WKB ANALYSIS FOR NONLINEAR SCHRÖDINGER EQUATIONS WITH POTENTIAL

RÉMI CARLES

Abstract. We justify the WKB analysis for the semiclassical nonlinear Schrödinger equation with a subquadratic potential. This concerns subcritical, critical, and supercritical cases as far as the geometrical optics method is concerned. In the supercritical case, this extends a previous result by E. Grenier; we also have to restrict to nonlinearities which are defocusing and cubic at the origin, but besides subquadratic potentials, we consider initial phases which may be unbounded. For this, we construct solutions for some compressible Euler equations with unbounded source term and unbounded initial velocity.

1. Introduction

Consider the initial value problem, for $x \in \mathbb{R}^n$ and $\kappa \geq 0$:

$$i\varepsilon\partial_t u^\varepsilon + \frac{\varepsilon^2}{2}\Delta u^\varepsilon = V(t,x)u^\varepsilon + \varepsilon^\kappa f\left(|u^\varepsilon|^2\right)u^\varepsilon \tag{1.1}$$

$$u^\varepsilon_{|t=0} = a_0^\varepsilon(x)e^{i\phi_0(x)/\varepsilon}. \tag{1.2}$$

The aim of WKB methods is to describe u^ε in the limit $\varepsilon \to 0$, when ϕ_0 does not depend on ε, and a_0^ε has an asymptotic expansion of the form:

$$a_0^\varepsilon(x) \sim a_0(x) + \varepsilon a_1(x) + \varepsilon^2 a_2(x) + \dots \tag{1.3}$$

The parameter $\kappa \geq 0$ describes the strength of a coupling constant, which makes nonlinear effects more or less important in the limit $\varepsilon \to 0$; the larger κ, the weaker the nonlinear interactions. In this paper, we describe the asymptotic behavior of u^ε at leading order, when the potential V and the initial phase ϕ_0 are smooth, and subquadratic in the space variable.

Such equations as (1.1) appear in physics: see e.g. [37] for a general overview. For instance, they are used to model Bose-Einstein condensation when V is an harmonic potential (isotropic or anisotropic) and the nonlinearity is cubic or quintic (see e.g. [14, 26, 33]). In most of this paper, the initial data we consider are in Sobolev spaces H^s. We outline a WKB analysis for the Gross-Pitaevskii equation in Appendix A. We refer to [4] for numerics on the semi-classical limit of (1.1).

We shall not recall the results concerning the Cauchy problem for (1.1)-(1.2), and refer to [9] for an overview on the semilinear Schrödinger equation.

2000 *Mathematics Subject Classification.* 35B30, 35B33, 35B40, 35C20, 35Q55, 81Q20.

Key words and phrases. Nonlinear Schrödinger equation, semiclassical analysis, critical indices, supercritical WKB analysis, Euler equation, Gross-Pitaevskii equation.

Supports by European network HYKE, funded by the EC as contract HPRN-CT-2002-00282, and by Centro de Matemática e Aplicações Fundamentais (Lisbon), funded by FCT as contract POCTI-ISFL-1-209, are acknowledged.

In the one-dimensional case $n = 1$, the cubic nonlinear Schrödinger equation is completely integrable, in the absence of an external potential, $V \equiv 0$. Several tough papers analyze the semi-classical limit in that case, for $\kappa = 0$: see e.g. [24, 32, 25, 38, 39]. We shall not use this approach in this paper, but rather work in the spirit of [23].

An interesting feature of (1.1) is that one does not expect the creation of harmonics, provided that only one phase is present initially, like in (1.2). The WKB methods consist in seeking an approximate solution to (1.1) of the form:

$$u^\varepsilon(t,x) \sim \left(\mathsf{a}_0(t,x) + \varepsilon \mathsf{a}_1(t,x) + \varepsilon^2 \mathsf{a}_2(t,x) + \ldots\right) e^{i\Phi(t,x)/\varepsilon} . \tag{1.4}$$

One must not expect this approach to be valid when caustics are formed: near a caustic, all the terms Φ, a_0, a_1, ... become singular. Past the caustic, several phases are necessary in general to describe the asymptotic behavior of the solution (see e.g. [16] for a general theory in the linear case). In this paper, we restrict our attention to times preceding this break-up.

For such an expansion to be available with profiles a_j independent of ε, it is reasonable to assume that κ is an integer. However, we do not assume that κ is an integer: we study the asymptotic behavior of u^ε *at leading order* (strong limits in $L^2 \cap L^\infty$ for instance), including cases where other powers of ε would come into play.

We distinguish two families of assumptions: "geometrical" assumptions, on the potential V and the initial phase ϕ_0, and "analytical" assumptions, on f and the initial amplitude a_0^ε. In all the cases, we shall not try to seek the optimal regularity; we focus our interest on the limit $\varepsilon \to 0$.

Assumption 1 (Geometrical)**.** *We assume that the potential and the initial phase are smooth and subquadratic:*

- $V \in C^\infty(\mathbb{R}_t \times \mathbb{R}^n_x)$, *and* $\partial_x^\alpha V \in L^\infty_{\text{loc}}(\mathbb{R}_t; L^\infty(\mathbb{R}^n_x))$ *as soon as* $|\alpha| \geq 2$.
- $\phi_0 \in C^\infty(\mathbb{R}^n)$, *and* $\partial_x^\alpha \phi_0 \in L^\infty(\mathbb{R}^n)$ *as soon as* $|\alpha| \geq 2$.

The assumption of V being subquadratic is classical in other contexts; for instance, locally in time, the dispersion for $e^{-i\frac{t}{\varepsilon}(-\varepsilon^2\Delta + V)}$ is the same as without potential (see [18, 19]),

$$\left\| e^{-i\frac{t}{\varepsilon}(-\varepsilon^2\Delta + V)} \right\|_{L^1 \to L^\infty} \leq \frac{C}{|\varepsilon t|^{n/2}}, \quad \forall |t| \leq \delta,$$

hence yielding the same local Strichartz estimates as in the free case. Global in time Strichartz estimates must not be expected in general, as shown by the example of the harmonic oscillator, which has eigenvalues. For positive superquadratic potentials, the smoothing effects and Strichartz estimates are different (see [40, 41]). This is related to the properties of the Hamilton flow, which also imply:

Lemma 1.1. *Under Assumption 1, there exists* $T > 0$ *and a unique solution* $\phi_{\text{eik}} \in C^\infty([0,T] \times \mathbb{R}^n)$ *to:*

$$\partial_t \phi_{\text{eik}} + \frac{1}{2}|\nabla_x \phi_{\text{eik}}|^2 + V(t,x) = 0 \quad ; \quad \phi_{\text{eik}|t=0} = \phi_0 . \tag{1.5}$$

This solution is subquadratic: $\partial_x^\alpha \phi_{\text{eik}} \in L^\infty([0,T] \times \mathbb{R}^n)$ *as soon as* $|\alpha| \geq 2$.

This result is proved in Section 2, where other remarks on Assumption 1 are made.

Assumption 2 (Analytical)**.** *We assume that the nonlinearity is smooth, and that the initial amplitude converges in Sobolev spaces:*

- $f \in C^\infty(\mathbb{R};\mathbb{R})$.
- *There exists* $a_0 \in H^\infty := \cap_{s\geq 0} H^s(\mathbb{R}^n)$*, such that* a_0^ε *converges to* a_0 *in* H^s *for any* s*, as* $\varepsilon \to 0$.

Remark. Some of the results we shall prove remain valid when f is complex-valued. In that case, the conservation of mass associated to the Schrödinger equation, $\|u^\varepsilon(t)\|_{L^2} = \|a_0^\varepsilon\|_{L^2}$, no longer holds. On the other hand, when $0 \leq \kappa < 1$, this assumption is necessary in our approach, and we even assume $f' > 0$.

1.1. **Subcritical and critical cases:** $\kappa \geq 1$**.** When the initial data is of the form (1.3), the usual approach consists in plugging a formal expansion of the form (1.4) into (1.1). Ordering the terms in powers of ε, and canceling the cascade of equations thus obtained yields Φ, a_0, a_1, ...

Assume in this section that $\kappa \geq 1$, and apply the above procedure. To cancel the term of order $\mathcal{O}(\varepsilon^0)$, we find that Φ must solve (1.5): $\Phi = \phi_{\rm eik}$. Canceling the term of order $\mathcal{O}(\varepsilon^1)$, we get:

$$\partial_t \mathsf{a}_0 + \nabla\phi_{\rm eik}\cdot\nabla \mathsf{a}_0 + \frac{1}{2}\mathsf{a}_0\Delta\phi_{\rm eik} = \begin{cases} 0 & \text{if } \kappa > 1,\\ -if\left(|\mathsf{a}_0|^2\right)\mathsf{a}_0 & \text{if } \kappa = 1.\end{cases}$$

We see that the value $\kappa = 1$ is critical as far as nonlinear effects are concerned: if $\kappa > 1$, no nonlinear effect is expected at leading order, since formally, $u^\varepsilon \sim \mathsf{a}_0 e^{i\phi_{\rm eik}/\varepsilon}$, and $\phi_{\rm eik}$ and a_0 do not depend on the nonlinearity f. If $\kappa = 1$, then a_0 solves a nonlinear equation involving f.

