ON THE LADYZHENSKAYA-SMAGORINSKY TURBULENCE MODEL OF THE NAVIER-STOKES EQUATIONS. REGULARITY UP TO THE BOUNDARY.

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Abstract

We establish regularity results *up to the boundary* for solutions to generalized Stokes and Navier-Stokes systems of equations in the stationary and in the evolutive cases. Generalized here means the presence of a shear dependent viscosity. We treat here the case $p \geq 2$. Actually, we are interested in proving regularity results in $L^q(\Omega)$ spaces for all the second order derivatives of the velocity and all the first order derivatives of the pressure.

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1 Introduction and main results

The Navier-Stokes system of equations with shear dependent viscosity has been studied in the last forty years by a great number of researchers, not only in Pure and Applied Mathematics, but also in engineering, physics and biology. A typical model of generalized evolution Navier-Stokes system of equations with shear dependent viscosity is the well known *Ladyzhenskaya model*

$$\left\{ \begin{array}{l} \partial_t u + u \cdot \nabla u - \nabla \cdot T(u,\pi) = f, \\ \nabla \cdot u = 0\,, \end{array} \right. \tag{1.1}$$

where T denotes the stress tensor

$$T = -\pi\, I + \nu_T(u)\, \mathcal{D}\, u\,, \tag{1.2}$$

$$\mathcal{D}\, u = \frac{1}{2}\,(\nabla\, u + \nabla\, u^T)\,,$$

$$\nu_T(u) = \nu_0 + \nu_1 |\mathcal{D}\, u|^{p-2}\,, \tag{1.3}$$

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and ν_0, ν_1 are strictly positive constants. In general we set

$$T_* = \frac{1}{2}(T + T^T), \tag{1.4}$$

where T is a generic tensor field and T^T is it transpose. In particular, $\mathcal{D}\,u = (\nabla\,u)_*$. We remark that the system (1.2) satisfies the Stokes Principle, see [34] and [31] page 231.
In the sequel, we assume that $p \geq 2$. The case $2 < p \leq 3$ has been applied in the last forty years to model turbulence phenomena in fluid flows, a central problem in theoretical and numerical fluid mechanics, meteorology, oceanography, aeronautics, etc. See, for instance, [32], [10] and [25]. In particular, by setting $p = n = 3$, we get the classical Smagorinsky model, introduced by him in [32], as a turbulence model. See [10] and references therein.

Nonlinear shear dependent viscosities are also used to model properties of materials. The cases $p > 2$ and $p < 2$ captures shear thickening and shear thinning phenomena, respectively. See, for instance, [26].
The first mathematical studies on the above class of equations go back to O.A. Ladyzhenskaya in a series of remarkable contributions. See [12], [13], [14] and [15]. In reference [18], Chap.2, n.5, J.-L. Lions considers the case in which $\mathcal{D}\,u$ is replaced by ∇u. However in this case the Stokes principle is not satisfied.

L^q regularity results, up to the boundary, for the second order derivatives of the solutions u to Ladyzhenskaya type models in bounded domains Ω, under the non-slip boundary condition

$$u_{|\Gamma} = 0, \tag{1.5}$$

are stated in [20]. Sharper regularity results, under slip and non-slip boundary conditions, were proved in reference [3] for the stationary problem in the half-space case $\Omega = \mathbb{R}^n_+$. Note that the boundary is flat. Below we consider generalized Stokes and Navier-Stokes equations in the stationary and in the evolution case. The very basic results are those proved for the generalized Stokes stationary problem

$$\begin{cases} -\frac{\nu_0}{2}\,\Delta\,u - \nu_1\,\nabla\cdot(|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u) + \nabla\,\pi = f, \\ \nabla\cdot u = 0\,. \end{cases} \tag{1.6}$$

Note that $\nabla\cdot(\mathcal{D}\,u) = \frac{1}{2}\,\Delta\,u$.

Our first aim here is to show that the regularity results proved in reference [3] for the solutions of the generalized Stokes system (1.6), in the case of a flat boundary, hold in any regular bounded domain Ω. Then we prove that suitable extensions to the stationary and the evolution Navier-Stokes equations hold.
In the sequel Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of class C^2.
It is well known that the extension of the regularity results from flat to non-flat boundaries is not at all trivial for the problems under consideration here. Actually some authors believe that the regularity results could be essentially weaker for non-flat boundaries. In the sequel we succeed in proving that this is not the case. Before stating our main results let us introduce some simplified notation. Weak solutions satisfy the estimates (2.3) and (2.5) below. Hence,

given $f \in L^2(\Omega)$ the quantities $\|\nabla u\|_p$ and $\|\pi\|_{p'}$ are bounded. For convenience we denote by P

$$P = P(\|\nabla u\|_p \,,\, \|\pi\|_{p'}) \tag{1.7}$$

very simple expressions that depend only on the two quantities indicated above. In particular these quantities are independent of the exponent q that appears in equation (1.9). Explicit expressions for these quantities follow immediately from our calculations (we will write explicit expressions up to a certain point). Concerning the lack of dependence on the exponent q, it is worth noting that in the following $2 \leq p \leq 3$ and $\leq q \leq 6$ (more precisely, given a fixed p the exponent q lies in the range $p \leq q \leq q_\infty = 3\,(4-p)$). In particular, Sobolev embedding constants and similar (used in the following) are bounded by constants which are independent of q in the above ranges.
Below we start by proving the following result.

Theorem 1.1. *Assume that*

$$f \in L^2(\Omega) \tag{1.8}$$

and let u, π be the weak solution to problem (1.6) *under the boundary condition* (1.5).

Assume, in addition, that

$$\nabla u \in L^q(\Omega) \tag{1.9}$$

for some $p \leq q \leq 6$. Then

$$\|u\|_{W^{2,r}(\Omega)} + \|\pi\|_{L^r(\Omega)} \leq C\,(1 + \|\nabla u\|_q^{\frac{p-2}{2}})\,(P + \|f\|_2) \tag{1.10}$$

and

$$\|\nabla \pi\|_{L^{\overline{p}}(\Omega)} \leq C\,(1 + \|\nabla u\|_q^{p-2})\,(P + \|f\|_2) \tag{1.11}$$

where r and $\overline{p}$ are given by

$$\frac{1}{r} = \frac{p-2}{2\,q} + \frac{1}{2}\,, \qquad \frac{1}{\overline{p}} = \frac{p-2}{q} + \frac{1}{2}\,. \tag{1.12}$$

Note that weak solutions satisfy the assumption (1.9) if $q = p$. In this case $r = p'$, the dual exponent of p, and $\overline{p} = p_0 = \frac{2p}{3p-4}$. Hence the follwing result holds.

Theorem 1.2. *Assume that*

$$f \in L^2(\Omega) \tag{1.13}$$

and let u, π be the weak solution to problem (1.6) *under the boundary condition* (1.5).

Then

$$\|u\|_{W^{2,p'}(\Omega)} + \|\pi\|_{L^{p'}(\Omega)} \leq C\,(1 + \|\nabla u\|_p^{\frac{p-2}{2}})\,(P + \|f\|_2) \tag{1.14}$$

and

$$\|\nabla \pi\|_{L^{p_0}(\Omega)} \leq C\,(1 + \|\nabla u\|_p^{p-2})\,(P + \|f\|_2)\,. \tag{1.15}$$

Note that both the above right hand sides are bounded by an expression of the form $P\,(1 + \|f\|)$.

By Theorem 1.1 for $q = p$ it follows that $u \in W^{2,p'}$. A Sobolev embedding theorem shows that $u \in W^{1,q_2}$, where $q_2 = (p')^* = \frac{3p}{2p-3}$. If $p < 3$, then q_2 is larger than p. This fact opens the way to a bootstrap argument by applying again Theorem 1.1, now with $q = q_2$. The bootstrap leads to a chain of "intermediate" W^{2,l_n} regularity results, by applying Theorem 1.1 at each stage to the previous value of the parameter q. The Theorem 1.2 is just the first element of this chain. By the above argument we prove an infinite sequence of regularity results (we do not explicitly write these results). A crucial point here is succeeding in "passing to the limit" (as $l_n \to \infty$) in this sequence of regularity results and proving in this way that $u \in W^{2,l}$, where l is the upper bound of the exponents l_n for which $u \in W^{2,l_n}$. This leads to the Theorem 1.3 below. In this theorem the exponent l turns out to be just the exponent for which theorem 1.1 with $q = l^*$ yields $u \in W^{2,l}$. Then, by a Sobolev embedding Theorem, $u \in W^{1,l^*}$. In other words, $q = l^*$ is the (unique) fixed point of the map $q \to r \to r^*$. So, further regularity cannot be obtained by appealing to Theorem 1.1.

Theorem 1.3. *Let f, u and π be as in Theorem 1.2. Then*

$$\|u\|_{W^{2,l}(\Omega)} \leq C\,(P + \|f\|_2) + (P + \|f\|_2)^{\frac{2}{4-p}} \leq P\,(1 + \|f\|_2^{\frac{2}{4-p}})\,, \tag{1.16}$$

where

$$l = 3\,\frac{4-p}{5-p}\,, \tag{1.17}$$

and

$$\|\nabla\,\pi\,\|_m \leq P\,(1 + \|f\|^{\frac{p}{4-p}})\,, \tag{1.18}$$

where

$$m = \frac{6(4-p)}{8-p}\,. \tag{1.19}$$

Remarks.

Note that (1.16) improves (1.14) since $p' < l$ if $2 < p < 3$. Moreover

$$u \in W^{1,l^*}(\Omega)\,,$$

where $l^* = 3\,(4-p)$. Clearly $l^* > p$ for $2 < p < 3$. In addition, $u \in C^{0,\alpha}(\Omega)$, where $\alpha = \frac{3-p}{4-p}$. Also note that $m > p'$ if $p < 2 + \frac{2}{5}$.

It is significant that, when $p = 2$, the statements and estimates established in Theorems 1.2 and 1.3 coincide with the classical results for the linear Stokes problem.

The extension to the stationary generalized Navier-Stokes system (1.20) below is straightforward.

Theorem 1.4. *All the regularity results stated in the Theorems 1.1, 1.2 and 1.3 hold for the generalized Navier-Stokes equations*

$$\begin{cases} -\frac{\nu_0}{2}\,\Delta\,u - \nu_1\,\nabla\cdot(|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u) + u\,\cdot\,\nabla\,u + \nabla\,\pi = f, \\ \nabla\cdot u = 0\,. \end{cases} \tag{1.20}$$

Moreover the estimates in the above statements hold provided that $\|f\|$ is replaced by $P\,\|f\|$.

As shown in the proofs below the above regularity theorems hold locally. More precisely, if (u, π) is a weak solution in some neighborhood of a point $x_0 \in \Gamma$ then the regularity results hold, say, in a neighborhood of one-half the radius. We also remark that stronger regularity results hold for derivatives of u and π in the tangential directions.

Concerning the evolution problem

$$(1.21)\qquad \begin{cases} \partial_t\, u + (u\cdot\nabla)\, u - \nu_0\, \nabla\cdot\mathcal{D}\, u - \nu_1\, \nabla\cdot\big(|\mathcal{D}\, u|^{p-2}\,\mathcal{D}\, u\big) + \nabla\,\pi = f, \\ \nabla\cdot u = 0\,, \\ u(0) = u_0(x)\,. \end{cases}$$

we prove the following results.

Theorem 1.5. *Let u be a weak solution to problem* (1.21) *under the boundary condition* (1.5)*, where $u_0 \in V_p$ and $f \in L^2(0,T;L^2)$. Assume that p satisfies* (11.8)*. Then*

$$(1.22)\qquad \begin{cases} u \in L^2(0,T;W^{2,p'}) \cap L^\infty(0,T;W^{1,p})\,, \\ \nabla\,\pi \in L^2(0,T;L^{p_0})\,, \\ \partial_t\, u \in L^2(0,T;L^2)\,. \end{cases}$$

The assumption (11.8) *is not necessary if the convective term is not present in the equations.*

Theorem 1.6. *Under the assumptions of Theorem 1.5 one has*

$$(1.23)\qquad \begin{cases} u \in L^{4-p}(0,T;W^{2,\,l}) \cap L^\infty(0,T;W^{1,p})\,, \\ \nabla\,\pi \in L^{\frac{2(4-p)}{p}}(0,T;L^m)\,, \\ \partial_t\, u \in L^2(0,T;L^2)\,. \end{cases}$$

The assumption (11.8) *is not necessary if the convective term is not present in the equations.*

Remark 1.1. ***On the regularity up to the boundary.*** *In going from interior to boundary regularity, when $p \neq 2,$, quite specific obstacles appear, obstacles which are unusual in other typical problems in fluid mechanics. An indication of this situation in our results is the lower regularity obtained for the second order derivatives of the velocity (and for the first order derivatives of the pressure) in the normal direction in comparison to that in the tangential directions. Let us illustrate the main reason for this situation. In proving interior regularity by the classical translation method, the translations are admissible in all the n independent directions. This allows suitable L^2-estimates for $\nabla\,\mathcal{D}\, u$, where the* *full gradient ∇ is obtained thanks to the possibility of appealing to translations in all the directions. On the other hand, it is easily shown that $c\,|\nabla\,\nabla\,u| \leq |\nabla\,\mathcal{D}\,u| \leq C\,|\nabla\,\nabla\,u|$. These two facts together lead to a small distinction if we replace $\mathcal{D}u$ by ∇u in the expression of the stress tensor. However, in proving regularity up to the boundary, the two cases are completely distinct, as is well known to authors acquainted with these b.v. problems (in fact, solutions to the J.-L. Lions model belong to $W^{2,2}$ up to the boundary). It seems not accidental that there is a very extensive literature on interior regularity for the above class of problems but, as far as we know, scant literature concerning regularity up to the boundary, in the $3-D$ case.*

Remark 1.2. ***The slip boundary condition.*** *In reference [3] we prove regularity results also for solutions to the slip (or Navier) boundary condition. The same extension can be done here in the case of an arbitrary, regular, boundary Γ. However this requires more careful transformation formulae for vector fields under local changes of coordinates* (3.2)*. In fact, in the sequel, we change each of the three components of a vector field as if they where scalars. This is sufficient under the simpler boundary condition* (1.5) *since the constraint $u = 0$ on the boundary is preserved by the above transformation. However, in the case of the slip boundary condition, the constraint that should be preserved is tangency to the boundary. Hence, in order to extend our results to the slip boundary condition we should use the covariant transform of a vector field v given by*

$$\widetilde{v}_j(y) = v_j\,; \quad \widetilde{v}_3(y) = v_3 - (\partial_1\,h)\,v_1 - (\partial_2\,h)\,v_2\,, \tag{1.24}$$

where $j = 1, 2$, and the v_i functions are calculated at the point $(y', y_3 + h(y'))$ (the notation is that introduced in the sequel). See Equation (4.5) in reference [3]. We believe that in this case there is not a substantial difference in the calculations to be done. However the situation becomes more elaborate.
For the use of the transformation (1.24) *in order to work under the slip boundary condition, we also refer the reader to reference [2].*

Remark 1.3. ***On the case $p < 2$.*** *We believe that the main lines of the method followed here can be adapted to the case in which $p < 2$. As for the case $p > 2$, the main point should be the study of the generalized stationary Stokes equations, since regularity results for the evolution problem, as well as in the presence of the convective term $(u \cdot \nabla)\,u$, may be essentially obtained from the regularity results for the Stokes stationary problem. For $p < 2$, it seems clear that the convective term precludes sharp regularity results for the evolution equations (as for the classical case $p = 2$). However, for the classical stationary Navier-Stokes equations, we get regularity results even for $n = 4$. This clearly means that, by assuming $n = 3$, sharp regularity results up to the boundary are to be expected for values of $p < 2$, provided that these values remain sufficiently near the value* 2.

