DYNAMICS ALONG THE EDGES OF SIMPLE POLYHEDRONS

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Abstract: From an analytic flow ϕ^t on a simple polyhedron Γ^d we construct an abstract piecewise-linear dynamical system that encapsulates the limit behaviour of the initial flow along heteroclinic cycles on the boundary $\partial\Gamma^d$. Examples and an application are given.

1. INTRODUCTION

In Game Dynamical Theory there is a handful of interesting ODE's defined on simple polyhedrons like the simplex Δ^d, or on products of simplexes $\Delta^k \times \Delta^{d-k}$. The replicator equation,

$$x_i' = x_i \left(\sum_{j=0}^{d} a_{i\,j}\, x_j - \sum_{k,\,s=0}^{d} a_{k\,s} x_k\, x_s \right) \qquad (0 \leq i \leq d) \,, \tag{1}$$

plays a central role in this theory. The stage where its dynamics unfolds is the simplex Δ^d. Think of a population where individuals interact according to a set of $d+1$ possible strategies, indexed from 0 to d. Let $x(t) = (x_0(t), \ldots, x_d(t)) \in \Delta^d$ represent the population state with the ratio of each strategy usage within the population. The $(d+1)\times(d+1)$ matrix $A = (a_{i\,j})$ is called the interaction matrix. Each coefficient $a_{i\,j}$ is the average payoff for an individual with strategy i when interacting with another individual with strategy j. Notice that equation (1) says that the population growth rate of strategy i, $\frac{x_i'}{x_i}$, is exactly the difference from the average payoff of strategy i to the global average payoff of all population strategies. Quite some effort has already been put to study special classes of this equation. See [12] for a good introduction on the subject. Despite its simple appearance, the general dynamics of (1) is far too rich to be understood. One reason is that the replicator equation contains, in some sense, another general ODE, the Lotka-Volterra equation

$$y_i' = y_i \left(r_i + \sum_{j=1}^{d} a_{i\,j} y_j \right) \qquad (1 \leq i \leq d) \,, \tag{2}$$

which models competition within an ecosystem formed by d species. Let $y(t) = (y_1(t), \ldots, y_d(t)) \in \mathbb{R}^d_+$ be the population vector with each species size in the ecosystem. The coefficient $a_{i\,j}$ governs the interaction between species i and j, while r_i stands for the interaction of species i with the environment. Every Lotka-Volterra system is equivalent to a replicator system in the sense that the underlying vector fields are equivalent. The equivalence is the algebraic map defined by the relations $y_i = \frac{x_i}{x_0}$ $(1 \leq i \leq d)$, which maps the interior of the simplex Δ^d onto the the interior of $\mathbb{R}^d_+$. Recall that, given two vector fields X on M, and Y on N, a smooth equivalence is any smooth map $\varphi : M \to N$ such that for some function $\lambda : M \to (0, +\infty)$, $D\varphi_x(X_x) = \lambda(x)\, Y_{\varphi(x)}$ for all $x \in M$. This means, in particular, that the equivalence φ takes orbits of X onto orbits of Y, preserving the time flow orientation. This algebraic equivalence, between replicator and Lotka-Volterra equations, was shown by J. Hofbauer in [8]. Intuitively, it comes from letting the environment play the role of another species, while looking at species as strategies. Two important subclasses of these systems, already considered by Volterra, are the so called dissipative and conservative Lotka-Volterra systems. A Lotka-Volterra system, with interaction matrix A, is said to be conservative if there is a diagonal matrix $D > 0$ such that $A\,D$ is skew-symmetric. This implies, in particular, that the system is Hamiltonian with respect to some symplectic structure. A Lotka-Volterra is called dissipative if there is a diagonal matrix $D > 0$ such that $A\,D \leq 0$. In this case, the system admits a global Lyapounov function. We have proved in [5] a result which further motivates the study of conservative Lotka-Volterra systems: Every stably dissipative Lotka-Volterra system, with a singularity interior to $\mathbb{R}^d_+$, has a global attractor where the dynamics is that of a conservative Lotka-Volterra system. In that work we have also studied a specific conservative Lotka-Volterra model with the following coefficient matrices

$$A = (a_{i\,j}) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & \delta & 0 \\ 0 & -\delta & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad r = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} . \tag{3}$$

This system models a four species chain food obtained coupling together two independent predator-prey systems, where the coupling strength is controlled through parameter δ. We prove, there, it is non-integrable for any $\delta \neq 0$. The Hamiltonian function for this system is

$$h(y) = -\delta\, \log y_1 \,-\, \delta\, \log y_4 \,+\, \sum_{i=1}^{4} y_i - \log y_i \,. \tag{4}$$

There, we pay special attention to a family of periodic orbits $\Gamma = \Gamma(\delta, E)$ defined, for all δ, as the intersection of the energy level $\{h = E\}$ with the following invariant 2-plane

$$\Pi = \{\, (y_1, y_2, y_3, y_4) \,:\, y_1 = (1+\delta)\, y_3,\; y_4 = (1+\delta)\, y_2 \,\}\ .$$

We show that there is a 3-plane containing Π, slicing transversally all energy levels into 2-spheres. The orbit Γ splits each of these 2-spheres into two disks transversal to the flow. The first return map, along the flow, to any of these disks determines the dynamics in that energy level. This geometry is carefully described there. Finally, we show that the periodic orbit Γ has rotation number which tends to $+\infty$ with the energy level E, and its character alternates between stable (elliptic) and unstable (hyperbolic). Furthermore, as $E \to +\infty$ there are parameter regions where the hyperbolicity of Γ becomes arbitrarily strong.

Here we go a little further proving the following:

Theorem A *For $\delta \in (0,1)$, and all sufficiently large energy level E, the Lotka-Volterra system defined by matrices (3) has a nontrivial invariant hyperbolic basic set in the level surface $\{h = E\}$.*

The strategy for this theorem's proof is to analyze Poincaré first return maps along heteroclinic cycles on the boundary of Δ^d, for the equivalent replicator model. Recall that, given a vector field X, a heteroclinic cycle of X is a circular sequence of orbits $\gamma_0, \gamma_1, \ldots, \gamma_{n+1} = \gamma_0$ of X for which there are hyperbolic singularities $p_0, p_1, \ldots, p_{n+1} = p_0$ such that $\gamma_i \subseteq W^u(p_i) \cap W^s(p_{i+1})$ for every $i = 0, 1, \ldots, n$. The use of cross-sections, and return maps, to analyze dynamics along heteroclinic cycles is an old tool going back to Poincaré. In the context of Game Dynamics there is already an extensive literature on the study of boundary heteroclinic cycles. See for instance [2] [3], [4],[9], [11] and [15]. One difficulty here is the fact that all heteroclinic cycles of this model go through the vertex corresponding to the environment, which is a topological saddle with zero eigenvalues. We take a global approach constructing an abstract piecewise-linear dynamical system that encapsulates the behaviour of a replicator system along heteroclinic cycles on the boundary of Δ^d. The technique developed is general, applying to any analytic vector field on a simple polyhedron, which is tangent to each of the polyhedron faces. These vector fields are those which induce complete flows, leaving all faces invariant. We shall refer them as complete vector fields on a polyhedron. The same procedure can also be applied to other ODE's studied in Game Dynamical Theory. Theorem A, addressed in section 12, follows as an application of this technique. Since we are not aware of such dynamical limit construction in the literature, we consider it to be the main novelty of this work.

Given a polyhedron Γ^d, we construct a non-compact piecewise-linear manifold $\mathcal{C}^*(\Gamma^d)$, which we call the dual cone of Γ^d, by gluing together sectors isomorphic to $\mathbb{R}^d_+$, one for each vertex of Γ^d, where the boundary faces of two sectors, corresponding to a pair of adjacent vertexes, are identified. Then we introduce a simple kind of dynamical systems on dual cones: piecewise-constant vector fields, that we refer as skeleton vector fields. For skeleton vector fields, we can as well define heteroclinic cycles, and first return maps to a given cross section. We characterize such Poincaré return maps along heteroclinic cycles. In section 5 we provide some useful formulas for explicit computation of Poincaré maps. Finally, in section 6 we introduce two kind of invariant structures for conservative, or Hamiltonian, skeleton dynamics: skeleton 1-forms and skeleton symplectic forms.

In section 7 we introduce rescaling co-ordinates, mapping a small neighbourhood of all edges and vertexes onto the dual cone of the polyhedron. We denote by $\mathcal{X}(\Gamma^d)$ the space of analytic complete vector fields on Γ^d. Notice that a vertex of Γ^d is always a singularity for any vector field $X \in \mathcal{X}(\Gamma^d)$. For typical vector fields, an edge of Γ^d consists of a single orbit flowing from a vertex to the other. Such edges will be referred as flowing edges. To each vector field $X \in \mathcal{X}(\Gamma^d)$ we associate a skeleton vector field χ on the dual cone $\mathcal{C}^*(\Gamma^d)$. The components of χ are related with the eigenvalues at the vertex singularities. For some degenerated singularities, with zero eigenvalues, the components of χ may also relate to higher order derivatives of the vector field along edge eigendirections. In section 8 we define a general subclass of $\mathcal{X}(\Gamma^d)$, formed by what we call regular vector fields. A vector field $X \in \mathcal{X}(\Gamma^d)$ is regular if its dynamics along the edges of Γ^d is, qualitatively, determined by χ. In section 9 we give the ingredients for Hamiltonian dynamics. We define a space $\mathcal{H}(\Gamma^d)$ of analytic functions on the polyhedron Γ^d, with a pole on each $d-1$-dimensional face, and a space $\Omega_0^2(\Gamma^d)$ of algebraic 2-forms on Γ^d with a double pole on each $d-2$-dimensional face of the polyhedron. Then, in proposition 9.3, we show that

Theorem B *The symplectic gradient of any function $h \in \mathcal{H}(\Gamma^d)$, with respect to an algebraic symplectic structure $\omega \in \Omega_0^2(\Gamma^d)$, is always equivalent to a regular vector field X on Γ^d.*

The function $h \in \mathcal{H}(\Gamma^d)$, and the symplectic structure $\omega \in \Omega_0^2(\Gamma^d)$, also determine associated skeleton objects on $\mathcal{C}^*(\Gamma^d)$, a skeleton 1-form λ, and a a skeleton 2-form Ω. In some natural sense the relation between X, h and ω is preserved by their skeleton versions χ, λ and Ω. In particular the skeleton vector field χ preserves the skeleton 1-form λ. Finally, in propositions 10.3, 10.4 and 10.6, we prove that

Theorem C *Given $X \in \mathcal{X}(\Gamma^d)$, $h \in \mathcal{H}(\Gamma^d)$ and $\omega \in \Omega_0^2(\Gamma^d)$, as in theorem B, let χ, λ and Ω be their respective skeletons. Then χ, λ and Ω are the rescaling limits of X, h and ω, respectively, along the edges of Γ^d.*

In section 11, corollary 11.2, we characterize the regularity of the replicator equation. Then, in proposition 11.4 we show that

Theorem D *Conservative Lotka-Volterra systems are equivalent to regular vector fields obtained from symplectic gradients of functions in $\mathcal{H}(\Delta^d)$ with respect to algebraic symplectic forms in $\Omega_0^2(\Delta^d)$.*

In sections 13 and 14 we prove the technical lemmas 9.1 and 9.2 needed in section 9. We also provide an intrinsic characterization of functions in $\mathcal{H}(\Gamma^d)$. The last section contains a few remarks and lemmas on Pfaffians, and determinants of skew symmetric matrices.

There are some obvious generalizations. Theorems B and C hold for compact manifolds with boundary, instead of simple polyhedrons. Recall that a compact manifold with boundary, say M^d of dimension d, is one which at every point is locally diffeomorphic to a model $\left(\mathbb{R}^k \times \mathbb{R}_+^{d-k}, 0\right)$, for some $0 \leq k \leq d$. The integer k is called the index of M^d at that point. The set of all points with index k is denoted by $\partial_k(M^d)$. This set is exactly the union of all interiors of k-dimensional faces of the manifold M^d. The analyticity assumption is mainly aesthetic. The theorems B and C can easily be generalized to finite smoothness class assumptions. Finally, we believe that algebraic symplectic structures in section 9, and proposition (10.6), could be replaced by 'algebraic Poisson structures', a more general concept.

We end this section with some ideas for further work. First, we should understand the dynamics of skeleton vector fields. Take a $d-1$-dimensional face Π of the dual cone $\mathcal{C}^*(\Gamma^d)$. Such faces are associated with edges of Γ^d. Any regular skeleton vector, on the dual cone, determines a first return map $\pi : \Pi \to \Pi$. The domain of π splits into a disjoint union of open convex cones, each corresponding to a heteroclinic cycle starting and ending with the edge associated with Π. The restriction of π to each of these components is a linear mapping with non zero determinant, and its image is another open convex cone. Thus, we can associate to π two partitions of Π, the domain and image partitions. A way of understanding skeleton dynamics would be to derive information on the geometry of these partitions, or on the dynamical character of π in any one of its domain components, from simple relations on the skeleton vector field data. This would enable us to easily construct skeleton vector fields with prescribed dynamical behaviour. Another interesting subject is the study of bifurcations in the context of skeleton vector fields. These skeleton bifurcations are caused by changes in the geometry and combinatorics of the domain and image partitions of π. Can we relate them to bifurcations of the underlying vector field? We believe so.

Given a skeleton vector field, can we realize it as the edge asymptotics of some regular vector field? This realization is important to construct examples with complicated dynamical behaviour. For general regular vector fields the answer to this problem is yes. For conservative skeleton vector fields, the answer is yes locally, in neighbourhood of the 1-dimensional skeleton of Γ^d. The global difficulty is that a skeleton symplectic structure can be realized as a global 2-form, but it is not clear if the realized form is non degenerated everywhere, in other words, if it is a symplectic structure on the interior of Γ^d. Perhaps we could avoid this difficulty dealing with Poisson structures instead of symplectic structures.

Part 1. SKELETON DYNAMICS ON POLYHEDRONS

2. Polyhedrons

Let Γ^d be a simple convex polyhedron of dimension d, where *simple* means that for each vertex of Γ^d there are exactly d incident edges. For the sake of simplicity we shall assume that Γ^d is embedded in the euclidean space $\mathbb{R}^d$ and it is defined by a finite number of linear inequalities. More precisely,

Definition 2.1. *A compact convex subset $\Gamma^d \subseteq \mathbb{R}^d$ is called a* simple convex polyhedron of dimension d *if there is a family of affine functions* $\left\{ f_i : \mathbb{R}^d \to \mathbb{R} \right\}_{i\in I}$ *such that:*

(a) $\Gamma^d = \bigcap_{i\in I} (f_i)^{-1}[0, +\infty)$,
(b) *given $J \subseteq I$, if there is a point $p \in \Gamma^d$ such that $f_j(p) = 0$, $\forall j \in J$, then the linear forms associated with the functions in $\{f_j\}_{j\in J}$ are linearly independent,*
(c) *for all $i \in I$, $\Gamma^d \cap (f_i)^{-1}(0) \neq \emptyset$.*

Then we say that the family $\{f_i\}_{i\in I}$ defines the polyhedron Γ^d .

Observe that condition (c) in the definition of a simple polyhedron is unnecessary because discarding those functions for which (c) does not hold we obtain a subfamily fulfilling all conditions (a)-(c).

Definition 2.2. *Let Γ^d be a simple polyhedron defined by a family $\{f_i\}_{i\in I}$. A non-empty subset $\rho \subseteq \Gamma^d$ is called a $r-$face if there are $d - r$ functions $f_{i_1}, \cdots, f_{i_{d-r}}$, in the defining family, such that $\rho = \Gamma^d \cap (f_{i_1})^{-1}(0) \cap \cdots \cap (f_{i_{d-r}})^{-1}(0)$. We denote by $K^r(\Gamma^d)$ the set of all $r-$faces of Γ^d.*

Elements in $V = K^0(\Gamma^d)$ are called *vertexes* and will be denoted by capital Latin letters like A, B or C; elements in $E = K^1(\Gamma^d)$ are named *edges* and will be denoted by small Greek letters like α, β and γ; finally, $(d-1)-$faces in $F = K^{d-1}(\Gamma^d)$ will be referred, shortly, as *faces* and will also be denoted by small creek letters like σ, σ' or ρ. Given a vertex $A \in V$ we denote by F_A, respectively E_A, the $d-$element sets of all faces in F, resp. edges in E, which contain the vertex A.

We will call *corner* of Γ^d to any triple $(A, \sigma, \gamma) \in V \times E \times F$ such that $\gamma \cap \sigma = \{A\}$. The set of all corners of Γ^d will be denoted by C. Notice that any pair of elements in a corner, vertex-face, vertex-edge or edge-face, determines uniquely the third one. Therefore, we shall sometimes refer to the corner (A, σ, γ) shortly as (A, γ) or (A, σ). An edge γ with vertex endpoints A and B determines two corners (A, σ, γ) and (B, ρ, γ), or simply (A, γ) and (B, γ), which we shall refer as the *end corners* of γ. The faces σ and ρ will be referred as the *opposite faces* to γ.

Given a family $\{f_i\}_{i \in I}$ defining a polyhedron Γ^d, by virtue of condition (c) in definition 2.1, the correspondence $i \mapsto \Gamma^d \cap (f_i)^{-1}(0)$ induces a one to one map between I and F. Thus, from now on we shall assume that the family defining a polyhedron Γ^d is always indexed in F in such a way that $\sigma = \Gamma^d \cap (f_\sigma)^{-1}(0)$.

3. The dual cone of a polyhedron

The dual cone $\mathcal{C}^*(\Gamma^d)$ of a polyhedron Γ^d can be described roughly as follows: Let Γ^* be the dual polyhedron, and $\partial\Gamma^*$ be its boundary, i.e. the union of all faces σ^k of Γ^* with dimension $k < d$. Take any point O outside the affine hyperplane spanned by Γ^*. Then $\mathcal{C}^*(\Gamma^d)$ is the cone through $\partial\Gamma^*$ with origin at O (see figure 1), i.e.

$$\mathcal{C}^*(\Gamma^d) = \{ \, O + \lambda\, v \, : \, v \in \partial\Gamma^* \, \} \, .$$

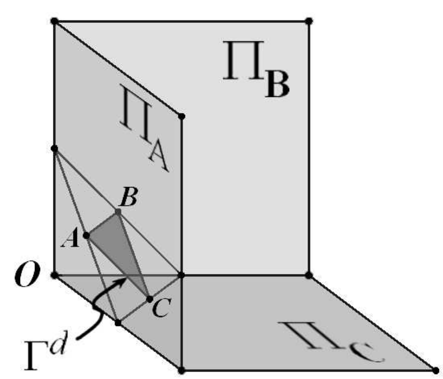

Figure 1. The dual cone $\mathcal{C}^*(\Gamma^d)$.

For each face σ of $\partial\Gamma$ let σ^* be its dual face of Γ^*. We shall say that $\Pi_\sigma = \{ \, O + \lambda\, v \, : \, v \in \sigma^* \, \}$ is a face of the dual cone. By duality,

(1) for each $r-$dimensional face σ of $\partial\Gamma$, the face Π_σ is a $(d-r)-$dimensional sector. Notice that σ^* has dimension $d-1-r$,
(2) Given faces ρ and σ of Γ^d, $\quad \rho \subseteq \sigma \iff \Pi_\sigma \subseteq \Pi_\rho$.
(3) $\mathcal{C}^*(\Gamma^d) = \bigcup_{A \in V} \Pi_A$.

We now give an alternative algebraic definition, embedding the dual cone of Γ^d into the Euclidean space $\mathbb{R}^F$, which will be more convenient for analytic treatment. First we introduce the dual cone faces as sectors in $\mathbb{R}^F$. Recall that $F = K^{d-1}(\Gamma^d)$ is the set of all $(d-1)-$dimensional faces of Γ^d.

Definition 3.1. *For each $r-$dimensional face ρ of Γ^d set*

$$\Pi_\rho = \left\{ u = (u_\sigma)_{\sigma \in F} \in \mathbb{R}_+^F \,:\, u_\sigma = 0 \text{ for all } \sigma \in F \text{ such that } \rho \not\subseteq \sigma \right\} ,$$

where $\mathbb{R}_+ = [0, +\infty[$. We denote by Π_ρ^{sp} the linear span of Π_ρ.

When $\rho = A$ is a vertex, because Γ^d is a simple polyhedron,

$$\Pi_A = \left\{ u \in \mathbb{R}_+^F \,:\, u_\sigma = 0 \text{ for all } \sigma \notin F_A \right\}$$

is a sector of dimension d. Given an edge $\gamma \in E$ with endpoints A and B, let (A, σ, γ) and (B, ρ, γ) be the corresponding end corners. Then we have

$$\Pi_\gamma = \{ u \in \Pi_A \,:\, u_\sigma = 0 \} = \{ u \in \Pi_B \,:\, u_\rho = 0 \} = \Pi_A \cap \Pi_B .$$

In other words, if A and B are connected by an edge γ, the d-dimensional sectors Π_A and Π_B intersect along the lower $(d-1)-$dimensional sector Π_γ. In general, it is not difficult to see that properties (1) and (2) above hold for the sectors Π_σ just defined. Therefore,

Definition 3.2. *the* dual cone of Γ^d *is defined to be* $\mathcal{C}^*(\Gamma^d) = \bigcup_{A \in V} \Pi_A$.

Given a face σ of Γ^d. we shall refer to the interior, resp. boundary, of Π_σ, denoted by intΠ, resp. $\partial\Pi$, to mean its topological interior, resp. topological boundary, relative to the linear span Π_σ^{sp}. We shall denote by $\mathcal{C}_r^*(\Gamma^d)$ the r-dimensional skeleton of the dual cone $\mathcal{C}^*(\Gamma^d)$, which is the union of all r-dimensional faces of $\mathcal{C}^*(\Gamma^d)$, i.e. the union of all faces Π_ρ with $\rho \in K^{d-r}(\Gamma^d)$. In the sequel we shall refer to the skeletons of $\mathcal{C}^*(\Gamma^d)$ with dimensions $d-1$ and $d-2$ respectively.

4. Piecewise linear flows on $\mathcal{C}^*(\Gamma^d)$

We shall call *skeleton vector field* on $\mathcal{C}^*(\Gamma^d)$ to any family $\chi = \left(\chi_\sigma^A\right)_{(A,\sigma) \in V \times F}$ such that $\chi_\sigma^A = 0$ for all $\sigma \notin F_A$. Notice this condition is equivalent to say that for each $A \in V$ the vector $\chi^A = \left(\chi_\sigma^A\right)_{\sigma \in F}$ is tangent to the sector Π_A, i.e., $\chi^A \in T\Pi_A = \Pi_A^{sp}$. A skeleton vector field is then as a piecewise constant vector field, which induces a piecewise linear flow on $\mathcal{C}^*(\Gamma^d)$. To understand this flow we are going to analyse the Poincaré return maps on faces of the $(d-1)-$skeleton $\mathcal{C}_{d-1}^*(\Gamma^d)$. We begin with some preliminary definitions. Given a vertex $A \in V$, we say that it is $\chi-$*attracting* if $\chi^A \in \Pi_A$; A is $\chi-$*repelling* if $-\chi^A \in \Pi_A$, and A is of *saddle type* if otherwise $\chi^A \notin \Pi_A$ and $-\chi^A \notin \Pi_A$. Notice that a vertex A is $\chi-$repelling, respectively $\chi-$attracting, if the sector Π_A is forward, respectively backward, invariant by the flow of the constant vector field χ^A.