We will see in Section 3 that a_0 solves a transport equation that turns out to be a ordinary differential equation along the rays of geometrical optics, as is usual in the hyperbolic case (see e.g. [34]). More typical of Schrödinger equation is the fact that this ordinary differential equation can be solved explicitly: the nonlinear effect is measured by a nonlinear phase shift (see the example of [28] for a similar result in the hyperbolic setting). We prove the following result in Section 3:

Proposition 1.2. *Let Assumptions 1 and 2 be satisfied. Let* $\kappa \geq 1$*. Then for any* $\varepsilon \in]0,1]$*, (1.1)-(1.2) has a unique solution* $u^\varepsilon \in C^\infty([0,T]\times\mathbb{R}^n)\cap C([0,T];H^s)$ *for any* $s > n/2$ *(*T *is given by Lemma 1.1). Moreover, there exist* $a, G \in C^\infty([0,T]\times\mathbb{R}^n)$*, independent of* $\varepsilon \in]0,1]$*, where* $a \in C([0,T];L^2\cap L^\infty)$*, and* G *is real-valued with* $G \in C([0,T];L^\infty)$*, such that:*

$$\left\| u^\varepsilon - a e^{i\varepsilon^{\kappa-1}G} e^{i\phi_{\rm eik}/\varepsilon}\right\|_{L^\infty([0,T];L^2\cap L^\infty)} \to 0 \quad \text{as } \varepsilon \to 0.$$

The profile a *solves the initial value problem:*

$$\partial_t a + \nabla\phi_{\rm eik}\cdot\nabla a + \frac{1}{2}a\Delta\phi_{\rm eik} = 0 \quad ; \quad a_{|t=0} = a_0, \tag{1.6}$$

and G *depends nonlinearly on* a *through* f*. In particular, if* $\kappa > 1$*, then*

$$\left\| u^\varepsilon - a e^{i\phi_{\rm eik}/\varepsilon}\right\|_{L^\infty([0,T];L^2\cap L^\infty)} \to 0 \quad \text{as } \varepsilon \to 0,$$

and no nonlinear effect is present in the leading order behavior of u^ε. If $\kappa = 1$, nonlinear effects are present at leading order, measured by G.

The dependence of G upon a and f is made more explicit in Section 3, in terms of the Hamilton flow determining $\phi_{\rm eik}$ (see (3.4)). Note that in the above result, we do not assume that κ is an integer.

1.2. **Supercritical case:** $\kappa = 0$**.** It follows from the above analysis that the case $0 \leq \kappa < 1$ is supercritical. We restrict our attention to the case $\kappa = 0$. We present an analysis of the range $0 < \kappa < 1$ in Section 6. Plugging an asymptotic expansion of the form (1.4) into (1.1) yields a shifted cascade of equations:

$$\begin{aligned} &\mathcal{O}\left(\varepsilon^0\right): \quad \partial_t \Phi + \frac{1}{2}|\nabla\Phi|^2 + V + f\left(|\mathsf{a}_0|^2\right) = 0,\\ &\mathcal{O}\left(\varepsilon^1\right): \quad \partial_t \mathsf{a}_0 + \nabla\Phi\cdot\nabla\mathsf{a}_0 + \frac{1}{2}\mathsf{a}_0\Delta\Phi = 2if'\left(|\mathsf{a}_0|^2\right)\operatorname{Re}\left(\mathsf{a}_0\overline{\mathsf{a}_1}\right). \end{aligned}$$

Two comments are in order. First, we see that there is a strong coupling between the phase and the main amplitude: a_0 is present in the equation for Φ. Second, the above system is not closed: Φ is determined in function of a_0, and a_0 is determined in function of a_1. Even if we pursued the cascade of equations, this phenomenon would remain: no matter how many terms are computed, the system is never closed (see [21]). This is a typical feature of supercritical cases in nonlinear geometrical optics (see [11, 12]).

In the case when $V \equiv 0$ and $\phi_0 \in H^s$, this problem was resolved by E. Grenier [23], by modifying the usual WKB methods; this approach is recalled in Section 4. Note that even though a_1 is not determined by the above system, the pair $(\rho, v) := (|\mathsf{a}_0|^2, \nabla\Phi)$ solves a compressible Euler equation:

$$\begin{aligned} \partial_t v + v\cdot\nabla v + \nabla V + \nabla f(\rho) = 0\ ; &\quad v\big|_{t=0} = \nabla\phi_0\\ \partial_t\rho + \nabla\cdot(\rho v) = 0\ ; &\quad \rho\big|_{t=0} = |a_0|^2. \end{aligned} \tag{1.7}$$

Using techniques introduced in the study of quasilinear hyperbolic equations, E. Grenier justified a WKB expansion for nonlinearities which are defocusing, and cubic at the origin ($f' > 0$). We shall not change this assumption, but show how to treat the case of a subquadratic potential with a subquadratic initial phase. Note that even the construction of solution to (1.7) under Assumption 1 is not standard: the source term ∇V may be unbounded, as well as the initial velocity $\nabla\phi_0$.

Assumption 3. *In addition to Assumption 2, we assume:*

- $f' > 0$.
- *There exists $a_0, a_1 \in H^\infty$, with $xa_0, xa_1 \in H^\infty$, such that:*

$$\|a_0^\varepsilon - a_0 - \varepsilon a_1\|_{H^s} + \|xa_0^\varepsilon - xa_0 - \varepsilon xa_1\|_{H^s} = o(\varepsilon), \quad \forall s \geq 0.$$

We can then describe the asymptotic behavior of the solution to (1.1)-(1.2) for small times:

Theorem 1.3. *Let Assumptions 1, 2 and 3 be satisfied. Let $\kappa = 0$. There exists $T_* > 0$ independent of $\varepsilon \in]0,1]$ and a unique solution $u^\varepsilon \in C^\infty([0,T_*]\times$*

$\mathbb{R}^n) \cap C([0,T_*];H^s)$ for any $s > n/2$ to (1.1)-(1.2). *Moreover, there exist $a, \phi \in C([0,T_*];H^s)$ for every $s \geq 0$, such that:*

$$\limsup_{\varepsilon\to 0} \left\| u^\varepsilon - ae^{i(\phi+\phi_{\rm eik})/\varepsilon} \right\|_{L^2\cap L^\infty} = \mathcal{O}(t) \quad \text{as } t \to 0. \tag{1.8}$$

Here, a and ϕ are nonlinear functions of $\phi_{\rm eik}$ and a_0, given by (5.9). *Finally, there exists $\phi^{(1)} \in C([0,T_*];H^s)$ for every $s \geq 0$, real-valued, such that:*

$$\limsup_{\varepsilon\to 0} \sup_{0\leq t\leq T_*} \left\| u^\varepsilon - ae^{i\phi^{(1)}} e^{i(\phi+\phi_{\rm eik})/\varepsilon} \right\|_{L^2\cap L^\infty} = 0. \tag{1.9}$$

The phase shift $\phi^{(1)}$ is a nonlinear function of $\phi_{\rm eik}, a_0$ and a_1.

This result can be understood as follows. At leading order, the amplitude of u^ε is given by $ae^{i\phi^{(1)}}$, which can be approximated for small times by a, because $\phi^{(1)}\big|_{t=0} = 0$. The rapid oscillations are described by the phase $\phi + \phi_{\rm eik}$. The function ϕ is constructed as a perturbation of $\phi_{\rm eik}$, but must not be considered as negligible: its H^s-norms are not small in general, see (5.9) (at time $t = 0$ for instance). As a consequence of our analysis, the pair

$$(\rho, v) = \left(|a|^2, \nabla(\phi + \phi_{\rm eik})\right) = \left(\left|ae^{i\phi^{(1)}}\right|^2, \nabla(\phi + \phi_{\rm eik})\right)$$

solves the system (1.7).

Remark 1.4. With this result, we could deduce instability phenomena for (1.1)-(1.2) in the same fashion as in [7]. Note that because of Assumption 1, it seems that the approaches of [5, 6, 13] cannot be adapted to the present case: the Laplacian can never be neglected, and apparently, WKB approach is really needed.

The rest of this paper is organized as follows. In Section 2, we prove Lemma 1.1. In Section 3, we prove Proposition 1.2, and explain how G is obtained. We recall the approach of [23] in Section 4, and show how to adapt it to prove Theorem 1.3 in Section 5. We present an analysis for the case $0 < \kappa < 1$ in Section 6. We sketch the proof of the analogue of Theorem 1.3 for the Gross-Pitaevskii equation in Appendix A.

2. Global in space Hamilton-Jacobi theory

In this section, we prove Lemma 1.1. Consider the classical Hamiltonian:

$$H(t,x,\tau,\xi) = \tau + \frac{1}{2}|\xi|^2 + V(t,x), \quad (t,x,\tau,\xi) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n.$$

It is smooth by Assumption 1. Therefore, it is classical (see e.g. [15]) that in the neighborhood of each point $x \in \mathbb{R}^n$, one can construct a smooth solution to the eikonal equation (1.5), on some time interval $[-t(x), t(x)]$, for some $t(x) > 0$ depending on x. The fact that in Lemma 1.1, we can find some $T > 0$ uniform in $x \in \mathbb{R}^n$ is due to the fact that the potential and the initial phase are subquadratic.