2 Weak Solutions

NOTATION. The symbol $\|\,.\,\|_p$ denotes the canonical norm in $L^p(\Omega)$. Further, $\|\,.\,\| = \|\,.\,\|_2$. We denote by $W^{k,p}(\Omega), k$ a positive integer and $1 < p < \infty$, the usual Sobolev space of order k, by $W^{1,p}_0(\Omega)$ the closure in $W^{1,p}(\Omega)$ of $C^\infty_0(\Omega)$

and by $W^{-1,p'}(\Omega)$ the strong dual of $W_0^{1,p}(\Omega)$, where $p' = p/(p-1)$. The canonical norms in these spaces are denoted by $\| \,.\, \|_{k,p}$. $L^p_{\#}(\Omega)$ denotes the subspace of L^p consisting of functions with vanishing mean value.

In notation concerning duality pairings and norms, we will not distinguish between scalar and vector fields. Very often we also omit from the notation the symbols indicating the domains Ω or Γ, provided that the meaning remains clear.

We set

$$\mathbb{L}^p(\Omega_0) = [L^p(\Omega_0)]^3, \quad \mathbb{W}^{k,p}(\Omega_0) = [W^{k,p}(\Omega_0)]^3\,, \quad \mathbb{W}_0^{1,p}(\Omega_0) = [W_0^{1,p}(\Omega_0)]^3\,,$$

for any open subset Ω_0 of $\mathbb{R}^3$. We remark that $\|\mathcal{D}\, v\|_p$ is a norm in $\mathbb{W}_0^{1,p}$, which is equivalent to the canonical norm.

We set

$$V_p = \left\{ v \in W^{1,p}(\Omega) : \, (\nabla \cdot v)_{|\Omega} = 0\,; \; v_{|\Gamma} = 0 \right\}\,.$$

Note that, by appealing to inequalities of Korn's type, one shows that there is a positive constant c such that

$$\|\nabla v\|_p + \|v\|_p \leq c\,\|\mathcal{D} v\|_p \tag{2.1}$$

for each $v \in V_p$. Hence the two quantities above are equivalent norms in V_p. We denote by c, $\overline{c}$, c_1, c_2, etc., positive constants that depend, at most, on Ω, ν_0, ν_1 and p. Nevertheless we allow the constants ν_0 and ν_1 to appear each time this helps to understand the passage from one equation to the next one.

The dependence of the constants c on p is not crucial provided that $1 < p_0 \leq p \leq p_1 < \infty$.

The same symbol c may denote different constants, even in the same equation.

Definition. We say that a pair (u, π) is a <u>weak solution</u> of problem (1.6), (1.5) if it belongs to $\mathbb{W}_0^{1,p}(\Omega) \times L^{p'}_{\#}(\Omega)$, and if it satisfies

$$\begin{aligned} &\tfrac{\nu_0}{2} \int_\Omega \nabla u \cdot \nabla \phi \, dx + \nu_1 \int_\Omega |\mathcal{D} u|^{p-2}\, \mathcal{D} u \cdot \mathcal{D} \phi \, dx \\ &- \int_\Omega \pi\, (\nabla \cdot \phi)\, dy + \int_\Omega (\nabla \cdot u)\, \psi \, dx = \int_\Omega f \cdot \phi \, dx\,. \end{aligned} \tag{2.2}$$

for each $(\phi, \psi) \in \mathbb{W}_0^{1,p}(\Omega) \times L^{p'}_{\#}(\Omega)$.

Since a solution u of (2.2) necessarily satisfies

$$\int_\Omega \nabla \cdot u \, dx = 0\,,$$

it readily follows that (2.2) holds for each $(\phi, \psi) \in \mathbb{W}_0^{1,p}(\Omega) \times L^{p'}(\Omega)$.

Existence and uniqueness of the above solution is well known. By setting $(\phi, \psi) = (u, \pi)$ in the above definition we show that weak solutions satisfy, in particular, the estimates

$$\begin{cases} \nu_0\, \|\nabla u\| \leq c\, \|f\|\,, \\ \\ \nu_1\, \|\nabla u\|_p \leq c\, \|f\|_{p'}^{\frac{1}{p-1}}\,. \end{cases} \tag{2.3}$$

Moreover, by setting in (2.2) $\psi = 0$ and by using test-functions $\phi \in C_0^\infty(\Omega)$ one gets

$$\nabla \pi = - \nabla \cdot \left[\nu_0 \nabla u + \nu_1 |\mathcal{D}u|^{p-2} \mathcal{D}u \right] + f \,. \tag{2.4}$$

By appealing to a classical result of Nečas we prove that

$$\|\pi\|_{L^{p'}} \leq c(\nu_0 \|\nabla u\| + \nu_1 \|\mathcal{D}u\|_p^{p-1} + \|f\|_{p'}) \,. \tag{2.5}$$

For convenience we fix π by assuming that its mean value in Ω vanishes.
Now let $\Omega_0 \subset \Omega$ be an arbitrary open subset of Ω (in particular Ω itself), assume that $\nabla u \in L^s(\Omega)$, set $\psi = 0$ and use test-functions $\phi \in C_0^\infty(\Omega_0)$. Then

$$\|\pi\|_{L_{\#}^{\frac{s}{p-1}}} \leq c(\nu_0 \|\nabla u\|_{\frac{s}{p-1}} + \nu_1 \|\mathcal{D}u\|_s^{p-1} + \|f\|_2) \,, \tag{2.6}$$

where norms relate to Ω_0 and $\frac{s}{p-1} \leq 6$.
In particular

$$\|\pi\|_{L^{\frac{s}{p-1}}} \leq c(\nu_0 \|\nabla u\|_{\frac{s}{p-1}} + \nu_1 \|\mathcal{D}u\|_s^{p-1} + \|f\|_2 + \|\pi\|_{p'}) \,, \tag{2.7}$$

in any $\Omega_0 \subset \Omega$.

3 The change of variables

In order to reduce our problem, by a suitable change of variables, to a problem involving a flat boundary, we need to consider functions with a sufficiently small support.

Let $x_0 \in \Gamma$ be given and let Π be the tangent plane to Γ at x_0. We assume that the axes of x_i, $i = 1, 2, 3$, are such that the origin coincides with x_0 and the x_3 axis has the direction of the inward normal to Γ at x_0. Hence the axes of x_i, $i = 1, 2$, lie in the plane Π. We may use this particular system of coordinates since the analytical expressions that appear on the left hand side of (2.2) are invariant under orthogonal transformations, due to the invariance of the analytical expressions of the divergence and gradient.

We assume that Γ is a manifold of class C^2. Let $x_0 \in \Gamma$ be given and let $(x', x_3) = (x_1, x_2, x_3)$, be the above system of coordinates. There is a positive real a and a real function $x_3 = \eta(x')$, of class C^2 defined on the sphere $\{x' : |x'| < a\}$, such that: the points x for which $x_3 = \eta(x')$ belong to Γ; and the points x for which $\eta(x') < x_3 < a + \eta(x')$ belong to Ω; the points x for which $-a + \eta(x') < x_3 < \eta(x')$ belong to $\mathbb{R}^3 - \Omega$. Without loss of generality, we assume that $a \leq 1$. We define

$$\begin{aligned} I_a &= \{x : |x'| < a, \ -a + \eta(x') < x_3 < a + \eta(x')\} \,, \\ \Omega_a &= \{x \in I_a \,:\, \eta(x') < x_3\} \,, \\ \Gamma_a &= \{x \in I_a \,:\, x_3 = \eta(x')\} \,. \end{aligned} \tag{3.1}$$

Clearly $\Omega_a = \Omega \cap I_a$ and $\Gamma_a = \Gamma \cap I_a$.

Actually, we extend the function $\eta(x')$ to the whole of Ω_a by setting $\eta(x', x_3) = \eta(x')$. Nevertheless, since η is independent of x_3, we use the notation $\eta(x')$.

Next we introduce the change of variables $y = T\,x$ given by

$$(y_1, y_2, y_3) = (x_1, x_2, x_3 - \eta(x')), \ \ (x_1, x_2, x_3) = (y_1, y_2, y_3 + \eta(y')) \,, \tag{3.2}$$

and set

$$
(3.3) \qquad \begin{aligned} J_a &= \{y : |y'| < a, \ -a < y_3 < a\}, \\ Q_a &= \{y \in J_a \ : \ 0 < y_3\}, \\ \Lambda_a &= \{y \in J_a \ : \ y_3 = 0\}, . \end{aligned}
$$

The map T is a C^2 diffeomorphism of I_a onto J_a, that maps Ω_a onto Q_a and Γ_a onto Λ_a. Note that the Jacobian determinant of the map T is equal to 1.

We define functions $\widetilde{g}$ by setting $\widetilde{g}(y) = g(x)$ or, more precisely, by

$$
(3.4) \qquad \widetilde{g}(y) = g(T^{-1}(y)),
$$

where g denotes here an arbitrary scalar or vector field. As a notation rule, $g = g(x)$ and $\widetilde{g} = \widetilde{g}(y)$. Moreover, partial derivatives and differential operators when applied to functions g concern the x variables and when applied to functions $\widetilde{g}$ concern the y variables. We primarily use the notation $\partial_k\, g$ instead of $\frac{\partial\, g}{\partial\, x_k}$. Hence

$$
\partial_k\, \widetilde{g} = \frac{\partial\, \widetilde{g}(y)}{\partial\, y_k}
$$

and

$$
\partial_k\, g = \frac{\partial\, g(x)}{\partial\, x_k}\,.
$$

Note the distinction between $\widetilde{\nabla f}$ and $\nabla \widetilde{f}$. Actually, $\widetilde{\nabla f}(y) = (\nabla_x f)(T^{-1}(y))$ and $(\nabla \widetilde{f})(y) = \nabla_y[f(T^{-1}(y))]$.

Since some expressions are quite long, in addition to the "tilde" notation we also use the symbol $\mathcal{T}$ to denote the map $f \to \widetilde{f}$. In other words,

$$
(\mathcal{T}\, f)(y) = \widetilde{f}(y)\,.
$$

Vector fields are transformed here coordinate by coordinate (as independent scalars). More precisely

$$
(3.5) \qquad \widetilde{v}_j(y) = v_j(x) = v_j(y',\, y_3 + \eta(y'))\,,
$$

where $j = 1, 2, 3$. Conversely,

$$
(3.6) \qquad v_j(x) = \widetilde{v}_j(y) = v_j(x', x_3 - \eta(x'))\,.
$$

Given x, if $y = T\,x$ then $y' = x'$. Hence $\widetilde{\eta}(y) = \eta(x) = \eta(x') = \eta(y')$, moreover $\frac{\partial\,\eta(x')}{\partial\, x_j} = \frac{\partial\,\eta(y')}{\partial\, y_j}$, and so on. In the sequel we identify the above functions and use the sole notation $\eta(y')$.

We set

$$
\begin{aligned} \mathbb{V}(\Omega_a) &= \{v : v \in \mathbb{W}^{1,p}_0(\Omega_a),\ supp\ v \subset I_a\}\,, \\ \mathbb{V}(Q_a) &= \{\widetilde{v} : \widetilde{v} \in \mathbb{W}^{1,p}_0(Q_a),\ supp\ \widetilde{v} \subset J_a\}\,. \end{aligned}
$$

Clearly, if a test function $\phi(x)$ belongs to $\mathbb{V}(\Omega_a)$ the transformed function $\widetilde{\phi}(y)$ belongs to $\mathbb{V}(Q_a)$.

A main point in the sequel is that

$$
(3.7) \qquad \partial_j\, \eta(0) = 0, \qquad j = 1, 2\,,
$$

which holds since Π is tangential to Γ at x_0. The following result is a consequence of (3.7) together with the continuity of $\nabla\,\eta$ over Γ.

Lemma 3.1. *Given an ϵ_0 there is an $a(\epsilon_0) > 0$ such that if $a \leq a(\epsilon_0)$ then*

$$|(\nabla \eta(y')| < \epsilon_0 \,, \qquad \forall\, y' \quad \textit{such that} \quad |y'| < a \,. \tag{3.8}$$

Moreover, $a(\epsilon_0)$ is independent of the point x_0.

Note that $a(\epsilon_0)$ depends on the $C^1(J_a)$ norm of η. Since Γ is compact the desired independence holds.

From (3.4) it follows that, for each y,

$$\widetilde{\partial_k \phi} = \partial_k \widetilde{\phi} - (\partial_k \eta)(y')\, \partial_3 \widetilde{\phi} \,, \tag{3.9}$$

where $\partial_k \eta$ is calculated at y'. Note that the first equation (3.9) holds for $k = 3$ since $\partial_3 \eta = 0$. By iteration, the above formula may be extended to higher order derivatives (not used in the sequel):

$$\mathcal{T}(\partial^2_{jk} \phi) = \partial^2_{jk} \widetilde{\phi} - (\partial_k \eta)\, \partial^2_{j3} \widetilde{\phi}$$

$$- (\partial_j \eta)\, \partial^2_{k3} \widetilde{\phi} + (\partial_j \eta)\, (\partial_k \eta)\, \partial^2_3 \widetilde{\phi} - (\partial^2_{jk} \eta)\, \partial_3 \widetilde{\phi} \,.$$

REMARK. We want to emphasize that, basically, our regularity results will be proved in the following local form. Let x_0 and Ω_a be as above. If $(u, \pi) \in \mathbb{W}^{1,p}(\Omega_a) \times L^{p'}(\Omega_a)$ satisfies (1.6) in the weak sense in Ω_a and satisfies (1.5) in Γ_a, then the regularity results hold in Ω_r for $r < a$ (for instance, for $r = \frac{a}{2}$). We prove this local result by assuming that $a > 0$ is sufficiently small. Our final value of a is not necessarily equal to the initial one. As we proceed through the proof we may need to consider smaller values of a. However we will show explicitly that each new (smaller) value of a depends only on an upper bound of the $C^2(J_a)$ norm of η. In particular, a positive lower bound for a, independent of the point x_0, exists since Γ is compact. This leads to the global result in the whole of Ω.

4 Translations and related properties

In the sequel we deal with translations of h_j in the y_j-direction, $j = 1, 2$. For notational convenience we consider the case $j = 1$ and set $h = h_1$. We use the following convention:

$$y + h = (y_1 + h, y_2, y_3), \qquad y' + h = (y_1 + h, y_2) \,.$$

The amplitude $|h|$ of the translations is always assumed to be smaller than the distance from the support of $\widetilde{\phi}$ to the set $(\partial\, Q_a) \setminus \Lambda_a$.

A test-function $\phi(x)$ is transformed into a function $\widetilde{\phi}(y)$. Since in the following we made translations in the y variables we need to determine (and study the differential properties) of the test function $\phi_h(x)$ such that $\widetilde{(\phi_h)}(y) = \widetilde{\phi}(y + h)$. This is the aim of this section.

Lemma 4.1. *Let $\phi(x) \in \mathbb{V}(\Omega_a)$. Define ϕ_h by*

$$\phi_h(x) = \phi(x_1 + h, x_2, x_3 - \eta(x') + \eta(x' + h)) \,. \tag{4.1}$$

Then

$$\widetilde{\phi}_h(y) = \widetilde{\phi}(y + h) \,. \tag{4.2}$$

The verification is left to the reader.

Next we want to establish the transformation law for derivatives of the "pseudo-translations" $\phi_h(x)$. One has the following result.

Lemma 4.2. *Let $\phi(x) \in \mathbb{V}(\Omega_a)$, let $\phi_h(x)$ be as in the previous lemma, and let $k \leq 3$ be fixed. Then*

$$(4.3)\quad (\widetilde{D_k\phi_h})(y) = (\widetilde{D_k\phi})(y+h) + (\widetilde{D_3\phi})(y+h)\,[(D_k\eta)(y'+h) - (D_k\eta)(y')]\,.$$

If $k = 3$ the second term on the right hand side vanishes identically.

