Existence of weak solutions for a quasilinear version of Benney equations

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Abstract

Benney introduced a general strategy for deriving systems of nonlinear partial differential equations associated with long- and short-wave solutions. The semi-linear Benney system was studied recently by Tsutsumi and Hatano. Here, we tackle the nonlinear version of it and using compensated compactness techniques we prove the global existence of weak solutions to the Cauchy problem, in the case that the equation for the amplitude of the long wave is a quasilinear conservation law with flux $f(v) = av^2 - bv^3$ where a, b are constants with $b > 0$.

AMS classification: 35L70, 35L65.

1. Introduction and main result.

In [1], Benney introduces new models of nonlinear time-dependent equations to study the interaction between short and long waves. The models are important in the theory of capillary-gravity waves, internal-surface waves and short and long gravity waves in fluids with finite depth. If $U = U(x,t)$ is a solution to a nonlinear partial differential equation which has both short and long waves, then geometrical considerations show that there are four length scales to be taken into account:

$$\begin{aligned} a_s &= \text{amplitude of the short wave,} \\ a_l &= \text{amplitude of the long wave,} \\ l_s &= \text{wave length of the short wave,} \\ l_s &= \text{wave length of the long wave.} \end{aligned}$$

Based on these scales, Benney introduces three dimensionless parameters, namely

$$\begin{aligned} \varepsilon_s &= \frac{a_s}{l_s} = \text{slope of the short wave,} \\ \varepsilon_l &= \frac{a_l}{l_l} = \text{slope of the long wave,} \\ \mu &= \frac{l_s}{l_l} = \text{ratio of short to long wave} \ll 1, \end{aligned}$$

and derives various models depending on the relative magnitude of the parameters $\varepsilon_s, \varepsilon_l$ and μ.

Benney treats the general model equation

$$\frac{\partial U}{\partial t} + AU = BU,$$

where A is a linear differential operator involving only spatial derivatives and B is a nonlinear differential operator. Noticing that

$$\frac{a_l}{a_s} = \frac{\varepsilon_l}{\varepsilon_s \mu},$$

and restricting attention to the case where the amplitudes of the short and long waves are comparable, we see that the long waves are substantially weaker that the short ones. In [1, eqs.(3.27), (3.28)], an example of this type of equations is given as:

$$\begin{aligned} i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} &= |u|^2 u + vu, \\ \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} f(v) &= \frac{\partial}{\partial x}|u|^2, \end{aligned} \tag{1.1}$$

where $x \in \mathbf{R},\ t \geq 0$, and $f = f(v)$ is a nonlinear polynomial real function, $u = u(x,t) \in \mathbf{C}$ and $v = v(x,t) \in \mathbf{R}$ represent the short wave and the long wave, respectively, and $i = \sqrt{-1}$.

The (semi-linear) case $f(v) = av$, $a \in \mathbf{R}$ was studied by Tsutsumi and Hatano in the pioneering papers [7,8]. In the present paper we consider the case

$$f(v) = av^2 - bv^3, \tag{1.2}$$

where $a \in \mathbf{R}$, $b > 0$ are given constants, with initial data

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x), \ \ x \in \mathbf{R}. \tag{1.3}$$

In order to simplify the presentation and without loss of generality, we assume that $a = b = 1$ in (1.2).

Using the general approach known as the vanishing viscosity method we start below by studying a regularized version ($\varepsilon > 0$)

$$\begin{aligned} i\frac{\partial u^\varepsilon}{\partial t} + \frac{\partial^2 u^\varepsilon}{\partial x^2} &= |u^\varepsilon|^2 u^\varepsilon + v^\varepsilon u^\varepsilon \\ \frac{\partial v^\varepsilon}{\partial t} + \frac{\partial}{\partial x} f(v^\varepsilon) &= \frac{\partial}{\partial x}|u^\varepsilon|^2 + \varepsilon \frac{\partial^2 v^\varepsilon}{\partial x^2} \end{aligned}, \tag{1.4}$$

in the domain $\mathbf{R} \times \mathbf{R}_+$ and with initial data

$$u(\cdot, 0) = u_0, \qquad v(\cdot, 0) = v_0. \tag{1.5}$$

Next, we will used arguments from the compensated compactness method by Murat and Tartar [4,6] generalized to the L^p-setting [2,3,5].