Recall that if there exist some constants $a, b > 0$ such that a potential $\mathtt{V}$ satisfies $\mathtt{V}(x) \geq -a|x|^2 - b$, then $-\Delta + \mathtt{V}$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$ (see [35, p. 199]). If $-\mathtt{V}$ has superquadratic growth, then it is not possible to define $e^{-it(-\Delta+\mathtt{V})}$ (see [17, Chap. 13, Sect. 6, Cor. 22] for the case $\mathtt{V}(x) = -x^4$

in space dimension one). This is due to the fact that classical trajectories can reach an infinite speed. We will see below that if the initial phase ϕ_0 is superquadratic, then focusing at the origin may occur "instantly" (see Example 1).

To construct the solution of (1.5), introduce the flow associated to H: let $x(t,y)$ and $\xi(t,y)$ solve

$$\begin{cases} \partial_t x(t,y) = \xi\,(t,y) \quad ; \quad x(0,y) = y, \\ \partial_t \xi(t,y) = -\nabla_x V\,(t, x(t,y)) \quad ; \quad \xi(0,y) = \nabla\phi_0(y). \end{cases} \tag{2.1}$$

Recall the result (valid under weaker conditions than Assumption 1):

Theorem 2.1 ([15], Th. A.3.2)**.** *Suppose that Assumption 1 is satisfied. Let $t \in [0,T]$ and θ_0 an open set of $\mathbb{R}^n$. Denote*

$$\theta_t := \{x(t,y) \in \mathbb{R}^n, y \in \theta_0\} \quad ; \quad \theta := \{(t,x) \in [0,T] \times \mathbb{R}^n, x \in \theta_t\}.$$

Suppose that for $t \in [0,T]$, the mapping

$$\theta_0 \ni y \mapsto x(t,y) \in \theta_t$$

is bijective, and denote by $y(t,x)$ its inverse. Assume also that

$$\nabla_x y \in L^\infty_{\rm loc}(\theta).$$

Then there exists a unique function $\theta \ni (t,x) \mapsto \phi_{\rm eik}(t,x) \in \mathbb{R}$ that solves (1.5)*, and satisfies $\nabla_x^2 \phi_{\rm eik} \in L^\infty_{\rm loc}(\theta)$. Moreover,*

$$\nabla_x \phi_{\rm eik}(t,x) = \xi(t, y(t,x)). \tag{2.2}$$

Proposition 2.2 ([36], Th. 1.22 and [15], Prop. A.7.1)**.** *Suppose that the function $\mathbb{R}^n \ni y \mapsto x(y) \in \mathbb{R}^n$ satisfies:*

$$|\det \nabla_y x| \geq C_0 > 0 \quad \text{and} \quad \left|\partial_y^\alpha x\right| \leq C, \ |\alpha| = 1, 2.$$

Then x is bijective.

Proof of Lemma 1.1. Lemma 1.1 follows from the above two results. From Assumption 1, we know that we can solve (2.1) locally in time in the neighborhood of any $y \in \mathbb{R}^n$. Differentiate (2.1) with respect to y:

$$\begin{cases} \partial_t \partial_y x(t,y) = \partial_y \xi\,(t,y) \quad ; \quad \partial_y x(0,y) = \mathrm{Id}, \\ \partial_t \partial_y \xi(t,y) = -\nabla_x^2 V\,(t, x(t,y))\, \partial_y x(t,y) \quad ; \quad \partial_y \xi(0,y) = \nabla^2 \phi_0(y). \end{cases} \tag{2.3}$$

Integrating (2.3) in time, we infer from Assumption 1 that for any $T > 0$, there exists C_T such that for $(t,y) \in [0,T] \times \mathbb{R}^n$:

$$|\partial_y x(t,y)| + |\partial_y \xi(t,y)| \leq C_T + C_T \int_0^t \left(|\partial_y x(s,y)| + |\partial_y \xi(s,y)|\right) ds.$$

Gronwall lemma yields:

$$\|\partial_y x(t)\|_{L^\infty_y} + \|\partial_y \xi(t)\|_{L^\infty_y} \leq C'(T). \tag{2.4}$$

Similarly,

$$\left\|\partial_y^\alpha x(t)\right\|_{L^\infty_y} + \left\|\partial_y^\alpha \xi(t)\right\|_{L^\infty_y} \leq C(\alpha, T), \quad \forall \alpha \in \mathbb{N}^n, \ |\alpha| \geq 1. \tag{2.5}$$

Integrating the first line of (2.3) in time, we have:

$$\det \nabla_y x(t,y) = \det \left(\mathrm{Id} + \int_0^t \nabla_y \xi\,(s,y)\,ds \right).$$

We infer from (2.4) that for $t \in [0,T]$, provided that $T > 0$ is sufficiently small, we can find $C_0 > 0$ such that:

$$|\det \nabla_y x(t,y)| \geq C_0, \quad \forall (t,y) \in [0,T] \times \mathbb{R}^n. \tag{2.6}$$

Applying Proposition 2.2, we deduce that we can invert $y \mapsto x(t,y)$ for $t \in [0,T]$.

To apply Theorem 2.1 with $\theta_0 = \theta = \theta_t = \mathbb{R}^n$, we must check that $\nabla_x y \in L^\infty_{\rm loc}(\mathbb{R}^n)$. Differentiate the relation

$$x\,(t, y(t,x)) = x$$

with respect to x:

$$\nabla_x y(t,x) \nabla_y x\,(t, y(t,x)) = \mathrm{Id}.$$

Therefore, $\nabla_x y(t,x) = \nabla_y x\,(t, y(t,x))^{-1}$ as matrices, and

$$\nabla_x y(t,x) = \frac{1}{\det \nabla_y x(t,y)} \mathrm{adj}\left(\nabla_y x\,(t, y(t,x))\right), \tag{2.7}$$

where $\mathrm{adj}\,(\nabla_y x)$ denotes the adjugate of $\nabla_y x$. We infer from (2.4) and (2.6) that $\nabla_x y \in L^\infty(\mathbb{R}^n)$ for $t \in [0,T]$. Therefore, Theorem 2.1 yields a smooth solution $\phi_{\rm eik}$ to (1.5), local in time and global in space: $\phi_{\rm eik} \in C^\infty([0,T] \times \mathbb{R}^n)$.

The fact that $\phi_{\rm eik}$ is subquadratic as stated in Lemma 1.1 then stems from (2.2), (2.5), (2.6) and (2.7). □

We now give some two examples showing that Assumption 1 is essentially sharp to solve (1.5) globally in space, at least when no assumption is made on the sign of V nor on the geometry of $\nabla \phi_0$. We already recalled that if $-V$ has a superquadratic growth, then $-\Delta + V$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$, so we shall rather study the dependence of $\phi_{\rm eik}$ on the initial phase ϕ_0.

Example 1. Assume that $V \equiv 0$ and

$$\phi_0(x) = -\frac{1}{(2+2\delta)T} \left(|x|^2 + 1 \right)^{1+\delta}, \quad T > 0,\ \delta \neq -1.$$

Then Assumption 1 is satisfied if and only if $\delta \leq 0$. When $\delta = 0$, then (1.5) is solved explicitly:

$$\phi_{\rm eik}(t,x) = \frac{|x|^2}{2(t-T)} - \frac{1}{2T}.$$

This shows that we can solve (1.5) globally in space, but only locally in time: as $t \to T$, a caustic reduced to a single point (the origin) is formed. Note that with $T < 0$, (1.5) can be solved globally in time *for positive times.*

When $\delta > 0$, then integrating (2.1) yields:

$$x(t,y) = y + \int_0^t \xi(s,y)ds = y + \int_0^t \xi(0,y)ds = y - \frac{t}{T}\left(|y|^2+1\right)^\delta y$$
$$= y\left(1 - \frac{t}{T}\left(|y|^2+1\right)^\delta\right).$$

For $R > 0$, we see that the rays starting from the ball $\{|y| = R\}$ meet at the origin at time

$$T_c(R) = \frac{T}{(R^2+1)^\delta}.$$

Since R is arbitrary, this shows that several rays can meet arbitrarily fast, thus showing that Theorem 2.1 cannot be applied uniformly in space.

Example 2. When $V(t,x) = \frac{1}{2}\sum_{j=1}^n \omega_j^2 x_j^2$ is an harmonic potential ($\omega_j \neq 0$), and $\phi_0 \equiv 0$, we have:

$$\phi(t,x) = -\sum_{j=1}^n \frac{\omega_j}{2} x_j^2 \tan(\omega_j t).$$

This also shows that we can solve (1.5) globally in space, but locally in time only. Note that if we replace formally ω_j by $i\omega_j$, then V is turned into $-V$, and the trigonometric functions become hyperbolic functions: we can then solve (1.5) globally in space *and* time.

Instead of invoking Theorem 2.1 and Proposition 2.2, one might try to differentiate (1.5) in order to prove that $\phi_{\rm eik}$ is subquadratic, in the same fashion as in [3, 2]. For $1 \leq j,k \leq n$, differentiate (1.5) with respect to x_j and x_k:

$$\partial_t \partial^2_{jk}\phi_{\rm eik} + \nabla_x\phi_{\rm eik}\cdot\nabla_x\left(\partial^2_{jk}\phi_{\rm eik}\right) + \sum_{l=1}^n \partial^2_{jl}\phi_{\rm eik}\partial^2_{lk}\phi_{\rm eik} + \partial^2_{jk}V(t,x) = 0\ ;$$
$$\partial^2_{jk}\phi_{{\rm eik}|t=0} = \partial^2_{jk}\phi_0\,.$$

We see that we obtain a system of the form

$$D_t y = Q(y) + R \quad ; \quad y_{|t=0} = y(0),$$

where y stands for the family $(\partial^2_{jk}\phi_{\rm eik})_{1\leq j,k\leq n}$, Q is quadratic, and R and $y(0)$ are bounded. The operator D_t is a well-defined transport operator provided that the characteristics given by:

$$\partial_t x(t,y) = \nabla_x\phi_{\rm eik}\left(t,x(t,y)\right) \quad ; \quad x(0,y) = y,$$

define a global diffeomorphism. Proving this amounts to using Proposition 2.2, for the rather general initial data we consider. So it seems that this approach does not allow to shorten the proof of Lemma 1.1.

