z'} X', Y') + \alpha(\nabla_{Z''} X', Y') = (\nabla_{Z'}\alpha)(X', Y') = (\nabla_{Y'}\alpha)(X', Z') = -\alpha(\nabla_{Y'} X', Z') = 0$, since $Z' \in \Delta \otimes C$. ■

For the study of this nullity foliation it is useful to consider the tensor $C_T : \Delta^{\perp} \to \Delta^{\perp}$ defined by

$$C_T(X) = -(\nabla_X T)^{\Delta^{\perp}},$$

where $T \in \Delta$ and $(\,)^{\Delta^{\perp}}$ denotes the orthogonal projection onto $\Delta^{\perp}$.

Proposition 6 *The follwoing conditions holds*

1. *C_T commutes with J, for all $T \in \Delta$.*
2. *$Q(C_T(Y), Z) = Q(Y, C_T(Z))$, for $T \in \Delta$ and $Y, Z \in \Delta^{\perp}$.*

Proof. Let $Y \in C(\Delta^{\perp})$. Clearly $\nabla_{Y"} T' \in C(\Delta')$ and $\nabla_{Y'} T" \in C(\Delta")$, for $Y \in C(TM)$. Indeed if $Z \in C(TM)$, $\alpha(\nabla_{Y"} T', Z') = (\nabla_{Y"}\alpha)(T', Z') - \nabla_{Y"}\alpha(T', Z') - \alpha(T', \nabla_{Y"} Z') = (\nabla_{T'}\alpha)(Y", Z') = 0$. One proves analogously that $\nabla_{Y'} T" \in C(\Delta")$. Hence $(\nabla_{Y'} T")^{\Delta^{\perp}} = (\nabla_{Y"} T')^{\Delta^{\perp}} = 0$ so that $C_T(Y') = -(\nabla_{Y'} T)^{\Delta^{\perp}} = -(\nabla_{Y'} T')^{\Delta^{\perp}} - (\nabla_{Y'} T")^{\Delta^{\perp}} = -(\nabla_{Y'} T')^{\Delta^{\perp}} \in C((\Delta^{\perp})')$ and $C_T(Y") = -(\nabla_{Y''} T'')^{\Delta^{\perp}} \in C((\Delta^{\perp})'')$. Thus $C_T(JY) = JC_T(Y)$.

The assertion in 2 is consequence of the following equality

$$(\nabla_T \alpha)(Y, Z) = Q(C_T(Y), Z)$$

since the symmetry of $(\nabla_T \alpha)(Y, Z)$ implies that $Q(C_T(Y), Z) = Q(Y, C_T(Z))$. Let us now prove the last equality. From Codazzi equation and the parallelism of $\alpha^{(1,1)}$ we get

$$(\nabla_T \alpha)(Y, Z) = (\nabla_Y \alpha)(T, Z) = (\nabla_{Y'}\alpha)(T', Z') + (\nabla_{Y''}\alpha)(T'', Z'') =$$
$$-\alpha(\nabla_{Y'} T', Z') - \alpha(\nabla_{Y''} T'', Z'') = \alpha(C_{T'}(Y'), Z') + \alpha(C_{T''}(Y''), Z'') =$$
$$Q(C_T(Y), Z)$$

since, as we have seen in the proof of assertion 1, $C_{T''}(Y') = C_{T'}(Y'') = 0$. ∎

When $\xi \in \overset{\perp}{T}_x M$ let A_ξ denote the Weingarten operator at x associated to α. We represent by $N_x(M)$ the first normal space of the immersion at x.

Lemma 7 *$A_{\alpha(S,T)} Y$, $A_{\alpha(S,Y)} T \in \Delta^{\perp}$ whenever $S, T \in \Delta$ and $Y \in \Delta^{\perp}$*

Proof. Let $x \in M$, $\dim \Delta_x = k$ and U be an open neighborhood of x, where $\dim \Delta$ is k. Let $\widetilde{M}$ be a maximal integral submanifold of Δ through x and $\widetilde{\varphi} = \varphi_{|\widetilde{M}} : \widetilde{M} \to R^p$. The second fundamental form $\widetilde{\alpha}$ of $\widetilde{\varphi}$ is parallel, since $\widetilde{\alpha}^{(2,0)} = 0$; hence $N_1(\Delta)$ is also parallel and we may reduce the codimension of $\widetilde{\varphi}$. $\widetilde{\varphi}(\widetilde{M})$ is then contained in a totally geodesic submanifold of R^p with dimension $\dim \widetilde{M} + \dim N_1(\Delta)$. Therefore, if $L \in \Delta$, $0 =< \nabla_L \alpha(S, T), Y >= - < A_{\alpha(S,T)} Y, L >$.