Proof. From (4.1) it readily follows that

$$(4.4)\quad \begin{aligned} (\partial_k\phi_h)(x) = &\ (\partial_k\phi)((x'+h,\, x_3 + \eta(x'+h) - \eta(x'))+ \\ &(\partial_n\phi)((x'+h,\, x_3 + \eta(x'+h) - \eta(x'))\,[(\partial_k\eta)(x'+h) - (\partial_k\eta)(x')]\,. \end{aligned}$$

Note that the last term is not taken into account if $k = 3$. By the definition of the "tilde" functions

$$(\widetilde{\partial_k\phi_h})(y) = (\partial_k\phi_h)(T^{-1}y) = (\partial_k\phi_h)(\overline{x})$$

where

$$\overline{x} = (\overline{x}_1, \overline{x}_2, \overline{x}_3) = (y', y_3 + \eta(y'))\,.$$

Hence from (4.4) with x replaced by $\overline{x}$ we get an expression for $(\widetilde{D_k\phi_h})(y)$ in terms of $\overline{x}$. By taking into account the definition of $\overline{x}$ we obtain

$$(4.5)\quad \begin{aligned} (\widetilde{\partial_k\phi_h})(y) = &\ (\partial_k\phi)(y'+h,\, y_3 + \eta(y'+h))+ \\ &(\partial_3\phi)(y'+h,\, y_3 + \eta(y'+h))\,[(\partial_k\eta)(y'+h) - (\partial_k\eta)(y')]. \end{aligned}$$

Since $(y'+h,\, y_3 + \eta(y'+h)) = T^{-1}(y+h)$ it follows that

$$(\partial_k\phi)(y'+h,\, y_3 + \eta(y'+h)) = (\widetilde{\partial_k\phi})(y+h)\,.$$

Consequently (4.3) follows from (4.5). $\square$

By setting in general

$$(\nabla\phi)_{ik} = \partial_k\phi_i$$

it follows from (4.3) that

$$(4.6)\quad (\widetilde{\nabla\phi_h})(y) = (\widetilde{\nabla\phi})(y+h) + (\widetilde{\partial_3\phi})(y+h) \otimes [(\nabla\eta)(y'+h) - (\nabla\eta)(y')]$$

where, since η does not depend on the 3-rd. variable, we set

$$\nabla\eta = (\partial_1\eta, \partial_2\eta)\,.$$

In particular, since $\mathcal{D}u = (\nabla u)_*$,

(4.7)
$$(\widetilde{\mathcal{D}\phi_h})(y) = (\widetilde{\mathcal{D}\phi})(y+h) + \Big\{(\widetilde{\partial_3\phi})(y+h) \otimes [(\nabla\eta)(y'+h) - (\nabla\eta)(y')]\Big\}_*\,.$$

Moreover,

$$(4.8)\quad (\widetilde{\nabla\cdot\phi_h})(y) = (\widetilde{\nabla\cdot\phi})(y+h) + (\widetilde{\partial_3\phi})(y+h)\cdot[(\nabla\eta)(y'+h) - (\nabla\eta)(y')]\,.$$

In the sequel we express the derivatives with respect to the y variables of functions $\widetilde{\phi}(y)$ in terms of the transformations of the derivatives of the original functions $\phi(x)$.

Lemma 4.3. *One has the following formulas*

$$(\partial_k \widetilde{\phi})(y) = (\widetilde{\partial_k \phi})(y) + (\partial_k \eta)(y')\, (\widetilde{\partial_3 \phi})(y)\,, \tag{4.9}$$

and

$$(\widetilde{\partial_k \phi})(y) = (\partial_k \widetilde{\phi})(y) - (\partial_k \eta)(y')\, (\partial_3 \widetilde{\phi})(y) \tag{4.10}$$

If $k = 3$ the second terms on the above right hand sides vanish identically.

Proof. Since

$$\widetilde{\phi}(y) = \phi(T^{-1} y)$$

it follows that

$$(\partial_k \widetilde{\phi})(y) = (\partial_k \phi)(T^{-1} y) + (\partial_3 \phi)(T^{-1} y)\, (\partial_k \eta)(y') =$$
$$(\widetilde{\partial_k \phi})(y) + (D\partial_k \eta)(y')\, (\widetilde{\partial_3 \phi})(y)\,.$$

Note that $\partial_3 \widetilde{\phi} = \widetilde{\partial_3 \phi}$. □

From the above Lemma it follows that

$$(\widetilde{\nabla \phi})(y) = (\nabla \widetilde{\phi})(y) - (\nabla \eta)(y') \otimes (\partial_3 \widetilde{\phi})(y) \tag{4.11}$$

and that

$$(\widetilde{\nabla \cdot \phi})(y) = (\nabla \cdot \widetilde{\phi})(y) - (\nabla \cdot \eta)(y')(\partial_3 \widetilde{\phi})(y)\,. \tag{4.12}$$

Lemma 4.4. *Given an $\epsilon_0 \in]0, 1[$ there is an $a(\epsilon_0) > 0$ such that if $a \leq a(\epsilon_0)$ then*

$$|(\nabla \widetilde{\phi})(y) - (\widetilde{\nabla \phi})(y)| \leq \epsilon_0\, |(\widetilde{\partial_3 \phi})(y)|\,, \qquad \forall\, y \in Q_a\,. \tag{4.13}$$

The same result holds if we replace y by $y - h$. Clearly we may replace ∇ by $\mathcal{D}$.

Proof. From (4.9) one shows that the left hand side of (4.13) is bounded by $|\nabla \eta(y')|\, |(\widetilde{\partial_3 \phi})(y)|$. Since $\nabla\eta(0) = 0$ it follows that $|\nabla\eta(y')| \leq \epsilon_0$ in a sufficiently small neighborhood of x_0. □

REMARK. Note that the identity $\nabla \phi(x) = (\widetilde{\nabla \phi})(y)$ together with (4.13) leads to a "point wise equivalence" between $|\nabla \phi(x)|$, $|(\widetilde{\nabla \phi})(y)|$ and $|(\nabla \widetilde{\phi})(y)|$. In particular, the L^q-norms of these quantities are equivalent.

Lemma 4.5. *Given an $\epsilon_0 \in]0, 1[$ there is an $a(\epsilon_0) > 0$ such that if $a \leq a(\epsilon_0)$ then*

$$\begin{aligned} &\Big|\Big((\widetilde{\nabla \phi})(y) - (\widetilde{\nabla \phi})(y-h)\Big) - \Big((\nabla \widetilde{\phi})(y) - (\nabla \widetilde{\phi})(y-h)\Big)\Big| \leq \\ &\epsilon_0\, |(\partial_3 \widetilde{\phi})(y) - (\partial_3 \widetilde{\phi})(y-h)| + |h|\, \|\eta\|_{C^2(Q_a)}\, |(\partial_3 \widetilde{\phi})(y-h)|\,. \end{aligned} \tag{4.14}$$

Proof. From (4.10) one has

$$\Big((\widetilde{\nabla \phi})(y) - (\widetilde{\nabla \phi})(y-h)\Big) - \Big((\nabla \widetilde{\phi})(y) - (\nabla \widetilde{\phi})(y-h)\Big) =$$
$$- \nabla \eta(y') \otimes \Big((\partial_3 \widetilde{\phi})(y) - (\partial_3 \widetilde{\phi})(y-h)\Big)$$
$$- (\nabla \eta(y') - \nabla \eta(y'-h)) \otimes (\partial_3 \widetilde{\phi})(y-h)\,.$$

Hence, in a sufficiently small neighborhood of x_0, (4.14) holds. □

5 Estimates for some second order derivatives of the velocity in terms of the pressure

FURTHER NOTATION. For convenience, in the sequel C denotes positive constants which are bounded from above provided that the quantities $\|\nabla\,\eta\|_{C^1(\Lambda_a)}$ and $\|\nabla\,\theta\|_{C^1(\Omega_a)}$ are bounded from above. Nevertheless, for the reader's convenience (and for completeness) we often write the explicit dependence on the above quantities before including them in a constant of type C. Multiplicative constants of type c will be incorporated in C.

In the sequel, in the absence of an explicit indication, tilde-functions inside integrals are calculated at the generic point y. Compare equations (5.1) and (5.2). Moreover, in the absence of an explicit indication, norms of functions of the x variable concern the domain Ω_a and norms of tilde-functions concern the domain Q_a.

From (2.2), by making the the change of variables $x \to T x = y$, it follows that

$$
\begin{aligned}
&\tfrac{\nu_0}{2}\int \widetilde{\nabla u}\cdot\widetilde{\nabla\phi}\,dy+\nu_1\int|\widetilde{\mathcal{D}u}|^{p-2}\widetilde{\mathcal{D}u}\cdot\widetilde{\mathcal{D}\phi}\,dy\\
&-\int\widetilde{\pi}\,(\widetilde{\nabla\cdot\phi})\,dy+\int(\widetilde{\nabla\cdot u})\,\widetilde{\psi}\,dy=\int\widetilde{f}\cdot\widetilde{\phi}\,dy\,.
\end{aligned}
\tag{5.1}
$$

for each $\widetilde{\phi}\in\mathbb{V}(Q_a)$ and each $\widetilde{\psi}\in L^p(Q_a)$. Recall that the Jacobian determinant of the T-transform is equal to one.

Next we consider the equation (5.1) with ϕ and ψ replaced by the admissible test functions ϕ_h and ψ_h respectively. Then by the change of variables $y\to y-h$ we show that

$$
\begin{aligned}
&\tfrac{\nu_0}{2}\int\widetilde{\nabla u}(y-h):\widetilde{\nabla\phi_h}(y-h)\,dy+\\
&\nu_1\int|\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h):\widetilde{\mathcal{D}\phi_h}(y-h)\,dy\\
&-\int\widetilde{\pi}(y-h)\,(\widetilde{\nabla\cdot\phi_h}(y-h))\,dy+\int(\widetilde{\nabla\cdot u}(y-h))\,\widetilde{\psi_h}(y-h)\,dy=\\
&\int\widetilde{f}(y-h)\cdot\widetilde{\phi}_h(y-h)\,dy\,.
\end{aligned}
\tag{5.2}
$$

for each $\widetilde{\phi}\in\mathbb{V}(Q_a)$ and each $\widetilde{\psi}\in L^p(Q_a)$.

By appealing to (4.2), (4.3), (4.6), (4.7) and (4.8) we may write the equation

(5.2) in the form

(5.3)

$$
\begin{aligned}
&\tfrac{\nu_0}{2}\int \widetilde{\nabla u}(y-h):\widetilde{\nabla\phi}(y)\,dy \\
&+\tfrac{\nu_0}{2}\int \widetilde{\nabla u}(y-h):\Big[(\widetilde{\partial_3\phi})(y)\otimes[(\nabla\eta)(y')-(\nabla\eta)(y'-h)]\Big]\,dy \\
&+\nu_1\int |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h):\widetilde{\mathcal{D}\phi}(y)\,dy \\
&+\nu_1\int |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h):\Big[(\widetilde{\partial_3\phi})(y)\otimes[(\nabla\eta)(y')-(\nabla\eta)(y'-h)]\Big]_*\,dy \\
&-\int \widetilde{\pi}(y-h)\,(\widetilde{\nabla\cdot\phi})(y)\,dy \\
&-\int \widetilde{\pi}(y-h)\,(\widetilde{\partial_3\phi})(y)\cdot[(\nabla\eta)(y')-(\nabla\eta)(y'-h)]\,dy \\
&+\int (\widetilde{\nabla\cdot u})(y-h)\,\widetilde{\psi}(y)\,dy=\int \widetilde{f}(y-h)\cdot\widetilde{\phi}(y)\,dy\,.
\end{aligned}
$$

Finally by taking the difference, side by side, between equation (5.1) and (5.3) we get

(5.4)

$$
\begin{aligned}
&\tfrac{\nu_0}{2}\int \Big(\widetilde{\nabla u}(y)-\widetilde{\nabla u}(y-h)\Big):\widetilde{\nabla\phi}(y)\,dy \\
&+\nu_1\int \Big(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y)-|\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\Big):(\widetilde{\mathcal{D}\phi})(y)\,dy \\
&-\int (\widetilde{\pi}(y)-\widetilde{\pi}(y-h))\,(\widetilde{\nabla\cdot\phi})(y)\,dy \\
&+\int ((\widetilde{\nabla\cdot u})(y)-(\widetilde{\nabla\cdot u})(y-h))\,\widetilde{\psi}(y)\,dy= \\
&-\int \widetilde{f}(y)\cdot(\widetilde{\phi}(y+h)-\widetilde{\phi}(y))\,dy \\
&+\tfrac{\nu_0}{2}\int \widetilde{\nabla u}(y-h):\Big[(\widetilde{\partial_3\phi})(y)\otimes[(\nabla\eta)(y')-(\nabla\eta)(y'-h)]\Big]\,dy \\
&+\nu_1\int |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h):\Big[(\widetilde{\partial_3\phi})(y)\otimes[(\nabla\eta)(y')-(\nabla\eta)(y'-h)]\Big]_*\,dy \\
&-\int \widetilde{\pi}(y-h)\,(\widetilde{\partial_3\phi})(y)\cdot[(\nabla\eta)(y')-(\nabla\eta)(y'-h)]\,dy\,.
\end{aligned}
$$

A MAIN REMARK. Now we would like to replace in (5.4) $\widetilde{\nabla\phi}(y)$ with $\widetilde{\nabla u}(y)-\widetilde{\nabla u}(y-h)$ and, by consequence, $\widetilde{\mathcal{D}\phi}(y)$ with $\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y-h)$. Unfortunately this is not allowed since $\widetilde{\nabla u}(y-h)$ is not the transformation of the gradient of an x-test function. However our goal will be obtained "up to a perturbation term" by setting in equation (5.4)

$$
\phi(x)=(u(x)-u_{-h}(x))\,\theta^2(x)\,, \tag{5.5}
$$

where θ is an arbitrary regular real function such that

$$
supp\quad \theta\subset I_a\,.
$$

To fix ideas also assume from now on that $0 < \theta(x) \le 1$. Note that $\widetilde{(\theta^2)} = (\widetilde{\theta})^2$ and $\widetilde{\nabla\,\theta^2} = 2\,\widetilde{\theta}\,\widetilde{\nabla\,\theta}$. Clearly

$$\widetilde{\phi}(y) = (\widetilde{u}(y) - \widetilde{u}(y-h))\,(\widetilde{\theta})^2(y)\,. \tag{5.6}$$

Lemma 5.1. *Let $\phi(x)$ be the admissible test-function given by* (5.5). *Then the y-transformed of $\nabla\,\phi(x)$, $\mathcal{D}\,\phi(x)$, $\partial_3\,\phi(x)$ and $\nabla\cdot\phi(x)$ are respectively given by* (5.7), (5.8), (5.9) *and* (5.10) *below.*

Proof. By taking the gradient of both sides of equation (5.5), by passing from the x to the y variables and by appealing to (4.6) it readily follows that

$$\begin{aligned}\widetilde{\nabla\phi}(y) = &\left(\widetilde{\nabla u}(y) - \widetilde{\nabla u}(y-h)\right)\,(\widetilde{\theta})^2(y)\\ &+(\widetilde{\partial_3 u})(y-h)\otimes\,[(\nabla\eta)(y') - (\nabla\eta)(y'-h)]\,(\widetilde{\theta})^2(y)\\ &+2\,\widetilde{\theta}(y)\,\left(\widetilde{u}(y) - \widetilde{u}(y-h)\right)\otimes\widetilde{\nabla\theta}(y)\,.\end{aligned} \tag{5.7}$$

In particular, one has

$$\begin{aligned}\widetilde{\mathcal{D}\phi}(y) = &\left(\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)\right)\,(\widetilde{\theta})^2(y)\\ &+\left\{(\widetilde{\partial_3 u})(y-h)\otimes\,[(\nabla\eta)(y') - (\nabla\eta)(y'-h)]\,(\widetilde{\theta})^2(y)\right\}_*\\ &+2\,\widetilde{\theta}(y)\,\left\{(\widetilde{u}(y) - \widetilde{u}(y-h))\otimes\widetilde{\nabla\theta}(y)\right\}_*\,,\end{aligned} \tag{5.8}$$

and also (since $\partial_3\,\eta = 0$),

$$\begin{aligned}\widetilde{\partial_3\,\phi}(y) = &\left(\widetilde{\partial_3\,u}(y) - \widetilde{\partial_3\,u}(y-h)\right)\,(\widetilde{\theta})^2(y)+\\ &2\,\widetilde{\theta}(y)\,\left(\widetilde{u}(y) - \widetilde{u}(y-h)\right)\widetilde{\partial_3\theta}(y)\,.\end{aligned} \tag{5.9}$$

Similarly, from (4.2) and (4.8) it readily follows that

$$\begin{aligned}\widetilde{\nabla\cdot\phi}(y) = &\left(\widetilde{\nabla\cdot u}(y) - \widetilde{\nabla\cdot u}(y-h)\right)\,(\widetilde{\theta})^2(y)+\\ &(\widetilde{\partial_3 u})(y-h)\cdot\,[(\nabla\eta)(y') - (\nabla\eta)(y'-h)]\,(\widetilde{\theta})^2(y)+\\ &2\,\widetilde{\theta}(y)\,\left(\widetilde{u}(y) - \widetilde{u}(y-h)\right)\cdot\widetilde{\nabla\theta}(y)\,.\end{aligned} \tag{5.10}$$

□

On the other hand, by setting

$$\psi(x) = (\pi(x) - \pi_{-h}(x))\,\theta^2(x)\,, \tag{5.11}$$

it follows (pux2)

$$\widetilde{\psi}(y) = (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,(\widetilde{\theta})^2(y)\,. \tag{5.12}$$

Next we replace in equation (5.4) the test functions ϕ and ψ by the expressions indicated in equations (5.5) and (5.11). We start by estimating each of the terms that appear in equation (5.4). In order to treat the second integral on the left hand side of (5.4) we appeal to the following well known result.