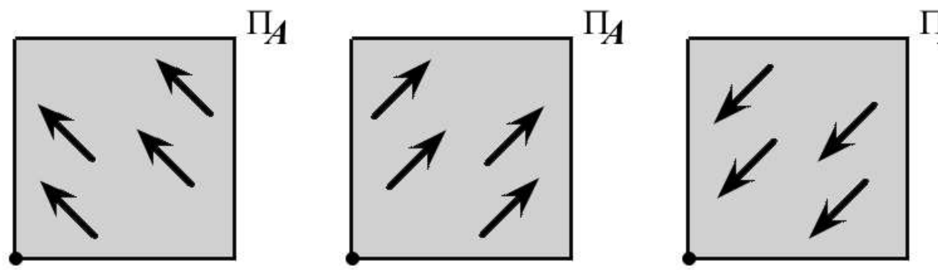

FIGURE 2. A saddle, a repelling and an attracting vertexes

The edges of Γ^d are also classified with respect to χ. Let $\gamma \in E$ be an edge with end corners (A, σ) and $(B, \rho) \in C$. We say that γ is of *saddle type*, or a *flowing edge*, if $\chi^A_\sigma \cdot \chi^B_\rho < 0$. When $\chi^A_\sigma < 0$ and $\chi^B_\rho > 0$ we shall write $A \xrightarrow{\gamma} B$. Otherwise, if $\chi^A_\sigma > 0$ and $\chi^B_\rho < 0$, then we write $A \xleftarrow{\gamma} B$. Flowing edges are naturally oriented. W shall sometimes refer to its source and target vertexes, or corners. We say that γ is a *neutral edge* if $\chi^A_\sigma = 0$ and $\chi^B_\rho = 0$, in which case we write $A \overset{\gamma}{=\!=} B$. In case we have $\chi^A_\sigma \cdot \chi^B_\rho > 0$, we say that γ is an *attracting*

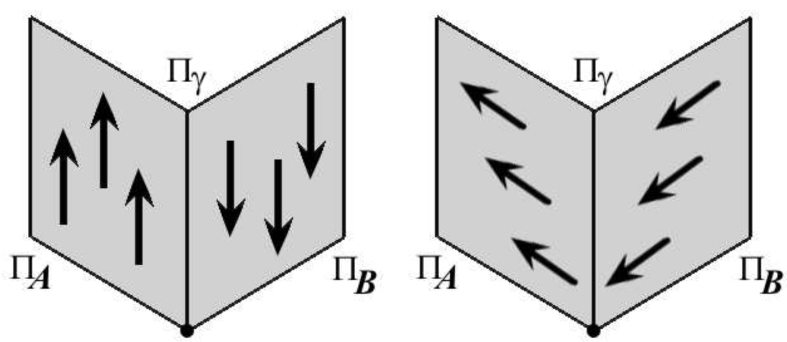

FIGURE 3. A neutral $A \overset{\gamma}{=\!=} B$, and a flowing edge $A \xleftarrow{\gamma} B$.

edge if $\chi^A_\sigma < 0$ and $\chi^B_\rho < 0$, in which case we write $A \overset{\gamma}{\rightarrow\!\leftarrow} B$. And, finally, we say that γ is a *repelling edge* if $\chi^A_\sigma > 0$ and $\chi^B_\rho > 0$. We write then $A \overset{\gamma}{\leftarrow\!\rightarrow} B$. A vertex $A \in V$ is $\chi-$attracting, respectively $\chi-$repelling, if and only if for all edges $\gamma \in E_A$, with endpoints $A, B \in V$, either $A \overset{\gamma}{=\!=} B$ or $A \xleftarrow{\gamma} B$, respectively either $A \overset{\gamma}{=\!=} B$ or $A \xrightarrow{\gamma} B$. Notice that these four cases are not

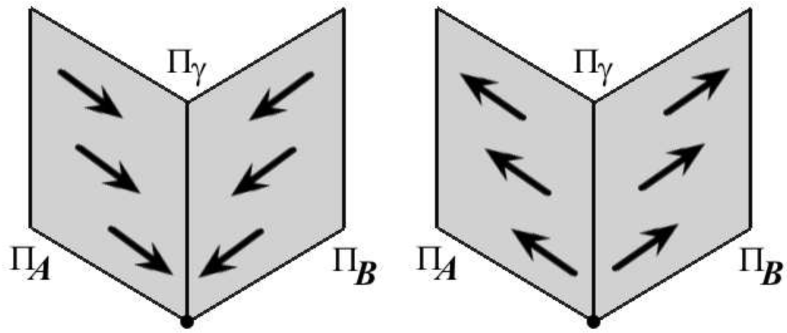

FIGURE 4. An attracting $A \overset{\gamma}{\rightarrow\leftarrow} B$, and a repelling edge $A \overset{\gamma}{\leftarrow\rightarrow} B$.

exhaustive. We say that an edge γ is *defined* if it is either neutral, flowing, attracting or repelling. Otherwise, when $\chi^A_\sigma \cdot \chi^B_\rho = 0$, but $\chi^A_\sigma \neq 0$ or $\chi^B_\rho \neq 0$ we say that γ is *undefined*.

Definition 4.1. *We say that a skeleton field χ is:*

(i) regular of type I *if all vertexes and edges are of saddle type.*
(ii) regular of type II *if all vertexes are of saddle type, and all edges are either neutral or of saddle type.*
(iii) regular of type III *if all edges are either neutral or of saddle type.*
(iv) regular of type IV *if all edges are defined.*

Obviously, we have the following chain of regularity relations

$$\text{type I} \;\Rightarrow\; \text{type II} \;\Rightarrow\; \text{type III} \;\Rightarrow\; \text{type IV}\,.$$

We assume throughout the rest of this section that χ is a skeleton vector field regular of type IV.

The flow of the skeleton vector field χ is encapsulated in the mappings

$$\pi^\chi,\, \pi^{-\chi} : \Sigma^{\pm\chi} \subseteq \mathcal{C}^*_{d-1}(\Gamma^d) \rightarrow \mathcal{C}^*_{d-1}(\Gamma^d)\,.$$

The first mapping, π^χ, is defined over the domain $\Sigma^\chi = \cup_\gamma \operatorname{int}\Pi_\gamma$, where the union is taken over all $\chi-$flowing edges $\gamma \in E$, whose target vertex is of saddle type. Analogously, $\pi^{-\chi}$ is defined over the union $\Sigma^{-\chi} = \cup_\gamma \operatorname{int}\Pi_\gamma$ of all $\chi-$flowing edges $\gamma \in E$ whose source vertex is of saddle type. For a given point $u \in \Sigma^\chi$ there is exactly one face Π_γ containing u in its interior. Since γ is a $\chi-$flowing edge, denoting by (A, σ), resp. (B, ρ), the source, respectively target, corner of γ, χ^A points outward Π_A, while χ^B points inward Π_B, on both sides of Π_γ. By definition of Σ^χ, the vertex B is of saddle type. Therefore, the half line $\{u + t\,\chi^B \;:\; t > 0\}$ intersects the boundary $\partial\Pi_B$ at a single point

u_1, and we set $\pi^\chi(u) = u_1$. Analogously, because A is of saddle type, the half line $\{u - t\,\chi^A \,:\, t > 0\}$ intersects the boundary $\partial\Pi_A$ at another single point u_{-1}, and we set $\pi^{-\chi}(u) = u_{-1}$.

These maps are inverse of each other in the sense that $\pi^\chi \circ \pi^{-\chi}$ and $\pi^{-\chi} \circ \pi^\chi$ are both equal to the identity, respectively over the domains $\Sigma^{-\chi}$ and Σ^χ.

We say that a point $u \in \Sigma^\chi$ *has finite forward orbit* if for some $n \in \mathbb{N}$, we have $(\pi^\chi)^i(u) \in \Sigma^\chi$, for all $i = 0, 1, \cdots, n-1$, but $(\pi^\chi)^n(u) \notin \Sigma^\chi$. Similarly, we say that a point $u \in \Sigma^{-\chi}$ *has finite backward, orbit* if for some $n \in \mathbb{N}$, we have $(\pi^{-\chi})^i(u) \in \Sigma^{-\chi}$, for all $i = 0, 1, \cdots, n-1$, but $(\pi^{-\chi})^n(u) \notin \Sigma^{-\chi}$. The sequences $\{\, (\pi^\chi)^i(u) \,:\, i = 0, 1, \cdots n-1 \,\}$ and $\{\, (\pi^{-\chi})^i(u) \,:\, i = 0, 1, \cdots n-1 \,\}$ are then called the *forward and backward orbits* of u.

Otherwise, if for all $n \in \mathbb{N}$, $(\pi^\chi)^n(u) \in \Sigma^\chi$, resp. $(\pi^{-\chi})^n(u) \in \Sigma^{-\chi}$, we say that u *has infinite forward, resp. backward, orbit.* The sequences $\{\, (\pi^\chi)^n(u) \,:\, n \in \mathbb{N} \,\}$ and $\{\, (\pi^{-\chi})^n(u) \,:\, n \in \mathbb{N} \,\}$, are called the *forward and backward orbits* of u.

We shall call *first return map* of χ to $\pi^\chi_{1^{\text{st}}} : \Sigma^\chi_{1^{\text{st}}} \subseteq \mathcal{C}^*_{d-1}(\Gamma^d) \to \mathcal{C}^*_{d-1}(\Gamma^d)$, the mapping with the first return along χ to each $(d-1)-$dimensional face Π_γ. The domain $\Sigma^\chi_{1^{\text{st}}}$ of this map is the set of all points $u \in \Sigma^\chi$ in some face Π_γ, associated with a $\chi-$flowing edge $\gamma \in E$, whose forward orbit eventually returns to the same face Π_γ. Assuming $u \in \operatorname{int} \Pi_\gamma$, of course we have

$$\pi^\chi_{1^{\text{st}}}(u) = \min\{\, n \geq 1 \,:\, (\pi^\chi)^n(u) \in \Pi_\gamma \,\} \,.$$

We define the *maximal invariant sets* $\Sigma^+_\infty(\chi)$, respectively $\Sigma^-_\infty(\chi)$, to be the set of all points in Σ with infinite forward, respectively backward, orbits. In other words

$$\Sigma^+_\infty(\chi) = \bigcap_{n\in\mathbb{N}} (\pi^\chi)^{-n}\,(\Sigma^\chi) \quad \text{and} \quad \Sigma^-_\infty(\chi) = \bigcap_{n\in\mathbb{N}} (\pi^{-\chi})^{-n}\left(\Sigma^{-\chi}\right) \,.$$

We also define $\Sigma_\infty(\chi) = \Sigma^+_\infty(\chi) \cap \Sigma^-_\infty(\chi)$.

An edge sequence $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ is called a *chain* of χ if there is a sequence of vertexes $(A_0, A_1, \cdots, A_n, A_{n+1})$ such that for all $i = 0, 1, \cdots, n$, $A_i \xrightarrow{\gamma_i} A_{i+1}$. We will also refer to a chain by the sequence of its vertexes, and write $\xi = (A_0, A_1, \cdots, A_n, A_{n+1})$. The integer n will be called the *length* of the chain ξ. Notice that the length of ξ is equal to the number of edges minus one, or else the number of vertexes minus two. Given a chain $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ we say that an orbit segment $(\, u, \pi^\chi(u), (\pi^\chi)^2(u), \cdots, (\pi^\chi)^n(u) \,)$ has *itinerary* ξ if for all $i = 0, 1, \cdots, n$, $(\pi^\chi)^i(u)$ is interior to the face Π_{γ_i}. Given any chain

$\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ we define Π_ξ as the set of all points interior to Π_{γ_0} with itinerary, up to the n-th iterate, equal to ξ, i.e.

$$\Pi_\xi = \{\, u \in \Pi_{\gamma_0} \,:\, (\pi^\chi)^i(u) \in \operatorname{int} \Pi_{\gamma_i} \quad \forall\, i = 0, 1, \cdots, n \,\} \,.$$

This set is the domain of the $\chi-$*Poincaré map along* ξ

$$\pi^\chi_\xi : \Pi_\xi \subseteq \Pi_{\gamma_0} \to \Pi_{\gamma_n} \,, \quad \text{defined by} \quad \pi^\chi_\xi = (\pi^\chi)^n \,.$$

Given two chains $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ and $\xi' = (\gamma'_0, \gamma'_1, \cdots, \gamma'_m)$, we say that ξ *connects with* ξ' if $\gamma_n = \gamma'_0$, and in this case define $\xi * \xi'$ to be the *concatenation* of the two sequences

$$\xi * \xi' = (\gamma_0, \gamma_1, \cdots, \gamma_n, \gamma'_1, \cdots, \gamma'_m) \,.$$

Notice that the length of $\xi * \xi'$ is the sum of both lengths of ξ and ξ'. It is quite obvious, from the definitions, that

Proposition 4.1. *Given two chains ξ and ξ', such that ξ connects with ξ',*

$$\Pi^\chi_{\xi * \xi'} = \{\, u \in \Pi_\xi \,:\, \pi^\chi_\xi(u) \in \Pi_{\xi'} \,\} \,,$$

and over this domain $\pi^\chi_{\xi * \xi'} = \pi^\chi_{\xi'} \circ \pi^\chi_\xi$.

We shall denote by $\Lambda^n = \Lambda^n(\chi)$ the set of all chains of length n, and by $\Lambda^n_\gamma = \Lambda^n_\gamma(\chi)$ the subset of all chains $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ with $\gamma_0 = \gamma$. Next proposition comes from the fact that every finite orbit segment of χ must have a definite chain ξ as its itinerary,

Proposition 4.2. *The domain of the composition mapping $(\pi^\chi)^n$ is equal to the disjoint union of all domains Π_ξ with $\xi \in \Lambda^n(\chi)$.*

A chain $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ is called a *cycle* when $\gamma_n = \gamma_0$. We denote by $\Lambda^\odot(\chi)$ the set of all cycles, and by $\Lambda^\odot_\gamma(\chi)$ the subset of all cycles $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ with $\gamma_0 = \gamma = \gamma_n$ but $\gamma_i \neq \gamma$ for every $i = 1, \cdots, n-1$. Then

Proposition 4.3. *Given a $\chi-$flowing edge $\gamma \in E$, the domain of $\pi^\chi_{1^{st}}$ restricted to the face Π_γ is equal to the disjoint union of all domains Π_ξ with $\xi \in \Lambda^\odot_\gamma(\chi)$. For each such ξ, $\pi^\chi_{1^{st}}(u) = \pi^\chi_\xi(u) \;\; \forall u \in \Pi_\xi$.*

5. Computing a piecewise linear flow

In this section we give formulas to compute explicitly the Poincaré maps along a chain $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$. With this purpose we introduce a family of linear maps $L^\chi_\gamma : \mathbb{R}^F \to \mathbb{R}^F$, one for each edge $\gamma \in E$. If γ is $\chi-$neutral, set $L^\chi_\gamma = \mathrm{Id}$ to be the identity map on $\mathbb{R}^F$. Otherwise, if γ is a $\chi-$flowing edge with source corner $(A, \sigma_0) \in C$, we define $L^\chi_\gamma : \mathbb{R}^F \to \mathbb{R}^F$ setting

$$L^\chi_\gamma(u_\sigma)_{\sigma\in F} = \left(u_\sigma - \frac{\chi^A_\sigma}{\chi^A_{\sigma_0}}\, u_{\sigma_0} \right)_{\sigma \in F} \tag{5}$$

Next proposition characterizes the Poincaré maps π^χ_ξ along chains $\xi = (\mu, \gamma)$ as restrictions of such linear endomorphisms.

Proposition 5.1. *Given chain $\xi = (\mu, \gamma) \in \Lambda^1(\chi)$, let $(A, \rho_0) \in C$ be the target corner of μ, and $(A, \sigma_0) \in C$ be the source corner of γ. The linear endomorphism L^χ_γ maps the linear span of Π_A onto the linear span of Π_γ, inducing an isomorphism between the linear spans of Π_μ and Π_γ. The domain Π_ξ, of the Poincaré map π^χ_ξ, is the open convex cone of all $u = (u_\sigma)_{\sigma\in F} \in int\,\Pi_\mu$ such that*

$$\frac{u_\sigma}{u_{\sigma_0}} > \frac{\chi^A_\sigma}{\chi^A_{\sigma_0}} \quad \text{for all faces } \sigma \supseteq \gamma \cup \mu \quad \text{with} \quad \chi^A_\sigma < 0\,.$$

Furthermore, $\pi^\chi_\xi = L^\chi_\gamma$ over Π_ξ.

Proof. By definition $v = L^\chi_\gamma(u)$ with

$$v_\sigma = u_\sigma - \frac{\chi^A_\sigma}{\chi^A_{\sigma_0}}\, u_{\sigma_0} = u_\sigma - \frac{u_{\sigma_0}}{\chi^A_{\sigma_0}}\, \chi^A_\sigma\,, \qquad \text{for all} \quad \sigma \in F\,.$$

This implies that

$$v \;=\; u + \tau\, \chi^A \quad \text{with} \quad \tau = -\frac{u_{\sigma_0}}{\chi^A_{\sigma_0}}\,.$$

Given $v \in \Pi^{sp}_A$, one has $u_\sigma = 0$ for all $\sigma \in F$ with $A \notin \sigma$. Let us prove that $v = u + \tau\,\chi^A \in \Pi^{sp}_\gamma$. If $\sigma \in F$ is such that $\gamma \not\subseteq \sigma$ then either $A \notin \sigma$, and in this case $u_\sigma = \chi^A_\sigma = 0$ which implies that $v_\sigma = 0$, or else $A \in \sigma$, which implies that $\sigma = \sigma_0$. In this case $v_{\sigma_0} = u_{\sigma_0} - \frac{\chi^A_{\sigma_0}}{\chi^A_{\sigma_0}}\, u_{\sigma_0} = 0$. Thus $v = L^\chi_\gamma(u) \in \Pi^{sp}_\gamma$, which shows that L^χ_γ maps Π^{sp}_A into Π^{sp}_γ.

If $u \in \Pi^{sp}_\mu$ then $u \in \Pi^{sp}_A$ and $u_{\rho_0} = 0$. Thus $v_{\rho_0} = -\frac{u_{\sigma_0}}{\chi^A_{\sigma_0}}\, \chi^A_{\rho_0}$, which implies that $\frac{v_{\rho_0}}{\chi^A_{\rho_0}} = -\frac{u_{\sigma_0}}{\chi^A_{\sigma_0}}$. Therefore

$$v = u - \frac{u_{\sigma_0}}{\chi^A_{\sigma_0}}\, \chi^A \quad \Longleftrightarrow \quad u = v - \frac{v_{\rho_0}}{\chi^A_{\rho_0}}\, \chi^A\,,$$

which shows that the restriction of L^χ_γ to Π^{sp}_μ is an isomorphism onto Π^{sp}_γ.

Given $u \in \operatorname{int}\Pi_\mu$, we have $u_{\sigma_0} > 0$, and so $\tau = -u_{\sigma_0}\,(\chi^A_{\sigma_0})^{-1} > 0$, since $\chi^A_{\sigma_0} < 0$. Therefore $L^\chi_\gamma(u) = u + \tau\,\chi^A$ lies in the intersection of the half-line $\{u + t\,\chi^A \,:\, t > 0\}$ with Π^{sp}_γ. Thus $\pi^\chi(u) = L^\chi_\gamma(u)$ as long as $\pi^\chi(u)$ or $L^\chi_\gamma(u)$ belong to $\operatorname{int}\Pi_\gamma$. In particular the domain Π_ξ is the set of all $u \in \operatorname{int}\Pi_\mu$ such that $L^\chi_\gamma(u) \in \operatorname{int}\Pi_\gamma$, this last condition being equivalent to say that

$$u_\sigma - \frac{\chi^A_\sigma}{\chi^A_{\sigma_0}}\, u_{\sigma_0} > 0 \quad \text{for all} \quad \sigma \supseteq \gamma\,,$$

or else

$$\frac{u_\sigma}{u_{\sigma_0}} > \frac{\chi^A_\sigma}{\chi^A_{\sigma_0}} \quad \text{for all} \quad \sigma \supseteq \gamma \qquad (*).$$

If $\chi^A_\sigma \geq 0$ then $\frac{\chi^A_\sigma}{\chi^A_{\sigma_0}} \leq 0$ and inequality $(*)$ is trivially fulfilled. We may, therefore, consider in $(*)$ only those faces σ for which $\chi^A_\sigma < 0$. But the only face $\sigma \supseteq \gamma$ that does not contain μ is ρ_0, for which one has $\chi^A_{\rho_0} > 0$. Thus, it is enough to check condition $(*)$ for σ such that $\sigma \supseteq \gamma \cup \mu$ and $\chi^A_\sigma < 0$. □

Let us describe now the restriction of the Poincaré map π^χ to an inner face Π_μ. The domain Π_μ is essentially divided as a disjoint union of convex cone regions, formed by points which are mapped together to the same inner face Π_γ.

Proposition 5.2. *Let μ be a $\chi-$flowing edge such that all edges incident with its target vertex are either neutral or of saddle type. For instance, this condition holds if χ is a skeleton vector field which is regular of type II. Then the interior of an inner face Π_μ is equal, up to a finite union of codimension-one hyperplanes, to the disjoint union of all cone regions Π_ξ with $\xi \in \Lambda^1(\chi)$.*

Proof. Let μ be a flowing edge with target vertex A. Consider the set F^χ_μ of all faces σ such that $\mu \subseteq \sigma$ and $\chi^A_\sigma < 0$. For each pair of distinct faces σ, $\sigma' \in F^\chi_\mu$ consider the hyperplane in Π_μ defined by

$$\frac{u_\sigma}{u_{\sigma'}} = \frac{\chi^A_\sigma}{\chi^A_{\sigma'}} \quad \Longleftrightarrow \quad -\frac{u_\sigma}{\chi^A_\sigma} = -\frac{u_{\sigma'}}{\chi^A_{\sigma'}}\,.$$

These hyperplanes through the origin divide $\operatorname{int}\Pi_\mu$ into a finite union of convex cone regions. In each one of these regions there is a face $\sigma_0 \in F^\chi_\mu$ such that

$$-\frac{u_{\sigma_0}}{\chi^A_{\sigma_0}} < -\frac{u_\sigma}{\chi^A_\sigma} \quad \text{for all other} \quad \sigma \in F^\chi_\mu\,.$$

Therefore, by proposition 5.1, this region coincides with the cone domain Π_ξ, where $\xi = (\mu, \gamma)$ and γ is the edge associated with the corner (A, σ_0). Notice that γ is a $\chi-$flowing edge. This follows because $\chi^A_{\sigma_0} < 0$ and by assumption γ can not be $\chi-$neutral. □

Corollary 5.3. *Given a chain $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n) \in \Lambda^n(\chi)$, the domain Π_ξ is a convex cone, which can be obtained as*

$$\Pi_\xi = \bigcap_{i=1}^{n} (L_{i-1})^{-1} \Pi_{(\gamma_{i-1},\gamma_i)} \ ,$$

where $L_0 = Id$ and $L_i = L^\chi_{\gamma_i} \circ \cdots \circ L^\chi_{\gamma_1}$ for every $i \geq 1$.
Furthermore,

$$\pi^\chi_\xi = L^\chi_{\gamma_n} \circ \cdots \circ L^\chi_{\gamma_1} \qquad over \quad \Pi_\xi \tag{6}$$

Proof. Notice that chain ξ can be decomposed as the concatenation of n length-one chains, $\xi = \xi_1 * \xi_2 * \cdots * \xi_n$, where $\xi_i = (\gamma_{i-1}, \gamma_i)$ for $i = 1, \cdots, n$. Use then propositions 4.1 and 5.1. □

Corollary 5.4. *Let χ be a skeleton vector field which is regular of type II, γ be any $\chi-$flowing edge, and $n \in \mathbb{N}$. Then the interior of Π_γ is equal to:*

1. *the disjoint union of all open cone regions Π_ξ, where ξ varies through length n chains starting with γ, i.e. $\xi \in \Lambda^n_\gamma(\chi)$, up to a finite union of codimension-one hyperplanes,*
2. *the disjoint union of all open cone regions Π_ξ, where ξ varies through cycles starting and ending with γ, i.e. $\xi \in \Lambda^{\odot}_\gamma(\chi)$, up to a countable union of codimension-one hyperplanes.*

Proof. Regular type II skeleton vector fields do not have attracting nor repelling vertexes. They also don't have attracting or repelling edges. If χ is regular of type II then the mappings $(\pi^\chi)^n$ and $\pi^\chi_{1^{\rm st}}$, on Π_γ, have domains equal to $\mathrm{int}(\Pi_\gamma)$ minus the set of all those points whose forward orbits end up in $\mathcal{C}^*_{d-2}(\Gamma^d)$. Both items of this corollary follow from proposition 5.2. □

For computational purposes it is convenient to work with the linear endomorphisms (5) instead of the maps $\pi^\chi_{(\mu,\gamma)}$, whose domain and range depend on μ and γ. The map L_γ is given by a square matrix $I - M_\gamma$, where

$$M_\gamma = \left(\frac{\chi^A_\sigma}{\chi^A_{\sigma_0}} \, \delta_{\sigma_0,\sigma'} \right)_{(\sigma,\sigma')\in F\times F}$$

is a matrix with a single non zero column. The expression $\delta_{\rho,\sigma'}$ stands for the Kronecker symbol.

