A Characterization of the standard embedding of \( \mathbb{C}P^2 \)

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Abstract

H. Hopf showed that any immersed constant mean curvature sphere \( S^2 \hookrightarrow \mathbb{R}^3 \) is the round sphere. Allowing higher dimensions for the domain manifold, the class of Kähler manifolds is an adequate framework to the generalization of Hopf’s theorem. In the Kähler domain setting the complexified second fundamental form \( \alpha \) splits according to types. The \((1,1)\) part of the second fundamental splitting plays the role of the mean curvature for surfaces and will be called the plurimean curvature. Therefore isometric immersions with parallel plurimean curvature (ppmc isometric immersions) generalize in a natural way the cmc immersions. It is a standard fact that \( \mathbb{R}^8 \) is the smallest space where \( \mathbb{C}P^2 \) can be embedded. The aim of this work is to generalize Hopf theorem proving that the only ppmc isometric immersion from \( \mathbb{C}P^2 \) into \( \mathbb{R}^8 \) is the standard immersion.

1 Introduction and statement of results

The smallest \( \mathbb{R}^k \) into which \( S^2 = \mathbb{C}P^1 \) may be embedded is \( \mathbb{R}^3 \). H. Hopf ([9]) showed that, up to a congruence, the only constant mean curvature (cmc) isometric immersion from the sphere into \( \mathbb{R}^3 \) is the standard immersion. Affording higher dimensions in the domain manifold, an adequate setting is the class of Kähler manifolds. When \( M \) is a Kähler manifold and \( \varphi : M \longrightarrow \mathbb{R}^k \) is an isometric immersion, the coupling of the second fundamental form \( \alpha \) of \( \varphi \) with the complex structure \( J \) of \( M \) originates two operators. To describe this operators we denote respectively by \( T^C M, T'M \) and \( T''M \) the complexification of \( TM \) and the eigenbundles of \( J \) corresponding to the eigenvalues \( i \) and \( -i \); \( \pi' \) and \( \pi'' \) will denote respectively the orthogonal projections of \( T^C M \) onto \( T'M \) and \( T''M \), where \( X = \pi'(X) + \pi''(X) \) for all \( X \in TM \). Then the complexification of \( \alpha \) decomposes according to types giving rise to

\[
\alpha^{(1,1)}(X,Y) = \alpha(\pi'(X),\pi''(Y)) + \alpha(\pi''(X),\pi'(Y))
\]

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and to
\[ \alpha^{(2,0)}(X, Y) = \alpha(X', Y') \]
where we have considered \( Z' = \pi'(Z), \ Z'' = \pi''(Z) \) for every \( Z \in C(T^C M) \).

When \( M \) is a Riemann surface observe that \( \alpha^{(1,1)} = \langle , \rangle H \), where \( H \) denotes the mean curvature of the immersion. Therefore, in this framework, an immersion is \( cmc \) if, and only if, \( \alpha^{(1,1)} \) is parallel. Then, \( cmc \) isometric immersions from Riemann surfaces are naturally generalized by isometric immersions from Kähler manifolds with \( \alpha^{(1,1)} \) parallel. The operator \( \alpha^{(1,1)} \) is given the name of plurimean curvature (\( pmc \)) (see [2] for a justification) and immersions with \( \alpha^{(1,1)} \) parallel are called \( ppmc \) immersions. When \( \alpha^{(1,1)} \) vanishes identically the immersion is called pluriminimal ([5], [3], [4]).

H. Hopf discovered that the traceless part of the second fundamental form of an immersed \( cmc \) surface is a holomorphic quadratic differential on the surface. This observation was the key to his well known theorem referred above. This holomorphic differential is nothing but the operator \( \alpha^{(2,0)} \). For \( ppmc \) isometric immersions \( \alpha^{(2,0)} \) is also a normal bundle valued holomorphic quadratic differential. Ferus ([8]) classified the isometric immersions with \( \alpha^{(2,0)} = 0 \) (the \( (2,0) \) - geodesic immersions).

