A Characterization of the standard embedding of $\mathbb{C}P^2$

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Abstract

H. Hopf showed that any immersed constant mean curvature sphere $S^2 \hookrightarrow \mathbb{R}^3$ is the round sphere. Allowing higher dimensions for the domain manifold, the class of Kähler manifolds is an adequate framework to the generalization of Hopf' s theorem. In the Kähler domain setting the complexified second fundamental form α splits according to types. The $(1,1)$ part of the second fundamental splitting plays the role of the mean curvature for sufaces and will be called the plurimean curvature. Therefore isometric immersions with parallel plurimean curvature (*ppmc* isometric immersions) generalize in a natural way the cmc immersions. It is a standard fact that $\mathbb{R}^8$ is the smallest space where $\mathbb{C}P^2$ can be embedded. The aim of this work is to generalize Hopf theorem proving that the only *ppmc* isometric immersion from $\mathbb{C}P^2$ into $\mathbb{R}^8$ is the standard immersion.

1 Introduction and statement of results

The smallest $\mathbb{R}^k$ into which $S^2 = \mathbb{C}P^1$ may be embedded is $\mathbb{R}^3$. H. Hopf ([9]) showed that, up to a congruence, the only constant mean curvature (*cmc*) isometric immersion from the sphere into $\mathbb{R}^3$ is the standard immersion. Affording higher dimensions in the domain manifold, an adequate settiong is the class of Kähler manifolds. When M is a Kähler manifold and $\varphi : M \longrightarrow \mathbb{R}^k$ is an isometric immersion, the coupling of the second fundamental form α of φ with the complex structure J of M originates two operators. To decribe this operators we denote respectively by $T^C M$, $T'M$ and $T''M$ the complexification of TM and the eigenbundles of J corresponding to the eigenvalues i and $-i$; π' and π'' will denote respectively the orthogonal projections of $T^C M$ onto $T'M$ and $T''M$, where $X = \pi'(X) + \pi''(X)$ for all $X \in TM$. Then the complexification of α decomposes according to types giving rise to

$$\alpha^{(1,1)}(X,Y) = \alpha(X', Y'') + \alpha(X'', Y')$$

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and to

$$\alpha^{(2,0)}(X,Y) = \alpha(X',Y')$$

where we have considered $Z' = \pi'(Z)$, $Z'' = \pi''(Z)$ for every $Z \in C(T^C M)$.

When M is a Riemann surface observe that $\alpha^{(1,1)} = \langle\ ,\ \rangle H$, where H denotes the mean curvature of the immersion. Therefore, in this framework, an immersion is *cmc* if, and only if, $\alpha^{(1,1)}$ is parallel. Then, *cmc* isometric immersions from Riemann surfaces are naturally generalized by isometric immersions from Kähler manifolds with $\alpha^{(1,1)}$ parallel. The operator $\alpha^{(1,1)}$ is given the name of plurimean curvature (*pmc*) (see [2] for a justification) and immersions with $\alpha^{(1,1)}$ parallel are called *ppmc* immersions. When $\alpha^{(1,1)}$ vanishes identically the immersion is called pluriminimal ([5], [3], [4]).

H. Hopf discovered that the traceless part of the second fundamental form of an immersed *cmc* surface is a holomorphic quadratic differential on the surface. This observation was the key to his well known theorem refered above. This holomorphic differential is nothing but the operator $\alpha^{(2,0)}$. For *ppmc* isometric immersions $\alpha^{(2,0)}$ is also a normal bundle valued holomorphic quadratic differential. Ferus ([8]) classified the isometric immersions with $\alpha^{(2,0)} = 0$ (the $(2,0)$ - geodesic immersions).

ppmc isometric immersions into space-forms display some special geometric features of the parallel mean curvature surfaces, namely the existence of a $1-$parameter deformation through a smooth family of *ppmc* isometric immersions which, up to a parallel isomorphism, have the same normal bundle ([2]). Just as in the case of immesions with parallel mean curvature, *ppmc* isometric iommersions can also be characterized by the pluriharmonicity of its Gauss map ([6], [2]).

It is a standard fact that $\mathbb{R}^8$ is the smallest space where $\mathbb{C}P^2$ can be immersed. The aim of this work is to generalize Hopf theorem proving that the only *ppmc* isometric immersion from $\mathbb{C}P^2$ into $\mathbb{R}^8$ is the standard immersion.