Proposition 5.5. *Let $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n) \in \Lambda^n(\chi)$. Setting $M_i = M_{\gamma_i}$ for each i, the map π^χ_ξ is represented by the product of matrices*

$$(I - M_n) \cdots (I - M_1) = I + \sum_{1\leq i_1 < \cdots < i_k \leq n} (-1)^k \, M_{i_k} \cdots M_{i_1},$$

where the sum above is taken over all sequences $1 \leq i_1 < \cdots < i_k \leq n$. *The domain* Π_ξ *can be characterized as the interior, in* Π_{γ_0}, *of the set of all* $u \in \Pi_{\gamma_0}$ *such that for each* $i = 1, \cdots, n$,

$$(I - M_i) \cdots (I - M_1)\, u \in \mathbb{R}^F_+.$$

Let $A_0, A_1, \cdots, A_n, A_{n+1}$ be the sequence of vertexes determined by the chain $\xi \in \Lambda^n(\chi)$, and $(A_0, \sigma_0, \gamma_0)$, $(A_1, \sigma_1, \gamma_1)$, $\cdots$, $(A_n, \sigma_n, \gamma_n)$ be the corresponding sequence of corners. It follows that for each $u \in \Pi_\xi \subseteq \Pi_{\gamma_0}$, the image $\pi^\chi_\xi(u) = u'$ is given by

$$u'_\sigma = u_\sigma + \sum_{k=1}^{n} (-1)^k \sum_{1 \leq i_1 < \cdots < i_k \leq n} \frac{\chi_\sigma^{A_{i_k}}\, \chi_{\sigma_{i_k}}^{A_{i_{k-1}}} \cdots \chi_{\sigma_{i_3}}^{A_{i_2}}\, \chi_{\sigma_{i_2}}^{A_{i_1}}}{\chi_{\sigma_{i_k}}^{A_{i_k}}\, \chi_{\sigma_{i_{k-1}}}^{A_{i_{k-1}}} \cdots \chi_{\sigma_{i_2}}^{A_{i_2}}\, \chi_{\sigma_{i_1}}^{A_{i_1}}}\, u_{\sigma_{i_1}}, \tag{7}$$

where $u' = (u'_\sigma)_{\sigma \in F}$ and in the sum above all nonzero summands are such that

$$A_0 \in \sigma_{i_1},\ A_{i_1} \in \sigma_{i_2},\ \cdots,\ A_{i_{k-1}} \in \sigma_{i_k}, \quad \text{and} \quad A_{i_k}, A_n, A_{n+1} \in \sigma.$$

6. Piecewise linear invariant structures on $\mathcal{C}^*(\Gamma^d)$

A continuous function $\lambda : \mathcal{C}^*(\Gamma^d) \to \mathbb{R}$ whose restriction to each sector Π_A is linear will be called a *skeleton* $1-$*form on* $\mathcal{C}^*(\Gamma^d)$. The function λ is determined by a family $\{\lambda^A\}_{A \in V}$ of linear $1-$forms $\lambda^A : \Pi^{sp}_A \to \mathbb{R}$ such that for all pairs of distinct vertexes $A, B \in V$, with common inner face $\Pi_\gamma = \Pi_A \cap \Pi_B$, we have ${\lambda^A}_{|\Pi^{sp}_\gamma} = {\lambda^B}_{|\Pi^{sp}_\gamma}$. Similarly, we call *skeleton* $2-$*form on* $\mathcal{C}^*(\Gamma^d)$ to any family $\{\omega^A\}_{A \in V}$ of linear $2-$forms $\omega^A : \Pi^{sp}_A \times \Pi^{sp}_A \to \mathbb{R}$ such that for all pairs of distinct vertexes $A, B \in V$, with common inner face $\Pi_\gamma = \Pi_A \cap \Pi_B$, we have ${\omega^A}_{|\Pi^{sp}_\gamma \times \Pi^{sp}_\gamma} = {\omega^B}_{|\Pi^{sp}_\gamma \times \Pi^{sp}_\gamma}$. Finally, we say that $\{\omega^A\}_{A \in V}$ is a *skeleton symplectic structure* if for each $A \in V$, ω^A is non-degenerated.

Given a skeleton vector field $\chi = (\chi^A_\sigma)$, we say that a skeleton $1-$form λ is $\chi-$invariant, or in other words that χ preserves λ, if and only if $\lambda^A(\chi^A) = 0$, for all $A \in V$. We say that χ is the *symplectic gradient of* λ *with respect to* $\{\omega^A\}$, if $\omega^A(\chi^A, u) = \lambda^A(u)$ for all $u \in \Pi^{sp}_A$ and $A \in V$. Clearly, if $\{\omega^A\}$ is a skeleton symplectic structure, given any skeleton $1-$form $\lambda = \{\lambda^A\}$, there is a unique such vector field.

A little symplectic linear geometry shows that:

Proposition 6.1. *If* χ *preserves the* $1-$*form* λ, *then* $\lambda \circ \pi^{\pm\chi} = \pi^{\pm\chi}$.

Proposition 6.2. *If* χ *is the symplectic gradient of* λ *w.r.t.* $\{\omega^A\}$ *then* λ *is* $\chi-$*invariant and the mappings* $\pi^{\pm\chi}$ *are symplectic for the induced symplectic structures on the level sets* $\{u \in \Sigma^{\pm\chi} : \lambda(u) = c\}$.

With a simple argument one can see that every skeleton 1 or 2−form can be uniquely extended to a global linear form on $\mathbb{R}^F$. More precisely,

Proposition 6.3. *Given a skeleton 1−form $\lambda : \mathcal{C}^*(\Gamma^d) \to \mathbb{R}$ there is a unique vector $(\lambda_\sigma)_{\sigma\in F} \in \mathbb{R}^F$ such that*

$$\lambda(u) = \sum_{\sigma\in F} \lambda_\sigma \, u_\sigma \ , \qquad \forall\, u \in \mathcal{C}^*(\Gamma^d) \ .$$

Proposition 6.4. *Given a skeleton 2−form $\{\omega^A\}_{A\in V}$ on $\mathcal{C}^*(\Gamma^d)$ there is a unique skew symmetric matrix $\Omega = (\,\omega_{\sigma_1,\sigma_1}\,) \in \mathbb{R}^{F\times F}$ such that*

(1) $\omega_{\sigma_1\,\sigma_2} = 0$ *whenever* $\sigma_1 \cap \sigma_2 = \emptyset$,
(2) *for all* $u = (u_\sigma)_{\sigma\in F}$, $v = (v_\sigma)_{\sigma\in F}$ *in* Π_A,

$$\omega^A\,(u,v) = \sum_{(\sigma_1,\sigma_2)\in F\times F} \omega_{\sigma_1\,\sigma_2}\, u_{\sigma_1}\, v_{\sigma_2} \ .$$

From now on we shall also refer to any skeleton 1 or 2−form by its coefficient matrices $(\lambda_\sigma)_{\sigma\in F} \in \mathbb{R}^F$ and $\Omega = (\omega_{\sigma_1,\sigma_1}) \in \mathbb{R}^{F\times F}$.

Part 2. SKELETON ASYMPTOTICS OF REGULAR VECTOR FIELDS ON POLYHEDRONS

7. Rescaling polyhedrons onto their dual cones

Assume that a family $\{f_\sigma\}_\sigma$ defining Γ^d is given. For each $A \in V$ define

$$N_A = \left\{ p \in \Gamma^d \ : \ f_\sigma(p) \le 1 \quad \text{for all} \quad \sigma \in F_A \,\right\} \ .$$

Multiplying all functions f_σ by some positive large constant we may assume the sets N_A to be pairwise disjoint, each set N_A being a small neighbourhood of A in Γ^d. The boundary of N_A consists of exterior faces in $\partial\Gamma^d$ and the *inner faces* Σ^A_γ defined below, one for each $\gamma \in E_A$. Given $\gamma \in E_A$ let $(A,\sigma,\gamma) \in C$ be the corresponding corner. Then we set

$$\Sigma^A_\gamma = \left\{ p \in \Gamma^d \ : \ f_\sigma(p) = 1 \quad \text{and} \quad f_{\sigma'}(p) \le 1 \quad \text{for all } \sigma' \supseteq \gamma \,\right\} \ . \tag{8}$$

Given an edge $\gamma \in E$ with end vertexes A and B we shall denote by τ_γ either one of the translation maps $\tau_\gamma : \Sigma^A_\gamma \to \Sigma^B_\gamma$, or $\tau_\gamma : \Sigma^B_\gamma \to \Sigma^A_\gamma$, defined by the fact that $p - \tau_\gamma(p)$ is parallel to γ for all p in Σ^A_γ or Σ^B_γ.

Consider now the following sequence of functions $h_n : (0,+\infty) \to \mathbb{R}$

$$h_1(x) = -\log x \quad \text{and} \quad h_n(x) = -\frac{1}{n-1}\left(1 - \frac{1}{x^{n-1}}\right) \quad \text{for } \ n \ge 2 \ . \tag{9}$$

Proposition 7.1. *For each $n \ge 1$ we have*

(1) $(h_n)'(x) = -x^{-n}$, *for all* $x > 0$,

(2) $h_n(1) = 0$ *and* $\lim_{x\to 0^+} h_n(x) = +\infty$,
(3) h_n *maps* $(0,1]$ *diffeomorphically onto* $[0,+\infty)$.

We shall call *order function* to any integer valued map $\nu : F \to \{1,2,3,\cdots\}$. Given any such function ν, and any $\epsilon > 0$ we define the *rescaling co-ordinates of order* ν to be $u = \Psi_\epsilon^\nu(p)$, where the mapping $\Psi_\epsilon^\nu : \Gamma^d \setminus \partial\Gamma^d \to \mathcal{C}^*(\Gamma^d)$ is defined as

$$u = \Psi_\epsilon^\nu(p) \quad \Leftrightarrow \quad (u_\sigma)_{\sigma\in F} = \left(\epsilon\, h_{\nu(\sigma)}\left(f_\sigma(p)\right) \right)_{\sigma\in F} \ . \tag{10}$$

These co-ordinates map a neighbourhood of the polyhedron 1-dimensional skeleton (vertexes and edges) onto the dual cone $\mathcal{C}^*(\Gamma^d)$. We also refer to the one-to-one affine map $\psi : \mathbb{R}^d \to \mathbb{R}^F$,

$$x = \psi(p) = (f_\sigma(p))_{\sigma\in F} \ , \tag{11}$$

whose image is an affine subspace $S \subseteq \mathbb{R}^F$ of dimension d, as a global system of co-ordinates. These co-ordinates chart Γ^d onto the convex polyhedron $\overline{\Gamma}^d = \{x \in S \,:\, x_\sigma \geq 0\,,\ \forall\, \sigma \in F\}$.

It is not difficult to see that

Proposition 7.2. *The maps* Ψ_ϵ^ν *satisfy:*

(1) *The restriction of* Ψ_ϵ^ν *to* $N_A \setminus \partial\Gamma^d$ *is one-to-one onto* Π_A.
(2) *For any edge* γ *with endpoints* A, $B \in V$,
$$\Psi_\epsilon^\nu(\Sigma_\gamma^A) = \Psi_\epsilon^\nu(\Sigma_\gamma^B) = \Pi_\gamma \quad \text{and} \quad (\Psi_\epsilon^\nu)^{-1}_{|\Sigma_\gamma^B} \circ (\Psi_\epsilon^\nu)_{|\Sigma_\gamma^A} = \tau_\gamma \ .$$
(3) *Given a curve* $p(t)$ *in* Γ^d *which in the co-ordinates (10) and (11) is respectively given by* $u(t)) = (u_\sigma(t))_{\sigma\in F}$ *and* $x(t) = (x_\sigma(t))_{\sigma\in F}$,
$$(u_\sigma)'(t) = -\epsilon\, \frac{(x_\sigma)'(t)}{x_\sigma(t)^\nu} = -\epsilon\, \frac{(df_\sigma)_p\, p'(t)}{f_\sigma\left(p(t)\right)^\nu} \ ,$$
where $\nu = \nu(\sigma)$ *and* $\sigma \in F$.
(4) *We can define inverse maps* $(\Psi_\epsilon^\nu)^{-1} : \mathcal{C}^*(\Gamma^d) \to \bigcup_{A\in F} N_A$ *which are single valued on* $\mathcal{C}^*(\Gamma^d) \setminus \mathcal{C}^*_{d-1}(\Gamma^d)$, *but double valued on* $\mathcal{C}^*_{d-1}(\Gamma^d)$.
(5) *Given any compact set* $K \subseteq \mathcal{C}^*(\Gamma^d)$ *which does not intersect* $\mathcal{C}^*_{d-2}(\Gamma^d)$, *the images* $(\Psi_\epsilon^\nu)^{-1}(K)$ *tend uniformly to the edge skeleton of* Γ^d, *the union of all edges in* E, *as* $\epsilon \to 0^+$.

One can easily compute inverse formulas for the diffeomorphisms h_n,

$$(h_1)^{-1}(u) = e^{-u} \quad \text{and} \quad (h_n)^{-1}(u) = (1 + (n-1)\,u)^{-\frac{1}{n-1}} \ .$$

The following is a technical lemma, which will be used to control the asymptotic behaviour of rescalings through (10). Notice that for $l, n \in \{1,\ldots,m\}$ with

$l < n$, the function $\left(h_l \circ (h_n)^{-1}\right)(u)$ is concave and its derivatives tend to zero as $u \to +\infty$. The proof of this lemma is straightforward.

Lemma 7.1. *Given positive integers $k, m \geq 1$ there is $r = r(k,m) > 0$ such that for all $1 \leq l < n \leq m$ and all $0 \leq i \leq k$,*

$$\text{(1)} \qquad \lim_{\epsilon \to 0^+} \sup_{u \geq \epsilon^r} \left| \frac{d^i}{du^i} \left[(h_n)^{-1} \left(\frac{u}{\epsilon} \right) \right] \right| = 0 \ .$$

$$\text{(2)} \qquad \lim_{\epsilon \to 0^+} \sup_{u \geq \epsilon^r} \left| \frac{d^i}{du^i} \left[\epsilon \, \left(h_l \circ (h_n)^{-1} \right) \left(\frac{u}{\epsilon} \right) \right] \right| = 0 \ .$$

This shows that for $l \leq n \leq m$, the families of functions $\{u \mapsto (h_n)^{-1}\left(\frac{u}{\epsilon}\right)\}_{\epsilon>0}$ and $\{u \mapsto \epsilon \, \left(h_l \circ (h_n)^{-1}\right)\left(\frac{u}{\epsilon}\right) - \delta_{l\,n}\, u\}_{\epsilon>0}$ converge to 0 as $\epsilon \to 0^+$, in the space of smooth functions $C^\infty(0,+\infty)$. In other words, every derivative converges uniformly to 0 over compact subsets $K \subseteq (0,+\infty)$. Item (1) is crucial in all rescalings. Item (2) will be used in the proof of proposition 10.4 alone.

8. Regular vector fields on Γ^d

Let $\mathcal{A}(\Gamma^d)$ be the space of all functions $f : \Gamma^d \to \mathbb{R}$ which have analytic extensions to a neighbourhood of Γ^d, and $\mathcal{X}(\Gamma^d)$ be the space of all vector fields X on Γ^d satisfying:

(1) X has an analytic extension to some neighbourhood of Γ^d,
(2) for each face $\rho \in K^r(\Gamma^d)$ and all $x \in \rho$, the vector $X(x)$ is tangent to ρ at x, i.e. $X(x) \in T_x\rho$.

Item 2. is equivalent to say that X induces a complete flow ϕ^t on Γ^d and all faces of Γ^d are ϕ^t-invariant. This invariance condition is also equivalent to say that for each $\sigma \in F$, $d(f_\sigma) \cdot X$ vanishes along $\sigma = \{\, f_\sigma = 0\,\}$. This implies that either $d(f_\sigma) \cdot X$ is identically zero, or there is a positive integer $\nu = \nu^X(\sigma)$, that shall refer to as the *order of σ with respect to X*, and a non-identically zero function $H_\sigma \in \mathcal{A}(\Gamma^d)$ such that

$$d(f_\sigma) \cdot X = (f_\sigma)^\nu \, H_\sigma \ .$$

We also express this fact saying that X *tangent to σ, with tangency contact of order ν*. Let us fix a vertex A and consider the basis $\left\{e_{(A,\sigma)}\right\}_{\sigma \in F_A}$ of $T_A\Gamma^d$ which is dual to the basis of $1-$forms $\{\, df_\sigma \,\}_{\sigma \in F_A}$. It is clear that $e_{(A,\sigma)}$ is the unique vector parallel to the edge γ, associated with the corner (A,σ), for which $d(f_\sigma)_A \cdot e_{(A,\sigma)} = 1$. Since all edges are invariant by the flow of X the vectors $e_{(A,\sigma)}$ are eigenvectors of the derivative DX_A. Assuming that X is tangent to σ, with tangency of order ν, either $\nu = 1$ and $H_\sigma(A)$ is the eigenvalue associated with $e_{(A,\sigma)}$, or else $\nu \geq 2$. In this second case the eigenvector $e_{(A,\sigma)}$ has zero eigenvalue, while $H_\sigma(A)$ is given by the following formula

$$H_\sigma(A) = \frac{1}{\nu!}\, d(f_\sigma)_A \cdot D^\nu X_A\, (\overbrace{e_{(A,\sigma)}, \cdots, e_{(A,\sigma)}}^{\nu}) \,.$$

Notice that, since the $(\nu-1)-$jet of X vanishes at the vertex A, the derivative $D^\nu X_A$ has an intrinsic meaning which is independent of the chosen co-ordinates.

We shall call *character of* X *at a corner* $(A,\sigma) \in C$ to the number

$$\chi_\sigma^A = -H_\sigma(A) \tag{12}$$

Setting $\chi_\sigma^A = 0$ whenever $A \notin \sigma$, $\chi = \left(\chi_\sigma^A\right)_{(A,\sigma)\in V\times F}$ is a skeleton field, which we shall call the *skeleton character of* X.

The character sign at a corner $(A,\sigma,\gamma) \in C$ is related with the flow orientation along the edge γ near the singularity at A. $\chi_\sigma^A < 0$, respectively $\chi_\sigma^A > 0$, means that A is repelling, respectively attracting, along γ. We shall say that an edge γ is a $X-$*flowing edge* when it consists of a single orbit flowing between the singularities at the endpoints.

For each vertex $A \in V$, the composition of co-ordinates (11) with the standard projection $\pi_A : \mathbb{R}^F \to \mathbb{R}^{F_A}$ is an isomorphism that we shall call the *local* $A-$*co-ordinates*. In these co-ordinates vertex A is mapped to the origin, each face $\sigma \in F_A$ is described as the level set $\{x_\sigma = 0\}$, and each edge γ incident with A corresponds to $x_\sigma-$axis, where σ is the face associated with the corner (A,γ).

Condition 2. implies that the system $\dot{x} = X(x)$ is expressed, in the local $A-$co-ordinates, by

$$\frac{d\,x_\sigma}{dt} = x_\sigma\, f_\sigma^A(x) \qquad (\sigma \in F_A) \tag{13}$$

with $f_\sigma^A(x)$ being an analytic function in the local co-ordinates.

We now define several classes of vector fields.

Definition 8.1. *We will say that a vector field $X \in \mathcal{X}(\Gamma^d)$ is regular if for each edge $\gamma \in E$ either X vanishes along γ, or else X has no singularity interior to γ, and X has non-zero character on both of γ end corners.*

We say that X is regular of type J*, where $J = I, II, III$ or IV, if X is regular and its skeleton character χ is regular of the same type J.*

Clearly,

$$\text{reg. type I} \;\Rightarrow\; \text{reg. type II} \;\Rightarrow\; \text{reg. type III} \;\Rightarrow\; \text{reg. IV} \;\Rightarrow\; \text{regular}\,.$$

The condition defining regular vector fields is a dynamical restriction that guarantees the dynamics along edges is ruled by character of the vector field at the corners.

In the rest of this section we shall prove that any regular vector field $X \in \mathcal{X}(\Gamma^d)$ rescales to its skeleton character, when we use the rescaling maps in (10), with the same order as X.

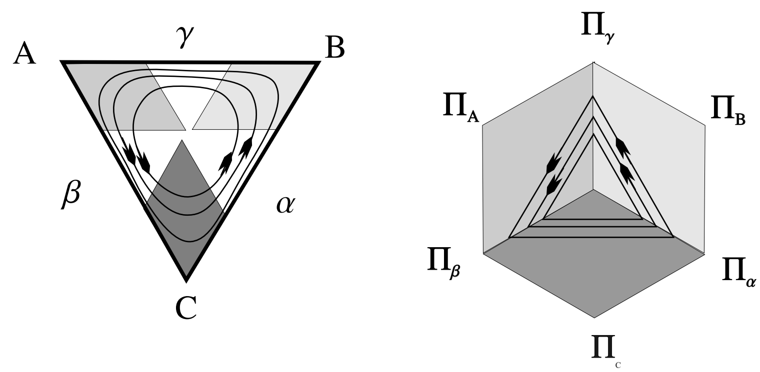

FIGURE 5. Rescaling of the vector field onto the dual cone $\mathcal{C}^*(\Gamma^d)$.

Actually things are not so simple. We need to split each sector Π_A into $d+1$ regions where we will use different asymptotic approximations. Fix any $k \geq 2$ (we shall control the derivatives of the rescaled vector field up to the order k) and fix also $m = \|\nu\| = \max_{\sigma\in F} \nu(\sigma)$. Take then $r = r(k,m) > 0$ according to lemma 7.1. For a given vertex $A \in V$ we define

$$\Pi_A^0(\epsilon) = \{\, u \in \Pi_A \,:\, u_\sigma \geq \epsilon^r \quad \text{for all} \quad \sigma \in F_A \,\} \,.$$

For each edge $\gamma \in E_A$, let $(A, \sigma, \gamma) \in C$ be the associated corner. Then we define

$$\Pi_\gamma(\epsilon) = \{\, u \in \Pi_A \,:\, u_\sigma = \epsilon^r \,\} \,, \quad \text{and}$$

$$\Pi_A^\gamma(\epsilon) = \{\, u \in \Pi_A \,:\, 0 \leq u_\sigma \leq \epsilon^r \ \text{ and } \ u_{\sigma'} \geq \epsilon^r \,,\, \forall\, \sigma' \in F_A - \{\sigma\} \,\} \,.$$

Finally we set $\Pi_A(\epsilon) = \bigcup_{\gamma\in E_A} \Pi_A^\gamma(\epsilon) \cup \Pi_A^0(\epsilon)$. If $\epsilon > 0$ is small enough, the region $\Pi_A(\epsilon)$ covers most of $\Pi_A \setminus \mathcal{C}^*_{d-2}(\Gamma^d)$. Actually $\Pi_A(\epsilon)$ increases, as ϵ tends to 0, eventually covering every point in this set. In other words, $\Pi_A \setminus \mathcal{C}^*_{d-2}(\Gamma^d) = \cup_{\epsilon>0} \Pi_A(\epsilon)$. The set $\Pi_A(\epsilon)$ splits into $d+1$ regions. The larger part is $\Pi_A^0(\epsilon)$ a convex cone parallel to Π_A which is bounded by the family of hyperplanes $\Pi_\gamma(\epsilon)$, with $\gamma \in E_A$. For each of the d edges $\gamma \in E_A$, the region $\Pi_A^\gamma(\epsilon)$ is a thin layer bounded between the hyperplanes Π_γ and $\Pi_\gamma(\epsilon)$. See figure 3.