One proves analogously that $< A_{\alpha(S,Y)} T, L >= 0$, hence $A_{\alpha(S,Y)} T \in \Delta^{\perp}$ ∎

Lemma 8 *$R^M(T, Y)S = A_{\alpha(T,Y)} S$ for $T, S \in \Delta$ and $Y \in \Delta^{\perp}$*

Proof. Gauss equation and lemma 10 tells that $R(S, Y)T \in \Delta^{\perp}$ for $S, T \in \Delta$ and $Y \in \Delta^{\perp}$ Using the fact that φ is *ppmc* we get that $\nabla_{Y''} X' \in \Delta''$ for any $X \in \Delta$, $Y, W \in TM$, since $\alpha(\nabla_{Y''} X', W') = 0$. This implies that $R(S', Y'')T' \in \Delta'$. Then we get from Gauss equation that, for any $Z \in \Delta^{\perp}$,

$$< R^M(T, Y)S, Z >=< R^M(T', Y'')S'', Z' > + < R^M(T'', Y')S', Z'' >=$$

$$=< \alpha(T', S''), \alpha(Y'', Z') > + < \alpha(T'', S'), \alpha(Y', Z'') >=$$

$$=< \alpha(T', Y''), \alpha(S'', Z') > + < \alpha(T'', Y'), \alpha(S', Z'') >=< \alpha(T, Y), \alpha(S, Z) >$$

since

$$0 =< R(S'', Y')T'', Z' >=< \alpha(Y', T''), \alpha(Z', S'') > .$$

■

Proposition 9 *The following equality holds:*

$$(\nabla_S C_T)(Y) = C_T C_S Y + C_{\nabla_S T} Y + R^M(S, Y)T \tag{1}$$

where $S, T \in \Delta$ and $Y \in \Delta^\perp$

Proof.

$$(\nabla_S C_T)(Y) = \overset{\perp}{\nabla}_S C_T(Y) - C_T(\overset{\perp}{\nabla}_S Y) = (-\nabla_S(\nabla_Y T)^{\Delta^\perp} + \nabla_{\overset{\perp}{\nabla}_S Y} T)^{\Delta^\perp} =$$

$$(-\nabla_S \nabla_Y T - A_{\alpha(Y,T)} S + \nabla_{\overset{\perp}{\nabla}_S Y} T)^{\Delta^\perp} = (-\nabla_S \nabla_Y T - A_{\alpha(Y,T)} S + \nabla_{\nabla_S Y} T)^{\Delta^\perp} =$$

$$(-\nabla_Y \nabla_S T - \nabla_{[S,Y]} + \nabla_S \alpha(T, Y) + \nabla_{\nabla_S Y^\perp} T)^{\Delta^\perp} =$$

$$C_{\nabla_S T}(Y) + C_T(C_S(Y)) + A_{\alpha(S,T)} Y - A_{\alpha(T,Y)} S$$

$$= C_T C_S(Y) + C_{\nabla_S T} Y + R^M(S, Y)T,$$

where we have used respectively the notation $(Z)^{\Delta^\perp}$ and $\overset{\perp}{\nabla}_X B$ to represent the images, through the orthogonal projection onto $\Delta^\perp$, of the vectors $Z \in R^p$ and $(\nabla_X B)$.

■

3 Proof of the main results

One has the following version of a splitting theorem of Moore ([11]):

Lemma 10 *Let $M = M_1 \times M_2$ be a product of Kähler manifolds and $\varphi : M_1 \times M_2 \to R^n$ be an isometric immersion. Then $\frac{1}{\sqrt{2}} \left|\alpha^{(1,1)}(X, Y)\right| = \left|\alpha^{(2,0)}(X, Y)\right| = \left|\alpha^{(0,2)}(X, Y)\right|$ for $X \in TM_1,\ Y \in TM_2$. Furthermore, if φ is either minimal or $(2, 0)$-geodesic, φ splitts as a product of immersions.*

Proof. The result is a direct consequence of Gauss equation. Indeed, if $X \in TM_1, Y \in TM_2$, we can write

$$0 =< R^M(X', Y'')Y', X'' >=< \alpha(X', Y'), \alpha(X'', Y'') > - < \alpha(X', X''), \alpha(Y', Y'') >$$

and

$$0 =< R^M(X', Y')X'', Y'' >=< \alpha(X', X''), \alpha(Y', Y'') > - < \alpha(X', Y''), \alpha(Y', X'') >,$$

from whence

$$| \alpha^{(2,0)} |^2 = < \alpha(X', Y'), \alpha(X'', Y'') > = < \alpha(X', Y''), \alpha(Y', X'') > = \frac{1}{2} | \alpha^{(1,1)} |^2 .$$

Therefore if φ is minimal, or φ is $(2,0)$-geodesic, we immediately infer that $\alpha(X,Y) = 0$ for all $X \in TM_1$ and $Y \in TM_2$ and φ splitts as a product of immersions ([11]). ■

From now on M is assumed to have codimension less or equal than two.