Our main result is as follows.

Theorem 1. *Given $u_0, v_0 \in H^1(\mathbf{R})$ with v_0 real-valued, there exist functions*

$$u \in L^\infty(\mathbf{R}_+; H^1(\mathbf{R})), \qquad v \in L^\infty(\mathbf{R}_+; (L^4 \cap L^2)(\mathbf{R}))$$

such that

$$i\int_0^\infty \int_{\mathbf{R}} u\frac{\partial \varphi}{\partial t}\,dx\,dt + \int_0^\infty \int_{\mathbf{R}} \frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\,dx\,dt + \int_{\mathbf{R}} u_0(x)\varphi(x,0)\,dx + \\ + \int_0^\infty \int_{\mathbf{R}} |u|^2 u\varphi\,dx\,dt + \int_0^\infty \int_{\mathbf{R}} vu\varphi\,dx\,dt = 0,$$

$$\int_0^\infty \int_{\mathbf{R}} v\frac{\partial \psi}{\partial t}\,dx\,dt + \int_0^\infty \int_{\mathbf{R}} f(v)\frac{\partial \psi}{\partial x}\,dx\,dt + \int_{\mathbf{R}} v_0(x)\psi(x,0)\,dx - \int_0^\infty \int_{\mathbf{R}} \frac{\partial}{\partial x}|u|^2\psi\,dx\,dt = 0,$$

for all functions $\varphi, \psi \in C_0^1(\mathbf{R} \times [0, +\infty[)$ (i.e. in the class of continuously differentiable functions with compact support), with φ being complex-valued and ψ real-valued.

To establish our main result we will first prove two lemmas, the first one being a direct generalization of the conservation laws derived in [7].

Lemma 2. *If $(u^\varepsilon, v^\varepsilon) \in C([0, +\infty[; H^1(\mathbf{R}))^2$ denotes a solution of the Cauchy problem (1.4), (1.5), then setting for all $t \geq 0$*

$$E_1(t) = \frac{1}{2}\int_{\mathbf{R}} \left|\frac{\partial u^\varepsilon}{\partial x}\right|^2 dx + \frac{1}{4}\int_{\mathbf{R}} |u^\varepsilon|^4 dx + \frac{1}{2}\int_{\mathbf{R}} v^\varepsilon |u^\varepsilon|^2 dx - \frac{1}{6}\int_{\mathbf{R}} (v^\varepsilon)^3 dx + \frac{1}{8}\int_{\mathbf{R}} (v^\varepsilon)^4 dx$$

and

$$E_2(t) = \frac{1}{2}\int_{\mathbf{R}} (v^\varepsilon)^2 dx + \operatorname{Im}\int_{\mathbf{R}} \left(u^\varepsilon \frac{\partial \bar{u}^\varepsilon}{\partial x}\right) dx,$$

it holds:

$$\frac{\partial}{\partial t}\int_{\mathbf{R}} |u^\varepsilon|^2 dx = 0, \tag{1.6}$$

$$\frac{\partial}{\partial t}E_1(t) + \frac{3}{2}\varepsilon\int_{\mathbf{R}} \left(v^\varepsilon \frac{\partial v^\varepsilon}{\partial x}\right)^2 dx = \varepsilon\int_{\mathbf{R}} v^\varepsilon \left(\frac{\partial v^\varepsilon}{\partial x}\right)^2 dx - \frac{\varepsilon}{2}\int_{\mathbf{R}} \frac{\partial v^\varepsilon}{\partial x}\frac{\partial}{\partial x}|u^\varepsilon|^2 dx, \tag{1.7}$$

$$\frac{\partial}{\partial t}E_2(t) + \varepsilon\int_{\mathbf{R}} \left(\frac{\partial v^\varepsilon}{\partial x}\right)^2 dx = 0. \tag{1.8}$$