With respect to the local $A-$co-ordinates in the sector Π_A, the system $p' = X(p)$ is expressed as

$$\frac{d\,x_\sigma}{dt} = (x_\sigma)^{\nu(\sigma)}\, H_\sigma^A(x) \qquad (\sigma \in F_A) \,, \tag{14}$$

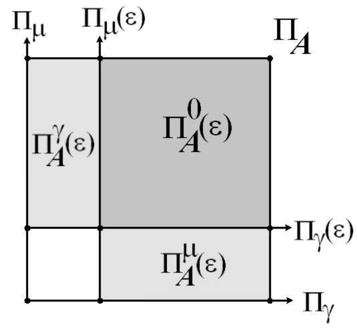

FIGURE 6. Partition of the face Π_A.

where H_σ^A is an analytic function defined in a neighbourhood of $N_A = [0,1]^d \subseteq \mathbb{R}^{F_A}$. Consider now the vector field on Π_A,

$$\tilde{X}^\epsilon(u) = \left(-H_\sigma^A(x)\right)_{\sigma\in F_A} \qquad \text{with} \qquad x = \left(h_{\nu(\sigma)}^{-1}\left(\frac{u_\sigma}{\epsilon}\right)\right)_{\sigma\in F_A} .$$

Proposition 8.1. *If ν is the order function of X, $(\Psi_\epsilon^\nu)_* X = \epsilon\, \tilde{X}^\epsilon$. Thus $\tilde{X}^\epsilon$ is, in the rescaled co-ordinates $u = \Psi_\epsilon^\nu(x)$, a time re-parametrization of X.*

In particular, in the co-ordinates $u = \Psi_\epsilon^\nu(x)$, X and $\tilde{X}^\epsilon$ share the same Poincaré maps. It is not difficult to see that for points $u \in \operatorname{int}\Pi_A$, $\tilde{X}^\epsilon(u)$ approximates the vector χ^A as $\epsilon \to 0$. Unfortunately, these approximations do not hold for points $u \in \operatorname{int}\Pi_\gamma$, with $\gamma \in E_A$. We define now, for each edge $\gamma \in E_A$, a different approximation for $\tilde{X}^\epsilon$ inside the region $\Pi_A^\gamma(\epsilon)$. Let $(A,\rho,\gamma) \in C$ be the associated corner and set $\tilde{X}_\gamma^\epsilon(u) = \left(-H_\sigma^A(x)\right)_{\sigma\in F_A}$ with $x = (x_\sigma)_{\sigma\in F_A}$, $x_\rho = h_{\nu(\rho)}^{-1}\left(\frac{u_\rho}{\epsilon}\right)$ and $x_\sigma = 0$, $\forall\, \sigma \neq \rho$. Using lemma 7.1, and the mean value theorem, we can establish that:

Proposition 8.2. *For every $1 \leq i \leq k$,*

(1) $\displaystyle \lim_{\epsilon\to 0^+} \sup_{u\in\Pi_A^0(\epsilon)} \left\| \tilde{X}^\epsilon(u) - \chi^A \right\| = 0$

(2) $\displaystyle \lim_{\epsilon\to 0^+} \sup_{u\in\Pi_A^\gamma(\epsilon)} \left\| \tilde{X}^\epsilon(u) - \tilde{X}_\gamma^\epsilon(u) \right\| = 0$

(3) $\displaystyle \lim_{\epsilon\to 0^+} \sup_{u\in\Pi_A(\epsilon)} \left\| D^i \tilde{X}^\epsilon(u) \right\| = 0$

(4) $\displaystyle \lim_{\epsilon\to 0^+} \sup_{u\in\Pi_A^\gamma(\epsilon)} \left\| D^i \tilde{X}_\gamma^\epsilon(u) \right\| = 0$

9. Conservative Invariants on Γ^d

For each face $\sigma \in F$, let $\mathcal{F}_\sigma = \{h_n \circ f_\sigma, : \; n \geq 1\}$, where the functions h_n were defined in (9), and then set $\mathcal{F} = \bigcup_{\sigma \in F} \mathcal{F}_\sigma$. We define $\mathcal{H}(\Gamma^d)$ to be the linear span of $\mathcal{A}(\Gamma^d) \cup \mathcal{F}$. Since functions in the set $\mathcal{F}$ are linearly independent, each function $h \in \mathcal{H}(\Gamma^d)$ can be uniquely decomposed as

$$h = g + \sum_{n=1}^{\infty} \sum_{\sigma \in F} \mu_{n\,\sigma} \left(h_n \circ f_\sigma\right) , \tag{15}$$

with $g \in \mathcal{A}(\Gamma^d)$, where only a finite number of coefficients $\mu_{n\,\sigma}$ are nonzero. Notice that the differential of h has a simpler algebraic expression:

$$dh = dg \;-\; \sum_{n=1}^{\infty} \sum_{\sigma \in F} \mu_{n\,\sigma} \frac{df_\sigma}{(f_\sigma)^n} \; . \tag{16}$$

We define the *order of h at σ* as the number

$$\nu^h(\sigma) = \max\{n \in \mathbb{N} \; : \; \mu_{n\,\sigma} \neq 0 \;\} \; ,$$

with $\nu^h(\sigma) = 0$ if all $\mu_{n\,\sigma} = 0$. We define the *character of h at σ* to be the coefficient $\lambda^h(\sigma) = \mu_{n\,\sigma}$ corresponding to the largest term $n = \nu^h(\sigma)$. The character is undefined when $\nu^h(\sigma) = 0$. We say that the linear form

$$\lambda^h : \mathcal{C}^*(\Gamma^d) \to \mathbb{R} \; , \quad \lambda^h(u) = \sum_{\sigma \in F} \lambda^h(\sigma)\, u_\sigma$$

is the *character skeleton* $1-$*form of* h.

In section 13 we shall give intrinsic definitions of the space $\mathcal{H}(\Gamma^d)$, as well as of the order and character of any face.

We define $\Omega_0^2(\Gamma^d)$ to be the finite dimensional space of $2-$forms

$$\omega = \sum_{(\sigma_1,\sigma_2) \in F \times F} \omega_{\sigma_1\,\sigma_2} \frac{df_{\sigma_1} \wedge df_{\sigma_2}}{f_{\sigma_1}\, f_{\sigma_2}} \; , \tag{17}$$

such that $\omega_{\sigma_1\,\sigma_2} = 0$ whenever $\sigma_1 \cap \sigma_2 = \emptyset$. We shall refer to elements in $\Omega_0^2(\Gamma^d)$ as *algebraic* $2-$*forms on* Γ^d. Each $2-$form $\omega \in \Omega_0^2(\Gamma^d)$ is associated with a skeleton $2-$form $\Omega = (\,\omega_{\sigma_1,\sigma_1}\,) \in \mathbb{R}^{F \times F}$.

The proof of the following lemmas will be given in sections 13 and 14, respectively. They give us representations in local A-co-ordinates, of 1-forms dh with $h \in \mathcal{H}(\Gamma^d)$, and 2-forms $\omega \in \Omega_0^2(\Gamma^d)$.

Lemma 9.1. *Given $h \in \mathcal{H}(\Gamma^d)$ and a vertex $A \in V$, there is a neighbourhood U of Γ^d and a vector valued function $\Lambda^A(p) = \big(\Lambda^A_\sigma(p)\big)_{\sigma\in F_A}$, analytic over $U \setminus \bigcup_{\sigma\notin F_A}(f_\sigma)^{-1}\{0\}$, such that: $\Lambda^A_\sigma(p) \equiv \lambda^h(\sigma)$ over σ, for each face $\sigma \in F_A$, and*

$$dh = \sum_{\sigma\in F_A} \Lambda^A_\sigma(p)\, \frac{df_\sigma}{(f_\sigma)^{\nu(\sigma)}}\ .$$

Lemma 9.2. *Given a vertex $A \in V$ and a form $\omega \in \Omega^2_0(\Gamma^d)$, there is a $d\times d$ matrix valued function $\Omega_A(p) = (\ (\Omega_A)_{\sigma_1\,\sigma_2}(p)\ )_{(\sigma_1,\sigma_2)\in F_A\times F_A}$ whose coefficients $(\Omega_A)_{\sigma_1\,\sigma_2}(p)$ are rational functions of $\{f_\sigma(p)\,:\,\sigma\in F_A\}$ such that:*

1. *the singularity set of each function $(\Omega_A)_{\sigma_1\,\sigma_2}(p)$ is contained in the zero set of $\prod_{\sigma\notin F_A} f_\sigma$,*
2. *$(\Omega_A)_{\sigma_1\,\sigma_2}(p)$ is constant over $\sigma_1\cap\sigma_2$, and equal to the coefficient $\omega_{\sigma_1\,\sigma_2}$ of the associated skeleton $2-$form. In particular the $2-$form ω^A, of the associated skeleton $2-$form, is determined by the skew symmetric matrix $\Omega_A(A)$,*
3. *ω can written as*

$$\omega = \sum_{\sigma_1,\sigma_2\in F_A} (\Omega_A)_{\sigma_1\,\sigma_2}(p)\, \frac{df_{\sigma_1}\wedge df_{\sigma_2}}{f_{\sigma_1}\, f_{\sigma_2}}\ .$$

Notice that, using the notation of the lemma above, for each vertex $A\in V$, the determinant $(\det\Omega_A)(p) = \det\,(\Omega_A(p))$ and the inverse matrix $\Omega_A^{-1}(p) = (\Omega_A(p))^{-1}$ are rational functions of $\{f_\sigma(p)\,:\,\sigma\in F_A\}$.

Definition 9.1. *We say that a $2-$form $\omega\in\Omega^2_0(\Gamma^d)$ is an* algebraic symplectic structure on Γ^d *if for each vertex $A\in V$, using the notation of the previous lemma, the zero set of $\det\Omega_A(p)$ in Γ^d is contained in the zero set of $\prod_{\sigma\notin F_A} f_\sigma$. This is equivalent to say that the inverse matrix function $\Omega_A^{-1}(p)$ is analytic in a neighbourhood of Γ^d minus the zero set of $\prod_{\sigma\notin F_A} f_\sigma$.*

Clearly, if $\omega\in\Omega^2_0(\Gamma^d)$ is an algebraic symplectic structure on Γ^d then the associated skeleton $2-$form is a skeleton symplectic structure on $\mathcal{C}^*(\Gamma^d)$.

Definition 9.2. *We will say that a function $h\in\mathcal{H}(\Gamma^d)$ is* regular *if for all faces $\sigma\in F$ $\nu^h(\sigma)\geq 1$, and all faces with $\nu^h(\sigma)\geq 2$ are pairwise disjoint.*

Given a function $h\in\mathcal{H}(\Gamma^d)$ with order $\nu=\nu^h$, a vertex $A\in V$ and faces $\sigma,\sigma'\in F$, we define the products:

$$p_\nu = \prod_{\rho\in F} (f_\rho)^{\nu(\rho)-1}\ , \tag{18}$$

$$p_\nu^\sigma = \prod_{\rho\neq\sigma} (f_\rho)^{\nu(\rho)-1} \,, \tag{19}$$

$$p_\nu^{\sigma\,\sigma'} = \prod_{\rho\notin\{\sigma,\sigma'\}} (f_\rho)^{\nu(\rho)-1} \,, \tag{20}$$

$$p_\nu^A = \prod_{\rho\notin F_A} (f_\rho)^{\nu(\rho)-1} \,. \tag{21}$$

Remark that (18) is strictly positive in $\mathrm{int}(\Gamma^d)$ and only vanishes at those faces $\sigma \in F$ with order $\nu(\sigma) \geq 2$. If h is regular then function (19) is strictly positive over the interior of face σ. Finally, (21) is strictly positive over the neighbourhood N_A of A.

Proposition 9.1. *Given a regular function $h \in \mathcal{H}(\Gamma^d)$ with order ν, and an algebraic symplectic structure $\omega \in \Omega_0^2(\Gamma^d)$, if X_h is the symplectic gradient of h w.r.t. ω then $p_\nu\, X_h$ is a vector field in $\mathcal{X}(\Gamma^d)$ with order function ν.*

Proof. Let h be a regular function in $\mathcal{H}(\Gamma^d)$ with order ν. In the sequel we shall make extensive use of the notation introduced in lemmas 9.1 and 9.2. The algebraic symplectic form ω is represented, in the 2−form basis $\{df_\sigma \wedge df_\rho\}_{(\sigma,\rho)\in F_A\times F_A}$, by the matrix $(\,(f_\sigma)^{-1}\,(f_\rho)^{-1}\,(\Omega_A)_{\sigma\,\rho}\,)$ with inverse $\big(\,f_\sigma\, f_\rho\,(\Omega_A^{-1})_{\sigma\,\rho}\,\big)$. Therefore the symplectic gradient X_h of h is given by

$$\begin{aligned}
(df_\sigma)(X_h) &= \sum_{\rho\in F_A} (\Omega_A^{-1})_{\sigma\,\rho}\, f_\sigma\, f_\rho\, \frac{\Lambda_\rho^A}{(f_\rho)^{\nu(\rho)}} \\
&= f_\sigma \sum_{\rho\in F_A} (\Omega_A^{-1})_{\sigma\,\rho}\, \frac{\Lambda_\rho^A}{(f_\rho)^{\nu(\rho)-1}} \\
&= \frac{f_\sigma}{p_\nu} \sum_{\rho\in F_A} (\Omega_A^{-1})_{\sigma\,\rho}\, \Lambda_\rho^A\, p_\nu^\rho \\
&= \frac{(f_\sigma)^{\nu(\sigma)}}{p_\nu} \sum_{\rho\in F_A-\{\sigma\}} (\Omega_A^{-1})_{\sigma\,\rho}\, \Lambda_\rho^A\, p_\nu^{\rho\,\sigma}
\end{aligned}$$

Thus there is a vector valued function $H^A(p) = \big(\,H_\sigma^A(p)\,\big)_{\sigma\in F_A}$ such that for each face $\sigma \in F_A$ with order $\nu(\sigma) = \nu(\sigma)$

$$(df_\sigma)(p_\nu\, X_h) = (f_\sigma)^{\nu(\sigma)}\, H_\sigma^A \,,$$

where for some neighbourhood U of Γ^d each function H_σ^A is analytic over $U \setminus \bigcup_{\sigma\notin F_A} (f_\sigma)^{-1}\{0\}$. This proves that $p_\nu\, X_h \in \mathcal{X}(\Gamma^d)$. ☐

Consider now the skeleton $2-$form $\Omega = (\,\omega_{\sigma_1\,\sigma_2}\,)$ associated with the algebraic symplectic structure ω. Let us identify the symplectic linear form ω^A in Π_A^{sp} with the sub-matrix of Ω, indexed in $F_A \times F_A$, which defines it. Denote by $\omega_A^{-1} = \left(\,(\omega_A^{-1})_{\sigma_1\,\sigma_2}\,\right)$ the inverse matrix of ω^A, $\omega_A^{-1} = (\omega^A)^{-1}$.

Proposition 9.2. *Under the assumptions of proposition 9.1, let $\chi = (\chi_\sigma^A)$ denote the skeleton character of $p_\nu\, X_h$. If all faces in F_A have order $\nu(\sigma) = 1$ then*

$$\chi_\sigma^A = -p_\nu^A(A) \sum_{\rho\in F_A} (\omega_A^{-1})_{\sigma\,\rho}\, \lambda^h(\rho)\ , \quad \forall\, \sigma \in F_A\ .$$

If σ is the unique face in F_A with order $\nu(\sigma) \geq 2$ then

$$\chi_\sigma^A = -p_\nu^A(A) \sum_{\rho\in F_A} (\omega_A^{-1})_{\sigma\,\rho}\, \lambda^h(\rho)\ , \quad \textit{and}$$

$$\chi_\rho^A = -p_\nu^A(A)\, (\omega_A^{-1})_{\rho\,\sigma}\, \lambda^h(\sigma)\ , \quad \forall\, \rho \in F_A - \{\sigma\}\ .$$

Proof. From definition (12) one has $\chi_\sigma^A = -H_\sigma^A(A)$, for each face $\sigma \in F_A$. Recalling the formula for H_σ^A deduced in the proof of proposition 9.1, notice that:

(1) if either $\nu(\sigma) = 1$ for all $\sigma \in F_A$, or else if $\sigma \in F_A$ is the only face with $\nu(\sigma) \geq 2$, then $p_\nu^{\rho\,\sigma} = p_\nu^A$ for all $\sigma,\, \rho \in F_A$, which implies that

$$H_\sigma^A = p_\nu^A \sum_{\rho\in F_A-\{\sigma\}} (\Omega_A^{-1})_{\sigma\,\rho}\; \Lambda_\rho^A\ .$$

(2) Assuming that $\sigma \in F_A$ is the unique face with order $\nu(\sigma) \geq 2$, and taking other faces ρ and τ in $F_A - \{\sigma\}$, we have $p_\nu^{\rho\,\sigma} = p_\nu^A$ and $p_\nu^{\rho\,\tau} = (f_\sigma)^{\nu(\sigma)}\; p_\nu^A$, for all $\tau \in F_A - \{\sigma, \rho\}$. Therefore

$$H_\rho^A = p_\nu^A \; \left((\Omega_A^{-1})_{\rho\,\sigma}\; \Lambda_\sigma^A + (f_\sigma)^{\nu(\sigma)} \sum_{\tau\in F_A-\{\sigma,\rho\}} (\Omega_A^{-1})_{\rho\,\tau}\; \Lambda_\tau^A \right)\ .$$

Finally remark that at $p = A$, $(\Omega_A)_{\sigma\,\rho}(p)$, $(\Omega_A^{-1})_{\sigma\,\rho}(p)$, $\Lambda_\rho^A(p)$ and $(f_\sigma)^{\nu(\sigma)}(p)$ are respectively equal to $(\omega^A)_{\sigma\,\rho}$, $(\omega_A^{-1})_{\sigma\,\rho}$, $\lambda^h(\rho)$ and 0, and it is enough making these substitutions. ∎

Lemma 9.3. *Given a corner $(A, \sigma, \gamma) \in C$, there is a strictly positive function $\varphi(q)$ defined over γ by*

$$\varphi(q) = \frac{p_\nu^A(q)}{p_\nu^A(A)}\, \frac{Pf(\Omega_A(A))}{Pf(\Omega_A(p))}\ ,$$

such that for all $q \in \{A\} \cup int(\gamma)$, $H^A(q) = -\varphi(q)\, \chi^A$.

Proof. Take an edge γ with end corners (A,σ), $(B,\rho)\in C$. Then, as q runs along γ, the skew symmetric matrix $\Omega_A(q)$ is a perturbation of $\Omega_A(A)$ which takes the the form (31). In fact we claim that there is a function $c_\tau(q)$ on γ such that for every $q\in\gamma$,

(a) $(\Omega_A)_{\tau\,\tau'}(q)=\omega_{\tau\,\tau'}$ is constant along γ, for τ, $\tau'\in F_\gamma$,
(b) $(\Omega_A)_{\sigma\,\tau}(q)=\omega_{\sigma\,\tau}+c_\tau(q)$, for $\tau\in F_\gamma$,
(c) $(\Omega_A)_{\tau\,\sigma}(q)=-\omega_{\sigma\,\tau}-c_\tau(q)$, for $\tau\in F_\gamma$.

Item (a) is obvious, since by (2) in lemma 9.2, we have $(\Omega_A)_{\tau\,\tau'}(q)$ constant and equal to $\omega_{\tau\,\tau'}$ along $\tau\cap\tau'$, which contains γ. Items (b) and (c) are just a matter of defining $c_\tau(q)$ and using that the matrix $\Omega_A(q)$ is skew symmetric.

By definition 9.1, the Pfaffians of the matrices $\Omega_A(q)$ keep a constant sign along γ. By lemma 15.1, we have along γ

$$\begin{aligned} H^A_\sigma(q) &= p^A_\nu(q)\sum_{\tau\in F_\gamma}(\Omega_A^{-1})_{\sigma\,\tau}(q)\,\Lambda^A_\tau(q)\\ &= \frac{p^A_\nu(q)}{\mathrm{Pf}\,(\Omega_A(q))}\sum_{\tau\in F_\gamma}(\omega^A)'_{\sigma\,\tau}\,\lambda^h_\tau\,, \end{aligned}$$

where $(\omega^A)'_{\sigma\,\tau}$ is a Pfaffian cofactor entry of the skew symmetric sub-matrix ω^A of Ω. Thus

$$\begin{aligned} H^A_\sigma(q) &= \frac{p^A_\nu(q)}{\mathrm{Pf}\,(\Omega_A(q))}\sum_{\tau\in F_\gamma}(\omega^A)'_{\sigma\,\tau}\,\lambda^h_\tau\\ &= -\varphi(q)\left(-\frac{p^A_\nu(A)}{\mathrm{Pf}\,(\Omega_A(A))}\sum_{\tau\in F_\gamma}(\omega^A)'_{\sigma\,\tau}\,\lambda^h_\tau\right)=-\varphi(q)\,\chi^A_\sigma\,, \end{aligned}$$

at least under the case (1) assumptions in the proof of proposition 9.2. But if case (2) holds instead, in the same proof, a similar argument derives the same conclusion. □

Lemma 9.4. *Given a corner $(A,\sigma,\gamma)\in C$, for all points $q\in int(\gamma)$,*

(1) $(df_\rho)\,(\,p_\nu\,X_h\,)_q=0$ *for each* $\rho\in F_A-\{\sigma\}$,
(2) $(df_\sigma)\,(\,p_\nu\,X_h\,)_q$ *has the same sign as* $-\chi^A_\sigma$.

Proof. Statement (1) is an obvious consequence of the invariance condition (2), in definition of the vector field space $\mathcal{X}(\Gamma^d)$. Observing that $(df_\sigma)(p_\nu\,X_h)_q$ and $-H^A_\sigma(q)$ have opposite signs, statement (2) follows from the previous lemma. □

This lemma shows that the character χ of $p_\nu\,X_h$ determines the global flow orientation along each edge γ. In particular this vector field can not have singularities in the middled of any edge γ, unless it vanishes completely along

γ. Moreover, if $\chi_\sigma^A = 0$ then $p_\nu\, X_h$ vanishes completely along the edge γ opposed to (A, σ). Finally, given two corners (A, σ), $(B, \rho) \in C$ opposed to an edge γ where $p_\nu\, X_h$ does not vanish completely, we must have $\chi_\sigma^A \cdot \chi_\rho^B < 0$. This proves that $p_\nu\, X_h$ is regular vector field of type III.

Proposition 9.3. *Given a regular function $h \in \mathcal{H}(\Gamma^d)$ with order ν, and an algebraic symplectic structure $\omega \in \Omega_0^2(\Gamma^d)$, let X_h be the symplectic gradient of h w.r.t. ω and consider the vector field $X = p_\nu\, X_h \in \mathcal{X}(\Gamma^d)$. Let χ be the character of X and $\lambda = \lambda^h$ the character of h. Then*

(1) *X is regular of type III.*
(2) *If all components of λ^h are nonzero with the same sign, then X is regular of type II.*
(3) *The skeleton vector field χ preserves λ.*

Proof. Form proposition 9.2 it follows that χ preserves $\lambda = \lambda^h$. In other words, $\sum_{\sigma \in F_A} \lambda_\sigma\, \chi_\sigma^A = 0, \;\; \forall\; A \in V$. If all components of λ_σ are nonzero with the same sign then it follows that for each vertex $A \in V$ there are at least two faces σ, $\rho \in F_A$ such that χ_σ^A and χ_ρ^A have opposite signs. Thus A is a saddle type vertex of the skeleton vector field χ, which proves that X is regular of type II. □

10. Flow Asymptotics along the edges

Throughout this section $X \in \mathcal{X}(\Gamma^d)$ will denote a regular vector field with skeleton character χ. For each $X-$flowing edge γ with source vertex $A \in V$, let $\Sigma_\gamma = \Sigma_\gamma^A$ be the inner face defined in (8). This is a cross-section, transversal to X, which intersects γ at a single point q_γ. Define

$$\Sigma^X = \Sigma^X(\Gamma^d) = \bigcup_\gamma \Sigma_\gamma \;,$$

where the union is taken over all $X-$flowing edges whose target vertex is of saddle type. Let $\mathcal{U}$ be a small neighbourhood, in $\mathbb{R}^d$, of the union of all $X-$flowing edges as above. The flow of X is encapsulated in the *Poincaré return map* to Σ^X

$$\pi^X : \mathcal{U} \cap \Sigma^X \to \Sigma \;, \tag{22}$$

defined for small enough $\mathcal{U}$ as follows: Given a point $p \in \mathcal{U} \cap \Sigma^X$, $\;\pi^X(p)$ is the first intersection of the forward orbit $\{\phi_X^t(p) : t \geq 0\}$ with Σ^X, where ϕ_X^t denotes the flow of the vector field X.