\( ppmc \) isometric immersions into space-forms display some special geometric features of the parallel mean curvature surfaces, namely the existence of a \( 1 \)-parameter deformation through a smooth family of \( ppmc \) isometric immersions which, up to a parallel isomorphism, have the same normal bundle ([2]). Just as in the case of immersions with parallel mean curvature, \( ppmc \) isometric immersions can also be characterized by the pluriharmonicity of its Gauss map ([6], [2]).

It is a standard fact that \( \mathbb{R}^8 \) is the smallest space where \( \mathbb{C}P^2 \) can be immersed. The aim of this work is to generalize Hopf theorem proving that the only \( ppmc \) isometric immersion from \( \mathbb{C}P^2 \) into \( \mathbb{R}^8 \) is the standard immersion.

Throughout the text \( M \) will represent a complete four dimensional, simply connected and connected, complex manifold with positive first Chern class. We will prove that

**Theorem 1.** Let \( \varphi : M \to \mathbb{R}^8 \) be a substancial immersion whose induced metric is Kähler.

If \( \varphi \) is \( ppmc \), then either

1. \( M = \mathbb{C}P^2 \) endowed with the Fubini-Study metric and \( \varphi \) is the standard embedding, or
2. \( M = M_1 \times M_2 \) and \( \varphi = \varphi_1 \times \varphi_2 \), where one of the \( M_i \) \((1 \leq i \leq 2)\) is \( S^2 \) with \( \varphi_i \) the standard immersion \( cmc \) into \( \mathbb{R}^3 \) and the other is a minimal immersion of a Riemann surface into \( S^4 \).

**Corollary 2.** Let \( \varphi : \mathbb{C}P^2 \to \mathbb{R}^8 \) be an immersion whose induced metric is Kähler. If \( \varphi \) is \( ppmc \), then \( \varphi \) is the standard embedding of \( \mathbb{C}P^2 \) endowed with the Fubini-Study metric.
2 Proof of results

We will use the notation $N(M)$ to represent simultaneously the normal bundle and its complexified bundle.

Clearly, from the $ppmc$ property we conclude, using Codazzi equation, that $\langle \alpha^{(2,0)}, H \rangle$ is a holomorphic differential, where $\langle \cdot, \cdot \rangle$ denotes the complex bilinear extension of the standard inner product of $\mathbb{R}^8$. Hence

$$\langle \alpha^{(2,0)}, H \rangle = \langle \alpha^{(0,2)}, H \rangle = 0$$

Analogously,

$$\langle \alpha^{(2,0)}, \alpha^{(2,0)} \rangle = \langle \alpha^{(0,2)}, \alpha^{(0,2)} \rangle = 0$$

In fact since $M$ has a metric with positive Ricci curvature, a Bochner type argument allows the conclusion that there are no non zero holomorphic differentials on $M$ ([1]). As above it is also easy to prove that $\langle \alpha^{(3,0)}, H \rangle = \langle \alpha^{(0,3)}, H \rangle = 0$ and $\langle \alpha^{(3,0)}, \alpha^{(2,0)} \rangle = 0$, where $\alpha^{(3,0)}$ (resp. $\alpha^{(0,3)}$) denotes the restriction of $\nabla \alpha$ to $\otimes^3 T'M$ (resp. to $\otimes^3 T''M$).

Let $Z = (Im)(\alpha^{(2,0)} + \alpha^{(0,2)}) \subset N(M)$. Since, at each point, $\dim Z$ is even and $H$ belongs to the orthogonal complement of $Z$ in $N(M)$, either $\dim Z = 2$, or $\dim Z = 0$.

If $\dim Z = 0$ everywhere, $\varphi$ is $(2,0)$-geodesic and the theorem is proved (see [8] for a classification of $(2,0)$ - geodesic isometric immersions). We will prove that, when $M$ is irreducible (i.e. it does not decompose as a product of two Riemann surfaces), $\varphi$ is, in fact, $(2,0)$-geodesic.

Assume then that $\dim Z = 2$ on an open subset $U$ of $M$. Let $L$ represent the orthogonal complement of $Z \oplus \langle H \rangle$ in $N(U)$, where $\langle H \rangle$ denotes the vector subspace spanned by $H$.

