A NOTE ON THE GLOBAL INTEGRABILITY, FOR ANY FINITE POWER, OF THE FULL GRADIENT FOR A CLASS OF POWER LAW MODELS, $p < 2$.

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Abstract

In the following we show that weak solutions to a class of systems of power law type, $p < 2$, have integrable gradient up to the boundary, with any finite exponent. The above class covers some well known generalized Navier-Stokes systems with shear dependent viscosity.

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1 Main result

In the following we prove the $W^{1,q}(\Omega)$-regularity up to the boundary, for any finite power q, for solutions of the system (1.4). The very weak assumptions made on the non-linear p-type term $g(x, \nabla u)$ are compensated by the presence of the Laplace operator. However this situation often appears in the literature, in the presence of more stringent assumptions on the p-term. Actually, these assumptions can be easily generalized and the Laplace operator replaced by a non symmetric (non variational) elliptic operator.
Our proof, based on a bootstrap argument and Stokes-elliptic regularization (see [4]) is elementary.
In the sequel Ω is a bounded, connected, open set in $\mathbb{R}^3$, locally situated on one side of its boundary Γ, a manifold of class C^2.
Below we consider solutions to the following class of stationary Navier-Stokes equations for flows with shear (more generally, gradient) dependent viscosity

$$\begin{cases} -\nabla \cdot T(u,\pi) + (u \cdot \nabla)\, u = f, \\ \nabla \cdot u = 0\,, \end{cases} \tag{1.1}$$

under suitable boundary conditions. T denotes the Cauchy stress tensor

$$T = -\pi\, I + \nu_0\, \mathcal{D}\, u + g(x, \nabla u) \tag{1.2}$$

and $\frac{1}{2}\, \mathcal{D}\, u$ is the symmetric gradient, i.e.,

$$\mathcal{D}\, u = \frac{1}{2}\, (\nabla\, u + \nabla\, u^T)\,.$$

Here ν_0 is a strictly positive constant and g is a tensor with components g_{ij}, $i, j = 1, 2, 3$. Note that $g = g(x, \mathcal{D}u)$ is a particular case of the above one. We set

$$|t|^2 = \sum t_{k,l}^2\,,$$

where $t = t_{kl}$ is a tensor. In the following we assume that $g(x,t)$ satisfies the classical Caratheodory conditions together with

$$|g(x,t)| \leq c\,(1+\,|t|)^{p-1} \tag{1.3}$$

for some $p \in (1,2)$. Note that $|g(x,t)|$ may behave like $(1+|t|)^{p(x)-1}$, provided that $p(x) \leq p$, almost everywhere in Ω.
Moreover,

$$g(x,\nabla u) = h(x,|\mathcal{D}u|)\,\mathcal{D}u\,,$$

where the real valued function h satisfies

$$|h(x,t)| \leq c\,(1+\,|t|)^{p-2}\,,$$

falls within the above picture.
For many salient results and a very interesting overview of the general theory, see [9].
Without loss of generality we assume that $\nu_0 = 1$. From (1.1) we get

$$\begin{cases} -\,\Delta\,u + \nabla\cdot\, g(x,\nabla\,u) + (u\,\cdot\,\nabla)\,u + \nabla\,\pi = f\,, \\ \\ \nabla\cdot u = 0\,. \end{cases} \tag{1.4}$$

In order to fix ideas we assume here the non-slip non-homogeneous boundary condition

$$u_{|\Gamma} = a(x)\,. \tag{1.5}$$

However, many other boundary conditions fall within the above scheme. Actually, it is sufficient that an estimate like (2.11) holds for the usual Stokes linear system (2.10) with the Dirichlet boundary condition replaced by the desired boundary condition.
In the sequel we take into account any possible weak solution u to our problem for which

$$u \in W^{1,p} \cap L^2\,. \tag{1.6}$$

If $a = 0$, or if the convective term is not present in (1.4), we replace (1.6) by $u \in W^{1,p}$.

Remark 1.1. *We are interested in the regularity properties of any possible weak solution to our problem. Hence our assumptions do not necessarily imply the existence of a solution. However, and this is the crucial point here, many well known existence theorems fall within the assumptions made here.*

Our main result is the following.