By the regularity of X, an edge sequence $\xi = (A_0, A_1, \cdots, A_n, A_{n+1})$ is a $\chi-$chain, in the sense of the definition given in section 4, if and only if it is a heteroclinic chain, i.e. for every $i = 0, 1, \cdots, n$, γ_i is a $X-$flowing edge with $\alpha-$limit A_i and $\omega-$limit A_{i+1}. Given a chain $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ we say that an orbit segment $\big(\, p,\, \pi^X(p),\, (\pi^X)^2(p),\, \cdots,\, (\pi^X)^n(p)\,\big)$ has *itinerary* ξ iff for every $i = 0, 1, \cdots, n$, $(\pi^X)^i(p) \in \mathcal{U} \cap \Sigma_{\gamma_i}$. Clearly, every orbit segment of X has some definite itinerary ξ, which must be a $\chi-$chain. Given any chain $\xi = (\gamma_0, \gamma_1, \cdots, \gamma_n)$ we define Σ_ξ as the set of all points in $\mathcal{U} \cap \Sigma_{\gamma_0}$ with itinerary, up to the n-th iterate, equal to ξ, i.e.

$$\Sigma_\xi = \{\, p \in \mathcal{U} \cap \Sigma_{\gamma_0} \;:\; (\pi^X)^i(p) \in \operatorname{int} \mathcal{U} \cap \Sigma_{\gamma_i} \quad \forall\, i = 0, 1, \cdots, n \,\}\ .$$

This set is the domain of the *$X-$Poincaré map along* ξ

$$\pi^X_\xi : \Sigma_\xi \subseteq \Sigma_{\gamma_0} \to \Sigma_{\gamma_n}\ , \quad \text{defined by} \quad \pi^X_\xi = (\pi^X)^n\ .$$

As in section 4 it is obvious that

Proposition 10.1. *Given two chains ξ and ξ', such that ξ connects with ξ',*

$$\Sigma_{\xi * \xi'} = \{\, u \in \Sigma_\xi \;:\; \pi^X_\xi(u) \in \Sigma_{\xi'} \,\}\ ,$$

and over this domain $\pi^X_{\xi*\xi'} = \pi^X_{\xi'} \circ \pi^X_\xi$.

Given an edge γ with target vertex $B \in V$, we set $\Sigma^+_\gamma = \Sigma^B_\gamma$ and then define $T^X_\gamma : \mathcal{U} \cap \Sigma_\gamma \to \Sigma^+_\gamma$ to be the Poincaré map, where $T^X_\gamma(p)$ is the first intersection of the forward orbit $\{\phi^t_X(p) : \, t \geq 0\}$ with Σ^+_γ.

Given a $\chi-$chain $\xi = (\mu, \gamma) \in \Lambda^1(\chi)$, we define $P^X_{\mu\,\gamma} : \mathcal{U} \cap \Sigma^+_\gamma \to \Sigma_\mu$ to be the Poincaré map along (μ, γ), where $P^X_{\mu\,\gamma}(p)$ is the first intersection of the forward orbit $\{\phi^t_X(p) : \, t \geq 0\}$ with Σ_μ.

Proposition 10.2. *Given $\xi = (\mu, \gamma) \in \Lambda^1(\chi)$, the domain Σ_ξ of π^X_ξ contains $\mathcal{U} \cap \Sigma_\gamma \cap T^{-1}_\gamma(\mathcal{U})$, and on this set, where the composition of $P^X_{\mu\,\gamma}$ with T^X_γ is defined, we have*

$$\pi^X_\xi = P^X_{\mu\,\gamma} \circ T^X_\gamma\ .$$

To shorten convergence statements in theorems bellow we first make their meaning precise. Suppose we are given a family of functions, or mappings, F_ϵ with varying domains $\mathcal{U}_\epsilon$. Let F be another function with domain $\mathcal{U}$. Assume that all these functions have the same target and source spaces, which are assumed to be linear spaces. We shall say that $\lim_{\epsilon \to 0^+} F_\epsilon = F$ *in the C^∞ topology*, to mean that:

(1) *domain convergence*: for every compact subset $K \subseteq \mathcal{U}$, we have $K \subseteq \mathcal{U}_\epsilon$ for all small enough $\epsilon > 0$, and

(2) *derivative uniform convergence on compacts*: for every $k \in \mathbb{N}$

$$\lim_{\epsilon\to 0^+} \sup_{u\in K\ 0\le i\le k} \left| D^i \left[F_\epsilon(u) - F(u)\right] \right| \;=\; 0 \;.$$

If in a statement F_ϵ is written as a composition of two or more mappings then its domain should be understood as the composition domain. For instance, suppose we are given maps $\pi : \mathcal{U} \subseteq \Sigma \to \Sigma'$ and $\tilde{\pi} : \tilde{\mathcal{U}} \subseteq \Pi \to \Pi'$, and also families of rescaling maps $\Psi_\epsilon : \Sigma \to \Pi$ and $\Psi'_\epsilon : \Sigma' \to \Pi'$, with values in some linear spaces Π and Π'. Then the *domain convergence statement* in the limit $\lim_{\epsilon\to 0^+} \Psi'_\epsilon \circ \pi \circ (\Psi_\epsilon)^{-1} = \tilde{\pi}$ will mean that for every compact subset $K \subseteq \tilde{\mathcal{U}}$ (the domain of $\tilde{\pi}$), we have $(\Psi_\epsilon)^{-1}(K) \subseteq \mathcal{U}$ (domain of π) for all small enough $\epsilon > 0$. When dealing with differential forms, convergence in the C^∞ topology will mean convergence of all coefficients, with respect to some fixed form basis.

Proposition 10.3. *Let $X \in \mathcal{X}(\Gamma^d)$ be a regular vector field with skeleton character χ, and order function $\nu = \nu^X$. Given any $\chi-$chain ξ we have,*

$$\lim_{\epsilon\to 0^+} \Psi^\nu_\epsilon \circ \pi^X_\xi \circ (\Psi^\nu_\epsilon)^{-1} = \pi^\chi_\xi \quad \textit{in the } \; C^\infty \; \textit{topology.}$$

Proof. By propositions 4.1 and 10.1, π^X_ξ and π^χ_ξ are compositions of length-one chain Poincaré maps. Thus it is enough to prove this theorem for such chains. This special case follows combining lemmas 10.1 and 10.2 bellow. □

Lemma 10.1. *Given any flowing edge γ,*

$$\lim_{\epsilon\to 0^+} \Psi^\nu_{\epsilon\,|\Sigma^+_\gamma} \circ T^X_\gamma \circ (\Psi^\nu_{\epsilon\,|\Sigma_\gamma})^{-1} = Id \quad \textit{in the } \; C^\infty \; \textit{topology,}$$

where Id stands for the identity mapping in the interior of Π_γ.

Proof. As pointed out in the begin of this section, the cross-section Σ_γ intersects γ at the point q_γ. Locally, we can identify Σ_γ with its tangent space $T_{q_\gamma}\Sigma_\gamma$ through the isomorphism $T_{q_\gamma}\Sigma_\gamma \equiv \Sigma_\gamma$, $v \mapsto q_\gamma + v$. Doing so the translation map $\tau_\gamma : \Sigma_\gamma \to \Sigma^+_\gamma$, defined in the beginning of section 7, can be identified with the tangent map of T^X_γ at q_γ, $\tau_\gamma = (DT^X_\gamma)_{q_\gamma}$. In other words τ_γ is precisely the degree one Taylor approximation of T^X_γ at q_γ. Notice that by item (2) of proposition 7.2,

$$\Psi^\nu_{\epsilon\,|\Sigma^+_\gamma} \circ \tau_\gamma \circ (\Psi^\nu_{\epsilon\,|\Sigma_\gamma})^{-1} = \mathrm{Id} \;.$$

On the other hand, by item (5) of proposition 7.2, given a compact set $K \subseteq \mathrm{int}\Pi_\gamma$, $(\Psi^\nu_{\epsilon\,|\Sigma_\gamma})^{-1}(K)$ approaches $\{q_\gamma\}$ as $\epsilon \to 0^+$. Therefore, a standard rescaling argument shows that $\Psi^\nu_{\epsilon\,|\Sigma^+_\gamma} \circ T^X_\gamma \circ (\Psi^\nu_{\epsilon\,|\Sigma_\gamma})^{-1}$ converges to $\Psi^\nu_{\epsilon\,|\Sigma^+_\gamma} \circ \tau_\gamma \circ (\Psi^\nu_{\epsilon\,|\Sigma_\gamma})^{-1} = \mathrm{Id}$, as $\epsilon \to 0^+$. □

Lemma 10.2. *Given any chain* $(\mu,\gamma)\in\Lambda^1(\chi)$,

$$\lim_{\epsilon\to 0^+}\Psi^{\nu}_{\epsilon\,|\Sigma_\mu}\circ P^X_{\mu\,\gamma}\circ(\Psi^{\nu}_{\epsilon\,|\Sigma^+_\gamma})^{-1}=L^{\chi}_{\gamma\,|\Pi_{(\mu,\gamma)}}\quad \textit{in the } C^\infty \textit{ topology.}$$

Proof. Use the considerations bellow, which end up in lemma 10.3. □

Take $A\in V$, and let γ and μ be two edges having A as target, respectively source, vertex. The rescaled map $\Psi^{\nu}_{\epsilon\,|\Sigma_\mu}\circ P^X_{\mu\,\gamma}\circ(\Psi^{\nu}_{\epsilon\,|\Sigma^+_\gamma})^{-1}$ is nothing but the Poincaré map, that we shall also denote by $P_{\mu\,\gamma}$, of the rescaled vector field $\tilde{X}_\epsilon=\frac{1}{\epsilon}(\Psi^\nu_\epsilon)_*X$, introduced in the end of section 8. We can factorize $P_{\mu\,\gamma}$ as a composition of three Poincaré maps,

$$P_{\mu\,\gamma}=(T^\epsilon_\mu)^{-1}\circ P^\epsilon_{\mu\,\gamma}\circ T^\epsilon_\gamma\ ,$$

where $P^\epsilon_{\mu\,\gamma}:\Pi_\gamma(\epsilon)\to\Pi_\mu(\epsilon)$, $T^\epsilon_\gamma:\Pi_\gamma\to\Pi_\gamma(\epsilon)$, and $(T^\epsilon_\mu)^{-1}$ is the inverse map of $T^\epsilon_\mu:\Pi_\mu\to\Pi_\mu(\epsilon)$. Note that, for notation simplicity, we did not made explicit references to the domains of these maps. Remark also that the hyperplanes $\Pi_\gamma(\epsilon)$ and $\Pi_\mu(\epsilon)$ depend implicitly on the choice of a number $k\geq 1$ (an upper bound for derivative orders) as they were defined in section 8. We can also define the Poincaré mappings $\tilde{P}^\epsilon_{\mu\,\gamma}:\Pi_\gamma(\epsilon)\to\Pi_\mu(\epsilon)$ and $\tilde{T}^\epsilon_\gamma:\Pi_\gamma\to\Pi_\gamma(\epsilon)$, associated, respectively, with the vector fields χ^A (constant vector field) and $\tilde{X}^\epsilon_\gamma$, this last defined in the end of section 8. There is a natural way of identifying $\Pi_\gamma(\epsilon)$ with Π_γ. Thus, we may think that all the mappings introduced above have source and target spaces which are not ϵ dependent. The following convergence statements are then meaningful.

Lemma 10.3. *All limits bellow exist in the* C^∞ *topology.*

(1) $\lim_{\epsilon\to 0^+} T^\epsilon_\gamma-\tilde{T}^\epsilon_\gamma=0$.

(2) $\lim_{\epsilon\to 0^+} P^\epsilon_{\mu\,\gamma}-\tilde{P}^\epsilon_{\mu\,\gamma}=0$.

(3) $\lim_{\epsilon\to 0^+} \tilde{P}^\epsilon_{\mu\,\gamma}=\pi^\chi_{(\mu,\gamma)}$.

(4) $\lim_{\epsilon\to 0^+} \tilde{T}^\epsilon_\gamma=Id$.

Proof. Items (1) and (2) follow by proposition 8.2.

Let (A,σ_0) be the source corner of μ. A simple computation shows that

$$\tilde{P}^\epsilon_{\mu\,\gamma}(u_\sigma)_{\sigma\in F_A}=\left(u_\sigma-\frac{\chi^A_\sigma}{\chi^A_{\sigma_0}}\,(u_{\sigma_0}-\epsilon^r)\right)_{\sigma\in F_A}\ ,$$

which proves item (3).

Let (A,ρ_0) be the target corner of γ and consider the vector field $\tilde{X}^\epsilon_\gamma(u)$. The fact that it only depends on the ρ_0 component of $u=(u_\sigma)_{\sigma\in F_A}$ implies that $\tilde{T}^\epsilon_\gamma:\Pi_\gamma\to\Pi_\gamma(\epsilon)$ is a translation map, $\tilde{T}^\epsilon_\gamma(u)=u+\theta^\epsilon_\gamma$, along some vector $\theta^\epsilon_\gamma=(\theta^\epsilon_\gamma(\sigma))_{\sigma\in F_A}\in\Pi^{sp}_A$. Since all components of $\tilde{X}^\epsilon_\gamma(u)$ are bounded, the ρ_0

component being positive and bounded from bellow, and since the distance between Π_γ and $\Pi_\gamma(\epsilon)$ is exactly ϵ^r, it follows that $\left\|\theta_\gamma^\epsilon\right\| = \mathcal{O}(\epsilon^r)$, which proves item (4). $\square$

Proposition 10.4. *Let $X \in \mathcal{X}(\Gamma^d)$ be a regular vector field with skeleton character χ, and $h \in \mathcal{H}(\Gamma^d)$ be a regular function with skeleton 1−form $\lambda : \mathcal{C}^*(\Gamma^d) \to \mathbb{R}$. Assume that X and h have the same order function $\nu : F \to \mathbb{N}$. If h is a first integral of X, i.e. $dh_p(X(p)) = 0$ for all $p \in \Gamma^d$, then for every $A \in V$,*

$$\lambda(u) = \lim_{\epsilon \to 0^+} \epsilon\, h \circ (\Psi_\epsilon^\nu)^{-1}(u) \quad \text{in the } C^\infty \text{ topology, over } \Pi_A .$$

In particular

$$\lambda = \lim_{\epsilon \to 0^+} \epsilon \,\left[(\Psi_\epsilon^\nu)^{-1} \right]^* (dh) \quad \text{in the } C^\infty \text{ topology, over } \Pi_A .$$

Proof. Let us work in the local A−co-ordinates. According to decomposition (15) we can write

$$h(x) = g(x) + \sum_{\sigma \in F_A} \sum_{n=1}^{\nu(\sigma)} \mu_{n\,\sigma}\, h_n(x_\sigma) ,$$

where $g(x)$ is analytic in $\Gamma^d \setminus \cup_{\sigma \notin F_A} (f_\sigma)^{-1}\{0\}$. Therefore, by lemma 7.1 (2),

$$\epsilon\; h \circ (\Psi_\epsilon^\nu)^{-1}(u) = \epsilon\, g \circ (\Psi_\epsilon^\nu)^{-1}(u) + \epsilon \sum_{\sigma \in F_A} \sum_{n=1}^{\nu(\sigma)} \mu_{n\,\sigma} \left(h_n \circ h_{\nu(\sigma)}^{-1} \right) \left(\frac{u_\sigma}{\epsilon} \right)$$

converges in the C^∞ topology to $\sum_{\sigma \in F_A} \mu_{\nu(\sigma)\,\sigma}\, u_\sigma = \lambda(u)$. $\square$

Corollary 10.5. *Under the assumptions of proposition 10.4, the skeleton 1−form λ is χ−invariant. In particular, for every χ−chain ξ, $\lambda \circ \pi_\xi^\chi = \lambda$.*

Given an order function $\nu : F \to \mathbb{N}$ and a skeleton 2−form $\Omega = (\omega_{\sigma_1\,\sigma_2})$ on $\mathcal{C}^*(\Gamma^d)$, let us define the skeleton 2−form $\Omega^\nu = \left(\omega^\nu_{\sigma_1\,\sigma_2}\right)$:

$$\omega^\nu_{\sigma_1\,\sigma_2} = \begin{cases} \omega_{\sigma_1\,\sigma_2} & \text{if } \nu(\sigma_1) = \nu(\sigma_2) = 1 \\ 0 & \text{otherwise} \end{cases} ,$$

which will referred to as the skeleton form *reduced by ν*.

Proposition 10.6. *Let $h \in \mathcal{H}(\Gamma^d)$ be a regular function with order ν, and $\omega \in \Omega^2_0(\Gamma^d)$ an algebraic symplectic structure with associated skeleton 2−form Ω. Then for every $A \in V$,*

$$\Omega^\nu = \lim_{\epsilon \to 0^+} \epsilon^2 \,\left[(\Psi_\epsilon^\nu)^{-1} \right]^* (\omega) \quad \text{in the } C^\infty \text{ topology, over } \Pi_A .$$

Ω^ν denotes the skeleton 2−form reduced by ν.

Proof. By proposition 7.1(1) we have $\left[(\Psi_\epsilon^\nu)^{-1}\right]^* \dfrac{dx_\sigma}{(x_\sigma)^{\nu(\sigma)}} = \dfrac{du_\sigma}{\epsilon}$, which implies $\left[(\Psi_\epsilon^\nu)^{-1}\right]^* \dfrac{dx_{\sigma_1}\wedge dx_{\sigma_2}}{(x_{\sigma_1})^{\nu(\sigma_1)}\,(x_{\sigma_2})^{\nu(\sigma_2)}} = \dfrac{du_{\sigma_1}\wedge du_{\sigma_2}}{\epsilon^2}$. Therefore $\left[(\Psi_\epsilon^\nu)^{-1}\right]^* \dfrac{dx_{\sigma_1}\wedge dx_{\sigma_2}}{x_{\sigma_1}\,x_{\sigma_2}} = (x_{\sigma_1})^{\nu(\sigma_1)-1}\,(x_{\sigma_2})^{\nu(\sigma_2)-1}\,\dfrac{du_{\sigma_1}\wedge du_{\sigma_2}}{\epsilon^2}$,
where $x_{\sigma_i} = h_{\nu(\sigma_i)}^{-1}\left(\frac{u_{\sigma_i}}{\epsilon}\right)$. Notice that, by lemma 7.1(1), $\epsilon^2\ \left[(\Psi_\epsilon^\nu)^{-1}\right]^* \dfrac{dx_{\sigma_1}\wedge dx_{\sigma_2}}{x_{\sigma_1}\,x_{\sigma_2}}$ tends to zero as $\epsilon\to 0^+$, unless $\nu(\sigma_1)=\nu(\sigma_2)=1$, in which case it is equal to $du_{\sigma_1}\wedge du_{\sigma_2}$. By lemma 9.2 we can write

$$\omega = \sum_{\sigma_1,\sigma_2\in F_A} (\Omega_A)_{\sigma_1\,\sigma_2}(x)\,\frac{dx_{\sigma_1}\wedge dx_{\sigma_2}}{x_{\sigma_1}\,x_{\sigma_2}} \ .$$

where

$$(\Omega_A)_{\sigma_1\,\sigma_2}(x) = \omega_{\sigma_1\,\sigma_2} + x_{\sigma_1}\,\Theta^A_{\sigma_1\,\sigma_2}(x) + x_{\sigma_2}\,\Xi^A_{\sigma_1\,\sigma_2}(x) \ ,$$

and each of $\Theta^A_{\sigma_1\,\sigma_2}$ and $\Xi^A_{\sigma_1\,\sigma_2}$ is a rational function whose singularity set is contained in the zero set of p^A_ν. Therefore, by lemma 7.1(1),

$$\lim_{\epsilon\to 0^+} (\Omega_A)_{\sigma_1\,\sigma_2}\circ(\Psi_\epsilon^\nu)^{-1} = \omega_{\sigma_1\,\sigma_2} \quad \text{in the } C^\infty \text{ topology.}$$

It follows easily that $\epsilon^2\ [(\Psi_\epsilon^\nu)^{-1}]^*\,\omega$ converges, in the C^∞ topology, to the reduced $2-$form Ω^ν. □

Corollary 10.7. *Under the assumptions of proposition 10.6, let λ be the skeleton $1-$form of h, let X_h be the symplectic gradient of h w.r.t. ω, and $\tilde X = p_\nu\,X_h \in \mathcal{X}(\Gamma^d)$ be the corresponding regular vector field, whose skeleton we denote by χ. Then for each vertex $A\in V$, and all $u\in\Pi_A$,*

$$\lambda(\chi^A) = 0 \quad \text{and} \quad \Omega^\nu(\chi^A,u) = p_\nu(A)\,\lambda(u) \ .$$

Furthermore, for each cycle ξ, defined by a circular sequence of edges starting and ending with γ,

$$\lambda\circ\pi^\chi_\xi = \lambda \quad \text{and} \quad \left(\pi^\chi_\xi\right)^*\Omega^\nu{}_{|\Pi_\gamma} = \Omega^\nu{}_{|\Pi_\gamma} \ .$$

Proof. We have, by definition of symplectic gradient, $\iota_{X_h}\omega = dh$, where $\iota_X\omega$ denotes the contraction of ω by X. Rescaling the identity

$$\iota_{p_\nu\,X_h}\omega = p_\nu\,dh \ ,$$

by taking pull-backs with $(\Psi_\epsilon^\nu)^{-1}$ and letting $\epsilon\to 0^+$, we obtain the identity

$$\Omega^\nu(\chi^A,u) = p_\nu(A)\,\lambda(u) \ .$$

Since the regular vector field $X = p_\nu\,X_h$ and the Hamiltonian vector field X_h have the same Poincaré maps, for any cycle ξ starting and ending with the edge γ,

$$(\pi^X_\xi)^*\omega_{|\Sigma_\gamma} = \omega_{|\Sigma_\gamma} \ .$$

Rescaling this identity we obtain

$$(\pi_\xi^\chi)^*\Omega^\nu = \Omega^\nu \ .$$

$\square$

Part 3. EXAMPLES AND AN APPLICATION

11. The Replicator Equation

Consider the $d-$dimensional simplex

$$\Delta^d = \left\{ x \in \mathbb{R}^d \ : \ \sum_{i=1}^d x_i \leq 1 \ , \quad x_i \geq 0 \ \forall\, i = 1, \cdots, d \right\} \ ,$$

which is the simplest of all simple polyhedrons. It has $d+1$ vertexes, denoted by $E_0, E_1, \cdots, E_d$, where $E_0 = (0, \cdots, 0)$ and $E_i = (0, \cdots, 1, \cdots, 0)$ is the canonical basis vector with a single "1" at the i^{th} entry, for every $i = 1, \cdots, d$. This polyhedron has $d\,(d+1)/2$ edges. We denote by $\gamma_{i\,j} = [E_i, E_j]$ the edge joining E_i with E_j, for each $1 \leq i \neq j \leq d+1$. We also denote by σ_i the unique face containing all edges except E_i. The dual cone of Δ^d is then

$$\mathcal{C}^*(\Delta^d) = \bigcup_{i=0}^d \Pi_i = \{ \ u \in \mathbb{R}_+^{d+1} \ : \ \exists\, 0 \leq i \leq d \, , \quad u_i = 0 \ \ \} \ ,$$

where $\Pi_i = \{u \in \mathbb{R}_+^{d+1} : u_i = 0\}$ is the sector of $\mathcal{C}^*(\Delta^d)$ associated with face σ_i.