**Definition 3.** $\varphi$ is said to be isotropic if:

$$\langle \alpha^{(2,0)}(\cdot, \cdot, \cdot), \alpha^{(1,1)}(\cdot, \cdot, \cdot) \rangle = \langle \alpha^{(0,2)}(\cdot, \cdot, \cdot), \alpha^{(1,1)}(\cdot, \cdot, \cdot) \rangle = 0$$

and

$$\langle \alpha^{(2,0)}(\cdot, \cdot, \cdot), \alpha^{(2,0)}(\cdot, \cdot, \cdot) \rangle = \langle \alpha^{(0,2)}(\cdot, \cdot, \cdot), \alpha^{(0,2)}(\cdot, \cdot, \cdot) \rangle = 0.$$

**Lemma 4.** When $\ker \alpha^{(2,0)} = \emptyset$, $\varphi$ is isotropic.

**Proof.** If $\dim Z = 0$ everywhere, $\alpha^{(2,0)}$ vanishes identically and the equality is trivial. Assume then that $\dim Z = 2$ on an open subset $U$ of $M$ and consider the fourth-form $\beta \in C(\otimes^4 T^*M)$ defined by:

$$\beta(X,Y,Z,W) = \langle \alpha(X,Y), \alpha(Z,W) \rangle$$

$\beta$ is symmetric in the two first and in the two last entries. We consider the complex multi-linear extension of $\beta$.
Since $R_{\otimes 2T'} = 0$ we get from Gauss equation that
\[
\beta(X', Y', Z', W) = \beta(X', Z', Y', W)
\]
so that, when $\beta$ is restricted to $\otimes^3 T' \otimes TM$, we can also interchange entries 2 and 3. Using Codazzi equation we easily see that $\beta|_{\otimes^4 T'M}$ is an holomorphic differential, hence, reasoning as above, $\beta|_{\otimes^4 T'M} = 0$.

We need to show that $\beta(X', Y', Z', W'') = 0$ for $X, Y, Z, W \in TU$.

Notice that since $Z$ is an isotropic two-dimensional complex vector space, it decomposes as a sum of two one-dimensional complex vector subspaces. These correspond, respectively, to the images of $\alpha^{(2,0)}$ and $\alpha^{(0,2)}$ and will be denoted by $N'$ and $N''$.

Now consider $X \in TU$. Since $N'$ is one-dimensional there exists $Z \in TU$ such that $\alpha(X', Z') = 0$. Then
\[
\beta(Z', X', W', Y'') = 0
\]
or
\[
\beta(Z', W', X', Y'') = 0
\]

Since $ker\alpha^{(2,0)} \neq \emptyset$ one can choose $W$ such that $\alpha(Z', W') \neq 0$. Now $\dim N' = 1$ implies that
\[
\beta(A', B', X', Y'') = 0 \quad \forall A, B, X, Y \in TU
\]

\[\square\]

**Lemma 5.** ([7]) Under the above conditions, assume that $\varphi$ is not totally geodesic and $ker\alpha^{(2,0)} \neq \emptyset$. Then $M = M_1 \times M_2$, and $\varphi = \varphi_1 \times \varphi_2$, where one of the $M_i$ ($1 \leq i \leq 2$) is $S^2$, with $\varphi_1 : S^2 \rightarrow \mathbb{R}^3$ extrinsically symmetric, and the other is a Riemann surface mimally immersed in $S^4$.

**Proof.** We outline a sketch of the proof presented in [7]. Let $U$ be the open subset of $M$ where $\dim ker\alpha^{(2,0)}$ is maximum. For each $x \in U$ let
\[
\Delta_x = \left\{ X \in T_xM : \alpha^{(2,0)}(X, Y) = 0, \forall Y \in T_xM \right\}
\]
Clearly $\Delta_x$ and $\Delta_x^\perp$ are $J_x$ invariant. Thus we get a smooth distribution $\Delta$ on $U$. Using Codazzi equation it is easily seen that $\Delta$ is an integrable distribution whose leaves are totally geodesic Kähler submanifolds of $M$.