Lemma 3. *Under the hypothesis of Lemma 2 the function*

$$g_\varepsilon(t) = \int_{\mathbf{R}} \left|\frac{\partial u^\varepsilon}{\partial x}\right|^2 dx + \int_{\mathbf{R}} |u^\varepsilon|^4 dx + \int_{\mathbf{R}} (v^\varepsilon)^4 dx$$

satisfies the following property: There exists $\varepsilon_0 > 0$ *such that for all* $\varepsilon \le \varepsilon_0$

$$g_\varepsilon(t) \le h(t), \quad t \ge 0,$$

where h *is a positive continuous function in* $[0, +\infty[$ *independent of* ε.

Hence, combining this result with (1.6) and (1.8) we obtain for all $\varepsilon \le \varepsilon_0$

$$\int_{\mathbf{R}} (v^\varepsilon)^2 dx + \varepsilon \int_0^t \int_{\mathbf{R}} \left(\frac{\partial v^\varepsilon}{\partial x}\right)^2 dx\, d\tau \le h_1(t), \quad t \ge 0,$$

where h_1 is a positive continuous function in $[0, +\infty[$ independent of ε.

To conclude this introduction let us point out that there is a large number of interesting open problems concerning the class of systems under consideration in this paper, especially:
i) The existence of local-in-time smooth solutions for the system associated with an arbitrary flux-function f. The finite-time blow-up of these solutions.
ii) The uniqueness of weak solutions to the Cauchy problem.
iii) The existence of stationary waves, and their linearized and nonlinear stability under small or large perturbations.

2. The Cauchy problem for the regularized model.

We start with a sketch of the proof of Lemma 2. The conservation law (1.6) is standard for the Schrödinder equation [7, eq.(43)]. The equation (1.7) is a variant of the conservation of energy established in [7, eq.(44)]. The difference is in computing the following term (we drop the superscript ε and assume "enough" regularity for u and v without loss of generality):

$$\frac{1}{2}\int_{\mathbf{R}} |u|^2 \frac{\partial v}{\partial t} dx = \frac{1}{2}\left[\int_{\mathbf{R}} |u|^2 \frac{\partial}{\partial x}|u|^2 dx - \int_{\mathbf{R}} |u|^2 \frac{\partial}{\partial x} f(v) dx + \varepsilon \int_{\mathbf{R}} |u|^2 \frac{\partial^2 v}{\partial x^2} dx\right]$$

$$= \frac{1}{2}\left[\int_{\mathbf{R}} \frac{\partial}{\partial x}|u|^2 f(v) dx \;+\; \varepsilon \int_{\mathbf{R}} |u|^2 \frac{\partial^2 v}{\partial x^2} dx\right]$$

$$= \frac{1}{2}\left[\int_{\mathbf{R}} \frac{\partial v}{\partial t} f(v) dx + \int_{\mathbf{R}} \left(\frac{\partial}{\partial x} f(v)\right) f(v) dx + \varepsilon \int_{\mathbf{R}} \frac{\partial^2 v}{\partial x^2} \left(|u|^2 - f(v)\right) dx\right]$$

thus

$$\frac{1}{2}\int_{\mathbf{R}} |u|^2 \frac{\partial v}{\partial t} dx$$
$$= \frac{1}{2}\left[\frac{\partial}{\partial t}\int_{\mathbf{R}} F(v)dx + \varepsilon\int_{\mathbf{R}} \frac{\partial v}{\partial x}\,(2v-3v^2)\frac{\partial v}{\partial x}dx\right] - \frac{1}{2}\varepsilon\int_{\mathbf{R}} \frac{\partial v}{\partial x}\frac{\partial}{\partial x}|u|^2 dx$$