Theorem 1.1. *Assume that* (1.3) *holds and that the data*

$$f \in L^3 \tag{1.7}$$

and

$$a \in W^{1,+\infty}(\Gamma) \tag{1.8}$$

satisfy the necessary compatibility condition

$$\int_\Gamma a \cdot n \, d\Gamma = 0 \,. \tag{1.9}$$

Let u *be a solution of problem* (1.4), (1.5) *in the class* (1.6). *Then*

$$u \in W^{1,q}(\Omega) \quad \forall \, q < +\infty \,. \tag{1.10}$$

In particular

$$u \in C^{0,\alpha}(\overline{\Omega}) \,, \quad \forall \, \alpha < 1 \,. \tag{1.11}$$

Moreover the estimate (2.8) *holds with* v *replaced everywhere by* u*.*

Actually the above assumptions on f and a may be replaced by the weaker assumptions

$$f \in W^{-1,q}(\Omega) \,, \qquad \forall \, q < +\infty \tag{1.12}$$

and

$$a \in W^{1-\frac{1}{q},\, q}(\Gamma) \,, \qquad \forall \, q < +\infty \,. \tag{1.13}$$

2 Proof of Theorem 1.1

We define, for each non-negative integer n, the exponent

$$p_n = \frac{p}{(p-1)^n} \,. \tag{2.1}$$

This is a strictly increasing sequence, which tends to infinity with n. Note that $p_0 = p$ and $p_1 = p'$, the dual exponent of p.
In the sequel we need to assume that

We assume, moreover, the necessary compatibility condition

$$\int_\Gamma a \cdot n \, d\Gamma = 0 \,. \tag{2.2}$$

By well known results, see [3], there is a linear continuous map R from $W^{1-\frac{1}{q},\, q}(\Gamma)$ to $W^{1,\, q}(\Omega)$ such that for each $a \in W^{1-\frac{1}{q},\, q}(\Gamma)$ one has $\nabla \cdot (R\, a) = 0$ and $\gamma\,(R\, a) = a$, where γ denotes the trace operator. We still denote by a the function $R\, a$. If we assume that a satisfies (1.8) then we may prove directly that $R\, a \in W^{1,\, +\infty}(\Omega)$.
In the sequel, in order to simplify the presentation, we assume that (1.7) and (1.8) hold, even if (1.12) and (1.13) are sufficient for our purposes.
As usual, we consider as a new unknown the function

$$v = u - a \,.$$

This leads to the problems

$$\begin{cases} -\Delta\, v + \nabla \cdot \, g(x, \nabla\, a + \nabla\, v) + (v \, \cdot \, \nabla)\, v + \\ (a \, \cdot \, \nabla)\, v + (v \, \cdot \, \nabla)\, a + \nabla\, \pi = f + \Delta a - (a \, \cdot \, \nabla)\, a \,; \\ \nabla \cdot v = 0 \,; \\ v_{|\Gamma} = 0 \,. \end{cases} \tag{2.3}$$

From (1.6) it follows that $v \in W_0^{1,p} \cap L^2$. The spaces $W_0^{1,q}$ are endowed here with the norm $\|\nabla u\|_q$.
From (2.3) we get the "energy estimate"
(2.4)

$$\begin{cases} \|\nabla v\|_2^2 \leq (\|f\|_{-1,2} + \|\nabla a\|_2 + \|a\|_3 \|\nabla a\|_2) \|\nabla v\|_2 + \|\nabla a\|_\infty \|v\|_2^2 + \\ \\ c \, \|1 + |\nabla a| \, \|_p + c \, \|\nabla v\|_p^p . \end{cases}$$

Note that

$$(2.5) \qquad |g(x, \nabla a + \nabla v)| \leq c \, (1 + \|\nabla a\|_\infty + |\nabla v| \,)^{p-1} .$$

The symbol c denotes, here and in the sequel, positive constants that may depend, at most, on Ω. The same symbol may denote distinct constants.
Note that (2.4) shows that v and u belong to $W^{1,2}(\Omega)$.
We may improve the estimate (2.4), however this is not necessary for our purposes. Note, however, that under the more usual assumptions that lead to an existence theorem the term $\|\nabla v\|_p^p$ appears in the left hand side instead of in the right hand side. Moreover, if $a = 0$, the term $\||\nabla a|\| \, \|v\|_2^2$ is not present.
One has the following result.