Consider now the tensor field $C : \Delta \rightarrow Hom(\Delta^\perp, \Delta^\perp)$ defined by $C_T(X) = -\langle \nabla_X T \rangle^\perp$, for $T \in \Delta$ and $X \in \Delta^\perp$, where $(\ )^\perp$ denotes the orthogonal projection of onto $\Delta^\perp$. We infer straightforward from Codazzi equations that, for smooth sections $Y$ of $TM$ and $T$ of $\Delta, \nabla_Y T'$ (respectively $\nabla_Y T''$) is a section of $\Delta'$ (respectively of $\Delta''$). Hence $C_T$ preserves types in the decomposition
$(\Delta^\perp)' + (\Delta^\perp)''$, that is $C_TJ = JC_T$. Furthermore $C_T$ satisfies the following differential equation:

$$(\nabla sC_T)(Y) = C_TC_SY + C_{\nabla s}TY + R^M(S,Y)T$$

where $S, T \in \Delta$ and $Y \in \Delta^\perp$. Let $x \in \Delta, \gamma$ a geodesic on the maximal leaf of $\Delta$ through $x$ and denote by $T$ its velocity field. Let $\lambda(t)$ represent the eigenvalue of $C_{T(t)}$ and $Y(t)$ a correspondent eigenvector. From the last equation one gets that $\lambda$ satisfies the following Ricatti type equation:

$$\lambda' = \lambda^2 + r$$

where $r(t) = \langle R^M(T,Y)T, Y(t) \rangle_{\gamma(t)} \geq 0$. It is well known that $0$ is the only real solution of this equation defined on the real line. Therefore $C_T$ has no real eigenvalues. If $\lambda = \mu + i\eta$, taking $S = \mu T - \eta JT$, one gets $C_SY = (\mu^2 + \eta^2)Y$, hence $\mu = \eta = 0$ as above. We conclude then that $C_T^2 = 0$ for all $T \in \Delta$, thus $C_T = 0$ since $\dim \Delta^\perp = 2$. Now since $\Delta$ and $\Delta^\perp$ are invariant under the action of the holonomy group of $M$, by the deRham decomposition theorem we know that $U$ is a product of two Kähler manifolds $U_1$ and $U_2$. To prove that $\varphi|_U$ is a product of immersions we first notice that $\alpha(S', Y') = 0$ for all $S \in \Delta$ and $Y \in \Delta^\perp$. Using Gauss equation one easily gets that $\langle \alpha(S', Y'), \alpha(S'', Y'') \rangle = \langle \alpha(S'', Y''), \alpha(S'', Y') \rangle$, henceforth $\alpha(S, Y) = 0$ whenever $S \in \Delta, Y \in \Delta^\perp$. $\varphi|_U$ splits then as a product of immersions [10]. An analyticity argument allows the conclusion that $M$ is a product of two Kähler manifolds and $\varphi$ is a product immersion.

\[\square\]

We proceed now proving theorem 1. Due to lemma 4 it is enough to assume that $\alpha^{(2,0)}$ has an empty kernel.