Throughout the text M will represent a complete four dimensional, simply connected and connected, complex manifold with positive first Chern class. We will prove that

Theorem 1. *Let $\varphi : M \to \mathbb{R}^8$ be a substancial immersion whose induced metric is Kähler.*

If φ is ppmc, then either

1. *M is $\mathbb{C}P^2$ endowed with the Funini-Study metric and φ is the standard embedding, or*

2. *$M = M_1 \times M_2$ and $\varphi = \varphi_1 \times \varphi_2$, where one of the M_i $(1 \leq i \leq 2)$ is S^2 with φ_i the standard immersion cmc into $\mathbb{R}^3$ and the other is a minimal immersion of a Riemann surface into S^4.*

Corollary 2. *Let $\varphi : \mathbb{C}P^2 \to \mathbb{R}^8$ be an immersion whose induced metric is Kähler. If φ is ppmc, then φ is the standard embedding of $\mathbb{C}P^2$ endowed with the Fubini-Study metric.*

2 Proof of results

We will use the notation $N(M)$ to represent simultaneously the normal bundle and its complexified bundle.

Clearly, from the *ppmc* property we conclude, using Codazzi equation, that $\langle \alpha^{(2,0)}, H \rangle$ is a holomorphic differential, where $\langle\ ,\ \rangle$ denotes the complex bilinear extension of the standard inner product of $\mathbb{R}^8$. Hence

$$\left\langle \alpha^{(2,0)}, H \right\rangle = \left\langle \alpha^{(0,2)}, H \right\rangle = 0$$

Analogously,

$$\left\langle \alpha^{(2,0)}, \alpha^{(2,0)} \right\rangle = \left\langle \alpha^{(0,2)}, \alpha^{(0,2)} \right\rangle = 0$$

In fact since M has a metric with positive Ricci curvature, a Bochner type argument allows the conclusion that there are no non zero holomorphic differentials on M ([1]). As above it is also easy to prove that $\langle \alpha^{(3,0)}, H \rangle = \langle \alpha^{(0,3)}, H \rangle = 0$. and $\langle \alpha^{(3,0)}, \alpha^{(2,0)} \rangle = 0$, where $\alpha^{(3,0)}$ (resp $\alpha^{(0,3)}$) . denotes the restriction of $\nabla\alpha$ to $\otimes^3 T'M$ (resp. to $\otimes^3 T''M$).

Let $Z = (Im)(\alpha^{(2,0)} + \alpha^{(0,2)}) \subset N(M)$. Since, at each point, $\dim Z$ is even and H belongs to the orthogonal complement of Z in $N(M)$, either $\dim Z = 2$, or $\dim Z = 0$.

If $\dim Z = 0$ everywhere, φ is $(2,0)$-geodesic and the theorem is proved (see [8] for a classification of $(2,0)$ - geodesic isometric immersions). We will prove that, when M is irreducible (i.e. it does not decompose as a product of two Riemann surfaces), φ is, in fact, $(2,0)$-geodesic.

Assume then that $\dim Z = 2$ on an open subset U of M. Let L represent the orthogonal complement of $Z \oplus \langle H \rangle$ in $N(U)$, where $\langle H \rangle$ denotes the vector subspace spanned by H.

Definition 3. *φ is said to be isotropic if:*

$$\left\langle \alpha^{(2,0)}(\ \cdot\ ,\ \cdot\ ),\ \alpha^{(1,1)}(\ \cdot\ ,\ \cdot\ ) \right\rangle = \left\langle \alpha^{(0,2)}(\ \cdot\ ,\ \cdot\ ),\ \alpha^{(1,1)}(\ \cdot\ ,\ \cdot\ ) \right\rangle = 0.$$

and

$$\left\langle \alpha^{(2,0)}(\ \cdot\ ,\ \cdot\ ),\ \alpha^{(2,0)}(\ \cdot\ ,\ \cdot\ ) \right\rangle = \left\langle \alpha^{(0,2)}(\ \cdot\ ,\ \cdot\ ),\ \alpha^{(0,2)}(\ \cdot\ ,\ \cdot\ ) \right\rangle = 0.$$

Lemma 4. *When* $\ker \alpha^{(2,0)} = \emptyset$*, φ is isotropic.*

Proof. If $\dim Z = 0$ everywhere, $\alpha^{(2,0)}$ vanishes identically and the equality is trivial. Assume then that $\dim Z = 2$ on an open subset U of M and consider the fourth-form $\beta \in C(\otimes^4 T^*M)$ defined by:

$$\beta(X, Y, Z, W) = \langle \alpha(X, Y), \alpha(Z, W) \rangle$$

β is symmetric in the two first and in the two last entries. We consider the complex multi-linear extension of β.

Since $R_{|\otimes^2 T'} = 0$ we get from Gauss equation that

$$\beta(X', Y', Z', W) = \beta(X', Z', Y', W)$$

so that, when β is restricted to $\otimes^3 T'M \otimes TM$, we can also interchange entries 2 and 3. Using Codazzi equation we easily see that $\beta_{|\otimes^4 T'M}$ is an holomorphic differential, hence, reasoning as above, $\beta_{|\otimes^4 T'M} = 0$.