Lemma 2.1. *Assume that* (1.7) *and* (1.8) *hold. Let* $v \in W_0^{1,p}$ *be a solution of problem* (1.4), (1.5)*. If, for some index* $n \geq 0$,

$$(2.6) \qquad \nabla v \in L^{p_n}$$

then

$$(2.7) \qquad \nabla v \in L^{p_{n+1}} .$$

Moreover

$$(2.8) \qquad \|\nabla v\|_{p_{n+1}} \leq C_{p_{n+1}} \, (1 + \|a\|_{1,\infty}^2 + \|\nabla v\|_2^2 + \|f\|_3 + \|\nabla v\|_{p_n}^{p-1}) .$$

We denote by C_q positive constants, defined in the sequel (see (2.11)). These constants depend on the particular value of the integrability exponent q. See the Remark 2.1.
The above result holds if the assumptions (1.7) and (1.8) are replaced by $f \in W^{-1,p_{n+1}}(\Omega)$ and $a \in W^{1-\frac{1}{p_{n+1}}, \, p_{n+1}}(\Gamma)$.

Proof. By appealing to (2.5), straightforward calculations show that

$$\|g(\,\cdot\,, \nabla a + \nabla v)\|_{p_{n+1}}$$

is bounded by the right hand side of equation (2.9) below. Hence

$$(2.9) \qquad \|\nabla \cdot g(\,\cdot\,, \nabla a + \nabla u)\|_{-1,\, p_{n+1}} \leq c_0 \, (1 + \|\nabla a\|_\infty^{p-1} + \|\nabla u\|_{p_n}^{p-1}) ,$$

where the constant c_0 can be chosen independent of p_n.
Next consider the linear Stokes system

$$(2.10) \qquad \begin{cases} -\Delta w + \nabla \widetilde{\pi} = F, \\ \\ \nabla \cdot w = 0 \, , \\ \\ w_{|\Gamma} = 0 \, . \end{cases}$$

Well know regularity results, see [4], show that

$$\|\nabla w\|_q \leq C_q \|F\|_{-1,q}\,, \tag{2.11}$$

where C_q denotes a suitable positive constant. By setting

$$F = f + \Delta a - (a \cdot \nabla)\, a - (\nabla \cdot g(\nabla a + \nabla v) + (v \cdot \nabla)\, v + (a \cdot \nabla)\, v + (v \cdot \nabla)\, a)$$

it readily follows, by appealing to (2.9) and (2.11), that (2.8) holds. The constant c_0 was incorporated inside $C_{p_{n+1}}$.
In order to prove our thesis for u it is clearly sufficient to prove the same thesis for v due to the relation $u = a + v$ together with the regularity of a. Since $v \in W^{1,p_0}(\Omega)$, the thesis follows by induction, by appealing to Lemma 2.1. $\square$

Remark 2.1. *It is worth noting that if the constants C_q were uniformly bounded for large values of q then the estimates* (2.8) *would immediately yield the uniform boundedness of the norms $\|\nabla v\|_{p_{n+1}}$, and hence the Lipschitz continuity of v and u up to the boundary (in this context, the assumption* (1.13) *would be strictly necessary). This result would hold even if C_q were to go to $+\infty$ sufficiently slowly. However, it may be proved, see [10], that for the scalar equation $-\Delta w = F$ under the homogeneous Dirichlet boundary condition, one has $\|w\|_{2,q} \leq C_q \|F\|_q$, where $C_q = O(q)$. At best, we expect this same behavior for the constants C_q in the case of the Stokes problem* (2.10). *Merely assuming this behavior, we cannot prove that K_n is bounded as $n \to \infty$.*

Remark 2.2. *Theorem 1.1 allows, in particular, the extension up to the boundary of some of the interior regularity results know in the literature. We refer to some fundamental results proved by M. Fuchs and G. Seregin in reference [6] in the context of the Stokes problem in the case $p < 2$. For the exact assumptions made in this last reference, and their overlap with our assumptions, see the original work [6].*
In Theorem 3.2.1 these authors prove (1.11) *locally in Ω. See also [5]. In Theorem 3.2.3 they prove that $\mathcal{D}u \in L^\infty_{loc}(\Omega)$. As pointed out in [6], Remark 3.2.5, this result implies that $u \in W^{1,q}_{loc}(\Omega) \quad \forall\, q < +\infty$.*
It remains open, in particular, the conjecture proposed by the above authors in their Remark 3.2.9, namely, to prove that $\mathcal{D}u \in L^\infty(\Omega)$. Actually, by taking into account (1.10), *we may even expect that $\nabla u \in L^\infty(\Omega)$.*
For interior partial regularity results we also refer the reader to [2] (see, in particular, Lemma 3.14) and to [1] and references therein.
Finally we refer to [7] where the authors prove the existence of globally smooth solutions in the two dimensional case under suitable conditions.

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