Let U, V be two arbitrary vectors in $\mathbb{R}^N, N \geq 1$, and $p \geq 2$. Then

$$\left(|U|^{p-2}U - |V|^{p-2}V\right) \cdot (U - V) \geq \frac{1}{2}\left(|U|^{p-2} + |V|^{p-2}\right)|U - V|^2,$$

(5.13)

$$\left|\,|U|^{p-2}U - |V|^{p-2}V\right| \leq \frac{p-1}{2}\left(|U|^{p-2} + |V|^{p-2}\right)|U - V|.$$

Proposition 5.1. *Let* $\widetilde{\phi}(y)$ *be given by* (5.5). *Then*

(5.14)

$$\nu_1 \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) - |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\right) : \widetilde{\mathcal{D}\phi}(y)\,dy \geq$$

$$\tfrac{\nu_1}{4} \int \left(|(\widetilde{\mathcal{D}u})(y)|^{p-2} + |(\widetilde{\mathcal{D}u})(y-h)|^{p-2}\right) |(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)|^2\,(\widetilde{\theta})^2(y)\,dy$$

$$-\,C\,\|\nabla\widetilde{u}\|_p^p\,h^2\,.$$

Proof. For convenience, denote by S_1 the left hand side of (5.14). By (5.8) one has

(5.15)

$$S_1 = \nu_1 \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) - |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\right) \cdot \left(\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)\right)(\widetilde{\theta})^2(y)\,dy$$

$$+\,\nu_1 \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) - |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\right)$$

$$\cdot \left\{(\widetilde{\partial_3 u})(y-h) \otimes [(\nabla h)(y') - (\nabla h)(y'-h)]\right\}_* (\widetilde{\theta})^2(y)\,dy$$

$$+\,2\,\nu_1 \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) - |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\right) \cdot \left\{(\widetilde{u}(y) - \widetilde{u}(y-h)) \otimes \widetilde{\nabla}\theta(y)\right\}_* \widetilde{\theta}(y)\,dy$$

From (5.13) it follows that

(5.16)

$$S_1 \geq \tfrac{\nu_1}{2} \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\right)\left|\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)\right|^2 (\widetilde{\theta})^2(y)\,dy$$

$$-\,\tfrac{\nu_1}{2}\,(p-1) \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\right)\left|\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)\right|$$

$$|(\widetilde{\partial_3 u})(y-h)|\;|(\nabla\eta)(y') - (\nabla\eta)(y'-h)|\;(\widetilde{\theta})^2(y)\,dy$$

$$-\,\nu_1\,(p-1) \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\right)\left|\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)\right|\,\widetilde{\theta}(y)\;|\widetilde{u}(y) - \widetilde{u}(y-h)|\;|\widetilde{\nabla}\theta(y)|\,dy\,.$$

By appealing to Cauchy-Schwartz inequality one easily shows that
(5.17)

$$S_1 \geq \frac{\nu_1}{4} \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2} \right) \left| \widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h) \right|^2 (\widetilde{\theta})^2(y)\, dy$$

$$- \frac{\nu_1}{2} (p-1)^2 \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2} \right) \left| (\widetilde{\partial_3 u})(y-h) \right|^2$$

$$|(\nabla\eta)(y') - (\nabla\eta)(y'-h)|^2 \, (\widetilde{\theta})^2(y)\, dy$$

$$- 2\nu_1 (p-1)^2 \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2} \right) |\widetilde{u}(y) - \widetilde{u}(y-h)|^2 \left| \widetilde{\nabla\theta}(y) \right|^2 dy\,.$$

The last two integrals are bounded by

$$c\, h^2 \, (\|D^2 \eta\|_\infty^2 + \|D^2 \theta\|_\infty^2)\, \|\widetilde{\nabla u}(y)\|_p^p\,.$$

□

Next we estimate the third integral on the right hand side of (5.4).

Proposition 5.2. *Let $\widetilde{\phi}(y)$ be given by* (5.5)*. Then*
(5.18)

$$\nu_1 \,| \int \, |\widetilde{\mathcal{D}u}(y-h)|^{p-2} \widetilde{\mathcal{D}u}(y-h) : \Big[(\widetilde{\partial_3 \phi})(y) \otimes [(\nabla\eta)(y') - (\nabla\eta)(y'-h)] \Big]_* \, dy| \leq$$

$$\frac{\nu}{8} \int (|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2})\, |\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y)|^2 \, (\widetilde{\theta})^2(y)\, dy +$$

$$C\, h^2 \, \|\nabla \widetilde{u}\|_p^p\,.$$

Proof. Denote by S the integral on the left hand side of (5.18). By (5.8) one has
(5.19)

$$\nu_1\, S =$$

$$\nu_1 \int \, |\widetilde{\mathcal{D}u}(y-h)|^{p-2} \widetilde{\mathcal{D}u}(y-h) \cdot \Big[\left(\widetilde{\partial_3\, u}(y) - \widetilde{\partial_3\, u}(y-h) \right) (\widetilde{\theta})^2(y) \otimes [(\nabla\eta)(y') - (\nabla\eta)(y'-h)] \Big]_* \, dy$$

$$+ 2\, \nu_1 \int \, |\widetilde{\mathcal{D}u}(y-h)|^{p-2} \widetilde{\mathcal{D}u}(y-h) \cdot \Big[(\widetilde{u}(y) - \widetilde{u}(y-h))\, \widetilde{\partial_3 \theta}(y) \otimes [(\nabla\eta)(y') - (\nabla\eta)(y'-h)] \Big]_* \, \widetilde{\theta}(y)\, dy\,.$$

The second integral on the right hand side of (5.19) is easily seen to be bounded by

$$C\, \|\nabla\theta\|_\infty \, \|D^2\eta\|_\infty \, \|\nabla\widetilde{u}\|_p^p \, h_1^2\,,$$

hence bounded by the last term in the right hand side of equation (5.18).

Denote by I_1 the first integral on the right hand side of (5.19). By splitting this integral into two integrals, the first one including the term $\widetilde{\partial_3\, u}(y)$ and the second one including the term $\widetilde{\partial_3\, u}(y-h)$; by appealing to the change of variables $y_1 - h \to y_1$ in the second integral; and, finally, by splitting this last

integral in a convenient and obvious way, we get
(5.20)

$$I_1 =$$

$$\int |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h) : \Big[\widetilde{\partial_3\, u}(y)\,(\widetilde{\theta})^2(y) \otimes [(\nabla\eta)(y') - (\nabla\eta)(y'-h_)]\Big]_* \,dy$$

$$- \int |\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) : \Big[\widetilde{\partial_3\, u}(y)\,(\widetilde{\theta})^2(y) \otimes [(\nabla\eta)(y') - (\nabla\eta)(y'-h_)]\Big]_* \,dy$$

$$- \int |\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) : \Big\{\Big[\widetilde{\partial_3\, u}(y)\,(\widetilde{\theta})^2(y+h) \otimes [(\nabla\eta)(y'+h) - (\nabla\eta)(y')]\Big]_*$$

$$- \Big[\widetilde{\partial_3\, u}(y)\,(\widetilde{\theta})^2(y) \otimes [(\nabla\eta)(y') - (\nabla\eta)(y'-h_)]\Big]_*\Big\}\,dy\,.$$

The last integral on the right hand side of (5.20) is bounded by

$$C\,\|\nabla\eta\|_\infty\,\|\nabla\widetilde{u}\|_p^p\,h^2\,,$$

hence is bounded by the last term in the right hand side of equation (5.18). It remains to estimate the absolute value of difference of the two first integrals on the right hand side of (5.20). By appealing to (5.13) one shows that this absolute value is bounded by

$$\frac{p-1}{2}\int (|\widetilde{\mathcal{D}u}(y)|^{p-2}+|\widetilde{\mathcal{D}u}(y-h)|^{p-2})\,|\widetilde{\mathcal{D}u}(y)-\widetilde{\mathcal{D}u}(y)|\;|\widetilde{\partial_3\, u}(y)|\,(\widetilde{\theta})^2(y)\,|(\nabla\eta)(y')-(\nabla\eta)(y'-h)|\,dy\,.$$

In turn, this quantity is bounded by

$$\frac{1}{8}\int (|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2})\,|\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y)|^2\,(\widetilde{\theta})^2(y)\,dy +$$

$$C\int |\widetilde{\partial_3\, u}(y)|^2\,|(\nabla\eta)(y')-(\nabla\eta)(y'-h)|^2(|\widetilde{\mathcal{D}u}(y)|^{p-2}+|\widetilde{\mathcal{D}u}(y-h)|^{p-2})|\,(\widetilde{\theta})^2(y)\,dy\,.$$

Since the last integral is bounded by

$$C\,\|D^2\eta\|_\infty^2\,\|\nabla\widetilde{u}\|_p^p\,h_1^2\,,$$

the estimate (5.18) follows. □

From (5.14) and (5.18) we get the following result.

Proposition 5.3. *Denote by $\mathcal{S}(\nu_1)$ the difference between the two ν_1 terms that appear in equation* (5.4) *and let $\widetilde{\phi}(y)$ be given by* (5.5). *Then*
(5.21)

$$\mathcal{S}(\nu_1) \geq \bar{c}\,\nu_1 \int \Big(|(\widetilde{\mathcal{D}u})(y)|^{p-2} + |(\widetilde{\mathcal{D}u})(y-h)|^{p-2}\Big)\,|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)|^2\,(\widetilde{\theta})^2(y)\,dy$$

$$- C\,\|\nabla\, u\|_p^p\,h^2\,.$$

Next we consider the "ν_0" terms that appear in (5.4). A simplification of the above argument (alternatively, set $p = 2$ in (5.21)) leads to the following result.

Proposition 5.4. *Denote by $\mathcal{S}(\nu_0)$ the difference between the two ν_0 terms that appear in equation* (5.4) *and let $\widetilde{\phi}(y)$ be given by* (5.5)*. Then*

$$
\begin{aligned}
\mathcal{S}(\nu_0) \geq\, &\bar{c}\,\nu_0 \int\ |(\widetilde{\nabla u})(y) -\ (\widetilde{\nabla u})(y-h)|^2\,(\widetilde{\theta})^2(y)\,dy \\
&- C\,\|\,\nabla u\|_2^2\,h_1^2\,.
\end{aligned}
\tag{5.22}
$$

Next we consider the f-term. A classical result shows that

$$
|\int \widetilde{f}(y)\cdot\,(\widetilde{\phi}(y+\,h) -\,\widetilde{\phi}(y))\,dy| \leq\, h\,\|\widetilde{f}\|_2\,\|\nabla\,\widetilde{\phi}\|_2\,. \tag{5.23}
$$

Since $\widetilde{\phi}(y)$ is given by (5.6), straightforward calculations yield (recall that $0 \leq \theta(x) \leq 1$)

$$
\begin{aligned}
&|\int \widetilde{f}(y)\cdot\,(\widetilde{\phi}(y+\,h) -\,\widetilde{\phi}(y))\,dy| \leq \\
&h\,\|\widetilde{f}\|_{2,Q_a}\,\Big(\int |\nabla\,\widetilde{u}(y) -\,\nabla\,\widetilde{u}(y-h)|^2\,(\widetilde{\theta})^2(y)\,dy\Big)^{\frac{1}{2}} + \\
&h^2\,\|\widetilde{f}\|_2\,\|\nabla\,\widetilde{u}\|_2\,\|\nabla\,(\widetilde{\theta})^2\|_\infty.
\end{aligned}
\tag{5.24}
$$

At this point it looks convenient to the reader to establish here a full stop. In this regard we write the equation that follows from (5.4) by appealing to Propositions 5.4 and 5.3, and to equation (5.24). One has

(5.25)

$$
\begin{aligned}
&\bar{c}\,\nu_0 \int\ |(\widetilde{\nabla u})(y) -\ (\widetilde{\nabla u})(y-h)|^2\,(\widetilde{\theta})^2(y)\,dy \\
&+\bar{c}\,\nu_1 \int\ \Big(|(\widetilde{\mathcal{D}u})(y)|^{p-2} +\ |(\widetilde{\mathcal{D}u})(y-h_)|^{p-2}\Big)\,|(\widetilde{\mathcal{D}u})(y) -\ (\widetilde{\mathcal{D}u})(y-h)|^2\,(\widetilde{\theta})^2(y)\,dy \leq \\
&+\int\ \widetilde{\pi}(y)\,(\widetilde{\partial_3 u})(y-h)\cdot\ [(\nabla\eta)(y') -\ (\nabla\eta)(y'-h)]\,(\widetilde{\theta})^2(y)\,dy \\
&+\int\ (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}(y)\ (\widetilde{u}(y) -\ \widetilde{u}(y-h))\cdot\widetilde{\nabla\theta}(y)\,dy \\
&-\int\ \widetilde{\pi}(y-h)\,(\widetilde{\partial_3\,u})(y)\cdot\ [(\nabla\eta)(y') -\ (\nabla\eta)(y'-h)]\,(\widetilde{\theta})^2(y)\,dy \\
&-\int\ \widetilde{\pi}(y-h)\ (\widetilde{u}(y) -\ \widetilde{u}(y-h))\,\widetilde{\partial_3\theta}(y)\cdot\ [(\nabla\eta)(y') -\ (\nabla\eta)(y'-h)]\,\widetilde{\theta}(y)\,dy \\
&+C\,\left(\|\nabla u\|_2^2 +\ \|\nabla u\|_p^p\right)\,h^2 + \\
&c\,\frac{h^2}{\nu_0}\,\|\widetilde{f}\|_2^2 +\,h^2\,\|\widetilde{f}\|_2\,\|\nabla\,\widetilde{u}\|_2\,\|\nabla\,(\widetilde{\theta})^2\|_\infty\,.
\end{aligned}
$$

By recalling, if necessary, (4.9) one easily shows that the fourth integral in the right hand side of the above equation is bounded by

$$
C\,\|\nabla\,\theta\|_\infty\,\|\pi\|_{p'}\,\|\nabla u\|_p\,h^2\,.
$$

Similar estimates hold for the first and the third integrals in the right hand side of the same equation. The above arguments prove the following result.