We need to show that

$$\beta(X', Y', Z', W'') = 0 \qquad X, Y, Z, W \in TU.$$

Notice that since Z is an isotropic two-dimensional complex vector space, it decomposes as a sum of two one-dimensional complex vector subspaces. These correspond, respectively, to the images of $\alpha^{(2,0)}$ and $\alpha^{(0,2)}$ and will be denoted by N' and N''.

Now consider $X \in TU$. Since N' is one-dimensional there exists $Z \in TU$ such that $\alpha(X', Z') = 0$. Then

$$\beta(Z', X', W', Y'') = 0$$

or

$$\beta(Z', W', X', Y'') = 0$$

Since $ker\alpha^{(2,0)} \neq \emptyset$ one can choose W such that $\alpha(Z', W') \neq 0$. Now $\dim N' = 1$ implies that

$$\beta(A', B', X', Y'') = 0 \qquad \forall \quad A, B, X, Y \in TU$$

$\square$

Lemma 5. *([7]) Under the above conditions, assume that φ is not totally geodesic and $\ker \alpha^{(2,0)} \neq \emptyset$. Then $M = M_1 \times M_2$, and $\varphi = \varphi_1 \times \varphi_2$, where one of the M_i $(1 \leq i \leq 2)$ is S^2, with $\varphi_i : S^2 \to \mathbb{R}^3$ extrinsically symetric, and the other is a Riemann surface mimally immersed in S^4.*

Proof. We outline a sketch of the proof presented in [7]. Let U be the open subset of M where $\dim \ker \alpha^{(2,0)}$ is maximum. For each $x \in U$ let

$$\Delta_x = \left\{ X \in T_x M : \alpha^{(2,0)}(X, Y) = 0, \forall Y \in T_x M \right\}$$

Clearly Δ_x and $\Delta_x^{\perp}$ are J_x invariant. Thus we get a smooth distribution Δ on U. Using Codazzi equation it is easily seen that Δ is an integrable distribution whose leaves are totally geodesic Káhler submanifolds of M.

Consider now the tensor field $C : \Delta \to Hom(\Delta^{\perp}, \Delta^{\perp})$ defined by $C_T(X) = -(\nabla_X T)^{\Delta^{\perp}}$, for $T \in \Delta$ and $X \in \Delta^{\perp}$, where $(\quad)^{\Delta^{\perp}}$ denotes the orthogonal projection of onto $\Delta^{\perp}$. We infer sraightway from Codazzi equations that, for smooth sections Y of TM and T of Δ, $\nabla_{Y''} T'$ (respectively $\nabla_{Y'} T''$) is a section of Δ' (respectively of Δ''). Hence C_T preserves types in the decomposition

$(\Delta^\perp)' \oplus (\Delta^\perp)''$, that is $C_T J = JC_T$. Furthermore C_T satisfies the following differential equation:

$$(\nabla_S C_T)(Y) = C_T C_S Y + C_{\nabla_S T} Y + R^M(S,Y)T$$

where $S, T \in \Delta$ and $Y \in \Delta^\perp$. Let $x \in \Delta$, γ a geodesic on the maximal leaf of Δ through x and denote by T its velocity field. Let $\lambda(t)$ represent the eigenvalue of $C_{T(t)}$ and $Y(t)$ a correspondent eigenvector. From the last equation one gets that λ satisfies the following Ricatti type equation:

$$\lambda' = \lambda^2 + r$$

where $r(t) = \left\langle R^M(T,Y)T, Y(t)\right\rangle_{\gamma(t)} \geq 0$. It is well known that 0 is the only real solution of this equation defined on the real line. Therefore C_T has no real eigenvalues. If $\lambda = \mu + i\eta$, taking $S = \mu T - \eta JT$, one gets $C_S Y = (\mu^2 + \eta^2)Y$, hence $\mu = \eta = 0$ as above. We conclude then that $C_T^2 = 0$ for all $T \in \Delta$, thus $C_T = 0$ since $\dim \Delta^\perp = 2$. Now since Δ and $\Delta^\perp$ are invariant under the action of the holonomy group of M, by the deRham decomposition theorem we know that U is a product of two Káhler manifolds U_1 and U_2. To prove that $\varphi_{|U}$ is a product of immersions we first notice that $\alpha(S', Y') = 0$ for all $S \in \Delta$ and $Y \in \Delta^\perp$. Using Gauss equation one easily gets that $\langle \alpha(S', Y'), \alpha(S'', Y'')\rangle = \langle \alpha(S', Y''), \alpha(S'', Y')\rangle$, henceforth $\alpha(S, Y) = 0$ whenever $S \in \Delta$, $Y \in \Delta^\perp$. $\varphi_{|U}$ splits then as a product of immersions [10]. An analiticity argument allows the conclusion that M is a product of two Káhler manifolds and φ is a product immersion.