3. Subcritical and critical cases

To establish Proposition 1.2, define

$$a^\varepsilon(t,x) := u^\varepsilon(t,x)e^{-i\phi_{\rm eik}(t,x)/\varepsilon}.$$

Then u^ε solves (1.1)-(1.2) if and only if a^ε solves:

$$\begin{aligned}\partial_t a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi_{\rm eik} &= i\frac{\varepsilon}{2}\Delta a^\varepsilon - i\varepsilon^{\kappa-1}f\left(|a^\varepsilon|^2\right)a^\varepsilon,\\ a^\varepsilon_{|t=0} &= a^\varepsilon_0.\end{aligned}\tag{3.1}$$

Proposition 3.1. *Let Assumptions 1 and 2 be satisfied. Let $\kappa\geq 1$. For any $\varepsilon\in]0,1]$, (3.1) has a unique solution $a^\varepsilon\in C^\infty([0,T]\times\mathbb{R}^n)\cap C([0,T];H^s)$ for any $s>n/2$. Moreover, a^ε is bounded in $L^\infty([0,T];H^s)$ uniformly in $\varepsilon\in]0,1]$, for any $s\geq 0$.*

Proof. Using a mollification procedure, we see that it is enough to establish energy estimates for (3.1) in H^s, for any $s\geq 0$. Let $s>n/2$, and $\alpha\in\mathbb{N}^n$, with $s=|\alpha|$. Applying ∂_x^α to (3.1), we find:

$$\partial_t\partial_x^\alpha a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla\partial_x^\alpha a^\varepsilon = i\frac{\varepsilon}{2}\Delta\partial_x^\alpha a^\varepsilon - i\varepsilon^{\kappa-1}\partial_x^\alpha\left(f\left(|a^\varepsilon|^2\right)a^\varepsilon\right) + R^\varepsilon_\alpha,\tag{3.2}$$

where

$$R^\varepsilon_\alpha = \left[\nabla\phi_{\rm eik}\cdot\nabla,\partial_x^\alpha\right]a^\varepsilon - \frac{1}{2}\partial_x^\alpha\left(a^\varepsilon\Delta\phi_{\rm eik}\right).$$

Take the inner product of (3.2) with $\partial^\alpha a^\varepsilon$, and consider the real part: the first term of the right hand side of (3.2) vanishes, and we have:

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\partial_x^\alpha a^\varepsilon\|^2_{L^2} + \mathrm{Re}\int_{\mathbb{R}^n}\overline{\partial_x^\alpha a^\varepsilon}\nabla\phi_{\rm eik}\cdot\nabla\partial_x^\alpha a^\varepsilon \leq& \varepsilon^{\kappa-1}\left\|f\left(|a^\varepsilon|^2\right)a^\varepsilon\right\|_{H^s}\|a^\varepsilon\|_{H^s}\\ &+\|R^\varepsilon_\alpha\|_{L^2}\|a^\varepsilon\|_{H^s}.\end{aligned}$$

Notice that we have

$$\begin{aligned}\left|\mathrm{Re}\int_{\mathbb{R}^n}\overline{\partial_x^\alpha a^\varepsilon}\nabla\phi_{\rm eik}\cdot\nabla\partial_x^\alpha a^\varepsilon\right| &= \frac{1}{2}\left|\int_{\mathbb{R}^n}\nabla\phi_{\rm eik}\cdot\nabla\left|\partial_x^\alpha a^\varepsilon\right|^2\right|\\ &= \frac{1}{2}\left|\int_{\mathbb{R}^n}\left|\partial_x^\alpha a^\varepsilon\right|^2\Delta\phi_{\rm eik}\right| \leq C\|a^\varepsilon\|^2_{H^s},\end{aligned}$$

since $\Delta\phi_{\rm eik}\in L^\infty([0,T]\times\mathbb{R}^n)$ from Lemma 1.1. Moser's inequality yields:

$$\left\|f\left(|a^\varepsilon|^2\right)a^\varepsilon\right\|_{H^s}\leq C\left(\|a^\varepsilon\|_{L^\infty}\right)\|a^\varepsilon\|_{H^s}.$$

Summing over α such that $|\alpha|=s$, we infer:

$$\frac{d}{dt}\|a^\varepsilon\|_{H^s}\leq C\left(\|a^\varepsilon\|_{L^\infty}\right)\|a^\varepsilon\|_{H^s} + \|R^\varepsilon_\alpha\|_{H^s}.$$

Note that the above locally bounded map $C(\cdot)$ is independent of ε if and only if $\kappa\geq 1$. To apply Gronwall lemma, we need to estimate the last term: we use the fact that the derivatives of order at least two of $\phi_{\rm eik}$ are bounded, from Lemma 1.1, to have:

$$\|R^\varepsilon_\alpha\|_{L^2}\leq C\|a^\varepsilon\|_{H^s}.$$

We can then conclude by a continuity argument and Gronwall lemma:

$$\|a^\varepsilon\|_{L^\infty([0,T];H^s)}\leq C\left(s,\|a^\varepsilon_0\|_{H^s}\right).$$

For the mollification procedure, this yields boundedness in the high norm. Contraction in the small norm then follows easily with classical arguments (see e.g. [1, 30]), completing the proof of Proposition 3.1. □

Corollary 3.2. *Let Assumptions 1 and 2 be satisfied. Let $\kappa \geq 1$. Then*

$$\|a^\varepsilon - \widetilde{a}^\varepsilon\|_{L^\infty([0,T];H^s)} \to 0 \quad \text{as } \varepsilon \to 0, \quad \forall s \geq 0,$$

where $\widetilde{a}^\varepsilon$ solves:

$$\partial_t \widetilde{a}^\varepsilon + \nabla \phi_{\rm eik} \cdot \nabla \widetilde{a}^\varepsilon + \frac{1}{2} \widetilde{a}^\varepsilon \Delta \phi_{\rm eik} = -i\varepsilon^{\kappa-1} f\left(|\widetilde{a}^\varepsilon|^2\right) \widetilde{a}^\varepsilon \quad ; \quad \widetilde{a}^\varepsilon_{|t=0} = a_0. \tag{3.3}$$

Proof. The proof of Proposition 3.1 shows that $\widetilde{a}^\varepsilon \in C^\infty([0,T]\times\mathbb{R}^n)$, and that $\widetilde{a}^\varepsilon$ is bounded in $L^\infty([0,T];H^s)$ uniformly in $\varepsilon \in]0,1]$, for any $s \geq 0$. Let $w^\varepsilon = a^\varepsilon - \widetilde{a}^\varepsilon$: it solves

$$\partial_t w^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla w^\varepsilon + \frac{1}{2} w^\varepsilon \Delta\phi_{\rm eik} = i\frac{\varepsilon}{2}\Delta a^\varepsilon - i\varepsilon^{\kappa-1}\left(F(a^\varepsilon) - F(\widetilde{a}^\varepsilon)\right),$$
$$w^\varepsilon_{|t=0} = a_0^\varepsilon - a_0,$$

where we have denoted $F(z) = f(|z|^2)z$. Proceeding as in the proof of Proposition 3.1, we have the following energy estimate:

$$\frac{d}{dt}\|w^\varepsilon\|_{H^s} \leq C\|w^\varepsilon\|_{H^s} + \varepsilon\|\Delta a^\varepsilon\|_{H^s} + \|F(a^\varepsilon) - F(\widetilde{a}^\varepsilon)\|_{H^s}.$$

Since H^s is an algebra and F is C^1, the Fundamental Theorem of Calculus yields:

$$\|F(a^\varepsilon) - F(\widetilde{a}^\varepsilon)\|_{H^s} \leq C\left(\|a^\varepsilon\|_{H^s}, \|\widetilde{a}^\varepsilon\|_{H^s}\right)\|w^\varepsilon\|_{H^s}.$$

Now since a^ε and $\widetilde{a}^\varepsilon$ are bounded in $L^\infty([0,T];H^{s+2})$ uniformly in $\varepsilon \in]0,1]$, we have an estimate of the form:

$$\frac{d}{dt}\|w^\varepsilon\|_{H^s} \leq C(s)\|w^\varepsilon\|_{H^s} + C(s)\varepsilon.$$

We conclude by Gronwall lemma, since $\|a_0^\varepsilon - a_0\|_{H^s} \to 0$ from Assumption 2. □

Remark 3.3. If we assume moreover that like in (1.3),

$$a_0^\varepsilon = a_0 + \mathcal{O}(\varepsilon) \quad \text{in } H^s,\ \forall s \geq 0,$$

then the above estimate can be improved to:

$$\|a^\varepsilon - \widetilde{a}^\varepsilon\|_{L^\infty([0,T];H^s)} = \mathcal{O}(\varepsilon), \quad \forall s \geq 0.$$