Theorem 5.2. *The following estimate holds.*
(5.26)
$$
\begin{aligned}
&\bar{c}\,\nu_0 \int |(\widetilde{\nabla u})(y) - (\widetilde{\nabla u})(y-h)|^2\,(\widetilde{\theta})^2(y)\,dy \\
&+\bar{c}\,\nu_1 \int \left(|(\widetilde{\mathcal{D}u})(y)|^{p-2} + |(\widetilde{\mathcal{D}u})(y-h)|^{p-2}\right)|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)|^2\,(\widetilde{\theta})^2(y)\,dy \leq \\
&|\int (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}(y)\,(\widetilde{u}(y) - \widetilde{u}(y-h))\cdot\widetilde{\nabla\theta}(y)\,dy| \\
&+C\,\left(1 + \|\nabla\theta\|_\infty^2\right)\left(\|\nabla u\|_2^2 + \|\nabla u\|_p^p + \|\pi\|_{p'}\,\|\nabla u\|_p\right)\,h^2 + \\
&+c\,\frac{h^2}{\nu_0}\,\|\widetilde{f}\|_2^2 + h^2\,\|\widetilde{f}\|_2\,\|\nabla\,\widetilde{u}\|_2\,\|\nabla\,(\widetilde{\theta})^2\|_\infty\,.
\end{aligned}
$$

For convenience we define the nonnegative quantities $\widetilde{A}_0$ and $\widetilde{A}_1$ by

$$
\widetilde{A}_0^2 = \int |(\widetilde{\nabla u})(y) - (\widetilde{\nabla u})(y-h)|^2\,(\widetilde{\theta})^2\,dy \tag{5.27}
$$

and
(5.28)
$$
\widetilde{A}_1^2 = \int \left(|(\widetilde{\mathcal{D}u})(y)|^{p-2} + |(\widetilde{\mathcal{D}u})(y-h)|^{p-2}\right)\,|(\widetilde{\mathcal{D}u})(y) - (\widetilde{\mathcal{D}u})(y-h)|^2\,(\widetilde{\theta})^2\,dy
$$

respectively, and also

$$
U(u,\pi) = \|\nabla u\|_p^2 + \|\nabla u\|_p^p + \|\pi\|_{p'}^{p'}\,. \tag{5.29}
$$

Theorem 5.2 yields the following result.

Theorem 5.3. *The following estimate holds.*

$$
\begin{aligned}
&\nu_0\,\widetilde{A}_0^2 + \nu_1\,\widetilde{A}_1^2 \leq \\
&C\,|h|\,\|\nabla\widetilde{u}\|_p\,\|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}(y)\|_{p'} + C\,U(u,\pi)\,h^2 + C\,\|f\|_2^2\,h^2\,.
\end{aligned} \tag{5.30}
$$

6 Estimates for the tangential derivatives of the pressure in terms of the velocity

In the sequel p denotes an exponent that lies in the interval

$$
2 \leq p \leq 3 \tag{6.1}
$$

and q an exponent that lies in the interval

$$
p \leq q \leq 6\,.
$$

Actually, for a fixed p, it is sufficient that $q \leq q_\infty$ where

$$
q_\infty = 3\,(4-p)\,. \tag{6.2}
$$

In general, for $1 < r < 3$ we define the Sobolev embedding exponent r^* by the equation

$$
\frac{1}{r^*} = \frac{1}{r} - \frac{1}{3}\,. \tag{6.3}
$$

Given p and q as above we define $r = r(q)$ by

$$\frac{1}{r} = \frac{p-2}{2\,q} + \frac{1}{2}\,. \tag{6.4}$$

Note that

$$\frac{1}{r'} = \frac{q-p+2}{2q}\,.$$

In the sequel we assume that $\nabla u \in L^q$. In particular (2.7) holds. In this section we prove the following result.

Theorem 6.1. *For sufficiently small positive values of a (which are independent of the particular point x_0) one has*

$$\begin{aligned}
&\|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}\|_r \leq \\
&c\,\widetilde{A}_0 + c\,\|\nabla u\|_q^{\frac{p-2}{2}}\,\widetilde{A}_1 + \\
&C\,|h|\,(\|\nabla u\|_2 + \|\nabla u\|_{(p-1)r}^{p-1} + \|\pi\|_r) + C\,|h|\,\|f\|_2\,.
\end{aligned} \tag{6.5}$$

This result follows as a corollary of the crucial estimate stated in the following theorem.

Theorem 6.2. *Given $\epsilon_0 > 0$ there is $a(\epsilon_0) > 0$ (independent of the point x_0) such that for $a \leq a(\epsilon_0)$, one has*

$$\begin{aligned}
&|\int \nabla\,[(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}]\,\cdot\,\widetilde{\phi}\,dy| \leq \\
&c\,(\widetilde{A}_0 + C\,|h|\,\|\nabla u\|_2)\,\|\nabla\,\widetilde{\phi}\|_2 + c\,\|\nabla u\|_q^{\frac{p-2}{2}}\,\widetilde{A}_1\,\|\nabla\,\widetilde{\phi}\|_{r'} + \\
&\epsilon_0\,\|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}\|_r\,\|\nabla\,\widetilde{\phi}\|_{r'} + \\
&C\,|h|\,(\|\nabla u\|_2 + \|\nabla u\|_{(p-1)r}^{p-1} + \|\pi\|_r + \|f\|_2)\,\|\nabla\widetilde{\phi}\|_{r'}\,,
\end{aligned} \tag{6.6}$$

for each $\widetilde{\phi} \in C_0^2(Q_a)$.

Proof. of Theorem 6.1.
We start by proving the Theorem 6.1 by assuming that (6.6) holds. Equation (6.6) shows that $\nabla\,[(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}] \in W^{-1,r}(Q_a)$ and that the corresponding norm is bounded by the right hand side of equation (6.7) bellow. A main point here is that $\widetilde{\theta}$ has compact support in J_a. To fix ideas the reader may assume, once and for all, that

$$supp \quad \widetilde{\theta} \subset \overline{Q}_{\frac{a}{2}}\,,$$

and that the translation amplitudes satisfies $|h| < \frac{a}{2}$. Next, by appealing to a well known Nečas result, see [23], one shows that

$$\begin{aligned}
&\|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}\|_r \leq \\
&\epsilon_0\,\|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}\|_r + \\
&c\,\widetilde{A}_0 + c\,\|\nabla u\|_q^{\frac{p-2}{2}}\,\widetilde{A}_1 + \\
&C\,|h|\,(\|\nabla u\|_2 + \|\nabla u\|_q^{p-1} + \|\pi\|_r + \|f\|_2)\,.
\end{aligned} \tag{6.7}$$

This proves the Theorem 6.1. $\square$

The remain of this section is devoted to the proof of (6.6). We start by proving the following Lemma.

Lemma 6.3. *For each* $\widetilde{\phi} \in C_0^2(Q_a)$ *one has*

(6.8)

$$
\begin{aligned}
&|\int \nabla\,[(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}]\cdot\widetilde{\phi}\,dy| \leq \\
&\nu_0\,|\int \left(\widetilde{\nabla u}(y) - \widetilde{\nabla u}(y-h)\right) : \widetilde{\nabla(\theta\phi)}(y)\,dy| + \\
&\nu_1\,|\int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) - |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\right)\cdot\widetilde{\nabla(\theta\phi)}(y)\,dy| + \\
&\epsilon_0\,\|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}\|_r\,\|\nabla\,\widetilde{\phi}\|_{r'} + \\
&|\int \widetilde{f}\cdot\left(\widetilde{(\theta\phi)}(y+h) - \widetilde{(\theta\phi)}(y)\right)\,dy| + \\
&C\,|h|\,(\|\nabla u\|_2 + \|\nabla u\|^{p-1}_{(p-1)r} + \|\pi\|_r)\,\|\nabla\widetilde{\phi}\|_{r'}\,,
\end{aligned}
$$

where ϵ_0 *and* a *are chosen below.*

Proof. From (5.4) with $\psi = 0$ and ϕ replaced by $\theta\,\phi$ one easily gets

$$
\begin{aligned}
&\int (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,(\nabla\cdot\widetilde{(\theta\phi)})(y)\,dy = \\
&\tfrac{\nu_0}{2}\int \left(\widetilde{\nabla u}(y) - \widetilde{\nabla u}(y-h)\right) : \widetilde{\nabla(\theta\phi)}(y)\,dy + \\
&\nu_1\int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) - |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\right)\cdot\widetilde{\nabla(\theta\phi)}(y)\,dy \\
&+\int \widetilde{f}(y)\cdot\left(\widetilde{(\theta\phi)}(y+h) - \widetilde{(\theta\phi)}(y)\right)\,dy + \mathcal{R}
\end{aligned}
\tag{6.9}
$$

where $\mathcal{R}$ satisfies

$$
\begin{aligned}
&|\mathcal{R}| \leq \nu_0\,|h|\,\|\eta\|_{C^2}\,\|\widetilde{\nabla u}\|_2\,\|\widetilde{\partial_3(\theta\phi)}\|_2 + \\
&\nu_1\,|h|\,\|\eta\|_{C^2}\,\|\widetilde{\nabla u}\|^{p-1}_{(p-1)r}\,\|\widetilde{\partial_3(\theta\phi)}\|_{r'} + |h|\,\|\eta\|_{C^2}\,\|\pi\|_r\,\|\widetilde{\partial_3(\theta\phi)}\|_{r'}\,.
\end{aligned}
\tag{6.10}
$$

Since $\widetilde{\partial_3(\theta\phi)} = \widetilde{\theta}\,\partial_3\,\widetilde{\phi} + \widetilde{\phi}\,\widetilde{\partial_3\theta}$ (recall, in particular (4.10) for $k = 3$) it follows that

$$
\|\widetilde{\partial_3(\theta\phi)}\|_{r'} \leq C\,\|\nabla\,\widetilde{\phi}\|_{r'}\,.
$$

Hence, with a simplified notation,

$$
|\mathcal{R}| \leq C\,|h|\,(\|\nabla u\|_2 + \|\nabla u\|^{p-1}_{(p-1)r} + \|\pi\|_r)\,\|\nabla\widetilde{\phi}\|_{r'}\,.
\tag{6.11}
$$

Note that $2 \leq r'$. On the other hand, by appealing to (4.12), one shows that

$$
(\nabla\cdot\widetilde{(\theta\phi)}) = \widetilde{\theta}\,\nabla\cdot\,\widetilde{\phi}) - \widetilde{\theta}\,(\nabla\cdot\,h)\,\partial_3\,\widetilde{\phi} + \widetilde{\phi}\cdot\widetilde{\nabla\theta}\,.
$$

Hence we may decompose the left hand side of (6.9) as

(6.12)
$$\int (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,(\nabla \cdot \widetilde{(\theta\phi)})\,dy = \int [(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}]\,(\nabla \cdot \widetilde{\phi})\,dy +$$
$$\int (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,(\widetilde{\nabla\theta}) \cdot \widetilde{\phi}\,dy + \int [(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}]\,(\nabla\,\eta)\,\partial_3\,\widetilde{\phi}\,dy\,.$$

By means of a suitable translation one shows that

(6.13)
$$\int (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,(\widetilde{\nabla\theta}) \cdot \widetilde{\phi}\,dy = -\int \widetilde{\pi}\,(\widetilde{\phi}(y+h) - \widetilde{\phi}(y)) \cdot (\widetilde{\nabla\theta})\,dy$$
$$-\int \widetilde{\pi}\,\widetilde{\phi}(y+h) \cdot (\widetilde{\nabla\theta}(y+h) - \widetilde{\nabla\theta}(y))\,dy\,.$$

Hence the second integral on the right hand side of (6.12) satisfies

$$|\int (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,(\widetilde{\nabla\theta}) \cdot \widetilde{\phi}\,dy| \leq \tag{6.14}$$
$$|h|\,\|\widetilde{\nabla\theta}\|_{C^1}\,\|\widetilde{\pi}\|_r\,\|\nabla\,\widetilde{\phi}\|_{r'}\,.$$

On the other hand

(6.15)
$$|\int [(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}]\,(\nabla\,\eta)\,\partial_3\,\widetilde{\phi}\,dy| \leq \|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}\|_r\,\|\nabla\,\eta\|_{C_0}\,\|\widetilde{\nabla\phi}\|_{r'}\,.$$

From (6.12), (6.14) and (6.15) it follows that

(6.16)
$$\int (\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,(\nabla \cdot \widetilde{(\theta\phi)})\,dy = -\int \nabla\,[(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}] \cdot \widetilde{\phi}\,dy + \mathcal{R}_2\,,$$

for each $\widetilde{\phi} \in C_0^2(Q_a)$, where $\mathcal{R}_2$ satisfies the estimate

$$|\mathcal{R}_2| \leq C\,|h|\,\|\widetilde{\pi}\|_r\,\|\nabla\,\widetilde{\phi}\|_{r'} + \epsilon_0\,\|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}\|_r\,\|\nabla\,\widetilde{\phi}\|_{r'}\,, \tag{6.17}$$

for an arbitrarily small positive ϵ_0, provided that $a \leq a(\epsilon_0)$. We applied to the fact that $\nabla\,\eta(0) = 0$. From (6.16), (6.17) and (6.9), (6.11) the estimate (6.8) follows. □

.

Next we estimate the ν_0 and the ν_1-terms in the right hand side of equation (6.8). We prove the following result.

Lemma 6.4. *The following estimates hold.*

(6.18)
$$\nu_1\,\Big|\int \Big(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) - |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\Big) : \nabla\widetilde{(\theta\,\phi)}\,dy\Big| \leq$$
$$c\,\|\nabla u\|_q^{\frac{p-2}{2}}\,(\widetilde{A}_1 + C\,|h|\,\|\nabla u\|_q^{\frac{p}{2}})\,\|\nabla\,\widetilde{\phi}\|_{r'}$$

and

$$\nu_0\,\Big|\int \Big(\widetilde{\nabla u}(y) - \widetilde{\nabla u}(y-h)\Big) : \nabla\widetilde{(\theta\,\phi)}\,dy\Big| \leq \tag{6.19}$$
$$c\,(\widetilde{A}_0 + C\,|h|\,\|\nabla u\|_2)\,\|\nabla\,\widetilde{\phi}\|_2\,.$$

Proof. From (4.11) it follows that

$$\nabla\widetilde{(\theta\phi)} = \widetilde{\theta}\,\nabla\,\widetilde{\phi} - \widetilde{\theta}\,[(\nabla\,\eta)\otimes\,\partial_3\,\widetilde{\phi}] + \widetilde{\phi}\otimes\,\widetilde{\nabla\theta}\,,$$

for each $y \in Q_a$. Moreover

$$|\widetilde{\theta}\,\nabla\,\widetilde{\phi} - \widetilde{\theta}\,[(\nabla\,\eta)\otimes\,\partial_3\,\widetilde{\phi}]| \leq C\,|\widetilde{\theta}|\,|\nabla\,\widetilde{\phi}|\,. \tag{6.20}$$

hence, by appealing to (5.13), it follows that
(6.21)

$$\nu_1 \left|\int \Big(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) \,-|\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\Big) :\, \nabla\widetilde{(\theta\,\phi)}\,dy\right| \leq$$

$$C\,\nu_1 \int \Big(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\Big)\, |\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)|\,|\widetilde{\theta}|\,|\nabla\,\widetilde{\phi}|\,dy +$$

$$\nu_1 \left|\int \Big(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) \,-|\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h)\Big) :\, (\widetilde{\phi}\otimes\,\widetilde{\nabla\theta})\,dy\right|\,.$$

Next
(6.22)

$$\int \Big(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\Big)\, |\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)|\,|\widetilde{\theta}|\,|\nabla\,\widetilde{\phi}|\,dy \leq$$

$$c\int \Big(\,|\widetilde{\mathcal{D}u}(y)|^{\frac{p-2}{2}} + |\widetilde{\mathcal{D}u}(y-h)|^{\frac{p-2}{2}}\Big) \times$$

$$\Big[\Big(\,|\widetilde{\mathcal{D}u}(y)|^{\frac{p-2}{2}} + |\widetilde{\mathcal{D}u}(y-h)|^{\frac{p-2}{2}}\Big)\, |\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)|\,\widetilde{\theta}\,\Big]\,|\nabla\,\widetilde{\phi}|\,dy\,.$$

By taking into account the definition of r and by appealing to Hőlder's inequality one shows that
(6.23)

$$\int \Big(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\Big)\, |\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)|\,|\widetilde{\theta}|\,|\nabla\,\widetilde{\phi}|\,dy \leq$$

$$c\,\|\widetilde{\nabla u}\|_q^{\frac{p-2}{2}}\,\widetilde{A}_1\,\|\nabla\,\widetilde{\phi}\|_{r'}\,,$$

which is the desired estimate for the first integral in the right hand side of (6.21).