□

We proceed now proving theorem 1. Due to lemma4 it is enough to assume that $\alpha^{(2,0)}$ has an empty kernel.

We assume first that M is irreducible to conclude that condition 1 holds. For each section ξ of the normal bundle, $A_\xi^{(1,1)}$ will represent the "Weigartten" operator associated to $\alpha^{(1,1)}$ in the direction of ξ, that is, $\left\langle A_\xi^{(1,1)}(X), Y\right\rangle = \left\langle \alpha^{(1,1)}(X,Y), \xi\right\rangle$ for all $X, Y \in TM$. $A_H^{(1,1)}$ is parallel since $\alpha^{(1,1)}$ and H are parallel. This entails that $A_H^{(1,1)}$ is a multiple of the identity, since M is irreducible. Now $\left\langle \alpha^{(2,0)}, H\right\rangle = 0$ implies that A_H is also a multilpe of the identity, hence the image of M is contained in a round sphere. If $\alpha^{(2,0)}$ does not vanish identically the normal space spanned by the image of $\alpha^{(1,1)}$ has to be one dimensional, otherwise in view of lemma3, that subspace would have dimension two. Taking a unitary section T of this subspace, orthogonal to H, we would conclude that $A_T^{(1,1)}$ vanishes identically, since it is parallel (T is parallel) and has trace 0, a contradiction. Therefore $\alpha^{(1,1)}$ is totally umbilical. Taking, at each $x \in M$, an orthonormal basis E_i, JE_i, $1 \leq i \leq 2$, we obtain using Guass equation that $\left\langle \alpha(E_i', E_i''), \alpha(E_j', E_j'')\right\rangle = \left\langle \alpha(E_i', E_j''), \alpha(E_i'', E_j')\right\rangle$, from whence $H = 0$ which cannot happen. Thus φ is $(2,0)$ - geodesic. Ferus classification tells then that M must be the complex projective space endowed with the Fubinni-study metric and that φ is the standard immersion.

Assume now that M is reducible and $A_H^{(1,1)}$ has two distinct eigenvalues λ_1 and λ_2. Let T^1 and T^2 represent the eigendistributions of $A_H^{(1,1)}$ associated respectively to λ_1 and λ_2. It is easily seen that T^1 and T^2 are parallel and J invariant. Since T^1 and T^2 are invariant by the action of the holonomy group of M, from the de Rham decomposition theorem we know that M is a product of two Kähler manifolds M_1 and M_2. One has $\lambda_1 \neq 0$ or $\lambda_2 \neq 0$. Suppose $\lambda_1 \neq 0$. Since $\left\langle \alpha^{(2,0)}, H \right\rangle = 0$, T^1 and T^2 are kept invariant by A_H and T^1 and T^2 are also the eigenspaces of A_H corresponding respectively to the eigenvalues λ_1 and λ_2. If the subspace of the normal bundle spanned by the image of $\alpha^{(1,1)}$ is one dimensional, one has $\alpha^{(1,1)}(X, Y) = 0$ for each $X \in T_1$, $Y \in T_2$. Now from

$$0 = \langle R(X', Y'')X'', Y' \rangle = \langle \alpha(X', X''), \alpha(Y', Y'') \rangle - \langle \alpha(X', Y'), \alpha(X'', Y'') \rangle$$

and

$$0 = \langle R(X', Y')X'', Y'' \rangle = \langle \alpha(X', X''), \alpha(Y', Y'') \rangle - \langle \alpha(X', Y''), \alpha(X'', Y') \rangle$$

we get that

$$\langle \alpha(X', Y'), \alpha(X'', Y'') \rangle = \langle \alpha(X', Y''), \alpha(X'', Y') \rangle \,,$$

hence $\alpha(X, Y) = 0$ whenever $X \in T_1$, $Y \in T_2$. Thus φ decomposes as a product immersion ([10]) Assume then that the subbundle spanned by the image of $\alpha^{(1,1)}$ has rank two. Consider a unitary secction T of L. A_H and A_T commute since H and T are parallel in the normal bundle, so that this two operators have the same eigenbundles. The equation $\left\langle \alpha^{(2,0)}, T \right\rangle = 0$ tells that A_T commutes also with J. Therefore A_T leaves T_1 and T_2 invariant. Thus $\alpha(X, Y) = 0$ and φ is a product of *cmc* immersions. One of the immersions, say φ_1, must be $(2, 0)$-geodesic, that is, a two - sphere in $\mathbb{R}^3$ ([8]). Then φ_2 is a *cmc* immersion from M_2 into $\mathbb{R}^5$. One can prove easily that the mean curvature H_2 of φ_2 is a umbilical direction, hence $\varphi_2(M_2)$ sits minimally in S^4.

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