We have reduced the study of the asymptotic behavior of u^ε to the understanding of $\widetilde{a}^\varepsilon$. To complete the proof of Proposition 1.2, we resume the framework of Section 2. With $x(t,y)$ given by the Hamilton flow (2.1), introduce the Jacobi determinant

$$J_t(y) = \det \nabla_y x(t,y).$$

Denote

$$A^\varepsilon(t,y) := \widetilde{a}^\varepsilon\left(t, x(t,y)\right)\sqrt{J_t(y)}.$$

We see that so long as $y \mapsto x(t,y)$ defines a global diffeomorphism (which is guaranteed for $t \in [0,T]$ by construction), (3.3) is equivalent to:

$$\partial_t A^\varepsilon = -i\varepsilon^{\kappa-1} f\left(J_t(y)^{-1}|A^\varepsilon|^2\right)A^\varepsilon \quad ; \quad A^\varepsilon(0,y) = a_0(y).$$

This ordinary differential equation along the rays of geometrical optics can be solved explicitly: since f is real-valued, we see that $\partial_t|A^\varepsilon|^2=0$, hence

$$A^\varepsilon(t,y)=a_0(y)\exp\left(-i\varepsilon^{\kappa-1}\int_0^t f\left(J_s(y)^{-1}\left|a_0(y)\right|^2\right)ds\right).$$

Back to the function $\widetilde{a}^\varepsilon$, Proposition 1.2 follows, with:

$$\begin{aligned} a(t,x)&=\frac{1}{\sqrt{J_t(y(t,x))}}a_0\left(y(t,x)\right),\\ G(t,x)&=-\int_0^t f\left(J_s(y(t,x))^{-1}\left|a_0(y(t,x))\right|^2\right)ds. \end{aligned} \tag{3.4}$$

4. A hyperbolic point of view

In this section, we study (1.1) in the case $\kappa=0$, with no potential:

$$i\varepsilon\partial_t u^\varepsilon+\frac{\varepsilon^2}{2}\Delta u^\varepsilon=f\left(|u^\varepsilon|^2\right)u^\varepsilon\quad;\quad u^\varepsilon_{|t=0}=a_0^\varepsilon(x)e^{i\varphi_0(x)/\varepsilon}. \tag{4.1}$$

We recall the method introduced by E. Grenier [23], which is valid for smooth nonlinearities which are defocusing and cubic at the origin. Throughout this section, we assume the following:

Assumption 4 (Study of (4.1))**.** *We have $f'>0$. In addition, $\varphi_0\in H^\infty$, and there exists $a_0,a_1\in H^\infty$ such that*

$$a_0^\varepsilon=a_0+\varepsilon a_1+o(\varepsilon)\quad \text{in } H^s,\ \forall s\geq 0.$$

Note that this assumption is closely akin to Assumption 3: nevertheless, we do not make any assumption on the momenta of a_0 and a_1, and the initial phase is bounded. We will see in Section 5 how to weaken this assumption.

4.1. Grenier's approach.

The principle is somehow to perform the usual WKB analysis "the other way round". First, write the *exact* solution as

$$u^\varepsilon(t,x)=a^\varepsilon(t,x)e^{i\Phi^\varepsilon(t,x)/\varepsilon}, \tag{4.2}$$

where Φ^ε is real-valued. Then show that the "amplitude" a^ε and the "phase" Φ^ε have asymptotic expansions as $\varepsilon\to 0$:

$$a^\varepsilon\sim a+\varepsilon a^{(1)}+\varepsilon^2a^{(2)}+\dots\quad;\quad \Phi^\varepsilon\sim\phi+\varepsilon\phi^{(1)}+\varepsilon^2\phi^{(2)}+\dots$$

Introducing two unknown functions to solve one equation yields a degree of freedom. The historical approach [27, Chap. III] consisted in writing

$$\begin{aligned} \partial_t\Phi^\varepsilon+\frac12\left|\nabla\Phi^\varepsilon\right|^2+f\left(|a^\varepsilon|^2\right)&=\varepsilon^2\frac{\Delta a^\varepsilon}{2a^\varepsilon}\quad;\quad \Phi^\varepsilon\big|_{t=0}=\varphi_0,\\ \partial_t a^\varepsilon+\nabla\Phi^\varepsilon\cdot\nabla a^\varepsilon+\frac12 a^\varepsilon\Delta\Phi^\varepsilon&=0\quad;\quad a^\varepsilon\big|_{t=0}=a_0^\varepsilon. \end{aligned}$$

Of course, this choice is not adapted when the amplitude a^ε vanishes (see [21]), so it must be left out when $a_0^\varepsilon\in L^2(\mathbb{R}^n)$ in general. The approach introduced by E. Grenier consists in imposing:

$$\begin{aligned} \partial_t\Phi^\varepsilon+\frac12\left|\nabla\Phi^\varepsilon\right|^2+f\left(|a^\varepsilon|^2\right)&=0\quad;\quad \Phi^\varepsilon\big|_{t=0}=\varphi_0,\\ \partial_t a^\varepsilon+\nabla\Phi^\varepsilon\cdot\nabla a^\varepsilon+\frac12 a^\varepsilon\Delta\Phi^\varepsilon&=i\frac{\varepsilon}{2}\Delta a^\varepsilon\quad;\quad a^\varepsilon\big|_{t=0}=a_0^\varepsilon. \end{aligned} \tag{4.3}$$

Before recalling the results of [23], observe that if a^ε and Φ^ε are bounded in some sufficiently small Sobolev spaces uniformly in ε, passing to the limit formally in (4.3) yields:

$$
\begin{aligned}
&\partial_t\phi + \frac{1}{2}|\nabla\phi|^2 + f\left(|a|^2\right) = 0 \quad ; \quad \phi\big|_{t=0} = \varphi_0\,,\\
&\partial_t a + \nabla\phi\cdot\nabla a + \frac{1}{2}a\Delta\phi = 0 \quad ; \quad a\big|_{t=0} = a_0\,.
\end{aligned} \tag{4.4}
$$

We see that when the nonlinearity is exactly cubic ($f(y)\equiv y$), $(\rho,v) := \left(|a|^2,\nabla\phi\right)$ solves the compressible, isentropic Euler equation

$$
\begin{aligned}
&\partial_t v + v\cdot\nabla v + \nabla\rho = 0 \quad ; \quad v\big|_{t=0} = \nabla\varphi_0\,,\\
&\partial_t\rho + \nabla\cdot(\rho v) = 0 \quad ; \quad \rho\big|_{t=0} = |a_0|^2\,.
\end{aligned} \tag{4.5}
$$

From this point of view, the formulation (4.4) is closely akin to the change of unknown function $\rho\to\sqrt{\rho}$ introduced in [31] (see also [10]) to study (4.5) when the initial density is compactly supported, a situation more or less similar to the present one. Note however that here, a is complex-valued in general.

Introducing the "velocity" $v^\varepsilon = \nabla\Phi^\varepsilon$, (4.3) yields

$$
\begin{aligned}
&\partial_t v^\varepsilon + v^\varepsilon\cdot\nabla v^\varepsilon + 2f'\left(|a^\varepsilon|^2\right)\operatorname{Re}\left(\overline{a^\varepsilon}\nabla a^\varepsilon\right) = 0 \quad ; \quad v^\varepsilon\big|_{t=0} = \nabla\phi_0\,,\\
&\partial_t a^\varepsilon + v^\varepsilon\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\nabla\cdot v^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,.
\end{aligned} \tag{4.6}
$$

Separate real and imaginary parts of a^ε, $a^\varepsilon = a_1^\varepsilon + i a_2^\varepsilon$. Then we have

$$
\partial_t\mathbf{u}^\varepsilon + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j\mathbf{u}^\varepsilon = \frac{\varepsilon}{2}L\mathbf{u}^\varepsilon\,, \tag{4.7}
$$

$$
\text{with}\quad \mathbf{u}^\varepsilon = \begin{pmatrix} a_1^\varepsilon\\ a_2^\varepsilon\\ v_1^\varepsilon\\ \vdots\\ v_n^\varepsilon\end{pmatrix}\quad ,\quad L = \begin{pmatrix} 0 & -\Delta & 0 & \dots & 0\\ \Delta & 0 & 0 & \dots & 0\\ 0 & 0 & & 0_{n\times n} & \end{pmatrix},
$$

$$
\text{and}\quad A(\mathbf{u},\xi) = \sum_{j=1}^n A_j(\mathbf{u})\xi_j = \begin{pmatrix} v\cdot\xi & 0 & \frac{a_1}{2}\,{}^t\xi\\ 0 & v\cdot\xi & \frac{a_2}{2}\,{}^t\xi\\ 2f'a_1\,\xi & 2f'a_2\,\xi & v\cdot\xi I_n\end{pmatrix},
$$

where f' stands for $f'(|a_1|^2+|a_2|^2)$. The matrix $A(\mathbf{u},\xi)$ can be symmetrized by

$$
S = \begin{pmatrix} I_2 & 0\\ 0 & \frac{1}{4f'}I_n\end{pmatrix}, \tag{4.8}
$$

which is symmetric and positive since $f'>0$. For an integer $s > 2+n/2$, we bound $(S\partial_x^\alpha\mathbf{u}^\varepsilon,\partial_x^\alpha\mathbf{u}^\varepsilon)$ where α is a multi index of length $\leq s$, and $(\cdot,\cdot)$ is the usual L^2 scalar product. We have

$$
\frac{d}{dt}\left(S\partial_x^\alpha\mathbf{u}^\varepsilon,\partial_x^\alpha\mathbf{u}^\varepsilon\right) = \left(\partial_t S\partial_x^\alpha\mathbf{u}^\varepsilon,\partial_x^\alpha\mathbf{u}^\varepsilon\right) + 2\left(S\partial_t\partial_x^\alpha\mathbf{u}^\varepsilon,\partial_x^\alpha\mathbf{u}^\varepsilon\right)
$$

since S is symmetric. For the first term, we consider the lower $n \times n$ block:

$$(\partial_t S \partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon) \leq \left\| \frac{1}{f'} \partial_t \left(f' \left(|a_1^\varepsilon|^2 + |a_2^\varepsilon|^2 \right) \right) \right\|_{L^\infty} (S\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon) \, .$$