In Game Dynamics, the replicator equation (1) is determined by the matrix $A = (a_{i\,j})_{0\leq i,j \leq d}$. Let $\overline{x} = (x_0, x_1, \cdots, x_d)$, where $x_0 = 1 - \sum_{k=1}^d x_k$. Then $\sum_{j=0}^d a_{i\,j}\, x_j = (A\,\overline{x})_i$ is the $i-$th component of $A\,\overline{x}$, and $\sum_{k,\,s=0}^d a_{k\,s} x_k\, x_s = \overline{x}^t\, A\,\overline{x}$. Thus (1) can be expressed as

$$x_i' = X_A^i(x) := x_i\, \left((A\,\overline{x})_i - \overline{x}^t\, A\,\overline{x} \right) \ , \quad 1 \leq i \leq d \ .$$

Let X_A denote the vector field on Δ^d with components X_A^i. This is a complete vector field, $X_A \in \mathcal{X}(\Delta^d)$. The matrix A is not uniquely determined by X_A. In fact, it is easy to see that $X_A = X_B$ if and only if the matrices A and B differ by some matrix C with constant columns, i.e. $a_{i\,j} - b_{i\,j} = c_j$ for each $1 \leq i, j \leq d+1$. Notice that the matrices $(a_{i\,j})$ and $(a'_{i\,j}) = (a_{i\,j} - a_{j\,j})$ represent the same replicator equation (1), and $(a'_{i\,j})$ is the unique representative matrix with zero diagonal. From now on we will assume the coefficients $a_{i\,j}$ in (1) to be normalized so that $a_{i\,i} = 0$ for all $i = 0, 1, \cdots, d$.

Proposition 11.1. *The character of X_A at the corner $(E_j, \sigma_i, \gamma_{i\,j})$ satisfies:*

a) *$a_{i\,j}$ is the eigenvalue at E_j along $\gamma_{i\,j}$; in case $a_{i\,j} \neq 0$ the character at (E_j, σ_i) is also $\chi_{\sigma_i}^{E_j} = a_{i\,j}$.*

b) *X_A vanishes along $\gamma_{i\,j}$ $\Leftrightarrow$ $a_{i\,j} = a_{j\,i} = 0$; when this happens the character at both ends, (E_j, σ_i) and (E_i, σ_j), is zero.*
c) *X_A has tangency contact with σ_i of order ≥ 2 $\Leftrightarrow$ $a_{i\,k} = 0$ for all $k \neq i$ and the $d \times d-$matrix $(a_{r\,s})_{r,s\neq i}$ is skew-symmetric; in this case, if $a_{j\,i} \neq 0$, then the character at both ends (E_j, σ_i) and (E_i, σ_j) is given by $\chi^{E_i}_{\sigma_j} = -\chi^{E_j}_{\sigma_i} = a_{j\,i}$.*

Proof. The $i-$th component of $X(x)$ is $x_i\, H_i(x)$ where

$$H_i(x) = \sum_{k=0}^{d} a_{i\,k}\, x_k - \sum_{r,s=0}^{d} a_{r\,s}\, x_r\, x_s \ .$$

To analyse the system in a neighbourhood of E_j we replace x_j by $1 - \sum_{k\neq j} x_k$, above, to obtain

$$\begin{aligned} H_i(x) = H_i^{E_j}(x) = a_{i\,j} \quad &+ \quad \sum_{k\neq j} (a_{i\,k} - a_{j\,k} - a_{k\,j} - a_{i\,j})\, x_k \\ &- \quad \sum_{r,s\neq j} (a_{r\,s} - a_{j\,s} - a_{r\,j})\, x_r\, x_s \ . \end{aligned}$$

Then it is clear that the eigenvalue at E_j along $\gamma_{i\,j}$ is equal to $a_{i\,j} = H_i^{E_j}(0)$.

Assuming $a_{i\,j} = 0$, and setting $x_k = 0$ for all $k \neq i, j$, we have

$$H_i^{E_j}(x) = -a_{j\,i}\, x_i\, (1 + x_i) \ .$$

Therefore X_A vanishes along $\gamma_{i\,j}$ if and only if $a_{j\,i} = a_{i\,j} = 0$.

When $a_{i\,j} = 0$, the vector field X_A has a tangency contact with σ_i of order ≥ 2 if and only if there is an analytic factorization $H_i^{E_j}(x) = x_i\, G_i^{E_j}(x)$, which in turn holds if and only if for all $x \in \mathbb{R}^{d+1}$

(1) $\sum_{k\neq i,\,j} (a_{i\,k} - a_{j\,k} - a_{k\,j})\, x_k = 0$, and
(2) $\sum_{r,s\neq i,j} (a_{r\,s} - a_{j\,s} - a_{r\,j})\, x_r\, x_s = 0$.

But (2) is equivalent to $(a_{r\,s} - a_{r\,j} - a_{j\,s})_{r,s\neq i}$ being a skew-symmetric matrix, while (1) implies that $a_{i\,k} - a_{j\,k} - a_{k\,j} = 0$, for all $k \neq i$. Notice that for $k = j$, $a_{j\,j} = 0$ and we are assuming that $a_{i\,j} = 0$. Taking $r = s = k$, the skew-symmetry of $(a_{r\,s} - a_{r\,j} - a_{j\,s})_{r,s\neq i}$ implies that $a_{k\,j} + a_{j\,k} = 0$ for all $k \neq i, j$. Therefore $(a_{r\,j} + a_{j\,s})_{r,s\neq i}$ is skew-symmetric, which implies that $(a_{r\,s})_{r,s\neq i}$ is also skew-symmetric. Finally, $a_{i\,k} - a_{j\,k} - a_{k\,j} = 0$ implies that $a_{i\,k} = 0$ for all k.

Conversely, if $a_{i\,k} = 0$ for all k, and the matrix $(a_{r\,s})_{r,s\neq i}$ is skew-symmetric, then $(a_{r\,j} + a_{j\,s})_{r,s\neq i}$ and $(a_{r\,s} - a_{r\,j} - a_{j\,s})_{r,s\neq i}$ are skew-symmetric matrices. Therefore (1) and (2) hold, which proves that X_A is tangent to σ_i with order 2.

In this case, assuming that $a_{i\,j} = 0$ but $a_{j\,i} \neq 0$, then by (1) and (2) above we have $H_i^{E_j}(x) = -a_{j\,i}\, x_i$. Thus, the i-th component of X_A takes the form

$X_A^i(x) = (x_i)^2\,(-a_{j\,i})$, which implies that $\chi_{\sigma_i}^{E_j} = -a_{j\,i}$. But from a) we also have $\chi_{\sigma_j}^{E_i} = a_{j\,i}$. $\square$

It follows from the proof that there are no tangencies of X_A with some face of Δ^d with order ≥ 3.

Corollary 11.2. *Assume A is a matrix with zero diagonal, consider the vector field X_A in (??), and let I denote the set of all $i \in \{0,1,\ldots,d\}$ such that X_A has a tangency of order ≥ 2 with the face σ_i.*

1. *If $|I| \geq 2$, then A is skew-symmetric and $a_{i\,j} = 0$ if $i \in I$ or $j \in I$.*
2. *If $I = \{i\}$, then the $d \times d-$ matrix $(a_{r\,s})_{r,s\neq i}$ is skew-symmetric, and $a_{i\,k} = 0$ for all $k \neq i$.*
3. *If $|I| \geq 1$, then X_A is regular of type II.*
4. *If If $|I| = 0$, then*
   a) *X_A is regular of type I $\Leftrightarrow$ $a_{j\,i}\,a_{i\,j} < 0$ for all $i \neq j$, and every column (row) of A has at least two entries with opposite signs.*
   b) *X_A is regular of type II $\Leftrightarrow$ $a_{j\,i} = a_{i\,j} = 0$ or $a_{j\,i}\,a_{i\,j} < 0$ for all $i \neq j$, and every column (row) of A has at least two entries with opposite signs.*
   c) *X_A is regular of type III $\Leftrightarrow$ $a_{j\,i} = a_{i\,j} = 0$ or $a_{j\,i}\,a_{i\,j} < 0$ for all $i \neq j$.*
   d) *X_A is regular of type IV $\Leftrightarrow$ $a_{j\,i} = a_{i\,j} = 0$ or $a_{j\,i}\,a_{i\,j} \neq 0$ for all $i \neq j$.*

The character of X_A at a corner (E_j, σ_i) is always $\chi_{\sigma_i}^{E_j} = a_{i\,j}$, except in case (2), when σ_i is the unique face of order 2, in which case $\chi_{\sigma_i}^{E_j} = -a_{j\,i}$.

Consider now a conservative Lotka-Volterra system (2), given by some non degenerated $d \times d$ skew-symmetric matrix $A = (a_{i\,j})$. The correspondent quadratic vector field can be written as

$$X(y) = X_{r,A}(y) = y * (r + A\,y)\ ,$$

where $*$ stands for component-wise multiplication of vectors in $\mathbb{R}^d$. Take $q \in \mathbb{R}^d$ to be the unique solution of $r + A\,q = 0$ and consider the function

$$h\colon \mathbb{R}_+^d \to \mathbb{R}\ , \quad h(y) = \sum_{i=1}^{d} y_i - q_i\,\log y_i\ . \tag{23}$$

The system (2) is Hamiltonian, and $X_{r,A}$ is the symplectic gradient of h with respect to the following "algebraic" symplectic structure on $\mathbb{R}^d_+$,

$$\omega = \sum_{i,j=1}^{d} a_{ij}^{-1}\, \frac{dx_i \wedge dx_j}{x_i\, x_j} \ ,$$

constructed from the inverse matrix coefficients $A^{-1} = (a_{ij}^{-1})$. See [5].

Let $\varphi : \Delta^d - \sigma_0 \to \mathbb{R}^d_+$ be the Hofbauer algebraic equivalence, from the replicator to the Lotka-Volterra equation, which is given by

$$\varphi(x_0, x_1, \cdots, x_d) = \left(\frac{x_1}{x_0}, \cdots, \frac{x_d}{x_0} \right) \ ,$$

where $x_0 = 1 - \sum_{k=1}^d x_k$. The inverse map is

$$\varphi^{-1}(y) = \frac{1}{1 + \sum_{j=1}^d y_j}\, (y_1, \cdots, y_d) \ .$$

It is obvious then that the vector field $\tilde{X} = \varphi^* (X_{r,A})$ on $\Delta^d - \sigma_0$ is the symplectic gradient of $\tilde{h} = h \circ \varphi$ with respect to the symplectic structure $\tilde{\omega} = \varphi^* \omega$. Let us now identify each of these pull-back objects.

First, it is clear that $\tilde{h} = h \circ \varphi$, given by

$$\tilde{h}(x) = \sum_{i=1}^{d} \frac{x_i}{x_0} - q_i\, \log x_i - q_i\, \log x_0 \ , \tag{24}$$

is a regular function in $\mathcal{H}(\Delta^d)$ with order function $\nu = \nu^{\tilde{h}}$ given by $\nu(\sigma_0) = 2$ and $\nu(\sigma_1) = \cdots = \nu(\sigma_d) = 1$, and having character $1-$form $\lambda = \lambda^{\tilde{h}}$ given by $\lambda(\sigma_0) = 1$ and $\lambda(\sigma_i) = q_i$ for every $1 \leq i \leq d$.

The ω's pull-back is described next

Proposition 11.3. *The $2-$form $\tilde{\omega} = \varphi^* \omega$ is an algebraic symplectic structure on Δ^d of the form*

$$\tilde{\omega} = \sum_{i,j=0}^{d} \Omega_{i\,j}\, \frac{dx_i \wedge dx_j}{x_i\, x_j} \ ,$$

where the coefficients of the associated skeleton symplectic structure $(\Omega_{i\,j})$ on $\mathcal{C}^*(\Delta^d)$ are given by

$$\begin{aligned} \Omega_{i\,j} &= a_{ij}^{-1} \quad \text{if} \quad 1 \leq i, j \leq d, \\ \Omega_{0\,j} &= -\sum_{k=1}^{d} a_{kj}^{-1} \quad \text{and} \quad \Omega_{i\,0} = -\sum_{k=1}^{d} a_{ik}^{-1} \ . \end{aligned}$$

Proof. A simple computation shows that

$$
\begin{aligned}
\tilde{\omega} &= \sum_{i,j=1}^{d} a_{ij}^{-1} \, \frac{d\frac{x_i}{x_0} \wedge d\frac{x_j}{x_0}}{\frac{x_i}{x_0}\,\frac{x_j}{x_0}} \\
&= \sum_{i,j=1}^{d} a_{ij}^{-1} \, \frac{x_0^2}{x_i\, x_j} \, \left(\frac{1}{x_0}\, dx_i - \frac{x_i}{x_0^2}\, dx_0 \right) \wedge \left(\frac{1}{x_0}\, dx_j - \frac{x_j}{x_0^2}\, dx_0 \right) \\
&= \sum_{i,j=1}^{d} a_{ij}^{-1} \, \left(\frac{dx_i \wedge dx_j}{x_i\, x_j} - \frac{dx_0 \wedge dx_j}{x_0\, x_j} - \frac{dx_i \wedge dx_0}{x_i\, x_0} \right) \\
&= \sum_{i,j=1}^{d} \Omega_{ij} \, \frac{dx_i \wedge dx_j}{x_i\, x_j} + \sum_{j=1}^{d} \Omega_{0j} \, \frac{dx_0 \wedge dx_j}{x_0\, x_j} + \sum_{i=1}^{d} \Omega_{i0} \, \frac{dx_i \wedge dx_0}{x_i\, x_0} \\
&= \sum_{i,j=0}^{d} \Omega_{ij} \, \frac{dx_i \wedge dx_j}{x_i\, x_j}
\end{aligned}
$$

This proves that $\tilde{\omega} \in \Omega_0^2(\Gamma^d)$. It remains to check conditions of definition 9.1. We start by representing $\tilde{\omega}$ as linear combination, with rational function coefficients, in the basis

$$
\left\{ \frac{dx_i \wedge dx_j}{x_i\, x_j} \;:\; 1 \leq i, j \leq d \right\}
$$

of 2−forms corresponding to faces containing vertex E_0.

$$
\begin{aligned}
\tilde{\omega} &= \sum_{i,j=1}^{d} \left(\Omega_{ij} - \Omega_{i0}\, \frac{x_j}{x_0} - \Omega_{0j}\, \frac{x_i}{x_0} \right) \, \frac{dx_i \wedge dx_j}{x_i\, x_j} \\
&= \sum_{i,j=1}^{d} \left(\Omega_{ij} + \frac{x_j}{x_0} \, \sum_{k=1}^{d} \Omega_{ik} + \frac{x_i}{x_0} \, \sum_{k=1}^{d} \Omega_{kj} \right) \, \frac{dx_i \wedge dx_j}{x_i\, x_j} \\
&= \sum_{i,j=1}^{d} (\Omega_{E_0})_{i\,j}(x) \, \frac{dx_i \wedge dx_j}{x_i\, x_j} \; .
\end{aligned}
$$

By lemma 15.3, the correspondent matrix $\Omega_{E_0}(x)$ has determinant,

$$
\begin{aligned}
\det \Omega_{E_0}(x) &= \left(1 + \sum_{k=1}^{d} \frac{x_k}{x_0} \right)^2 \, \det A^{-1} \\
&= \frac{1}{x_0^2} \, \det(A^{-1}) \neq 0 \; ,
\end{aligned}
$$

which does not vanish over $\Delta^d - \sigma_0$.

It still remains to prove that for all other vertexes E_k, when we represent $\tilde{\omega}$ as linear combination, with rational function coefficients, in the basis

$$\left\{\frac{dx_i \wedge dx_j}{x_i\, x_j} \;:\; i,j \neq k\right\}$$

of 2−forms associated with faces containing E_k, the determinant of $\Omega_{E_k}(x)$ is nonzero over $\Delta^d - \sigma_k$. Notice that the matrix Ω has the property that each row, resp. column, of Ω adds up to zero. Equivalently, each row, resp. column, equals minus the sum of all other rows, resp. columns. It follows that each minor Ω'_{kk}, the determinant of the sub-matrix obtained removing Ω's k^{th} row and column, is equal to $\det(A^{-1})$. Using these facts we can see, as above, that

$$\begin{aligned}
\tilde{\omega} &= \sum_{i,j\neq k}\left(\Omega_{ij} - \Omega_{ik}\,\frac{x_j}{x_k} - \Omega_{kj}\,\frac{x_i}{x_k}\right)\frac{dx_i \wedge dx_j}{x_i\, x_j} \\
&= \sum_{i,j\neq k}\left(\Omega_{ij} + \frac{x_j}{x_k}\sum_{r\neq k}\Omega_{ir} + \frac{x_i}{x_k}\sum_{r\neq k}\Omega_{rj}\right)\frac{dx_i \wedge dx_j}{x_i\, x_j} \\
&= \sum_{i,j\neq k}(\Omega_{E_k})_{i\,j}(x)\,\frac{dx_i \wedge dx_j}{x_i\, x_j}\ .
\end{aligned}$$

Thus, again by lemma 15.3, the matrix $\Omega_{E_k}(x)$ has determinant

$$\begin{aligned}
\det \Omega_{E_k}(x) &= \left(1 + \sum_{r\neq k}^{d}\frac{x_r}{x_k}\right)^2 \det A^{-1} \\
&= \frac{1}{x_k^2}\,\det(A^{-1}) \neq 0\ .
\end{aligned}$$

Therefore ω is an algebraic symplectic structure on Δ^d. $\square$

We can now characterize the pull-back $\tilde{X} = \varphi^*(X_{r,A})$. Clearly, $\tilde{X}(x)$ is the symplectic gradient of $\tilde{h}(x)$ with respect to $\tilde{\omega}$. The product function in (18) is $p_\nu(x) = x_0 = 1 - \sum_{k=1}^d x_k$. Therefore $x_0\,\tilde{X}(x)$ is a regular vector field in $\mathcal{X}(\Delta^d)$ with the same order as $\tilde{h}(x)$. In fact, defining the matrix with zero diagonal

$$\tilde{A} = (\tilde{a}_{i\,j}) = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ r_1 & a_{1\,1} & \cdots & a_{1\,n} \\ \vdots & \vdots & \cdots & \vdots \\ r_n & a_{n\,1} & \cdots & a_{n\,n} \end{pmatrix}\ .$$

we can easily prove that $x_0\,\tilde{X}(x) = X_{\tilde{A}}(x)$.

Proposition 11.4. *Every conservative Lotka-Volterra system is equivalent to replicator vector field in $\mathcal{X}(\Delta^d)$. If the solution q of $r + A\,q = 0$ is interior to the sector $(\mathbb{R}_+)^d$, then this vector field is regular of type II.*

Proof. If $q_i > 0$ for all $1 \leq i \leq d$, then all components of λ^h are positive. Thus, by item (2) of proposition 9.3, the vector field is regular of type II. □

12. An application

The aim of this section is to prove theorem A. Consider the coefficient matrices (3), and the corresponding conservative Lotka-Volterra system (2). We know that this Lotka-Volterra system is equivalent to a replicator system defined by a regular vector field $X \in \mathcal{X}(\Delta^4)$. From corollary 11.2, the correspondent skeleton character is given by

$$\chi = (\chi_{\sigma_j}^{E_i}) = \begin{pmatrix} 0 & -1 & 1 & -1 & 1 \\ 1 & 0 & -1 & 0 & 0 \\ -1 & 1 & 0 & -\delta & 0 \\ 1 & 0 & \delta & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 \end{pmatrix} . \tag{25}$$

The character of this system first integral, $\tilde{h}$ in (24), is

$$\lambda = (\lambda_{\sigma_j}) = (\,1\,, 1+\delta,\, 1,\, 1,\, 1+\delta\,)\,. \tag{26}$$

As we have seen, λ is an invariant 1-form of the skeleton vector field χ.

Among the ten edges of the 4−dimensional simplex Δ^4, there are three χ−neutral edges and the following seven χ−flowing edges: $0 \to 1$, $2 \to 0$, $0 \to 3$, $4 \to 0$, $1 \to 2$, $2 \to 3$ and $3 \to 4$. See figure bellow

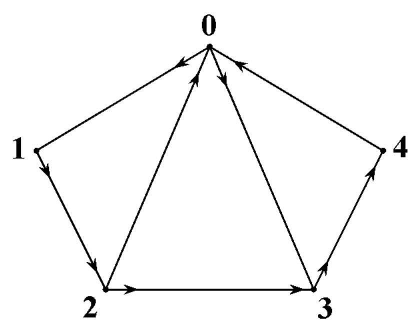

Figure 7. χ−flowing edges in Δ^4.

Therefore we have, as defined in (5), seven nontrivial endomorphisms $L^\chi_{i\,j} = L^\chi_{\gamma_{i\,j}} : \mathbb{R}^5 \to \mathbb{R}^5$, one for each χ−flowing edge, given by the following matrices:

$$L^{\chi}_{0\,1} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix} \qquad L^{\chi}_{0\,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

$$L^{\chi}_{2\,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -\delta & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad L^{\chi}_{4\,0} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L^{\chi}_{1\,2} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \qquad L^{\chi}_{2\,3} = \begin{pmatrix} 1 & 0 & 0 & -\delta^{-1} & 0 \\ 0 & 1 & 0 & \delta^{-1} & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L^{\chi}_{3\,4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & \delta \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For the sake of notation simplicity we shall write $\xi = (i_0\, i_1\, \cdots i_n\, i_{n+1})$ for $\xi = (E_{i_0}, E_{i_1}, \cdots, E_{i_n}, E_{i_{n+1}})$, $\Pi_{i\,j}$ instead of $\Pi_{\gamma_{i\,j}}$, and $u = (u_0, u_1, u_2, u_3, u_4)$ where u_i stands for u_{σ_i}, for each $i = 0, 1, \cdots 4$.

Consider the following chains

$$\begin{array}{lll} \xi^0 = (401) & \xi^1 = (01201) & \xi^4 = (40340) \\ \xi^2 = (012340) & \xi^3 = (0120340) & \\ \xi^5_n = \xi^0 * (\xi^1)^n * \xi^2 & \xi^6_n = \xi^0 * (\xi^1)^n * \xi^3 & (n \geq 0) \end{array}$$

We have $\Lambda^{\odot}_{40}(\chi) = \{\xi^4\} \cup \{\xi^5_n : n \geq 0\} \cup \{\xi^6_n : n \geq 0\}$. In other words, these are the cycles starting and ending with $\gamma_{4\,0}$, which in-between do not pass through this edge. Thus the restriction of the first return map to $\Pi_{4\,0}$ is given by

$$\pi^{\chi}_{1^{\rm st}}(u) = \begin{cases} \pi^{\chi}_{\xi^4}(u) & \text{if} \quad u \in \Pi_{\xi^4} \\ \pi^{\chi}_{\xi^5_n}(u) & \text{if} \quad u \in \Pi_{\xi^5_n}, \quad n \geq 0 \\ \pi^{\chi}_{\xi^6_n}(u) & \text{if} \quad u \in \Pi_{\xi^6_n}, \quad n \geq 0 \end{cases},$$

with domain $\Pi_{\xi^4} \cup \bigcup_{n=0}^{\infty} \Pi_{\xi^5_n} \cup \bigcup_{n=0}^{\infty} \Pi_{\xi^6_n}$.