We could appeal to similar devices to obtain as well an useful estimate for the second integral in the right hand side of (6.21). However, the lack of $\widetilde{\theta}(y)$ in this integral would imply some tricky arguments. We rather prefer to introduce a more elegant device to obtain the desired estimate. Denote by I the referred integral. An obvious translation shows that

$$I = \int\,|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) :\, (\widetilde{\phi}(y)\otimes\,\widetilde{\nabla\theta}(y))\,dy$$

$$-\int\,|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) :\, (\widetilde{\phi}(y+h)\otimes\,\widetilde{\nabla\theta}(y+h))\,dy\,.$$

By appealing to an obvious decomposition of $\Big(\widetilde{\phi}(y+h)\otimes\,\widetilde{\nabla\theta}(y+h)\Big) - \Big(\widetilde{\phi}(y)\otimes\,\widetilde{\nabla\theta}(y)\Big)$, it readily follows that

$$|I| \leq\, c\,|h|\,\|\widetilde{\nabla\theta}\|_{C^1}\,\|\widetilde{(\nabla u)}\|^{p-1}_{(p-1)r}\,(\|\widetilde{\phi}\|_{r'} + \|\nabla\,\widetilde{\phi}\|_{r'})\,.$$

Hence
(6.24)
$$\nu_1 \left| \int \left(|\widetilde{\mathcal{D}u}(y)|^{p-2}\widetilde{\mathcal{D}u}(y) - |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\widetilde{\mathcal{D}u}(y-h) \right) : (\widetilde{\phi} \otimes \widetilde{\nabla\theta})\, dy \right| \leq$$

$$C\, \nu_1\, |h|\, \|\nabla\, u\|^{p-1}_{(p-1)r}\, \|\nabla\, \widetilde{\phi}\|_{r'}\,.$$

By appealing to the equations (6.21), (6.23) and (6.24) one proves (6.18). Simpler devices show that (6.19) holds. Alternatively, we may set $p = q = 2$ in (6.18). □

Finally, by appealing to an obvious decomposition of the $\theta\phi$ terms, one shows that

$$|\int \widetilde{f} \cdot \left((\widetilde{\theta\phi})(y+h) - (\widetilde{\theta\phi})(y) \right) dy| \leq C\, |h|\, \|f\|_2\, \|\nabla\, \widetilde{\phi}\|_2\,. \tag{6.25}$$

Alternatively, we could replace the right hand side of (6.25) by $C\,|h|\,\|f\|_{p'}\,\|\nabla\,\widetilde{\phi}\|_p$.

From (6.8), (6.19), (6.18) and (6.25) the estimate (6.6) follows. The Theorem 6.2 is proved.

7 Estimates for the "tangential derivatives" in terms of the data

For convenience we set

$$U_0(u,\pi) = \|\nabla u\|_p^2 + \|\nabla u\|_p^{2(p-1)} + \|\pi\|_{p'}^2 + \|\pi\|_{p'}^{p'}\,. \tag{7.1}$$

In the sequel ∇_* denotes the gradient with respect to the variables $y_j\,, j = 1, 2$. Hence

$$|\nabla_*(\widetilde{\nabla u})(y)|^2 = \sum_{j=1,2} \sum_{i,k=1}^{3} (\partial_j\, (\widetilde{\nabla u})_{ik})^2\,,$$

$$|\nabla_*(\widetilde{\mathcal{D}u})(y)|^2 = \sum_{j=1,2} \sum_{i,k=1}^{3} (\partial_j (\widetilde{\mathcal{D}u})_{ik})^2\,,$$

and

$$|\nabla_* \widetilde{\pi}(y)|^2 = \sum_{j=1,2} (\partial_j \widetilde{\pi})^2\,.$$

In this section we prove the following result.

Theorem 7.1. *Let $x_0 \in \Gamma$ and let $a_0 > 0$ be as in section ...3. Assume that $(u,\pi) \in \mathbb{W}^{1,q}(Q_{a_0}) \times L^r(Q_{a_0})$ is a weak solution of problem* (1.6) *in Q_{a_0} which satisfies the boundary condition* (1.5) *in Γ_{a_0}. Then there is $a > 0$ (which depends only on the C^2-norm of η in the sphere $\{x' : |x'| < a_0\}$) such that*

$$\nabla_*(\widetilde{\nabla u}),\, |\widetilde{\mathcal{D}u}|^{p-2}\, \nabla_*\, \widetilde{\mathcal{D}u} \in L^2(Q_a)$$

and

$$\nabla_*\, \widetilde{\pi} \in L^r(Q_a)\,.$$

Moreover,

(7.2)

$$\|\nabla_*(\widetilde{\nabla u})\|_2^2 + \|\,|\widetilde{\mathcal{D}u}|^{\frac{p-2}{2}}\,\nabla^*\,\widetilde{\mathcal{D}u}\|_2^2 \leq C\,(\|\nabla u\|_p^2 + \|\nabla u\|_p^p + \|\pi\|_{p'}^{p'} + \|f\|_2^2)\,,$$

$$\|\nabla^*\,\widetilde{\pi}\|_r^2 \leq C\,(1 + \|\nabla u\|_q^{p-2})\,(\|f\|_2^2 + \|\nabla u\|_p^2 + \|\nabla u\|_p^p + \|\pi\|_{p'}^{p'})+$$

$$C\,(\|\nabla u\|_{(p-1)r}^{2(p-1)} + \|\pi\|_r^2)\,,$$

where the norms on the left hand sides concern the set $Q_{\frac{a}{2}}$ *and those in the right hand side concern* Ω_a*.*

REMARKS.

–For convenience the reader may assume that the norms $\|\nabla u\|_p$, $\|\pi\|_{p'}$ and $\|f\|_2$ concern the whole of Ω. On the other hand it is worth noting that in the proofs below the norms $\|\nabla u\|_{(p-1)r}$ and $\|\pi\|_r$ in Ω_a are obtained via their equivalence with the norms $\|\widetilde{\nabla u}\|_{(p-1)r}$ and $\|\widetilde{\pi}\|_r$ in Q_a respectively. In fact we may use these last quantities in the second equation (7.2).

–The constants C depend on the C^2-norms of η and θ in Q_a. However the C^2-norm of η is bounded from above on Γ, hence is independent of the particular point x_0. On the other hand the particular truncation function θ may be fixed once and for all in our proofs as a regular function equal to 1 for $|x'| \leq \frac{a}{2}$ and with compact support inside I_a. This shows that the dependence of the constants C on θ is just a dependence on a.

–The smallness of a in our proofs is required just to get a sufficiently small ϵ_0 in (3.8). This value does not depend on the point x_0. Hence a strictly positive lower bound for a exists, independently of x_0.

Proof of Theorem 7.1.

From equations (5.30) and (6.5), and by setting in this last equation $q = p$ (hence $r = p'$) it readily follows that

$$\widetilde{A}_0^2 + \widetilde{A}_1^2 \leq C\,|h|\,(\|\nabla u\|_p + \|\nabla u\|_p^{\frac{p}{2}})\,(\widetilde{A}_0 + \widetilde{A}_1)+$$
$$C\,h^2\,(\|\nabla u\|_p^2 + \|\nabla u\|_p^p + \|\pi\|_{p'}^{p'}) + C\,h^2\,\|f\|_2^2\,.$$

Hence

$$\widetilde{A}_0^2 + \widetilde{A}_1^2 \leq C\,h^2\,U(u,\pi) + C\,h^2\,\|f\|_2^2\,. \tag{7.3}$$

By appealing to equation (6.5), straightforward calculations show that

$$\|(\widetilde{\pi}(y) - \widetilde{\pi}(y-h))\,\widetilde{\theta}\|_r^2 \leq h^2\,\Lambda\,, \tag{7.4}$$

where, for convenience, Λ denotes the right hand side of the second equation (7.2) and constants C may be incorporated in Λ.

Let us write (7.3) in a more explicit form, by taking into account the definitions of A_0 and A_1. One has

$$\int_{Q_a} |\tfrac{\widetilde{\nabla}u(y) - \widetilde{\nabla}u(y-h)}{h^2}|^2\,\widetilde{\theta}^2(y)\,dy+$$

$$\int_{Q_a}\Big(|\widetilde{\mathcal{D}u}(y)|^{p-2} + |\widetilde{\mathcal{D}u}(y-h)|^{p-2}\Big)\,|\tfrac{\widetilde{\mathcal{D}u}(y) - \widetilde{\mathcal{D}u}(y-h)}{h^2}|^2\,\widetilde{\theta}^2(y)\,dy \leq \tag{7.5}$$

$$C\,(\|\nabla u\|_p^2 + \|\nabla u\|_p^p + \|\pi\|_{p'}^{p'}) + C\,\|f\|_2^2\,.$$

Similarly, from (7.4),

$$\left[\int_{Q_a}\left|\frac{\widetilde{\pi}(y)-\widetilde{\pi}(y-h)}{h^2}\right|^r\widetilde{\theta}^2(y)\,dy\right]^{\frac{2}{r}}\leq\Lambda\,. \tag{7.6}$$

By fixing θ as a regular function equal to 1 in $I_{a/2}$ we get equations like (7.5) and (7.6) but with the following modifications in the left hand sides: We drop the function $\widetilde{\theta}$ and take integrals over $Q_{a/2}$. This leads to the estimate of $\|D_*^2\,\widetilde{u}\|_2^2$ in Equation (7.2). In particular this last result implies the converge almost everywhere in $Q_{a/2}$ of the differential quotients that appear in the left hand side of the modified equation (7.5). By appealing to Fatou's lemma, we pass to the limit as h goes to zero and obtain the estimate of the second term in the first equation (7.2). The estimate concerning π, follows from the modified equation (7.6).

8 Normal derivatives

We set

$$|D_*^2u(x)|^2=|\nabla_*(\nabla u(x))|^2+|(\partial_3^2\,u_3)(x)|^2\,,$$

where the derivatives are with respect to the x-variables, and also

$$\xi(x)=\partial_3^2\,u(x)\,,$$

$$\xi'(x)=(\xi_1(x),\xi_2(x))$$

and

$$M(x)=|\mathcal{D}u(x)|\,.$$

Due to (3.8), we may replace (on "right hand sides" of estimates) derivatives $\partial_k\eta$, for $k=1,2$ simply by ϵ_0. Recall that $\partial_3\eta=0$. In the same line, $c\,\epsilon_0$ and ϵ_0^2 can be replaced by ϵ_0.

We will use without a particular warning that

$$\partial_3\widetilde{g}=\widetilde{\partial_3 g}\,. \tag{8.1}$$

Lemma 8.1. *One has a.e. in Q_a*

$$|\widetilde{\xi}_3|\leq|\nabla_*(\widetilde{\nabla u})|+\epsilon_0\,|\widetilde{\xi}'|\,. \tag{8.2}$$

Proof. From $\nabla\cdot u=0$ it follows that

$$\widetilde{\xi}_3=-\partial_3(\widetilde{\partial_1 u_1}+\widetilde{\partial_2 u_2})\,. \tag{8.3}$$

On the other hand, from (3.9),

$$\partial_3\,(\widetilde{\partial_m u_l})=\partial_m\widetilde{\partial_3 u_l}-(\partial_m\eta)\,\partial_3\,(\widetilde{\partial_3 u_l}). \tag{8.4}$$

Hence, for $m,l\neq3$,

$$|\partial_3\widetilde{\partial_m u_l}|\leq|\nabla_*(\widetilde{\nabla u})|+\epsilon_0\,|\widetilde{\xi}'|\,.$$

By taking into account (8.3), the thesis follows. □

Lemma 8.2. *One has a.e. in Q_a*

$$|\widetilde{D^2_* u(y)}| \leq |\nabla_* (\widetilde{\nabla u})| + \epsilon_0 \, |\widetilde{\xi'}| \,. \tag{8.5}$$

Proof. From (3.9)

$$\mathcal{T}\,(\partial_j\, \partial_k\, u_l) = \partial_k (\widetilde{\partial_j\, u_l}) - (\partial_k\, \eta)\, \partial_3 (\widetilde{\partial_j\, u_l}) \,.$$

By appealing to the above estimates the thesis follows easily. Note that if $j = k = l = 3$ the result follows from (8.2). □

Straightforward calculations show that

$$\begin{aligned} &\partial_k\, \left(|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, u \right) = \\ &|\mathcal{D}\, u|^{p-2}\, \mathcal{D}\, \partial_k u + (p-2)\, |\mathcal{D}\, u|^{p-4}\, (\mathcal{D}\, u \cdot \mathcal{D}\, \partial_k u)\, \mathcal{D}\, u \,. \end{aligned} \tag{8.6}$$

By using (8.6), the j.th equation (1.6) may be written in the form

$$\begin{aligned} &-\nu_0 \sum_{k=1}^{3} \partial_k^2\, u_j - \nu_1\, |\mathcal{D}\, u|^{p-2} \sum_{k=1}^{3} (\partial_k^2\, u_j + \partial_{jk}^2\, u_k) \\ &-(p-2)\, \nu_1\, |\mathcal{D}\, u|^{p-4} \sum_{l,m,k=1}^{3} \mathcal{D}_{lm}\, \mathcal{D}_{jk}\, (\partial_{mk}^2\, u_l + \partial_{lk}^2\, u_m) \\ &+ \partial_j \pi = f_j \,, \end{aligned} \tag{8.7}$$

where we set $\mathcal{D}_{ij} = (\mathcal{D}\, u)_{ij}$. Let us write the first 2 equations (8.7) as follows:

$$\begin{aligned} &\nu_0\, \partial_3^2\, u_j + \nu_1\, |\mathcal{D}\, u|^{p-2}\, \partial_3^2\, u_j \\ &+ 2\,(p-2)\, \nu_1\, |\mathcal{D}\, u|^{p-4}\, \mathcal{D}_{j3} \sum_{l=1}^{2} \mathcal{D}_{l3}\, \partial_3^2\, u_l = F_j(x) + \partial_j \pi - f_j \,, \end{aligned} \tag{8.8}$$

where the $F_j(x), j \neq 3$, are given by

$$\begin{aligned} F_j(x) :=\; &-\nu_0 \sum_{k=1}^{2} \partial_k^2\, u_j - \nu_1\, |\mathcal{D}\, u|^{p-2} \sum_{k=1}^{2} \partial_k^2\, u_j - \nu_1\, |\mathcal{D}\, u|^{p-2} \sum_{k=1}^{3} \partial_{jk}^2\, u_k \\ &-2\,(p-2)\, \nu_1\, |\mathcal{D}\, u|^{p-4} \Big(\mathcal{D}_{33}\, \mathcal{D}_{j3}\, \partial_3^2\, u_3 + \sum_{\substack{l,m,k=1 \\ (m,k)\neq(3,3)}}^{3} \mathcal{D}_{lm}\, \mathcal{D}_{jk}\, \partial_{mk}^2\, u_l \Big) \,. \end{aligned} \tag{8.9}$$