So long as $\|\mathbf{u}^\varepsilon\|_{L^\infty} \leq 2\|a_0^\varepsilon\|_{L^\infty}$, we have:

$$f'\left(|a_1^\varepsilon|^2 + |a_2^\varepsilon|^2\right) \geq \inf \left\{ f'(y) \ ; \ 0 \leq y \leq 4 \sup_{0<\varepsilon\leq 1} \|a_0^\varepsilon\|_{L^\infty}^2 \right\} = \delta_n > 0 \, ,$$

where δ_n is now fixed, since f' is continuous with $f' > 0$. We infer,

$$\left\| \frac{1}{f'} \partial_t \left(f' \left(|a_1^\varepsilon|^2 + |a_2^\varepsilon|^2 \right) \right) \right\|_{L^\infty} \leq C\|\mathbf{u}^\varepsilon\|_{H^s} \, ,$$

where we used Sobolev embeddings and (4.7). For the second term we use

$$(S\partial_t \partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon) = \frac{\varepsilon}{2} \left(SL(\partial_x^\alpha \mathbf{u}^\varepsilon), \partial_x^\alpha \mathbf{u}^\varepsilon \right) - \Big(S\partial_x^\alpha \Big(\sum_{j=1}^n A_j(\mathbf{u}^\varepsilon) \partial_j \mathbf{u}^\varepsilon \Big), \partial_x^\alpha \mathbf{u}^\varepsilon \Big).$$

We notice that SL is a skew-symmetric second order operator, so the first term is zero. For the second term, use the symmetry of $SA_j(\mathbf{u}^\varepsilon)$ and usual estimates on commutators to get finally:

$$\frac{d}{dt} \sum_{|\alpha|\leq s} (S\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon) \leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s}\right) \sum_{|\alpha|\leq s} (S\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon) \, ,$$

for $s > 2 + d/2$. Gronwall lemma along with a continuity argument yield:

Proposition 4.1 ([23], Th. 1.1)**.** *Let Assumption 4 be satisfied. Let $s > 2 + n/2$. There exist $T_s > 0$ independent of $\varepsilon \in]0,1]$ and $u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}$ solution to* (4.1) *on $[0, T_s]$. Moreover, a^ε and Φ^ε are bounded in $L^\infty([0,T_s]; H^s)$, uniformly in $\varepsilon \in]0,1]$.*

The solution to (4.3) formally converges to the solution of (4.4). Under Assumption 4, (4.4) has a unique solution $(a, \phi) \in L^\infty([0,T_*]; H^m)^2$ for any $m > 0$ for some $T_* > 0$ independent of m (see e.g. [1, 30]). We infer:

Proposition 4.2. *Let $s \in \mathbb{N}$. Then $T_s \geq T_*$, and there exists C_s independent of ε such that for every $0 \leq t \leq T_*$,*

$$\|a^\varepsilon(t) - a(t)\|_{H^s} \leq C_s \varepsilon \quad ; \quad \|\Phi^\varepsilon(t) - \phi(t)\|_{H^s} \leq C_s \varepsilon t. \tag{4.9}$$

Proof. We keep the same notations as above, (4.7). Denote by $\mathbf{v}$ the analog of $\mathbf{u}^\varepsilon$ corresponding to (a, ϕ). We have

$$\partial_t \left(\mathbf{u}^\varepsilon - \mathbf{v} \right) + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon) \partial_j \left(\mathbf{u}^\varepsilon - \mathbf{v} \right) + \sum_{j=1}^n \left(A_j(\mathbf{u}^\varepsilon) - A_j(\mathbf{v}) \right) \partial_j \mathbf{v} = \frac{\varepsilon}{2} L\mathbf{u}^\varepsilon \, .$$

Keeping the symmetrizer S corresponding to $\mathbf{u}^\varepsilon$, we can do similar computations to the previous ones. Note that we know that $\mathbf{u}^\varepsilon$ and $\mathbf{v}$ are bounded in $L^\infty([0, \min(T_s, T_*)]; H^s)$. Denote $\mathbf{w}^\varepsilon = \mathbf{u}^\varepsilon - \mathbf{v}$. Writing $L\mathbf{u}^\varepsilon = L\mathbf{w}^\varepsilon + L\mathbf{v}$, the term $L\mathbf{w}^\varepsilon$ disappears from the energy estimate, and we get, for $s > 2 + n/2$:

$$\frac{d}{dt} \sum_{|\alpha|\leq s} (S\partial_x^\alpha \mathbf{w}^\varepsilon, \partial_x^\alpha \mathbf{w}^\varepsilon) \leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s}, \|\mathbf{v}\|_{H^{s+2}}\right) \sum_{|\alpha|\leq s} (S\partial_x^\alpha \mathbf{w}^\varepsilon, \partial_x^\alpha \mathbf{w}^\varepsilon)$$
$$+ \varepsilon \|\mathbf{v}\|_{H^{s+2}} \|\mathbf{w}^\varepsilon(t)\|_{H^s} \, .$$

Gronwall lemma and a continuity argument show that $\mathbf{w}^\varepsilon$ (hence $\mathbf{u}^\varepsilon$) is defined on $[0,T_*]$. By Assumption 4, $\|\mathbf{w}^\varepsilon_{|t=0}\|_{H^s} = \mathcal{O}(\varepsilon)$, and we get:

$$\|\mathbf{w}^\varepsilon\|_{L^\infty([0,T_*];H^s)} = \mathcal{O}(\varepsilon).$$

The estimate for the phase (and not only its gradient) then follows from the above estimate and the integration in time of (4.3)-(4.4). □

Proposition 4.2 yields an approximation of u^ε *for small times only*:

$$\begin{aligned}\left\|u^\varepsilon(t)-a(t)e^{i\phi(t)/\varepsilon}\right\|_{L^2} &= \left\|a^\varepsilon(t)e^{i\Phi^\varepsilon(t)/\varepsilon} - a(t)e^{i\phi(t)/\varepsilon}\right\|_{L^2}\\ &= \mathcal{O}\left(\|a^\varepsilon(t)-a(t)\|_{L^2} + \left\|e^{i\Phi^\varepsilon(t)/\varepsilon} - e^{i\phi(t)/\varepsilon}\right\|_{L^\infty}\|a(t)\|_{L^2}\right)\\ &= \mathcal{O}(\varepsilon)+\mathcal{O}(t).\end{aligned}$$

For times of order $\mathcal{O}(1)$, the corrector a_1 must be taken into account:

Proposition 4.3. *Let Assumption 4 be satisfied. Define* $(a^{(1)},\phi^{(1)})$ *by*

$$\partial_t\phi^{(1)} + \nabla\phi\cdot\nabla\phi^{(1)} + 2\operatorname{Re}\left(\overline{a}a^{(1)}\right)f'\left(|a|^2\right) = 0,$$
$$\partial_t a^{(1)} + \nabla\phi\cdot\nabla a^{(1)} + \nabla\phi^{(1)}\cdot\nabla a + \frac12 a^{(1)}\Delta\phi + \frac12 a\Delta\phi^{(1)} = \frac{i}{2}\Delta a,$$
$$\phi^{(1)}\big|_{t=0} = 0 \quad ; \quad a^{(1)}\big|_{t=0} = a_1.$$

Then $a^{(1)},\phi^{(1)}\in L^\infty([0,T_*];H^s)$ *for every* $s\geq 0$*, and*

$$\|a^\varepsilon - a - \varepsilon a^{(1)}\|_{L^\infty([0,T_*];H^s)} + \|\Phi^\varepsilon - \phi - \varepsilon\phi^{(1)}\|_{L^\infty([0,T_*];H^s)} \leq C_s\varepsilon^2, \quad \forall s\geq 0\,.$$

The proof is a straightforward consequence of the above analysis, and is given in [23]. Despite the notations, it seems unadapted to consider $\phi^{(1)}$ as being part of the phase. Indeed, we infer from Proposition 4.3 that

$$\left\|u^\varepsilon - ae^{i\phi^{(1)}}e^{i\phi/\varepsilon}\right\|_{L^\infty([0,T_*];L^2\cap L^\infty)} = \mathcal{O}(\varepsilon).$$

Relating this information to the WKB methods presented in the introduction, we would have:

$$\mathsf{a}_0 = ae^{i\phi^{(1)}}.$$

Since $\phi^{(1)}$ depends on a_1 while a does not, we retrieve the fact that in supercritical régimes, the leading order amplitude in WKB methods depends on the initial first corrector a_1.