$$= \frac{1}{2}\left[\frac{\partial}{\partial t}\int_{\mathbf{R}} F(v)dx - 3\varepsilon\int_{\mathbf{R}} v^2\left(\frac{\partial v}{\partial x}\right)^2 dx + 2\varepsilon\int_{\mathbf{R}} v\left(\frac{\partial v}{\partial x}\right)^2 dx\right] - \frac{1}{2}\varepsilon\int_{\mathbf{R}} \frac{\partial v}{\partial x}\frac{\partial}{\partial x}|u|^2 dx,$$

where we have set $F(v) = \int_0^v f(\xi)d\xi$. Finally, we obtain (1.8) by the same calculations as in [7, eq.(45)] with the modification

$$\mathrm{Im}\frac{\partial}{\partial t}\int_{\mathbf{R}} u\frac{\partial \bar{u}}{\partial x}dx = -\int_{\mathbf{R}} v\frac{\partial}{\partial x}|u|^2 dx =$$
$$= -\int_{\mathbf{R}} v\left(\frac{\partial v}{\partial t} + \frac{\partial}{\partial x}f(v) - \varepsilon\frac{\partial^2 v}{\partial x^2}\right)dx = -\frac{1}{2}\frac{\partial}{\partial t}\int_{\mathbf{R}} v^2 dx - \varepsilon\int_{\mathbf{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx,$$

which completes the proof of Lemma 2. ■

Proof of Lemma 3.

The right-hand side in (1.7) can be estimated as follows. We have (after dropping out the superscript ε)

$$\varepsilon\int_{\mathbf{R}} v\left(\frac{\partial v}{\partial x}\right)^2 dx - \frac{\varepsilon}{2}\int_{\mathbf{R}} \frac{\partial v}{\partial x}\frac{\partial}{\partial x}|u|^2 dx$$

$$\leq \frac{\varepsilon}{2}\int_{\mathbf{R}}\left(\frac{\partial v}{\partial x}\right)^2 dx + \frac{\varepsilon}{2}\int_{\mathbf{R}}\left(v\frac{\partial v}{\partial x}\right)^2 dx + \varepsilon\int_{\mathbf{R}}\left|\frac{\partial v}{\partial x}\right|\,|u|\,\left|\frac{\partial u}{\partial x}\right| dx,$$

and therefore by $(1.6), (1.7)$ and (1.8) :

$$g(t) = \left(\left|\frac{\partial u}{\partial x}\right|_{L^2}^2 + |u|_{L^4}^4 + |v|_{L^4}^4\right)(t) \leq c + c\,\varepsilon\int_0^t \left|\frac{\partial u}{\partial x}\right|_{L^2}^{3/2}\left|\frac{\partial v}{\partial x}\right|_{L^2} d\tau.$$

Here we have used that

$$|u|_{L^\infty} \leq c|u|_{L^2}^{1/2}\left|\frac{\partial u}{\partial x}\right|_{L^2}^{1/2}$$

and $c > 0$ denote a constant independent of ε,

$$|v|_{L^2}^2(t) + \varepsilon\int_0^t \left|\frac{\partial v}{\partial x}\right|_{L^2}^2 d\tau \leq c + c\left|\frac{\partial u}{\partial x}\right|_{L^2}(t).$$

We derive

$$g(t) \le c + c\,\varepsilon \int_0^t g(\tau)^{3/4} \left|\frac{\partial v}{\partial x}\right|_{L^2} d\tau = \theta(t) \tag{1.9}$$

and

$$\varepsilon \int_{\mathbf{R}} \left|\frac{\partial v}{\partial x}\right|^2_{L^2} d\tau \le c + c\,g(t)^{1/2} \le c + c\,\theta(t)^{1/2}. \tag{1.10}$$