The measurable functions F_j satisfy

$$|F_j(x)| \leq c\, \left(\nu_0 + \nu_1\, |M(x)|^{p-2} \right)\, |D^2_*\, u(x)|$$

a.e. in Ω_a. Hence, from (8.5) it follows that

$$|\widetilde{F}_j| \leq \left(\nu_0 + \nu_1\, |\widetilde{M}|^{p-2} \right) \left(|\nabla_* (\widetilde{\nabla u})| + \epsilon_0\, |\widetilde{\xi'}| \right) \,. \tag{8.10}$$

Lemma 8.3. *One has a.e. in Q_a*

$$|\widetilde{\nabla_* \pi}| \leq |\nabla_* \widetilde{\pi}| + \epsilon_0 \, (\nu_0 + \nu_1 \, |\widetilde{M}|^{p-2}) \, (|\nabla_* (\widetilde{\nabla u})| + |\widetilde{\xi}'|) + \epsilon_0 \, |\widetilde{f}| \tag{8.11}$$

and

$$|\widetilde{\partial_3 \pi}| \leq (\nu_0 + \nu_1 \, |\widetilde{M}|^{p-2}) \, (|\nabla_* (\widetilde{\nabla u})| + |\widetilde{\xi}'|) + |\widetilde{f}| \, . \tag{8.12}$$

Proof. From equation (8.7) written for $j = 3$, we get an expression for $\partial_3 \pi$. In particular it follows that

$$\begin{aligned} |\partial_3 \pi| \leq \, & c \, \left(\nu_0 + (p-1) \, \nu_1 \, |M(x)|^{p-2}\right) \, |D_*^2 \, u(x)| + \\ & c \, (p-2) \, \nu_1 \, |\mathcal{D} \, u(x)|^{p-2} \sum_{l=1}^{2} |\partial_3^2 \, u_l| + |f_3(x)| \, , \end{aligned} \tag{8.13}$$

By transforming the inequality (8.13) from the x to the y variables and by appealing to (8.5) one gets (8.12). Since for $j = 1, 2$

$$|\widetilde{\partial_j \pi}| \leq |\partial_j \, \widetilde{\pi}| + \epsilon_0 \, |\widetilde{\partial_3 \pi}| \, ,$$

the estimate (8.11) holds. $\square$

Lemma 8.4. *One has a.e. in Q_a*

$$\sum_{l=1}^{3} |\partial_3^2 \, u_l| \leq c \, |\nabla_* (\widetilde{\nabla u})| + \frac{c}{\nu_0 + \nu_1 \, |\widetilde{M}|^{p-2}} (|\nabla_* \, \widetilde{\pi}| + |\widetilde{f}|) \, . \tag{8.14}$$

Proof. Consider the system (8.8) in terms of the y variables, i.e., the system

$$\begin{aligned} & \nu_0 \, \widetilde{\xi_j} + \nu_1 \, |\widetilde{M}|^{p-2} \, \widetilde{\xi_j} + 2 \, (p-2) \, \nu_1 \, |\widetilde{M}|^{p-4} \, \widetilde{\mathcal{D}}_{j3} \sum_{l=1}^{2} \widetilde{\mathcal{D}}_{l3} \, \widetilde{\xi_l} = \\ & \qquad \widetilde{F_j} + \widetilde{\partial_j \, \pi} - \widetilde{f_j} \, . \end{aligned} \tag{8.15}$$

Next we show that the 2×2 linear system (8.15) can be solved for the unknowns $\widetilde{\xi_j}$, $j = 1, 2$, for almost all $y \in Q_{a/2}$. The elements $\widetilde{a_{j\,l}}$ of the matrix system $\widetilde{A}$ are given by

$$\widetilde{a}_{j\,l} = (\nu_0 + \nu_1 \, |\widetilde{M}|^{p-2}) \, \delta_{j\,l} + 2 \, (p-2) \, \nu_1 \, |\widetilde{M}|^{p-4} \, \widetilde{\mathcal{D}}_{l3} \, \widetilde{\mathcal{D}}_{j3} \, ,$$

for $j, l \neq 3$. Note that $\widetilde{a}_{j\,l} = \widetilde{a}_{l\,j}$. One easily shows that

$$\sum_{j,l=1}^{n-1} \widetilde{a}_{j\,l} \lambda_j \lambda_l = (\nu_0 + \nu_1 \, |\widetilde{M}|^{p-2}) \, |\lambda|^2 + 2 \, (p-2) \, \nu_1 \, |\widetilde{M}|^{p-4} \, \left[(\widetilde{\mathcal{D} u}) \cdot \lambda \right]_3^2 \, .$$

Hence the matrix $\widetilde{A}$ is symmetric and positive definite. Moreover, the above identity shows that all the eigenvalues are larger than or equal to $\nu_0 + \nu_1 \, |\widetilde{M}|^{p-2}$. In particular

$$det \widetilde{A} \geq (\nu_0 + \nu_1 \, |\widetilde{M}|^{p-2})^2 \, .$$

Hence we get from (8.15), i.e. from

$$\sum_{l=1}^{2} \widetilde{a}_{j\,l}\, \widetilde{\xi}_l = \widetilde{F_j} + \widetilde{\partial_j\, \pi} - \widetilde{f}_j \,, \tag{8.16}$$

that

$$\sum_{l,j=1}^{2} \widetilde{a}_{j\,l}\, \widetilde{\xi}_l \widetilde{\xi}_j = \sum_{j=1}^{2} (\widetilde{F_j} + \widetilde{\partial_j\, \pi} - \widetilde{f}_j)\, \widetilde{\xi}_j \,. \tag{8.17}$$

holds. Consequently

$$(\nu_0 \, + \nu_1\, |\widetilde{M}|^{p-2})\, |\widetilde{\xi}| \leq \, |\widetilde{F_j} + \widetilde{\partial_j\, \pi} - \widetilde{f}_j|$$

a.e. in $Q_{a/2}$. By appealing to (8.10) and (8.11) we show that

$$(\nu_0 \, + \nu_1\, |\widetilde{M}|^{p-2})\, |\widetilde{\xi'}| \leq \, c\, \left(\nu_0 \, + \, \nu_1\, |\widetilde{M}|^{p-2} \right)\, |\nabla_*(\widetilde{\nabla u})| + c\, |\nabla_*\, \widetilde{\pi}| + c\, |\widetilde{f}| \,. \tag{8.18}$$

This estimate together with (8.2) proves (8.14). $\square$

Corollary 8.1. *One has*

(8.19)

$$\|\nabla u\|^2_{W^{1,r}(\Omega_{\frac{a}{2}})} \leq C\, (1 + \, \|\nabla u\|_q^{p-2})\, (\|f\|_2^2 + \, \|\nabla u\|_p^2 + \, \|\nabla u\|_p^p + \, \|\pi\|_{p'}^{p'}) +$$

$$C\, (\|\nabla u\|^{2(p-1)}_{(p-1)r} + \, \|\pi\|_r^2)\,,$$

where norms on the right hand side concern Ω_a.

PROOF. From (8.14) and (7.2) it follows that

$$\|\nabla (\widetilde{\nabla u})\|^2_{L^r(Q_{\frac{a}{2}})} \leq \, C\, \Lambda \,. \tag{8.20}$$

Clearly $\|\widetilde{\nabla u}\|^2_{W^{1,r}(Q_{\frac{a}{2}})}$ satisfies this last estimate. Hence (8.19) holds.

Next we appeal to the following algebraic lemma.

Lemma 8.5. *Let $2 < \, p < \, 3$ and $p \leq q \leq \, 6$ and set*

$$\beta_0 = \, \frac{3(p-2)}{6-p} \,.$$

Define α by the equation

$$\frac{1}{r} = \, \frac{1-\alpha}{p'} + \, \frac{\alpha}{r^*/(p-1)} \tag{8.21}$$

and set

$$\beta = \, (p-1)\, \alpha \,.$$

Then $\beta < \beta_0 < 1$ for each $q \geq p$.

The proof is left to the reader.

Lemma 8.6. *One has*
(8.22)

$$\|\pi\|_r \leq \epsilon^{-\frac{1}{1-\beta_0}} \|\pi\|_{p'}^{\frac{1-\alpha}{1-\beta_0}} + c\,\epsilon^{\frac{1}{\beta_0}} \big(\nu_0 \|\nabla u\|_p^{\frac{1}{p-1}} + \nu_1 \|\mathcal{D}u\|_{r*} + \|\pi\|_{p'}^{\frac{1}{p-1}} + \|f\|_2^{\frac{1}{p-1}}\big)$$

for each $\epsilon > 0$*, where the norms may be taken in any open regular subset* $\Omega_0 \subset \Omega$.

Proof. From (8.21) it follows that

$$\|\pi\|_r \leq \epsilon^{-1} \|\pi\|_{p'}^{1-\alpha}\, \epsilon\, \|\pi\|_{\frac{r*}{p-1}}^{\alpha} \tag{8.23}$$

By Young's inequality with exponents $\frac{1}{1-\beta_0}$ and $\frac{1}{\beta_0}$ we get

$$\|\pi\|_r \leq \epsilon^{-\frac{1}{1-\beta_0}} \|\pi\|_{p'}^{\frac{1-\alpha}{1-\beta_0}} + \epsilon^{\frac{1}{\beta_0}} \|\pi\|_{\frac{r*}{p-1}}^{\frac{1}{p-1}} \tag{8.24}$$

Finally, by appealing to (2.7) with $s = r^*$ one proves (8.22). For convenience, we take into account that $r^* \leq 6$, hence $\frac{r^*}{p-1} \leq p$. □

Lemma 8.7. *Let* $2 \leq p < 3$ *be fixed and define* r *and* r^* *as above, where* $p \leq q \leq 6$*. Set*

$$\alpha_0 = 1 - \frac{2}{15}(3-p)\,, \qquad \gamma_0 = \frac{1+\alpha_0}{2}\,. \tag{8.25}$$

Then, for an arbitrary g*,*

$$\begin{aligned} \|g\|_{(p-1)r}^{p-1} &\leq \big(\epsilon^{-1}(1 + \|g\|_p^{p-1}\big)^{(\frac{\gamma_0}{\alpha_0})'} + \\ &(2\epsilon)^{\frac{\gamma_0}{\alpha_0}}(1 + \|g\|_{r^*}^{\gamma_0})\,. \end{aligned} \tag{8.26}$$

Proof. Define α by

$$\frac{1}{(p-1)r} = \frac{1-\alpha}{p} + \frac{\alpha}{r^*}$$

Then

$$(p-1)\,\alpha = 1 - \frac{2(\frac{1}{p} - \frac{1}{3})}{\frac{1}{p} - (\frac{p-2}{2q} + \frac{1}{6})}\,.$$

The maximum of the above quantity is obtained for $q = 6$ (we may get better exponents by assuming that $q \leq q_\infty$). By setting $q = 6$ one easily shows that $(p-1)\,\alpha \leq \alpha_0$.
From the definition of α and interpolation it follows that

$$\|g\|_{(p-1)r}^{p-1} \leq \|g\|_p^{(p-1)(1-\alpha)}\, \|g\|_{r^*}^{(p-1)\alpha}\,.$$

In particular

$$\|g\|_{(p-1)r}^{p-1} \leq \epsilon^{-1}(1 + \|g\|_p^{p-1}) \cdot \epsilon\,(1 + \|g\|_{r^*}^{\alpha_0})\,.$$

The thesis follows by appealing to Young's inequality with exponents $(\frac{\gamma_0}{\beta_0})'$ and $\frac{\gamma_0}{\beta_0}$. □

For convenience we denote in the sequel by

$$P = P(\|\nabla u\|_p, \|\pi\|_{p'})$$

very simple expressions (that can be reduced, for instance, to low order polynomials) that depend only on the two quantities indicated above. Explicit expressions follow immediately from our calculations.
From (8.22) we get, with an obvious simplified notation

$$\|\pi\|_r \leq C(\epsilon)\,(P + \|f\|) + c\,\epsilon^{\frac{1}{\beta_0}}\,\|\nabla u\|_{r^*}\,, \tag{8.27}$$

where the norms of π and ∇u may concern any Ω_0 as above.
By appealing to (8.26) with $g = \nabla u$, and by (8.27) we prove the following estimate

Lemma 8.8. *One has*

$$\|\nabla u\|^{p-1}_{(p-1)r} + \|\pi\|_r \leq C(\epsilon)\,(P + \|f\|_2) + c\,(\epsilon^{\frac{\gamma_0}{\alpha_0}} + \epsilon^{\frac{1}{\beta_0}})(1 + \|\nabla u\|_{r^*})\,, \tag{8.28}$$

where norms could be taken in any Ω_0 as above.

By taking into account the continuous immersion

$$W^{1,r} \subset L^r\,,$$

and by appealing to (8.19) and (8.27), we get the following result

Proposition 8.1. *One has*

$$\begin{aligned} &\|\nabla u\|_{W^{1,r}(\Omega_{\frac{a}{2}})} + \|\pi\|_{L^r(\Omega_{\frac{a}{2}})} \leq C(\epsilon)\,(1 + \|\nabla u\|_q^{\frac{p-2}{2}})\,(P + \|f\|_2) + \\ &c\,(\epsilon^{\frac{\gamma_0}{\alpha_0}} + \epsilon^{\frac{1}{\beta_0}})(1 + \|\nabla u\|_{W^{1,r}(\Omega_a)})\,. \end{aligned} \tag{8.29}$$

In particular

$$\begin{aligned} &\|u\|_{W^{2,r}(\Omega)} + \|\pi\|_{L^r(\Omega)} \leq N\,\Big[C(\epsilon)\,(1 + \|\nabla u\|_q^{\frac{p-2}{2}})\,(P + \|f\|_2) + \\ &c\,(\epsilon^{\frac{\gamma_0}{\alpha_0}} + \epsilon^{\frac{1}{\beta_0}})(1 + \|\nabla u\|_{W^{1,r}(\Omega)})\Big]\,, \end{aligned} \tag{8.30}$$

where N is the number of sets of type $\Omega_{\frac{a}{2}}$ plus the number of spheres contained in the interior of Ω sufficient to cover Ω. It is worth noting that $W^{2,2}$ interior regularity for u is trivial. Finally, by fixing a sufficiently small value of ϵ we prove that (1.10) holds.
Note that by appealing to the second equation (7.2) and to (8.28) it readily follows that

$$\|\nabla^*\,\widetilde{\pi}\|_{L^r(Q_{\frac{a}{2}})} \leq C\,(1 + \|\nabla u\|_q^{\frac{p-2}{2}})\,(P + \|f\|_2)\,. \tag{8.31}$$

Finally from (8.12) and (8.18) one gets

$$|\widetilde{\partial_3\,\pi}| \leq c\,\left(\nu_0 + \nu_1\,|\widetilde{M}|^{p-2}\right)\,|\nabla_*(\widetilde{\nabla u})| + c\,|\nabla_*\,\widetilde{\pi}| + c\,|\widetilde{f}| \tag{8.32}$$

a.e. in Q_a. Set

$$\overline{p} = \frac{2q}{2(p-2)+q}\,. \tag{8.33}$$

From Hőlder's inequality

$$\| |\widetilde{M}|^{p-2}\,(\nabla_*(\widetilde{\nabla u}))\|_{\overline{p}} \leq \|\widetilde{M}\|_q^{p-2}\,\|\nabla_*(\widetilde{\nabla u}))\|_2\,.$$

It readily follows that

$$\|\widetilde{\partial_3\,\pi}\|_{\overline{p}} \leq c\,(1+\|\widetilde{M}\|_q^{p-2})\,\|\nabla_*(\widetilde{\nabla u}))\|_2 + c\,\|\nabla_*\,\widetilde{\pi}\|_{\overline{p}} + c\,\|\widetilde{f}\|_2\,, \tag{8.34}$$

where all the norms are taken in $Q_{\frac{a}{2}}$. From (8.34) it follows that $\|\nabla\,\widetilde{\pi}\|_{L^{\overline{p}}(Q_{\frac{a}{2}})}$ is bounded by the right hand side of (8.34) (replace c by $c+1$ and recall (8.1)). By appealing to the first estimate (7.2) one shows that the right hand side of (8.34) is bounded by the right hand side of equation (8.35) below. Hence

$$\|\nabla\,\pi\|_{L^{\overline{p}}(\Omega_{\frac{a}{2}})} \leq C\,(1+\|\nabla u\|_q^{p-2})\,(P+\|f\|_2) + \|\nabla_*\,\widetilde{\pi}\|_{\overline{p}}\,. \tag{8.35}$$

where a simple expression for the term $P = P(\|\pi\|_{p'}, \|\nabla\,u\|_p)$ is easily obtained. By appealing to (8.31) and by taking into account the interior regularity of the weak solutions one proves that (1.11) holds. The Theorem 1.1 is completely proved. Moreover the Theorem 1.2 follows by setting $q=p$ in the Theorem **??**. Remark. It is quite easy to show that $\nabla\,\pi \in L^{p_1}_{loc}(\Omega)$ where $p_1 = \frac{12}{p+4}$. Note that p_1 is larger than $\overline{p}$ (actually stronger results hold).