Remark 4.4. The term $e^{i\phi^{(1)}}$ does not appear in the Wigner measure of $ae^{i\phi^{(1)}}e^{i\phi/\varepsilon}$. Thus, from the point of view of Wigner measures, the asymptotic behavior of the exact solution is described by the Euler-type system (4.4).

Remark 4.5 (Introducing an isotropic harmonic potential). The above method makes it possible to consider the semi-classical of (1.1) when $V(t,x)=\frac12|x|^2$ is an isotropic harmonic potential, and Assumption 4 is satisfied. Let

$$U^\varepsilon(t,x) = \frac{1}{(1+t^2)^{n/4}}e^{i\frac{t}{1+t^2}\frac{|x|^2}{2\varepsilon}}u^\varepsilon\left(\arctan t, \frac{x}{\sqrt{1+t^2}}\right).$$

Then U^ε solves:

$$\begin{cases} i\varepsilon\partial_t U^\varepsilon + \dfrac{\varepsilon^2}{2}\Delta U^\varepsilon = \dfrac{1}{1+t^2} f\left(\left(1+t^2\right)^{n/2}|U^\varepsilon|^2\right)U^\varepsilon\,, \\ U^\varepsilon(0,x) = a_0^\varepsilon(x)e^{i\varphi_0(x)/\varepsilon}. \end{cases}$$

We can then proceed as above. The only difference is the presence of time in the nonlinearity, which changes very little the analysis.

Remark 4.6 (Momenta). If in Assumption 4, we replace H^s with

$$\Sigma^s = H^s \cap \mathcal{F}(H^s) = \left\{ w \in L^2 \ ; \ (1-\Delta)^{k/2}\langle x\rangle^{s-k} w \in L^2, \ 0 \leq k \leq s\right\},$$

then the above analysis can be repeated in Σ^s. The main difference is due to the commutations of the powers of x with the differential operators; it is easy to check that they introduce semilinear terms, which can be treated as source terms when applying Gronwall lemma.

4.2. **Remarks about some conserved quantities.** Consider the case of the cubic, defocusing Schrödinger equation: $f(y) \equiv y$. Recall three important evolution laws for (1.1):

$$\begin{aligned} \text{Mass: } & \frac{d}{dt}\|u^\varepsilon(t)\|_{L^2} = 0\,. \\ \text{Energy: } & \frac{d}{dt}\left(\|\varepsilon\nabla_x u^\varepsilon\|_{L^2}^2 + \|u^\varepsilon\|_{L^4}^4\right) = 0\,. \\ \text{Momentum: } & \frac{d}{dt}\operatorname{Im}\int \overline{u^\varepsilon}(t,x)\varepsilon\nabla_x u^\varepsilon(t,x)dx = 0\,. \\ \text{Pseudo-conformal law: } & \frac{d}{dt}\left(\|J^\varepsilon(t)u^\varepsilon\|_{L^2}^2 + t^2\|u^\varepsilon\|_{L^4}^4\right) = t(2-n)\|u^\varepsilon\|_{L^4}^4, \end{aligned}$$

where $J^\varepsilon(t) = x + i\varepsilon t\nabla_x$. These evolutions are deduced from the usual ones ($\varepsilon = 1$, see e.g. [9, 37]) via the scaling $\psi(t,x) = u(\varepsilon t, \varepsilon x)$. Using (4.2) and passing to the limit formally in the above formulae yields:

$$\frac{d}{dt}\|a(t)\|_{L^2} = 0\,.$$

$$\frac{d}{dt}\int\left(|a(t,x)|^2|\nabla\phi(t,x)|^2 + |a(t,x)|^4\right)dx = 0\,.$$

$$\frac{d}{dt}\int |a(t,x)|^2\nabla\phi(t,x)dx = 0\,.$$

$$\frac{d}{dt}\int\Big(|(x-t\nabla\phi(t,x))\,a(t,x)|^2 + t^2|a(t,x)|^4\Big)dx = = (2-n)t\int |a(t,x)|^4 dx\,.$$

Note that we also have the conservation ([8]):

$$\frac{d}{dt}\operatorname{Re}\int \overline{u^\varepsilon}(t,x)J^\varepsilon(t)u^\varepsilon(t,x)dx = 0\,,$$

which yields:

$$\frac{d}{dt}\int\left((x-t\nabla\phi(t,x))\,|a(t,x)|^2\right)dx = 0\,.$$

All these expressions involve only $(|a|^2, \nabla\phi)$, that is, the solution of (4.5). We thus retrieve formally some evolution laws for the Euler equation.

5. A generalization when $\kappa = 0$

In this section, we prove Theorem 1.3. First, we point out that the uniqueness for u^ε in $C([0,T_*];H^s)$ is straightforward for $s > n/2$. We thus have to prove that there exists such a solution, and that it is smooth.

As suggested by the statements of Theorem 1.3, the idea consists in resuming Grenier's method, and in writing the phase Φ^ε as

$$\Phi^\varepsilon = \phi_{\rm eik} + \phi^\varepsilon.$$

We take ϕ^ε as a new unknown function. Recall that the system (4.3) reads, with the present notations:

$$\partial_t \Phi^\varepsilon + \frac{1}{2}|\nabla\Phi^\varepsilon|^2 + V + f\left(|a^\varepsilon|^2\right) = 0 \quad ; \quad \Phi^\varepsilon\big|_{t=0} = \phi_0\,,$$
$$\partial_t a^\varepsilon + \nabla\Phi^\varepsilon\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\Phi^\varepsilon = i\frac{\varepsilon}{2}\Delta a^\varepsilon \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,.$$

This system becomes, in terms of ϕ^ε, and given (1.5):

$$\tag{5.1} \begin{aligned} &\partial_t\phi^\varepsilon + \frac{1}{2}|\nabla\phi^\varepsilon|^2 + \nabla\phi_{\rm eik}\cdot\nabla\phi^\varepsilon + f\left(|a^\varepsilon|^2\right) = 0,\\ &\partial_t a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi_{\rm eik} = i\frac{\varepsilon}{2}\Delta a^\varepsilon,\\ &\phi^\varepsilon\big|_{t=0} = 0 \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,. \end{aligned}$$

Like in Section 4.1, we work with $v^\varepsilon = \nabla\phi^\varepsilon$ instead of ϕ^ε, to begin with. The new terms are the factors where $\phi_{\rm eik}$ is present. The point is to check that they are semilinear perturbations, which can be treated as source terms in view of Gronwall lemma. Again, separate real and imaginary parts of a^ε, $a^\varepsilon = a_1^\varepsilon + i a_2^\varepsilon$, and introduce:

$$\mathbf{u}^\varepsilon = \begin{pmatrix} a_1^\varepsilon \\ a_2^\varepsilon \\ v_1^\varepsilon \\ \vdots \\ v_n^\varepsilon \end{pmatrix} \quad , \quad L = \begin{pmatrix} 0 & -\Delta & 0 & \dots & 0 \\ \Delta & 0 & 0 & \dots & 0 \\ 0 & 0 & & 0_{n\times n} & \end{pmatrix},$$

$$\text{and} \quad A(\mathbf{u},\xi) = \sum_{j=1}^n A_j(\mathbf{u})\xi_j = \begin{pmatrix} v\cdot\xi & 0 & \frac{a_1}{2}\,{}^t\xi \\ 0 & v\cdot\xi & \frac{a_2}{2}\,{}^t\xi \\ 2f'a_1\,\xi & 2f'a_2\,\xi & v\cdot\xi I_n \end{pmatrix},$$

where f' stands for $f'(|a_1|^2+|a_2|^2)$. Instead of (4.7), we now have a system of the form

$$\tag{5.2} \partial_t\mathbf{u}^\varepsilon + \sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j\mathbf{u}^\varepsilon + \sum_{j=1}^n B_j(\nabla\phi_{\rm eik})\partial_j\mathbf{u}^\varepsilon + M\left(\nabla^2\phi_{\rm eik}\right)\mathbf{u}^\varepsilon = \frac{\varepsilon}{2}L\mathbf{u}^\varepsilon\,,$$

where the matrices B_j depend linearly on their argument, and the matrix M is smooth, locally bounded. The quasilinear part of (5.2) is the same as in Section 4.1, and involves the matrices A_j. In particular, we keep the same symmetrizer S given by (4.8). The matrices B_j have a semilinear contribution, as we see below. The term corresponding to the matrix M can obviously be considered as a source term, since $\phi_{\rm eik}$ is subquadratic.