We assume first that $M$ is irreducible to conclude that condition 1 holds. For each section $\xi$ of the normal bundle, $A^\xi_{\alpha^{(1,1)}}$ will represent the "Weigertten" operator associated to $\alpha^{(1,1)}$ in the direction of $\xi$, that is, $\langle A^\xi_{\alpha^{(1,1)}}(X), Y \rangle = \langle \alpha^{(1,1)}(X, Y), \xi \rangle$ for all $X, Y \in TM$. $A^H_{\alpha^{(1,1)}}$ is parallel since $\alpha^{(1,1)}$ and $H$ are parallel. This entails that $A^H_{\alpha^{(1,1)}}$ is a multiple of the identity, since $M$ is irreducible. Now $\langle \alpha^{(2,0)}, H \rangle = 0$ implies that $A_H$ is also a multiple of the identity, hence the image of $M$ is contained in a round sphere. If $\alpha^{(2,0)}$ does not vanish identically the normal space spanned by the image of $\alpha^{(1,1)}$ has to be one dimensional, otherwise in view of lemma 3, that subspace would have dimension two. Taking a unitary section $T$ of this subspace, orthogonal to $H$, we would conclude that $A^H_{\alpha^{(1,1)}}$ vanishes identically, since it is parallel ($T$ is parallel) and has trace 0, a contradiction. Therefore $\alpha^{(1,1)}$ is totally umbilical. Taking, at each $x \in M$, an orthonormal basis $E_i, JE_i, 1 \leq i \leq 2$, we obtain using Grass equation that $\langle \alpha(E_i', E_i''), \alpha(E_j', E_j'') \rangle = \langle \alpha(E_i'', E_j''), \alpha(E_i', E_j') \rangle$, from whence $H = 0$ which cannot happen. Thus $\varphi$ is $(2,0)$ - geodesic. Ferus classification tells then
that $M$ must be the complex projective space endowed with the Fubinni-study metric and that $\varphi$ is the standard immersion.

Assume now that $M$ is reducible and $A^{(1,1)}_H$ has two distinct eigenvalues $\lambda_1$ and $\lambda_2$. Let $T^1$ and $T^2$ represent the eigendistributions of $A^{(1,1)}_H$ associated respectively to $\lambda_1$ and $\lambda_2$. It is easily seen that $T^1$ and $T^2$ are parallel and $J$ invariant. Since $T^1$ and $T^2$ are invariant by the action of the holonomy group of $M$, from the de Rham decomposition theorem we know that $M$ is a product of two Kähler manifolds $M_1$ and $M_2$. One has $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. Suppose $\lambda_1 \neq 0$. Since $\langle \alpha^{(2,0)}, H \rangle = 0$, $T^1$ and $T^2$ are kept invariant by $A_H$ and $T^1$ and $T^2$ are also the eigenspaces of $A_H$ corresponding respectively to the eigenvalues $\lambda_1$ and $\lambda_2$. If the subspace of the normal bundle spanned by the image of $\alpha$ is a product of $\lambda_1$ and $\lambda_2$, one has $\alpha^{(1,1)}(X, Y) = 0$ for each $X \in T_1, Y \in T_2$. Now from

$$0 = \langle R(X', Y''), X'', Y'' \rangle = \langle \alpha(X', X''), \alpha(Y', Y'') \rangle - \langle \alpha(X', Y'), \alpha(X'', Y'') \rangle$$

and

$$0 = \langle R(X', Y'')X'', Y'' \rangle = \langle \alpha(X', X''), \alpha(Y', Y'') \rangle - \langle \alpha(X', Y''), \alpha(X'', Y') \rangle$$

we get that

$$\langle \alpha(X', Y'), \alpha(X'', Y'') \rangle = \langle \alpha(X', Y''), \alpha(X'', Y') \rangle,$$

hence $\alpha(X, Y) = 0$ whenever $X \in T_1, Y \in T_2$. Thus $\varphi$ decomposes as a product immersion ([10]) Assume then that the subbundle spanned by the image of $\alpha^{(1,1)}$ has rank two. Consider a unitary section $T$ of $L$. $A_H$ and $A_T$ commute since $H$ and $T$ are parallel in the normal bundle, so that this two operators have the same eigenbundles. The equation $\langle \alpha^{(2,0)}, T \rangle = 0$ tells that $A_T$ commutes also with $J$. Therefore $A_T$ leaves $T_1$ and $T_2$ invariant. Thus $\alpha(X, Y) = 0$ and $\varphi$ is a product of cmc immersions. One of the immersions, say $\varphi_1$, must be $(2, 0)$-geodesic, that is, a two - sphere in $\mathbb{R}^3$ ([8]). Then $\varphi_2$ is a cmc immersion from $M_2$ into $\mathbb{R}^5$. One can prove easily that the mean curvature $H_2$ of $\varphi_2$ is a umbilical direction, hence $\varphi_2(M_2)$ sits minimally in $S^4$.

References


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