9 Proof of Theorem 1.3.

Since we know that weak solutions belong to $W^{1,p}$, the above results are effective if we set $q = p$. In this particular case $r = p'$ and by (8.19) and a Sobolev immersion theorem one gets $u \in W^{1,(p')*}$. If $0 < p < 3$ it follows that $(p')* > p$. hence the above results also hold for $q = (p')*$. This same argument, used again and again, leads to a bootstrap argument.
In the sequel $\| \quad \|_{k,s}$ denotes the norm in the Sobolev space $W^{k,s}(\Omega)$.

We define $r = r(q)$ by (6.4), and the Sobolev embedding exponent r^* by (6.3). Hence $r^* = r^*(q)$ is defined by

$$r^*(q) = \frac{6\,q}{3(p-2)+q}\,, \tag{9.1}$$

for $p \leq q \leq 6$. In the following $r = r(q)$ and $r^* = r^*(q)$.

Theorem 1.1 shows that if $u \in W^{1,q}$ then $u \in W^{2,r}$. Moreover, by (1.10),

$$\|u\|_{2,r} \leq C\,(1+\|\nabla u\|_q^{\frac{p-2}{2}})\,(P+\|f\|_2)\,.$$

Hence, by a Sobolev embedding theorem, $u \in W^{1,r^*}$ and

$$\|u\|_{1,r^*} \leq c_0\,\|u\|_{2,r} \leq C\,(1+\|\nabla u\|_q^{\frac{p-2}{2}})\,(P+\|f\|_2).$$

Since $1+\frac{2}{p-2} \leq r \leq 2$, the distinct values of the embedding constants c_0 are bounded from above by a constant independent of r.

This shows the following result.

Lemma 9.1. *If a solution u belongs to $W^{1,q}$ then u belongs to W^{1,r^*}, where $r^*(q)$ is given by* (9.1), *moreover*

$$\|u\|_{1,r^*} \leq C\,(1+\|\nabla u\|_q^{\frac{p-2}{2}})\,(P+\|f\|_2)\,. \tag{9.2}$$

Since $p \geq 2$ the function $r^*(q)$ is increasing and bounded from above (for instance, by 6). Next we define the increasing sequence

$$\begin{cases} q_1 = p\,, \\ q_{n+1} = r^*(q_n)\,. \end{cases} \tag{9.3}$$

Clearly

$$q_\infty = 3\,(4-p) \tag{9.4}$$

is a fixed point of r^*, $r^*(q_\infty) = q_\infty$, moreover

$$\lim_{n\to\infty} q_n = q_\infty\,. \tag{9.5}$$

From (9.2) it follows that

$$\|u\|_{1,q_{n+1}} \leq c\,(1+\|u\|_{1,q_n}^{\frac{p-2}{2}})(P+\|f\|)\,. \tag{9.6}$$

Next we appeal to an induction argument. Note that for $n=1$ one has

$$\|u\|_{1,q_1} = \|u\|_{1,p}\,.$$

If we are able to show that the quantities $a_n = \|u\|_{1,q_n}$, at least for large values of n, are uniformly bounded by a finite number L then well know results in Functional Analysis, together with (9.5), yield

$$\|u\|_{1,q_\infty} \leq L\,. \tag{9.7}$$

For convenience we write (9.6) in the form

$$\|u\|_{1,q_{n+1}} \leq b + b\,\|u\|_{1,q_n}^{\alpha}\,, \tag{9.8}$$

where $b = c(P+\|f\|)$ and

$$\alpha = \frac{p-2}{2}\,.$$

Note that $0 \leq \alpha < 1$ provided that $2 \leq p < 4$. Denote by λ the (unique) solution of the equation $\lambda = b + b\,\lambda^\alpha$. By (9.6) one has $a_{n+1} \leq b + b\,a_n^\alpha$. Set $b_1 = a_1$ and $b_{n+1} = b + b\,b_n^\alpha$. Clearly $a_n \leq b_n$ for each n. It is easily seen that if $b_1 < \lambda$ then the sequence b_n is strictly increasing an converges to the fixed point λ. If $b_1 > \lambda$ then the sequence decreases to the value λ. Hence the sequence b_n converges to λ, so $a_n < 2\,\lambda$ for large values of n. On the other hand one easily shows that

$$\lambda \leq 2b + (2\,b)^{\frac{1}{1-\alpha}}\,.$$

Hence, under the hypothesis of Theorem 1.3, one has

$$\|u\|_{1,q_\infty} \leq c(P+\|f\|) + c\,(P+\|f\|)^{\frac{2}{4-p}}\,. \tag{9.9}$$

The Theorem 1.3 follows now by applying once more the Theorem 1.1, now with $q = q_\infty$ given by (9.4). In this case the equation (6.4) shows that $r = r(q_\infty) = l$, with l given by (1.17). Hence, from (1.10), it follows that

$$\|u\|_{W^2_r(\Omega)} \leq P\,(1 + \|f\|^{\frac{2}{4-p}})\,. \tag{9.10}$$

Finally, from (1.11) written with $q = q_\infty$, together with (9.9), one proves (1.18).

10 Proof of Theorem 1.4.

We show here the a priori estimate that leads to the desired results. A complete proof is done by following standard devices.
Since

$$\int_\Omega (u \cdot \nabla) u \cdot u\, dx = 0\,,$$

it readily follows that all the estimates for weak solutions, stated in section 2, hold here.
On the other hand, by Hőlder's inequality,

$$\|(u \cdot \nabla)\, u\| \leq \|u\|_{p^*} \|\nabla\, u\|_s$$

where $s = \frac{6p}{5p-6}$. By well know embedding theorems it follows that

$$\|(u \cdot \nabla)\, u\| \leq \|u\|_{W^{1,p}} \|u\|_{W^{\frac{3}{2},p'}}\,.$$

By appealing, in particular, to the compact embedding of $W^{2,p'}$ into $W^{\frac{3}{2},p'}$ one proves that to each positive real ϵ it corresponds a positive C_ϵ such that

$$\|(u \cdot \nabla)\, u\|_2 \leq \|u\|_{W^{1,p}} (C_\epsilon \|u\|_{W^{1,p}} + \epsilon \|u\|_{W^{2,p'}})\,. \tag{10.1}$$

Next we treat the term $(u \cdot \nabla)\, u$ as a "right hand side", by adding it to the external forces f in the estimate (1.14). This gives

$$\|u\|_{2,p'} \leq P\,(1 + \|f\|) + C_\epsilon\, P + \epsilon\, C_0\, (1 + \|u\|^{\frac{p}{2}}_{1,p})\, \|u\|_{2,p'}\,, \tag{10.2}$$

where the quantities $P = P(\|\nabla\, u\|_p\,, \|\pi\|_{p'})$ may change from equation to equation and C_0 denotes a particular constant C. By fixing a sufficiently small value of ϵ we easily show that

$$\|u\|_{2,p'} \leq P\,(1 + \|f\|)$$

for some P. From (10.1) it follows that

$$\|(u \cdot \nabla)\, u\| \leq P\,(1 + \|f\|)\,.$$

Consequently, if in the estimates stated in the Theorems 1.1, 1.2 and 1.3 we replace f by $f + (u \cdot \nabla)\, u$ this simply leads to replace $\|f\|$ by $P\,\|f\|$.

11 The evolution Navier-Stokes equation

In the sequel we merely prove the a priori estimates that lead to Theorems 1.5 and 1.6. Complete proofs are done by applying the estimates to the approximate solutions obtained by the Faedo-Galerkin method. By now this is a well known device. See, for instance, [25] section 2 where this method is followed for the evolution Ladyzhenskaya model.

Multiplication by u, integration in Ω followed by suitable integrations by parts show that

$$\frac{1}{2}\frac{d}{d\,t}\,\|u(t)\|^2 + \frac{\nu_0}{2}\,\|\mathcal{D}\,u\|^2 + \frac{\nu_1}{2}\,\|\mathcal{D}\,u\|_p^p = \int_\Omega f\,u\,dx\,. \tag{11.1}$$

By integration of (11.1) with respect to time, one gets the following result:

Lemma 11.1. *Let u be a weak solution to problem* (1.21) *under the boundary condition* (1.5) *plus x'-periodicity. Then u satisfies the estimate*

$$\begin{aligned}&\|u(t)\|^2_{L^\infty(0,T;L^2)} + \nu_0\,\|u\|^2_{L^2(0,T;H^1)} + \nu_1\,\|u\|^p_{L^p(0,T;W^{1,p})} + \\ &\leq c\,\left(\|u(0)\|^2 + \tfrac{1}{\nu_0}\,\|f\|^2_{L^2(0,T;H^{-1})}\right)\,.\end{aligned} \tag{11.2}$$

Next we prove a stronger estimate "in time". See (11.4). A complete proof of this estimate is done by passing through the solutions of a suitable family of approximate problems. This can be done by appealing to a Faedo-Galerkin procedure as, for instance, in Theorem 2.2 in reference [25].

We define $\mathcal{M}$ by the equation

$$\begin{aligned}&\mathcal{M}^2 = \,2\,exp\left\{\tfrac{c}{\nu_1}\,\int_0^T\|\mathcal{D}\,u\|_p^{4-\,p}\,dt\right\}\cdot \\ &\left\{\nu_0\,\|\mathcal{D}\,u_0\|^2 + \,\nu_1\,\|\mathcal{D}\,u_0\|_p^p + \,c\,\int_0^T\|f(t)\|^2\,dt\right\}\,.\end{aligned} \tag{11.3}$$

Note that, by (11.2), the first integral in the right hand side of (11.3) can be estimated in terms of the data since $4-\,p \leq\, p$.

One has the following result:

Lemma 11.2. *Let u be as in Lemma 11.1 and assume that $u_0 \in\, V_p$,* (11.8) *holds and $f \in L^2(0,T;L^2)$. Then*

$$\|\partial_t\,u\|^2_{L^2(0,T;L^2)} + \,\nu_0\,\|\nabla\,u\|^2_{L^\infty(0,T;L^2)} + \,\nu_1\,\|\nabla\,u\|^p_{L^\infty(0,T;L^p)} \leq \,c\,\mathcal{M}^2\,. \tag{11.4}$$

Proof. By suitable integrations by parts, it follows that

$$\begin{aligned}&-\int_\Omega\left[\nabla\cdot\left(\nu_0\,\nabla\,u + \,\nu_1\,|\mathcal{D}\,u|^{p-2}\,\mathcal{D}\,u\right) + \,\nabla\,\pi\right]\cdot\,\tfrac{\partial\,u}{\partial\,t}\,dx = \\ &\tfrac{\nu_0}{2}\,\tfrac{d}{d\,t}\|\mathcal{D}\,u\|^2 + \,\tfrac{\nu_1}{2\,p}\,\tfrac{d}{d\,t}\|\mathcal{D}\,u\|_p^p\,.\end{aligned} \tag{11.5}$$

On the other hand

$$\int_\Omega|(u\cdot\,\nabla)\,u|^2\,dx \leq\, c\,\|u\|^2_{\frac{2\,p}{p-2}}\,\|\nabla\,u\|_p^2\,. \tag{11.6}$$

Furthermore

$$\|u\|_{\frac{2p}{p-2}} \leq c\,\|u\|_{p^*} \tag{11.7}$$

provided that

$$p \geq 2 + \frac{2}{5}\,. \tag{11.8}$$

Remark. The assumption (11.8) is superfluous if we drop the term $(u\cdot\nabla)\,u$ from equation (1.1).

By appealing to a Sobolev embedding theorem together with (2.1), one shows that

$$\|(u\cdot\nabla)\,u\| \leq c\,\|\mathcal{D}\,u\|_p^2\,. \tag{11.9}$$

Hence, from (1.21) and (11.5), one gets

$$\begin{aligned} &\|\partial_t\,u\|^2 + \nu_0\,\tfrac{d}{dt}\|\nabla\,u\|^2 + \nu_1\,\tfrac{d}{dt}\|\mathcal{D}\,u\|_p^p \leq \\ &c\,\left(\|f\|^2 + \|\mathcal{D}\,u\|_p^{4-p}\,\|\mathcal{D}\,u\|_p^p\right)\,. \end{aligned} \tag{11.10}$$

From (11.10) straightforward, well known, manipulations show that

$$\|\partial_t\,u\|^2_{L^2(0,T;L^2)} + \nu_0\,\|\mathcal{D}\,u\|^2_{L^\infty(0,T;L^2)} + \nu_1\,\|\mathcal{D}\,u\|^p_{L^\infty(0,T;L^p)} \leq \mathcal{M}^2\,. \tag{11.11}$$

Finally, by (2.1), (11.4) follows for some constants c. □

In particular, the following result holds.

Corollary 11.1. *The constants of type P, see* (1.7), *are now time depending uniformly bounded functions $P = P(t)$ in $(0,T)$.*

Proof of theorem 1.5.

One has, almost everywhere in $]0,\mathrm{T}[$,

$$-\nu_0\Delta\,u - \nu_1\,\nabla\cdot(|\mathcal{D}u|^{p-2}\,\mathcal{D}u) + \nabla\,\pi = f(x) - (u\cdot\nabla)\,u - \partial_t\,u\,.$$

Hence, by taking into account (1.14), one shows that for each $t\in]0,T[$

$$\|u\|_{2,p'} \leq P(t)\,\left(1 + \|f\| + \|(u\cdot\nabla)\,u\| + \|\partial_t\,u\|\right)\,. \tag{11.12}$$

Consequently, by appealing to (11.4) and (11.9), straightforward calculations show that $\|u\|_{L^2(0,T;\,W^{2,p'})}$ is bounded.

Similarly, by appealing to (1.15), one proves that $\|\nabla\,\pi\|_{L^2(0,T;\,L^{p_0})}$ is bounded.

Proof of theorem 1.6.

Now p' is replaced by l and we combine (11.4) with (1.16). One gets

$$\|u\|_{2,l} \leq P(t)\,\left(1 + \|f\|^{\frac{2}{4-p}} + \|\mathcal{D}\,u\|^{\frac{4}{4-p}} + \|\partial_t\,u\|^{\frac{2}{4-p}}\right)\,, \tag{11.13}$$

a.e. in $]0,T[$. Hence, by taking the $(4-p).^{th}$ power of both sides of (11.13) and by integrating in Ω, one shows that $\|u\|_{L^{4-p}(0,T;\,W^{2,l})}$ is bounded. By appealing to (1.18) one proves that $\|\nabla\,\pi\|_{L^{\frac{2(4-p)}{p}}(0,T;\,L^m)}$ is bounded.

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