Proposition 11 $C_T^2 = 0$*, for all* $T \in \Delta$

Proof. Let U be the open dense subset of M where $\zeta = \dim \Delta$ is minimal. Let $x_0 \in M$ and take $T_{x_0} \in \Delta_{x_0}$. Consider a geodesic γ on the maximal leaf L of Δ through x_0 and let T denote the velocity field of γ .

Notice that, since C_T commutes with J it has , at each $t \in R$, only one eigenvalue $\lambda(t)$. Take a unitary eigenvector Y. From equation (1) we easily get:

$$\lambda' = \lambda^2 + \langle R(T,Y)T, Y\rangle$$

Now, from Lemma 8 we know that $\langle R(T,Y)T, Y\rangle = \langle \alpha(T,Y), \alpha(T,Y)\rangle \geq 0$

We deduce now from this Ricatti type equation that, for each $T \in \Delta$, 0 is the only eigenvalue of C_T. Notice first that there exists $S \in \Delta$, such that C_S has a real eigenfunction. In fact if $\lambda = \mu + iv$, taking $S = \mu T - vJT$ one gets $C_S(Y) = (\mu^2 + v^2)Y$. Then $\mu^2 + v^2$ must be identically zero, otherwise, since the leaves of Δ are complete ([8]), there would exist a solution of the above Ricatti equation defined on the real line which cannot happen. ■

Proof of Theorem 2 Suppose that $\zeta = 2m - 2$. Since C_T is a complex operator we know that the dimensions of $\ker C_T$ and imC_T are even. Equality $C_T C_T = 0$ says that $imC_T \subset \ker C_T$, hence $C_T = 0$.

Since Δ and $\Delta^\perp$ are invariant by the action of the holonomy group of M, from the de Rham decomposition theorem we know that U is a product of two Kähler manifolds U_1 and U_2. To prove that $\varphi_{|U}$ is a product of immersions we first notice that $\alpha(S', Y') = 0 \ \forall S \in \Delta \ \forall Y \in \Delta^\perp$. Using Lemma 10, $\varphi_{|U} = \varphi_1 \times \varphi_2$. An analiticity argument allows the conclusion that M is the product of two Kahler manifolds and the immersion is a product of immersions.

Proof of Corollaries 3 and 4 When $c > 0$, observe that if $\varphi : M^m \to S_c^n \subset R^{n+1}$ is *ppmc*, $io\varphi$ is *ppmc*, where $i : Q_c \to R^{n+1}$ represents the inclusion map. The results now follows directly from from theorems 2 and 5.

As a model for the hyperbolic space we use the hyperbolic space model.Again if $\varphi : M^m \to H_c^n \subset R_1^{n+1}$ is *ppmc*, $io\varphi$ is also *ppmc*. All above arguments work for *ppmc* maps from M^m into the Minkowski space. Using an adapted version of lemma 13 we know also that, when $M = M_1 \times M_2$, $io\varphi$ is a product immersion $\varphi_1 \times \varphi_2$, that is, there exists two two non-degenerate orthogonal affine subspaces

E_1 and E_2 of the Minkoswski space R_1^n such that $\varphi_1(M_1) \subset E_1$ and $\varphi_2(M_2) \subset E_2$. Consider $\dim E_1 = r_1$ and $\dim E_2 = r_2$. Then $\varphi_i(M_i)$ is either contained in a hyperbolic space $H^{r_1-1}(c_i)$ or in an euclidean sphere $S^{r_1-1}(c_i)$, giving rise to the following situations: $\varphi_1 \times \varphi_2 : M_1 \times M_2 \to H_{c_1}^{r_1-1} \times S_{c_2}^{r_2-1}$ $(\frac{1}{c_1} + \frac{1}{c_2} = \frac{1}{c})$, where φ_1 is extrinsically symmetric, or $\varphi_1 \times \varphi_2 : M_1 \times M_2 \to S_{c_1}^{r_1-1} \times H_{c_2}^{r_2-1}$, where φ_1 is the extrinsically symmetric ([14]).

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