We see that

$$\theta'(t) \le c\,\varepsilon^{1/2}\theta(t)^{3/4}\left(\varepsilon^{1/2}\left|\frac{\partial v}{\partial x}\right|_{L^2}(t)\right)$$

$$\theta'(t)\,\theta^{-3/4}(t) \le c\varepsilon^{1/2}\left(\varepsilon^{1/2}\left|\frac{\partial v}{\partial x}\right|_{L^2}(t)\right). \tag{1.11}$$

Let us now multiply (1.11) by e^t and integrate by parts over the interval $[0,t]$. After applying Cauchy-Schwarz inequality we obtain

$$-\int_0^t \theta^{1/4}(\tau)\,e^{\tau} d\tau + \left[\theta^{1/4}(\tau)e^{\tau}\right]_0^t \le c\,\varepsilon^{1/2}\left(\int_0^t e^{2\tau} d\tau\right)^{1/2}\left(\int_0^t \varepsilon\left|\frac{\partial v}{\partial x}\right|^2_{L^2} d\tau\right)^{1/2}$$

and so, by (1.10),

$$\theta^{1/4}(t)\,e^t \le c + c\int_0^t \theta^{1/4}(\tau)\,e^{\tau} d\tau + c\,\varepsilon^{1/2}\,e^t\,\theta^{1/4}(t) + c\,\varepsilon^{1/2}e^t.$$

Hence, for $\varepsilon \le \varepsilon_0$ such that $c\,\varepsilon_0^{1/2} = 1/2$, it follows that

$$\theta^{1/4}(t)\,e^t \le c + c\int_0^t \theta^{1/4}(\tau) d\tau + c\,\varepsilon_0^{1/2}e^t$$

and the conclusion follows by Gronwall's inequality and (1.9). ■

We are in a position to establish the global existence for the approximating problem $(1.4), (1.5)$:

Proposition 4. *For each $\varepsilon > 0$ the Cauchy problem* $(1.4), (1.5)$ *admits a unique solution* $(u,v) \in C([0,+\infty[;H^1(\mathbf{R}))^2$. *Moreover, for all $T > 0$*

$$v \in L^2(0,T;H^2(\mathbf{R})), \qquad \frac{\partial v}{\partial t} \in L^2(0,T;L^2(\mathbf{R})).$$

Proof. First we prove the local-in-time existence and uniqueness of the solution. To this end we consider the product space $\mathbf{B}_R^T \times \mathrm{B}_R^T$, where

$$\mathbf{B}_R^T = \{w \in X_T = C([0,T];H^1(\mathbf{R}))\ :\ \|w\|_{L^\infty(0,T;H^1)} \le R\},$$

and B_R^T denotes the corresponding real-valued version for the second factor, and pairs of functions $(\tilde{u}, \tilde{v})$. We search for the map

$$(\tilde{u}, \tilde{v}) \longrightarrow \Phi(\tilde{u}, \tilde{v}) = (u, v) \tag{2.1}$$

where $(u, v) \in X_T \times X_T$ is the solution of the linear problem (we put $\varepsilon = 1$ for simplicity)

$$\begin{aligned} & i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = |\tilde{u}|^2 \tilde{u} + \tilde{v}\tilde{u} \\ & \frac{\partial v}{\partial t} + \frac{\partial}{\partial x} f(\tilde{v}) = \frac{\partial}{\partial x} |\tilde{u}|^2 + \frac{\partial^2 v}{\partial x^2} \\ & u(0) = u_0, \quad v(0) = v_0 \end{aligned} \tag{2.2}$$

Recall that, in the usual semi-group framework,

$$u(t) = e^{i\Delta t} u_0 - i \int_0^t e^{i\Delta(t-s)} \left[|\tilde{u}|^2 \tilde{u} + \tilde{v}\tilde{u} \right](s) ds$$

and

$$\frac{\partial}{\partial x} |\tilde{u}|^2 - \frac{\partial}{\partial x} f(\tilde{v}) \in C([0, T]; L^2(\mathbf{R})).$$