Both source and target space, for all the Poincaré mappings along these chains, are among the two faces $\Pi_{4\,0}$ and $\Pi_{0\,1}$. The chains ξ^0, ξ^4, ξ^5_n and ξ^6_n have domain $\Pi_{4\,0}$, while ξ^1, ξ^2 and ξ^3 have domain $\Pi_{0\,1}$. The chains ξ^2, ξ^3,ξ^4, ξ^5_n and ξ^6_n have target space $\Pi_{4\,0}$, while ξ^0 and ξ^1 have target space $\Pi_{0\,1}$. Since

$\Pi_{4\,0} = \{u\,:\,u_0 = u_4 = 0\}$ and $\Pi_{0\,1} = \{u\,:\,u_0 = u_1 = 0\}$, we consider coordinates (u_1, u_2, u_3) on $\Pi_{4\,0}$, and (u_2, u_3, u_4) on $\Pi_{0\,1}$. Given $A \subseteq \mathbb{R}^d$, let us denote by A^o the set

$$A^o = \{u \in \mathbb{R}^d\,:\, u\cdot a > 0, \quad \text{for all}\;\; a \in A\}\,.$$

Notice that A^o is always an open convex cone. The domain Π_ξ of a Poincaré mapping along a chain ξ is an open convex cone which can be written in this form for some finite set $A = \{w_1, \ldots, w_k\}$, i.e., $\Pi_\xi = \{w_1, \ldots, w_k\}^o$. We shall say that the vectors in A *define the cone* Π_ξ. In the next table we have the matrices, and the corresponding domain defining vectors for the Poincaré mappings along the chains ξ^0, ξ^1, $(\xi^1)^n$, ξ^2, ξ^3 and ξ^4.

ξ	Π_ξ	L^χ_ξ
$\xi^0 = 401$	$\begin{cases}(-1,\,0,\,1)\\(1,\,0,\,0)\\(0,\,1,\,0)\end{cases}$	$\begin{pmatrix}1&1&0\\-1&0&1\\1&0&0\end{pmatrix}$
$\xi^1 = 01201$	$\begin{cases}(-1-\delta,\,1,\,0)\\(1,\,0,\,0)\\(0,\,0,\,1)\end{cases}$	$\begin{pmatrix}1&0&0\\-1-\delta&1&0\\1&0&1\end{pmatrix}$
$(\xi^1)^n = 01\,\overbrace{201\cdots201}^{n}$	$\begin{cases}(-n-n\delta,\,1,\,0)\\(1,\,0,\,0)\\(0,\,0,\,1)\end{cases}$	$\begin{pmatrix}1&0&0\\-n-n\delta&1&0\\n&0&1\end{pmatrix}$
$\xi^2 = 012340$	$\begin{cases}(1,\,-\delta^{-1},\,0)\\(0,\,1,\,0)\\(0,\,0,\,1)\end{cases}$	$\begin{pmatrix}0&\delta^{-1}&0\\0&0&\delta\\1&-\delta^{-1}&1\end{pmatrix}$
$\xi^3 = 0120340$	$\begin{cases}(-\delta,\,1,\,0)\\(1+\delta,\,-1,\,0)\\(0,\,0,\,1)\end{cases}$	$\begin{pmatrix}1+\delta&-1&0\\-\delta-\delta^2&1+\delta&\delta\\-\delta&1&1\end{pmatrix}$
$\xi^4 = 40340$	$\begin{cases}(1,\,0,\,-1)\\(0,\,1,\,0)\\(0,\,0,\,1)\end{cases}$	$\begin{pmatrix}1&0&-1\\0&1&1+\delta\\0&0&1\end{pmatrix}$

A simple computation shows that $\pi^\chi_{\xi^5_n}$ is given by the matrix,

$$\begin{pmatrix} -n-(n+1)\delta^{-1} & -n-n\delta^{-1} & \delta^{-1}\\ (n+1)\delta & n\delta & 0\\ 2n+2+(n+1)\delta^{-1} & 2n+1+n\delta^{-1} & -\delta^{-1}\end{pmatrix},$$

while its domain $\Pi_{\xi^5_n}$ is defined by the vectors (a^5_n, b^5_n, c^5_n), (d^5_n, e^5_n, f^5_n), $(1,\,0,\,0)$, and $(0,\,1,\,0)$, where

$$\begin{pmatrix} a^5_n & b^5_n & c^5_n\\ d^5_n & e^5_n & f^5_n\end{pmatrix} = \begin{pmatrix} -n-1-n\delta & -n-n\delta & 1\\ n+1+(n+1)\delta & n+(n+1)\delta & -1\end{pmatrix}.$$

Another simple computation gives for $\pi^{\chi}_{\xi^6_n}$ the following matrix,

$$\begin{pmatrix} n+2+(n+1)\delta & n+1+(n+1)\delta & -1 \\ -(n+1)-(n+1)(\delta+\delta^2) & -n-(n+1)(\delta+\delta^2) & 1+\delta \\ -(n+1)\delta & -(n+1)\delta & 1 \end{pmatrix}$$

while its domain $\Pi_{\xi^6_n}$ is defined by the vectors (a^6_n, b^6_n, c^6_n), (d^6_n, e^6_n, f^6_n), $(1, 0, 0)$, and $(0, 1, 0)$, where

$$\begin{pmatrix} a^6_n & b^6_n & c^6_n \\ d^6_n & e^6_n & f^6_n \end{pmatrix} = \begin{pmatrix} -n-1-(n+1)\delta & -n-(n+1)\delta & 1 \\ n+2+(n+1)\delta & n+1+(n+1)\delta & -1 \end{pmatrix} .$$

Notice that $(a^5_0, b^5_0, c^5_0) = (-1, 0, 1)$, and for all $n \geq 1$ we have

$$(a^5_n, b^5_n, c^5_n) = -(d^6_{n-1}, e^6_{n-1}, f^6_{n-1}) \text{ and } (d^5_n, e^5_n, f^5_n) = -(a^6_n, b^6_n, c^6_n) .$$

From the defining vectors, enumerated above, we obtain that

Lemma 12.1. *The open cones Π_{ξ^4}, $\Pi_{\xi^5_n}$ and $\Pi_{\xi^6_n}$ $(n \geq 0)$ are defined by the following inequalities:*

(1) Π_{ξ^4} *is defined by $-u_3 + u_1 < 0$, $u_1 > 0$ and $u_2 > 0$.*
(2) $\Pi_{\xi^5_n}$ *is defined by $u_1 > 0$, $u_2 > 0$ and*

$$\frac{-u_1+u_3}{(1+\delta)(u_1+u_2)} - \frac{\delta}{1+\delta} < n < \frac{-u_1+u_3}{(1+\delta)(u_1+u_2)} .$$

(3) $\Pi_{\xi^6_n}$ *is defined by $u_1 > 0$, $u_2 > 0$ and*

$$\frac{-u_1+u_3}{(1+\delta)(u_1+u_2)} - 1 < n < \frac{-u_1+u_3}{(1+\delta)(u_1+u_2)} - \frac{\delta}{1+\delta} .$$

Lemma 12.2. *Let $\lambda \cdot u = \sum_{j=0}^d \lambda_j u_j$ denote a linear 1-form in $\mathbb{R}^{d+1}$, with $\lambda_j > 0$ for all $0 \leq j \leq d$, and consider on the d-dimensional hyperplane*

$$S^d = \{ u \in \mathbb{R}^{d+1} \,:\, \lambda \cdot u = 1 \} ,$$

linear co-ordinates $(u_1, \ldots, u_d) \in \mathbb{R}^d$ by setting $u_0 = \left(1 - \sum_{j=1}^d \lambda_j u_j\right)/\lambda_0$. In these co-ordinates:

(a) *The half-space $\{\omega\}^o$, where $\omega \in \mathbb{R}^{d+1}$, is defined by the inequality*

$$\sum_{j=1}^{d} \left(\omega_j - \omega_0 \frac{\lambda_j}{\lambda_0} \right) u_j + \frac{\omega_0}{\lambda_0} > 0 .$$

(b) *Given a a linear isomorphism $L : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ with matrix $(a_{i\,j})$, which preserves the 1-form λ, i.e., $\lambda_j = \sum_{i=0}^{d} \lambda_i\, a_{i\,j}$ for all $0 \le j \le d$, then $v = Lu$ if and only if, for all $1 \le i \le d$,*

$$v_i = \frac{a_{i\,0}}{\lambda_0} + \sum_{j=1}^{d} \left(a_{i\,j} - a_{i\,0}\, \frac{\lambda_j}{\lambda_0} \right) u_j \ .$$

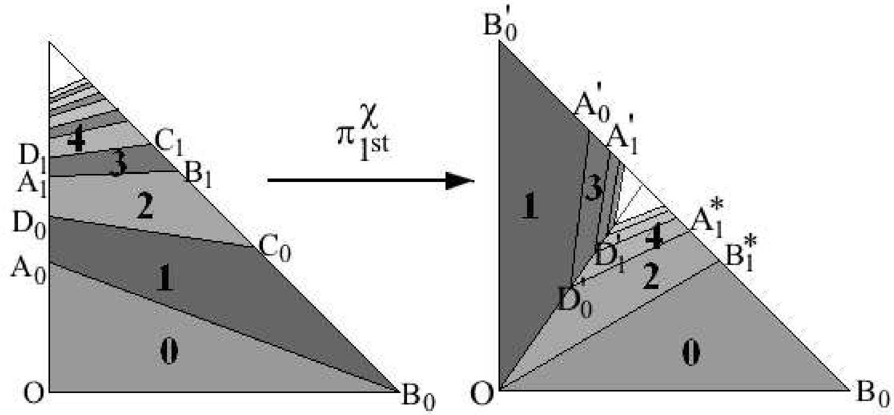

Figure 8. First return map to Δ_{40} for $\delta = 0.7$

The restriction of (26) to $\Pi_{4\,0}$ is the 1-form, invariant by the first return map $\pi^{\chi}_{1^{\rm st}} : \Pi_{4\,0} \to \Pi_{4\,0}$,

$$\lambda \cdot u = (1+\delta)\, u_1 + u_2 + u_3 \ .$$

Denote by Δ^2 the 2-dimensional simplex obtained intersecting $\Pi_{4\,0}$ with the hyperplane $\{\, u \,:\, \lambda \cdot u = 1 \,\}$. Consider the linear co-ordinates (u_2, u_3) in Δ^2, which correspond to set $u_1 = (1 - u_2 - u_3)/(1+\delta)$. In these co-ordinates, Δ^2 is identified with the canonical simplex defined by the inequalities $u_2 > 0$, $u_3 > 0$, and $u_2 + u_3 < 1$. For each cone $\Pi_\xi \subseteq \Pi_{4\,0}$, associated with a cycle $\xi \in \Lambda^{\odot}_{4\,0}(\chi)$, we set $\Delta_\xi = \Delta^2 \cap \Pi_\xi$. With this notation, Δ^2 is, up to a zero measure set, the disjoint union of the polygons

$$\Delta_{\xi^4},\ \Delta_{\xi^5_0},\ \Delta_{\xi^6_0},\ \Delta_{\xi^5_1},\ \Delta_{\xi^6_1},\ \Delta_{\xi^5_2},\ \Delta_{\xi^6_2},\ \cdots \ .$$

The picture above shows these polygons, as well as their $\pi^{\chi}_{1^{\rm st}}$-images, labeled in this order. These domains, as well as their corresponding linear maps, can be characterized using lemma 12.2.

The polygon Δ_{ξ^4} is a triangle defined by the inequalities $u_2 > 0$, $u_3 > 0$ and $u_3 < \frac{1}{2+\delta} - \frac{u_2}{2+\delta}$, with vertexes $O = (0,0)$, $A_0 = \left(\frac{1}{2+\delta}, 0\right)$ and $B_0 = (1,0)$. In this triangle, the first return map is given by

$$\pi^{\chi}_{\xi^4}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} 1 & 1+\delta \\ 0 & 1 \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} ,$$

and ranges over the triangle with vertexes $O = (0,0)$, $B_1^* = \left(\frac{1+\delta}{2+\delta}, \frac{1}{2+\delta}\right)$ and $B_0 = (1,0)$.

The polygon $\Delta_{\xi_0^5}$ is a trapezium defined by the inequalities $u_2 > 0$, $u_3 > 0$, $u_2 + u_3 < 1$ and

$$\frac{1}{2+\delta} - \frac{u_2}{2+\delta} \;<\; u_3 \;<\; \frac{1}{2} - \frac{(1-\delta)\,u_2}{2} .$$

This trapezium can also be described as $\Delta_{\xi_0^5} = [\,A_0,\ B_0,\ C_0,\ D_0\,]$, where $A_0 = \left(0, \frac{1}{2+\delta}\right)$, $B_0 = (1,0)$, $C_0 = \left(\frac{1}{1+\delta}, \frac{\delta}{1+\delta}\right)$, and $D_0 = \left(0, \frac{1}{2}\right)$. In $\Delta_{\xi_0^5}$, the first return map is given by

$$\pi^{\chi}_{\xi_0^5}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{\delta}{1+\delta} \\ \\ \frac{2+\delta^{-1}}{1+\delta} \end{pmatrix} + \begin{pmatrix} -\frac{\delta}{1+\delta} & -\frac{\delta}{1+\delta} \\ \\ 1-\frac{2+\delta^{-1}}{1+\delta} & -\frac{2+3\delta}{\delta(1+\delta)} \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} .$$

Then, the image polygon is $\pi^{\chi}_{\xi_0^5}\left(\Delta_{\xi_0^5}\right) = [\,A_0',\ B_0',\ O,\ D_0'\,]$, where $A_0' = \left(\frac{\delta}{2+\delta}, \frac{2}{2+\delta}\right)$, $B_0' = (0,1)$ and $D_0' = \left(\frac{\delta}{2+2\delta}, \frac{1}{2+2\delta}\right)$.

Notice that, when mapped by $\pi^{\chi}_{\xi_0^5}$, the trapezium $\Delta_{\xi_0^5}$ is rotated 180^o, squeezed horizontally, and then stretched vertically. The effect of this map is clearly hyperbolic, at least for $\delta \in (0,1)$. We can also check that the 2×2 matrix above has trace less than -2, when δ ranges from 0 to 1. This affine map has a single hyperbolic fixed point at $P_0 = \left(\frac{\delta}{2+3\delta}, \frac{1+\delta}{2+3\delta}\right) \in \Delta_{\xi_0^5}$.

The polygon $\Delta_{\xi_1^5}$ is another trapezium, defined by the inequalities $u_2 > 0$, $u_3 > 0$, $u_2 + u_3 < 1$ and

$$\frac{2+\delta}{3+2\delta} - \frac{(1-\delta-\delta^2)\,u_2}{3+2\delta} \;<\; u_3 \;<\; \frac{2}{3} - \frac{(1-2\delta)\,u_2}{3} .$$

Alternatively, $\Delta_{\xi_1^5}$ may be described as $\Delta_{\xi_1^5} = [\,A_1,\ B_1,\ C_1,\ D_1\,]$, where $A_1 = \left(0, \frac{2+\delta}{3+2\delta}\right)$, $B_1 = \left(\frac{1}{2+\delta}, \frac{1+\delta}{2+\delta}\right)$, $C_1 = \left(\frac{1}{2+2\delta}, \frac{1+2\delta}{2+2\delta}\right)$ and $D_1 = \left(0, \frac{2}{3}\right)$. Over $\Delta_{\xi_1^5}$, the first return map is

$$\pi^{\chi}_{\xi_1^5}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} \frac{2\delta}{1+\delta} \\ \\ \frac{4+2\delta^{-1}}{1+\delta} \end{pmatrix} + \begin{pmatrix} -\frac{\delta(1-\delta)}{1+\delta} & -\frac{2\delta}{1+\delta} \\ \\ -\frac{1-3\delta^2}{\delta(1+\delta)} & -\frac{3+5\delta}{\delta(1+\delta)} \end{pmatrix}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} .$$

Then the image of $\Delta_{\xi_1^5}$ by this map is $\pi^{\chi}_{\xi_1^5}\left(\Delta_{\xi_1^5}\right) = [\,A_1',\, A_0',\, D_0',\, D_1'\,]$, where $A_1' = \left(\frac{2\delta}{3+2\delta}, \frac{3}{3+2\delta}\right)$ and $D_1' = \left(\frac{2\delta}{3+3\delta}, \frac{2}{3+3\delta}\right)$. Again, the effect of mapping $\pi^{\chi}_{\xi_0^5}$, on the trapezium $\Delta_{\xi_0^5}$ is a 180^o rotation, followed by horizontal squeezing and vertical stretching. Therefore, this map is also hyperbolic, for all $\delta \in (0,1)$, and has a single hyperbolic fixed point at $P_1 = \left(\frac{2\delta}{3+4\delta-\delta^2}, \frac{2+2\delta}{3+4\delta-\delta^2}\right) \in \Delta_{\xi_1^5}$.

The polygon $\Delta_{\xi_0^6}$ is a trapezium defined by the inequalities $u_2 > 0$, $u_3 > 0$, $u_2 + u_3 < 1$ and

$$\frac{1}{2} - \frac{(1-\delta)\,u_2}{2} \quad u_3 \;<\; \frac{2+\delta}{3+2\delta} - \frac{1-\delta-\delta^2}{3+2\delta}\, u_2 \;.$$

Alternatively, $\Delta_{\xi_0^6} = [\,D_0,\, C_0,\, B_1,\, A_1\,]$. In $\Delta_{\xi_0^6}$, the first return map is

$$\pi^{\chi}_{\xi_0^6}\begin{pmatrix} u_2 \\ u_3 \end{pmatrix} = \begin{pmatrix} -\frac{1+\delta+\delta^2}{1+\delta} \\ \\ -\frac{\delta}{1+\delta} \end{pmatrix} + \begin{pmatrix} \frac{1-\delta^2-\delta^3}{1+\delta} & \frac{2+3\delta+2\delta^2}{1+\delta} \\ \\ -\frac{\delta^2}{1+\delta} & \frac{1+2\delta}{1+\delta} \end{pmatrix} \begin{pmatrix} u_2 \\ u_3 \end{pmatrix} .$$

Then, the image polygon is $\pi^{\chi}_{\xi_0^6}\left(\Delta_{\xi_0^6}\right) = [\,D_0',\, O,\, B_1^*,\, A_1^*\,]$, where $A_1^* = \left(\frac{1+2\delta}{3+2\delta}, \frac{2}{3+2\delta}\right)$ and $B_1^* = \left(\frac{1+\delta}{2+\delta}, \frac{1}{2+\delta}\right)$. This map has no fixed point in $\Delta_{\xi_0^6}$.

We set $W^s_{\rm loc}(P_i)$ to be the intersection of the contracting eigenspace of P_i with the polygon $\Delta_{\xi_i^5}$. Analogously, we set $W^u_{\rm loc}(P_i)$ to be the intersection of the expanding eigenspace of P_i with the image polygon $\pi^{\chi}_{\xi_i^5}\left(\Delta_{\xi_i^5}\right)$. Then we define

$$W^s(P_i) = \bigcup_{n\geq 0} \left(\pi^{\chi}_{1^{\rm st}}\right)^{-n} W^s_{\rm loc}(P_i) \quad \text{and} \quad W^u(P_i) = \bigcup_{n\geq 0} \left(\pi^{\chi}_{1^{\rm st}}\right)^{n} W^u_{\rm loc}(P_i) \;.$$

Lemma 12.3. *For all $\delta \in (0,1)$,*

$$W^s_{loc}(P_0) \cap W^u(P_1) \neq \emptyset \quad \textit{and} \quad W^s_{loc}(P_1) \cap W^u_{loc}(P_0) \neq \emptyset \;,$$

with transversal intersections.

Proof. Let ℓ_i^s and ℓ_i^u denote the lines through P_i with the directions of the contracting and expanding eigenspaces of the $\pi^{\chi}_{\xi_i^5}$ fixed point P_i. The hyperbolicity of the branches $\pi^{\chi}_{\xi_i^5}$ guarantees that:

(a) ℓ_0^u cuts the line segment $[A_0', B_0']$.
(b) ℓ_1^s cuts the line segment $[A_1, D_1]$.
(c) ℓ_0^s cuts the line segment $[B_0, C_0]$.
(d) ℓ_1^u cuts the line segment $[D_0', D_1']$.

It's also easy to check analytically that these facts hold for every $\delta \in (0,1)$. From (a) and (b) it follows that ℓ_0^u and ℓ_1^s intersect, the intersection is transversal, and the intersection point belongs to $\pi^{\chi}_{\xi_0^5}\left(\Delta_{\xi_0^5}\right) \cap \Delta_{\xi_1^5}$. Therefore, the local manifolds $W^u_{\rm loc}(P_0)$ and $W^s_{\rm loc}(P_1)$ intersect each other.

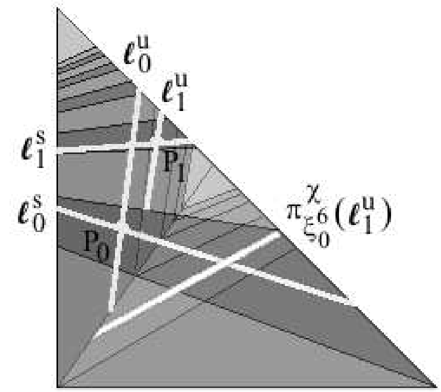

FIGURE 9. A heteroclinic cycle

Analogously, ℓ_1^u and ℓ_0^s also intersect transversally, but the intersection point does not belong to $\pi^{\chi}_{\xi_1^5}\left(\Delta_{\xi_1^5}\right)$ for every $\delta \in (0,1)$, which implies that $W^u_{\rm loc}(P_1)$ and $W^s_{\rm loc}(P_0)$ do not intersect for all $\delta \in (0,1)$. Actually, these local manifolds intersect for every $\delta \in (\frac{1}{2}, 1)$, but not when $\delta > 0$ is small. We have to iterate $W^u_{\rm loc}(P_1)$ forward once more to find an intersection with $W^s_{\rm loc}(P_0)$. See the drawing above. Notice that

(A) $W^u_{\rm loc}(P_1)$ cuts the trapezium $\Delta_{\xi_0^6} = [\,D_0,\, C_0,\, B_1,\, A_1\,]$ through the opposite edges $[A_1, B_1]$ and $[D_0, C_0]$.

This follows from (d) plus the fact that both D_0' and D_1' are bellow the line $u_3 = \frac{1}{2} - \frac{1-\delta}{2}\, u_2$ supporting the lower edge of $\Delta_{\xi_0^6}$, $[D_0, C_0]$. Thus $\pi^{\chi}_{\xi_0^6}\,(W^u_{\rm loc}(P_1))$ cuts the image trapezium $[\,D_0',\, O,\, B_1^*,\, A_1^*\,]$ through the opposite edges $[D_0', O]$ and $[A_1^*, B_1^*]$. To finish the proof we are going to see that

(B) $\pi^{\chi}_{\xi_0^6}\,(W^u_{\rm loc}(P_1))$ intersects transversally the line ℓ_0^s.