It is now easy to prove that there exists $T > 0$ and $R > 0$ such that the map Φ defined in (2.1) is a strict contraction in the complete metric space $\mathbf{B}_R^T \times \mathrm{B}_R^T$ and so, by the Banach fixed point theorem, we obtain a unique local in time solution (u, v) of the Cauchy problem (1.4), (1.5) in the space $X_T \times X_T$.
To conclude that there is a unique global solution (u, v) in the space $C([0, +\infty[; H^1(\mathbf{R}))^2$ we can apply (1.6), Lemma 3 (in $[0, T]$) and, to estimate $\left|\frac{\partial v}{\partial x}\right|_{L^2}(t)$, the formula (in the semi-group framework)

$$v(t) = e^{\Delta t} v_0 + \int_0^t e^{\Delta(t-s)} \left[\frac{\partial}{\partial x} |u|^2 - \frac{\partial}{\partial x} f(v) \right](s) ds.$$

Since we have

$$\left| \frac{\partial}{\partial x} e^{\Delta t} \rho \right|_{L^2} \leq \frac{c}{t^{3/4}} \, |\rho|_{L^1}, \quad t > 0,$$

we deduce

$$\left|\frac{\partial v}{\partial x}\right|_{L^2}(t) \leq \left|\frac{\partial v_0}{\partial x}\right|_{L^2} + c \int_0^t \frac{1}{(t-s)^{3/4}} \left[\left| u \frac{\partial \bar{u}}{\partial x} \right|_{L^1} + \left| v^2 \frac{\partial v}{\partial x} \right|_{L^1} + \left| v \frac{\partial v}{\partial x} \right|_{L^1} \right](s) ds \leq$$

$$\leq \left|\frac{\partial v_0}{\partial x}\right|_{L^2} + c \int_0^t \frac{1}{(t-s)^{3/4}} ds + c \int_0^t \frac{1}{(t-s)^{3/4}} \left|\frac{\partial v}{\partial x}\right|_{L^2}(s) ds.$$

Hence, using Gronwall's inequality, we arrive at the desired estimate. Since, for each $T > 0$, $\frac{\partial}{\partial x}[\,|u|^2 - f(v)] \in L^2(0,T;L^2(\mathbf{R}))$, we conclude that $v \in L^2(0,T;H^2(\mathbf{R}))$ and $\frac{\partial v}{\partial t} \in L^2(0,T;L^2(\mathbf{R}))$. ■

3. Proof of Theorem 1.

Let $(u_\varepsilon, v_\varepsilon)$ (for $\varepsilon \le \varepsilon_0$) be the approximating solutions in $C([0,+\infty[;H^1(\mathbf{R}))^2$ for the problem (1.4), (1.5). By Lemma 3 we have that

$$\{u_\varepsilon\}_\varepsilon \text{ bounded in } L^\infty(\mathbf{R}_+;H^1(\mathbf{R})), \qquad \left\{\frac{\partial u_\varepsilon}{\partial t}\right\}_\varepsilon \text{ bounded in } L^\infty(\mathbf{R}_+;H^{-1}(\mathbf{R})).$$

Recall that, for each $R > 0$, the injection $H^1(I_R) \hookrightarrow L^2(I_R)$ is compact, where $I_R = \{x \in \mathbf{R} : |x| \le R\}$. So, for each $T > 0$, the injection of the space

$$W_R^T = \left\{w \in L^2(0,T;H^1(I_R)) \ : \ \frac{\partial w}{\partial t} \in L^2(0,T;H^{-1}(I_R)\right\}$$

in $L^2(0,T;L^2(I_R))$ is compact, by a well known result.

By applying a standard diagonalization procedure, there exists $u \in L^\infty(\mathbf{R}_+;H^1(\mathbf{R}))$ and a subsequence of $\{u_\varepsilon\}_\varepsilon$ still denoted by $\{u_\varepsilon\}_\varepsilon$ such that

$$u_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} u \quad \text{in} \quad L^\infty(\mathbf{R}_+;H^1(\mathbf{R})) \text{ weak}* \quad \text{and a.e. in} \quad \mathbf{R}_+ \times \mathbf{R}. \tag{3.1}$$

We can also deduce that there exist $v \in L^\infty(\mathbf{R}_+;(L^4 \cap L^2)(\mathbf{R}))$ and $w \in L^\infty(\mathbf{R}_+;L^1(\mathbf{R}))$ such that

$$\begin{aligned} &v_\varepsilon \underset{\varepsilon\to 0}{\rightharpoonup} v \quad \text{in} \quad L^\infty(\mathbf{R}_+;(L^4 \cap L^2)(\mathbf{R})) \text{ weak}*, \\ &f(v_\varepsilon) \underset{\varepsilon\to 0}{\rightharpoonup} w \quad \text{in} \quad L^\infty(\mathbf{R}_+;L^{4/3}(\mathbf{R})) \text{ weak}*. \end{aligned} \tag{3.2}$$

Moreover we have, with $p = 3$,

$$|f'(\xi)| \le c(1 + |\xi|^{p-1}), \quad \xi \in \mathbf{R},$$

and for each real convex C^2 entropy function η with compact support we deduce from (1.4) with $q \in C^2(\mathbf{R})$ such that $q' = \eta' f'$ (q is the corresponding entropy flux)

$$\begin{aligned} \frac{\partial}{\partial t}\eta(v^\varepsilon) + \frac{\partial}{\partial x}q(v^\varepsilon) &= \eta'(v^\varepsilon)\,|u^\varepsilon|_x^2 + \varepsilon\,\eta'(v^\varepsilon)\,\frac{\partial^2 v^\varepsilon}{\partial x^2} \\ &= \varepsilon\,\frac{\partial^2}{\partial x^2}(\eta(v^\varepsilon)) - \varepsilon\,\eta''(v^\varepsilon)\left(\frac{\partial v^\varepsilon}{\partial x}\right)^2. \end{aligned} \tag{3.3}$$

For each open bounded set Ω in $\mathbf{R}\times]0,+\infty[$, $\{v_\varepsilon\}_\varepsilon$ is bounded in $L^p(\Omega)$ with $p=3$, and, by Lemma 3, $\left\{\varepsilon\,\eta''(v^\varepsilon)\left(\frac{\partial v^\varepsilon}{\partial x}\right)^2\right\}_\varepsilon$ is bounded in $L^1(\Omega)$ (since η'' is continuous with compact support) and so is bounded in the space of measures with finite mass $\mathcal{M}(\Omega)$. Hence, by (3.3)

$$\frac{\partial}{\partial t}\eta(v^\varepsilon)+\frac{\partial}{\partial x}q(v^\varepsilon)\in[\text{compact set of } H^{-1}(\Omega)]\ +\ [\text{bounded set of } \mathcal{M}(\Omega)].$$

We can now apply the Corollary 3.1 of Theorem 3.2 in [5] and, by a suitable diagonalization procedure, we can deduce $f(v_\varepsilon)\underset{\varepsilon\to 0}{\longrightarrow} f(v)$ in $\mathcal{D}'(\mathbf{R}\times]0,+\infty[)$. Hence, by (3.2), we have $f(v_\varepsilon)\underset{\varepsilon\to 0}{\longrightarrow} f(v)$ in $L^\infty(\mathbf{R}_+;L^{4/3}(\mathbf{R}))$ weak$*$. Finally, in view of (1.4), (1.5) and passing to the limit $\varepsilon\to 0$, we obtain the desired result. ■

Acknowledgements: This research was partially supported by FCT under program POCTI (Portugal/FEDER-EU).

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