Clearly this intersection point is in Δ^2. But since $\ell_0^s \cap \Delta^2 \subseteq \Delta_{\xi_0^5}$, it also belongs to $W^u(P_1) \cap W^s_{\rm loc}(P_0)$. Finally, (B) comes from the fact that ℓ_0^s cuts the line segment $[B_1^*, B_0]$. In order to check this, let s_0 be the slope of ℓ_0^s, and s be slope of the line through P_0 and B_1^*. It is enough to prove that $s_0 < s$. A simple computation shows that $s = -\frac{\delta^2}{2+3\delta+2\delta^2}$. Of course we can also compute s_0 explicitly, but the resulting expression is messy. To bypass this, we can make a simple argument instead. Let $T : \mathbb{R}\cup\{\infty\} \to \mathbb{R}\cup\{\infty\}$ be the fractional linear transformation with the inverse linear action of $\pi^{\chi}_{\xi_0^5}$ on $\mathbb{P}^1 = \mathbb{R} \cup \{\infty\}$. Another simple computation gives us that $T(s) < s$, for every $\delta \in (0,1)$, where

$$T(m) = -\frac{1 + \delta - \delta^2 - \delta^2\, m}{2 + 3\,\delta - \delta^2\, m}\ .$$

But T is strictly increasing, since the linear part of $\pi^{\chi}_{\xi_0^5}$ has determinant 1, while s_0 is a contractive fixed point of T. Therefore, $T(s) < s$ implies $s_0 < s$, and the proof is finished. $\square$

Proof of theorem A. By lemma 12.3, the skeleton vector field χ in (25) has some transversal heteroclinic cycle formed of two heteroclinic orbits. We look at the first return map $\pi^{\chi}_{1^{\mathrm{st}}} : \Pi_{4\,0} \to \Pi_{4\,0}$. Because χ admits the invariant 1-form λ in (26), it is enough to look at the level set $\{\, u \in \Pi_{4\,0} \,:\, \lambda \cdot u = 1 \,\}$. Here, the domain of $\pi^{\chi}_{1^{\mathrm{st}}}$ splits into a countable disjoint union of open convex sets. It is clear that both heteroclinic orbits keep a positive, and bounded from bellow, distance of these domain component boundaries. Thus, by proposition 10.3, each sufficiently high energy level of the Lotka-Volterra system has a transversal heteroclinic cycle, and by Smale's theorem, must have a hyperbolic basic set. $\square$

In theorem A we have assumed for simplicity that $\delta \in (0,1)$, but we believe that the same result holds for all $\delta > 0$. The reason for our restriction is that, for $\delta > 1$, the dynamics becomes harder to analyze due to the presence of an elliptic fixed point P_0 of the main branch $\pi^{\chi}_{\xi_0^5} : \Delta_{\xi_0^5} \to \Delta^2$.

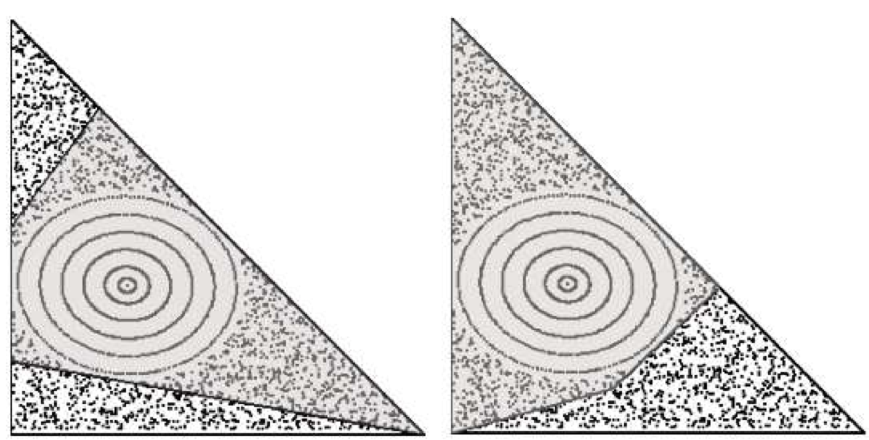

FIGURE 10. An elliptic fixed point P_0 at $\delta = 3.7$

The picture above shows ten different orbits, with 420 iterates each, for a particular parameter. The shaded regions represent the polygon $\Delta_{\xi_0^5}$, on the left, and its image $\pi^{\chi}_{\xi_0^5}\left(\Delta_{\xi_0^5}\right)$, on the right. The invariant curves break up as they touch the boundary of their domains. Outside these curves, the dynamics seems to be chaotic, which indicates the presence of hyperbolicity. Perhaps there are plenty of elliptic periodic points for many parameter values $\delta \in (0,1)$. We believe so, but the elliptic structures in this case should be too small to be easily detected numerically, and should only survive for very small intervals of the parameter δ. In the parameter interval $(0,1)$ hyperbolicity dominates. In this interval, at least in the asymptotic case $\delta \to 0^+$, one should be able to derive, from dynamics analysis, the existence of large uniformly hyperbolic structures, either in the sense of large thickness, or else meaning Hausdorff dimension close to 2. The Newhouse phenomenon, of persistent homoclinic tangencies associated with large thickness hyperbolic sets, is a mechanism for the appearance of many elliptic structures in the dynamics of the underlying Hamiltonian vector field. See for instance [6]. It would be interesting to understand, for conservative skeleton vector fields, the mechanism for the creation of elliptic structures, and then relate it to a corresponding bifurcation mechanism of the underlying dynamics, be it Newhouse phenomenon, or other.

Part 4. **APPENDIXES**

13. The function space $\mathcal{H}(\Gamma^d)$

Denote by $\Omega^1(\Gamma^d)$ the space of $1-$forms on Γ^d which extend analytically to a neighbourhood of Γ^d; by $\mathcal{A}(\Gamma^d, \partial\Gamma^d)$ the space of all analytic functions, defined on $\mathrm{int}(\Gamma^d)$, which have analytic extensions to some neighbourhood of Γ^d minus the union of all boundary level sets $(f_\sigma)^{-1}(0)$; and by $\Omega^1(\Gamma^d, \partial\Gamma^d)$ the space of all analytic $1-$forms on $\mathrm{int}(\Gamma^d)$ which extend analytically to a neighbourhood of Γ^d minus the boundary level sets $(f_\sigma)^{-1}(0)$.

Given $h \in \mathcal{A}(\Gamma^d, \partial\Gamma^d)$ and a family of faces $\{\sigma_i\}_i$, we will say that h *is essentially analytic on* $\cup_i\sigma_i$ when it is analytic in a neighbourhood of Γ^d minus the union of all the other boundary sets, $(f_\sigma)^{-1}(0)$ with $\sigma \neq \sigma_i$ for all i. A similar definition is given for $1-$forms in $\Omega^1(\Gamma^d, \partial\Gamma^d)$. Notice that essential analycity on $\cup_i\sigma_i$ implies analycity over $\mathrm{int}(\cup_i\sigma_i)$, but not on $\partial(\cup_i\sigma_i)$, where both interior and boundary are considered with respect to $\partial\Gamma^d$.

Proposition 13.1. *Given $h \in \mathcal{A}(\Gamma^d, \partial\Gamma^d)$ and a family $\{\sigma_i\}_i \subseteq F$, the following statements are equivalent:*

(i) *for each i, h is essentially analytic on σ_i,*
(ii) *h is essentially analytic on $\cup_i\sigma_i$.*

In particular, if $h \in \mathcal{A}(\Gamma^d, \partial\Gamma^d)$ is essentially analytic on each face $\sigma \in F$, then $h \in \mathcal{A}(\Gamma^d)$. The same statement also holds for $1-$forms in $\Omega^1(\Gamma^d, \partial\Gamma^d)$.

It is obvious that (ii) implies (i). The converse follows from a general principle for analytic functions of more than one complex variable: *A singularity set with codimension 2 is always removable.* See Theorem 8, Chap. I, in [7].

Given a function $h \in \mathcal{A}(\Gamma^d, \partial\Gamma^d)$, we say that the exact $1-$form $dh \in \Omega^1(\Gamma^d, \partial\Gamma^d)$ *has a pole of order $k \geq 1$ along the face σ* if there are $H \in \mathcal{A}(\Gamma^d, \partial\Gamma^d)$ and $\lambda \in \Omega^1(\Gamma^d, \partial\Gamma^d)$, both essentially analytic on σ, such that we can write

$$dh = \lambda + H \, \frac{df_\sigma}{(f_\sigma)^k} \,, \tag{27}$$

with $H \not\equiv 0$ non-identically zero over σ. In fact H must take a constant value c_σ over the level set $(f_\sigma)^{-1}(0)$. Since $0 = d\,dh = d\lambda + dH \wedge df_\sigma/(f_\sigma)^k$, the $2-$form $dH \wedge df_\sigma/(f_\sigma)^k$ must be essentially analytic on σ. This implies that $dH \wedge df_\sigma = 0$ on $(f_\sigma)^{-1}(0)$ and, therefore, H must be constant on this level set. Clearly, this definition does not depend on the choice of the affine functions $\{f_\sigma\}_\sigma$ defining the polyhedron Γ^d.

Proposition 13.2. *The space $\mathcal{H}(\Gamma^d)$, defined as the class of all finite linear combination of the form (15), is characterized as follows: $h \in \mathcal{H}(\Gamma^d)$ if and only if $h \in \mathcal{A}(\Gamma^d, \partial\Gamma^d)$ and for each boundary face $\sigma \in F$, either h is essentially analytic on σ, or dh has a pole of some finite order along σ.*

Moreover, for each $\sigma \in F$, the order $\nu^h(\sigma)$ is zero if h is essentially analytic on σ, or otherwise $\nu^h(\sigma)$ is the pole's order of dh along σ and the character $\lambda^h(\sigma)$ is equal to the coefficient c_σ associated with decomposition (27).

Proof. Let $\mathcal{G}(\Gamma^d)$ be the space of functions $h \in \mathcal{A}(\Gamma^d, \partial\Gamma^d)$ such that for each $\sigma \in F$, either h is essentially analytic on σ, or dh has a pole of some finite order along σ. Define the order $\mu^h(\sigma)$ to be zero if h is essentially analytic on σ, or else $\mu^h(\sigma)$ to be the pole's order of dh along σ and define its character as $\zeta^h(\sigma) = c_\sigma$. With these notations, we want to prove that:

1. $\mathcal{H}(\Gamma^d) = \mathcal{G}(\Gamma^d)$,
2. for each face σ, $\nu^h(\sigma) = \mu^h(\sigma)$, and $\lambda^h(\sigma) = \zeta^h(\sigma)$.

The proof goes on by regressive induction in the order $\mu^h(\sigma)$ of $h \in \mathcal{G}(\Gamma^d)$. The induction step is based on the following:

1. If $h \in \mathcal{G}(\Gamma^d)$ and $\mu^h(\sigma) = 1$ then $g = h - \zeta^h(\sigma)\, \log(f_\sigma) \in \mathcal{G}(\Gamma^d)$ and this function is essentially analytic on σ. Therefore $\mu^g(\sigma) = 0$.
2. If $h \in \mathcal{G}(\Gamma^d)$ and $\mu^h(\sigma) = k \geq 2$ then $g = h - \zeta^h(\sigma)\,(f_\sigma)^{-(k-1)} \in \mathcal{G}(\Gamma^d)$ and the differential of this function has a pole of order $\leq k-1$ along σ. Thus $\mu^g(\sigma) \leq k-1$.

In the end we obtain a remainder $G \in \mathcal{G}(\Gamma^d)$ with order $\mu^G = 0$, meaning that G is essentially analytic in each face $\sigma \in F$. By proposition 13.1 it follows that $G \in \mathcal{A}(\Gamma^d)$. The statements (1) and (2), in turn, follow from an elementary *division lemma*.

Given a regular level set $\Sigma = f^{-1}(0)$ *of an analytic function* f*, if* g *is another analytic function, which vanishes over* Σ*, then there is a function* h*, analytic in the common domain between* f *and* g*, such that* $g = f\,h$*.*

□

Lemma 13.1. *Given* $h \in \mathcal{H}(\Gamma^d)$ *we can write*

$$dh = \sum_{\sigma\in F_A} \Lambda^A_\sigma(p)\,\frac{df_\sigma}{(f_\sigma)^{\nu(\sigma)}}\ ,$$

where each coefficient $\Lambda^A_\sigma \in \mathcal{A}(\Gamma^d, \partial\Gamma^d)$ *is essentially analytic on* $\cup_{\sigma\in F_A}\sigma$ *and furthermore* $\Lambda^A_\sigma(p) \equiv \lambda^h(\sigma)$ *over* σ*.*

Proof. Given any 'order' function $\nu\colon F \to \mathbb{N}$, let $\Omega^1_{A\,\nu}(\Gamma^d)$ be the linear space of all $1-$forms which can be written as

$$\Lambda = \sum_{\sigma\in F_A} \Lambda^A_\sigma(p)\,\frac{df_\sigma}{(f_\sigma)^{\nu(\sigma)}}\ ,$$

where each coefficient $\Lambda^A_\sigma(p)$ is a function in $\mathcal{A}(\Gamma^d, \partial\Gamma^d)$, essentially analytic on $\cup_{\sigma\in F_A}\sigma$, and constant over σ. We want to prove that $dh \in \Omega^1_{A\,\nu}(\Gamma^d)$. By decomposition (16), in definition of space $\mathcal{H}(\Gamma^d)$, it is enough to see that for each $\rho \in F$ and each $0 \le n \le \nu(\rho)$, $\frac{df_\rho}{(f_\rho)^n} \in \Omega^1_{A\,\nu}(\Gamma^d)$.

If $\rho \in F_A$, $\frac{df_\rho}{(f_\rho)^n} = (f_\rho)^{\nu(\rho)-n}\,\frac{df_\rho}{(f_\rho)^{\nu(\rho)}}$ and $(f_\rho)^{\nu(\rho)-n}$ is analytic over Γ^d. Furthermore, this function is constant equal to zero, if $n < \nu(\rho)$, or constant equal to one, when $n = \nu(\rho)$.

If $\rho \notin F_A$, we can write df_ρ as a linear combination $df_\rho = \sum_{\sigma\in F_A} c^\rho_\sigma df_\sigma$. Therefore

$$\frac{df_\rho}{(f_\rho)^n} = \sum_{\sigma\in F_A} c^\rho_\sigma\,\frac{(f_\sigma)^{\nu(\sigma)}}{(f_\rho)^n}\,\frac{df_\sigma}{(f_\sigma)^{\nu(\sigma)}}\ ,$$

where each coefficient $\frac{(f_\sigma)^{\nu(\sigma)}}{(f_\rho)^n}$ is a function in $\mathcal{A}(\Gamma^d, \partial\Gamma^d)$, essentially analytic on $\cup_{\sigma\in F_A}\sigma$, and constant equal to 0 over σ. □

14. The 2−form space $\Omega_0^2(\Gamma^d)$

Let $\Omega_A^2(\Gamma^d)$ be the linear space of all 2−forms which can be written as

$$\omega = \sum_{\sigma,\rho\in F_A} (\Omega_A)_{\sigma\,\rho}(p)\,\frac{df_\sigma\wedge df_\rho}{f_\sigma\, f_\rho}\ ,$$

where each coefficient $(\Omega_A)_{\sigma\,\rho}(p)$ is a function in $\mathcal{A}(\Gamma^d,\partial\Gamma^d)$, essentially analytic on $\cup_{\sigma\in F_A}\sigma$, and constant over $\sigma\cap\rho$.

We want to prove Lemma 9.2 that can be synthesized in the inclusion

$$\Omega_0^2(\Gamma^d)\subseteq\Omega_A^2(\Gamma^d)\ .$$

Proof of Lemma 9.2. It is enough to see that

$$\forall\,\sigma,\,\rho\in F\ ,\quad \frac{df_\sigma\wedge df_\rho}{f_\sigma\, f_\rho}\in\Omega_A^2(\Gamma^d)\ .$$

If $\sigma,\,\rho\in F_A$, this statement is obvious.

If $\sigma_0\in F_A$, but $\rho\notin F_A$, we can write df_ρ as a linear combination $df_\rho = \sum_{\sigma\in F_A} c_\sigma^\rho df_\sigma$. Therefore

$$\frac{df_{\sigma_0}\wedge df_\rho}{f_{\sigma_0}\, f_\rho} = \sum_{\sigma\in F_A} c_\sigma^\rho\,\frac{f_\sigma}{f_\rho}\,\frac{df_{\sigma_0}\wedge df_\sigma}{f_{\sigma_0}\, f_\sigma}\ ,$$

where each coefficient $\frac{f_\sigma}{f_\rho}$ is a function in $\mathcal{A}(\Gamma^d,\partial\Gamma^d)$, essentially analytic on $\cup_{\sigma\in F_A}\sigma$, and constant equal to 0 over $\sigma_0\cap\sigma$. Therefore $\frac{df_{\sigma_0}\wedge df_\rho}{f_{\sigma_0}\, f_\rho}\in\Omega_A^2(\Gamma^d)$.

If $\sigma_0,\,\rho_0\notin F_A$, we can write both df_{σ_0} and df_{ρ_0} as linear combinations $df_{\sigma_0} = \sum_{\sigma\in F_A} c_\sigma^{\sigma_0} df_\sigma$ and $df_{\rho_0} = \sum_{\rho\in F_A} c_\rho^{\rho_0} df_\rho$. Therefore

$$\frac{df_{\sigma_0}\wedge df_{\rho_0}}{f_{\sigma_0}\, f_{\rho_0}} = \sum_{\sigma,\,\rho\in F_A} c_\sigma^{\sigma_0}\, c_\rho^{\rho_0}\,\frac{f_\sigma\, f_\rho}{f_{\sigma_0}\, f_{\rho_0}}\,\frac{df_\sigma\wedge df_\rho}{f_\sigma\, f_\rho}\ ,$$

where each coefficient $\frac{f_\sigma\, f_\rho}{f_{\sigma_0}\, f_{\rho_0}}$ is a function in $\mathcal{A}(\Gamma^d,\partial\Gamma^d)$, essentially analytic on $\cup_{\sigma\in F_A}\sigma$, and constant equal to 0 over $\sigma\cap\rho$. Thus $\frac{df_{\sigma_0}\wedge df_{\rho_0}}{f_{\sigma_0}\, f_{\rho_0}}\in\Omega_A^2(\Gamma^d)$.

□

Remark 14.1. *Theorem B, or proposition 9.3, can be easily generalized to the infinite dimensional space $\Omega^2(\Gamma^d)$ of all closed 2−forms in the intersection of all spaces $\Omega_A^2(\Gamma^d)$.*

15. Pfaffians, determinants and inverses of skew symmetric matrices

Each $d \times d$ skew symmetric matrix $\Omega = (\,\Omega_{i\,j}\,)$ determines the $2-$form

$$\omega = \frac{1}{2} \sum_{i,j=1}^{d} \Omega_{i\,j}\, dx_i \wedge dx_j = \sum_{i<j} \Omega_{i\,j}\, dx_i \wedge dx_j \ . \tag{28}$$

The *Pfaffian* of Ω is defined to be the coefficient $\mathrm{Pf}\,(\Omega)$ in the following development

$$\omega^{(d/2)} = (d/2)!\, \mathrm{Pf}\,(\Omega)\; dx_1 \wedge \cdots \wedge dx_d \ .$$

The Pfaffian $\mathrm{Pf}\,(\Omega)$ is a polynomial function in the coefficients $\Omega_{i\,j}$, which has degree $d/2$, and equals the square root of Ω's determinant,

$$\det \Omega = \mathrm{Pf}\,(\Omega)^2 \ .$$

For each pair of indices $\{i,\, j\} \subseteq \{1, \cdots, d\}$, denote by $\Omega^{i\,j}$ be the $(d-2) \times (d-2)$ skew symmetric sub-matrix of Ω obtained removing both its $i^{\rm th}$ and $j^{\rm th}$ rows, as well as its $i^{\rm th}$ and $j^{\rm th}$ columns. Let us define now the *Pfaffian cofactor entry* $\Omega'_{i\,j}$, of the skew symmetric matrix Ω. We set $\Omega'_{i\,j} = 0$ when $i = j$. Otherwise $\Omega'_{i\,j}$ is, up to some sign, the Pfaffian of the sub-matrix $\Omega^{i\,j}$:

$$\Omega'_{i\,j} = \pm\, (-1)^{i+j}\, \mathrm{Pf}\,(\Omega^{i\,j}) \ . \tag{29}$$

The sign above is chosen to be $+1$, resp. -1, when $i < j$, resp. $i > j$. There is a "Lapace rule" for Pfaffians of skew symmetric matrices. Denoting by $\delta_{i\,k}$ the Kronecker symbols, we have for all $1 \leq i,\, k \leq d$,

$$\delta_{i\,k}\, \mathrm{Pf}\,(\Omega) = \sum_{j=1}^{d} \Omega_{i\,j}\, \Omega'_{j\,k} \ . \tag{30}$$

This proves that the *Pfaffian cofactor matrix* $\Omega' = \left(\, \Omega'_{i\,j} \,\right)$, up to the factor $\mathrm{Pf}\,(\Omega)$, is the inverse matrix of Ω, i.e. $\Omega \cdot \Omega' = \mathrm{Pf}\,(\Omega)\, \mathrm{Id}$. Thus, if Ω in non degenerated then

$$\Omega^{-1} = \frac{1}{\mathrm{Pf}\,(\Omega)}\, \Omega' \ .$$

Both Ω' and Ω^{-1} are still skew symmetric.

Given a vector $c \in \mathbb{R}^d$, consider the following perturbation of the skew symmetric matrix Ω,

$$\Omega(c) = (\,\Omega_{i\,j}(c)\,) = (\,\Omega_{i\,j} + c_j\, \delta_{i\,k} - c_i\, \delta_{j\,k}\,) \ , \tag{31}$$

which is obtained from Ω summing vector c to its $k^{\rm th}$ column and subtracting the same vector to the $k^{\rm th}$ row.

Lemma 15.1. *The Pfaffian cofactor entries of Ω and $\Omega(c)$ are equal along the k^{th} row and column: For all $i = 1, \cdots, d$, $\Omega'_{k\,i}(c) = \Omega'_{k\,i}$ and $\Omega'_{i\,k}(c) = \Omega'_{i\,k}$. In particular, if the Pfaffians $Pf(\Omega)$ and $Pf(\Omega(c))$ have the same sign, then the inverse matrix entries, of Ω^{-1} and $\Omega(c)^{-1}$, also have the same sign along the k^{th} row and column.*

Proof. Because the sub-matrices $\Omega^{k\,i}$ and $\Omega^{i\,k}$ miss all c entries, we have $\Omega(c)^{k\,i} = \Omega^{k\,i}$ and $\Omega(c)^{i\,k} = \Omega^{i\,k}$. Thus, by the Pfaffian cofactors definition (29), we get $\Omega'_{k\,i}(c) = \Omega'_{k\,i}$ and $\Omega'_{i\,k}(c) = \Omega'_{i\,k}$. The inversion formula (30) then implies that $\Omega(c)^{-1}_{k\,i}$ and $\Omega^{-1}_{k\,i}$, resp. $\Omega(c)^{-1}_{i\,k}$ and $\Omega^{-1}_{i\,k}$, have the same sign. □

Lemma 15.2. *The matrix*

$$\Theta(x_1, \cdots, x_d) = \begin{pmatrix} 1 + x_1 & x_2 & \cdots & x_d \\ x_1 & 1 + x_2 & \cdots & x_d \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \cdots & 1 + x_d \end{pmatrix}$$

has determinant equal to $1 + x_1 + \cdots + x_d$.

Proof. The lemma follows by induction from the following equality

$$\det \Theta(x_1, \cdots, x_d) = x_1 + \det \Theta(x_2, \cdots, x_d) ,$$

which is easily checked.

$$\begin{aligned} \det \Theta(x_1, \cdots, x_d) &= x_1 \begin{vmatrix} 1 & x_2 & \cdots & x_d \\ 1 & 1 + x_2 & \cdots & x_d \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_2 & \cdots & 1 + x_d \end{vmatrix} + \begin{vmatrix} 1 & x_2 & \cdots & x_d \\ 0 & 1 + x_2 & \cdots & x_d \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x_2 & \cdots & 1 + x_d \end{vmatrix} \\ &= x_1 \begin{vmatrix} 1 & 0 & \cdots & 0 \\ 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 1 \end{vmatrix} + \begin{vmatrix} 1 + x_2 & \cdots & x_d \\ \vdots & \ddots & \vdots \\ x_2 & \cdots & 1 + x_d \end{vmatrix} \\ &= x_1 + \Theta(x_2, \cdots, x_d) \end{aligned}$$

□

Lemma 15.3. *Given a $d \times d$ skew symmetric matrix $\Omega = (\omega_{i\,j})$, and a vector $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$, the $d \times d$ matrix*

$$\Omega(x) = \left(\omega_{i\,j} + x_j \sum_{k=1}^{d} \omega_{i\,k} + x_i \sum_{k=1}^{d} \omega_{k\,j} \right)_{i,\,j} ,$$

has determinant $\det \Omega(x) = (1 + x_1 + \cdots + x_d)^2 \det \Omega$.

Proof. It is enough to check that $\Omega(x) = \Theta(x)^T \, \Omega \, \Theta(x)$, and then use the previous lemma. $\square$

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