Let s be an integer, $s > 2+n/2$, and let α be a multi index of length $\leq s$. We have:

$$\frac{d}{dt}\left(S\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right) = \left(\partial_t S\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right) + 2\left(S\partial_t\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right)$$

since S is symmetric. For the first term, we consider the lower $n\times n$ block:

$$\left(\partial_t S\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right) \leq \left\| \frac{1}{f'}\partial_t \left(f'\left(|a_1^\varepsilon|^2 + |a_2^\varepsilon|^2\right)\right)\right\|_{L^\infty} \left(S\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right) . \tag{5.3}$$

We consider times not larger than T given by Lemma 1.1, so that the function $\phi_{\rm eik}$ remains smooth and subquadratic. So long as $\|\mathbf{u}^\varepsilon\|_{L^\infty} \leq 2\|a_0^\varepsilon\|_{L^\infty}$, we have:

$$f'\left(|a_1^\varepsilon|^2 + |a_2^\varepsilon|^2\right) \geq \inf\left\{ f'(y)\ ;\ 0\leq y\leq 4\sup_{0<\varepsilon\leq 1}\|a_0^\varepsilon\|_{L^\infty}^2\right\} = \delta_n > 0\,.$$

We infer,

$$\left\| \frac{1}{f'}\partial_t \left(f'\left(|a_1^\varepsilon|^2 + |a_2^\varepsilon|^2\right)\right)\right\|_{L^\infty} \leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s} + \|x\mathbf{u}^\varepsilon\|_{H^{s-1}}\right) , \tag{5.4}$$

for some locally bounded map $C(\cdot)$. We used Sobolev embeddings, (5.2) and Lemma 1.1: the terms B_j are sublinear in x, hence the norm $\|x\mathbf{u}^\varepsilon\|_{H^{s-1}}$ which we did not consider in Section 4.1. We emphasize that this estimate explains why we assume $s > 2+n/2$, and not only $s > 1+n/2$: we control $\partial_t \mathbf{u}^\varepsilon$ in L^∞ using (5.2), so we need to estimate $L\mathbf{u}^\varepsilon$ in L^∞. For all the other terms, $s > 1+n/2$ would suffice. This also explains why we wrote $\|x\mathbf{u}^\varepsilon\|_{H^{s-1}}$ and not $\|x\mathbf{u}^\varepsilon\|_{H^s}$. For the second term we use

$$\begin{aligned}\left(S\partial_t\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right) =& \frac{\varepsilon}{2}\left(SL(\partial_x^\alpha \mathbf{u}^\varepsilon), \partial_x^\alpha \mathbf{u}^\varepsilon\right) - \Big(S\partial_x^\alpha\Big(\sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon\Big), \partial_x^\alpha \mathbf{u}^\varepsilon\Big)\\ &-\Big(S\partial_x^\alpha\Big(\sum_{j=1}^n B_j(\nabla\phi_{\rm eik})\partial_j \mathbf{u}^\varepsilon\Big), \partial_x^\alpha \mathbf{u}^\varepsilon\Big) - \Big(S\partial_x^\alpha\Big(M(\nabla^2\phi_{\rm eik})\mathbf{u}^\varepsilon\Big), \partial_x^\alpha \mathbf{u}^\varepsilon\Big).\end{aligned}$$

The first two terms of the right hand side are handled in the same way as in Section 4.1: the first one is zero, and the second can be estimated by:

$$\Big(S\partial_x^\alpha\Big(\sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon\Big), \partial_x^\alpha \mathbf{u}^\varepsilon\Big) \leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s}\right)\sum_{|\alpha|\leq s}\left(S\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right), \tag{5.5}$$

where we keep the convention that $C(\cdot)$ is a locally bounded map. Let us briefly explain this quasilinear estimate. First, write

$$\begin{aligned}\left(S\partial_x^\alpha\left(A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon\right), \partial_x^\alpha \mathbf{u}^\varepsilon\right) =& \left(SA_j(\mathbf{u}^\varepsilon)\partial_j\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right)\\ &+\left(S\left(\partial_x^\alpha(A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon) - A_j(\mathbf{u}^\varepsilon)\partial_j\partial_x^\alpha \mathbf{u}^\varepsilon\right), \partial_x^\alpha \mathbf{u}^\varepsilon\right).\end{aligned}$$

By symmetry of $SA_j(\mathbf{u}^\varepsilon)$,

$$\begin{aligned}\left(SA_j(\mathbf{u}^\varepsilon)\partial_j\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right) =& -\left(\partial_j(SA_j(\mathbf{u}^\varepsilon))\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right)\\ &-\left(SA_j(\mathbf{u}^\varepsilon)\partial_j\partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right).\end{aligned}$$

We infer:

$$\begin{aligned}\left|(SA_j(\mathbf{u}^\varepsilon)\partial_j\partial_x^\alpha \mathbf{u}^\varepsilon,\partial_x^\alpha \mathbf{u}^\varepsilon)\right| &\leq \left\|\partial_j\left(SA_j(\mathbf{u}^\varepsilon)\right)\right\|_{L^\infty}\|\partial_x^\alpha \mathbf{u}^\varepsilon\|_{L^2}^2\\ &\leq C\left(\|\mathbf{u}^\varepsilon\|_{L^\infty}\right)\|\nabla_x \mathbf{u}^\varepsilon\|_{L^\infty}\|\partial_x^\alpha \mathbf{u}^\varepsilon\|_{L^2}^2 .\end{aligned}$$

The usual estimates on commutators (see e.g. [30]) lead to

$$\left|\left(S\left(\partial_x^\alpha\left(A_j(\mathbf{u}^\varepsilon)\partial_j \mathbf{u}^\varepsilon\right)-A_j(\mathbf{u}^\varepsilon)\partial_j\partial_x^\alpha \mathbf{u}^\varepsilon\right),\partial_x^\alpha \mathbf{u}^\varepsilon\right)\right|\leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s}\right)\|\mathbf{u}^\varepsilon\|_{H^s}^2 ,$$

and (5.5) follows, since we consider times where S^{-1} is bounded.

For the third term of $(S\partial_t\partial_x^\alpha \mathbf{u}^\varepsilon,\partial_x^\alpha \mathbf{u}^\varepsilon)$, write:

$$\begin{aligned}\left(S\partial_x^\alpha\left(B_j(\nabla\phi_{\rm eik})\partial_j \mathbf{u}^\varepsilon\right),\partial_x^\alpha \mathbf{u}^\varepsilon\right) &= \int SB_j(\nabla\phi_{\rm eik})\partial_j\partial_x^\alpha \mathbf{u}^\varepsilon\partial_x^\alpha \mathbf{u}^\varepsilon dx\\ &\quad+\int S\left[\partial_x^\alpha,B_j(\nabla\phi_{\rm eik})\partial_j\right]\mathbf{u}^\varepsilon\partial_x^\alpha \mathbf{u}^\varepsilon dx.\end{aligned}$$

For the first term of the right hand side, an integration by parts yields:

$$\begin{aligned}\left|\int SB_j(\nabla\phi_{\rm eik})\partial_j\partial_x^\alpha \mathbf{u}^\varepsilon\partial_x^\alpha \mathbf{u}^\varepsilon dx\right| &\leq \left\|\partial_j\left(SB_j(\nabla\phi_{\rm eik})\right)\right\|_{L^\infty}\|\mathbf{u}^\varepsilon\|_{H^s}^2\\ &\leq C(\|a^\varepsilon\|_{L^\infty})\left\|\langle x\rangle a^\varepsilon\nabla a^\varepsilon\right\|_{L^\infty}\|\mathbf{u}^\varepsilon\|_{H^s}^2\\ &\leq C(\|\mathbf{u}^\varepsilon\|_{L^\infty})\Big(\|\mathbf{u}^\varepsilon\|_{L^\infty}+\|x\mathbf{u}^\varepsilon\|_{L^\infty}+\|\nabla \mathbf{u}^\varepsilon\|_{L^\infty}\Big)^2\|\mathbf{u}^\varepsilon\|_{H^s}^2,\end{aligned} \tag{5.6}$$

where we have used Lemma 1.1. Again from Lemma 1.1, the commutator

$$\left[\partial_x^\alpha,B_j(\nabla\phi_{\rm eik})\partial_j\right]$$

is a differential operator of degree $\leq s$, with bounded coefficients. We infer:

$$\left|\Big(S\partial_x^\alpha\Big(\sum_{j=1}^n B_j(\nabla\phi_{\rm eik})\partial_j \mathbf{u}^\varepsilon\Big),\partial_x^\alpha \mathbf{u}^\varepsilon\Big)\right|\leq C(\|\mathbf{u}^\varepsilon\|_{H^s}+\|x\mathbf{u}^\varepsilon\|_{H^{s-1}})\|\mathbf{u}^\varepsilon\|_{H^s}^2.$$

We have obviously

$$\left|\Big(S\partial_x^\alpha\Big(M(\nabla^2\phi_{\rm eik})\mathbf{u}^\varepsilon\Big),\partial_x^\alpha \mathbf{u}^\varepsilon\Big)\right|\leq C(\|\mathbf{u}^\varepsilon\|_{L^\infty})\|\mathbf{u}^\varepsilon\|_{H^s}^2.$$

This yields:

$$\frac{d}{dt}\left(S\partial_x^\alpha \mathbf{u}^\varepsilon,\partial_x^\alpha \mathbf{u}^\varepsilon\right)\leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s}+\|x\mathbf{u}^\varepsilon\|_{H^{s-1}}\right)\|\mathbf{u}^\varepsilon\|_{H^s}^2, \tag{5.7}$$

where the map $C(\cdot)$ is locally bounded. We now have to bound $x\mathbf{u}^\varepsilon$ in H^{s-1} to close our family of estimates: we consider

$$\frac{d}{dt}\Big(S\partial_x^\beta(x_k\mathbf{u}^\varepsilon),\partial_x^\beta(x_k\mathbf{u}^\varepsilon)\Big);\quad 1\leq k\leq n,\ |\beta|\leq s-1.$$

We can proceed as above, replacing $\mathbf{u}^\varepsilon$ with $x_k\mathbf{u}^\varepsilon$: $x_k\mathbf{u}^\varepsilon$ solves almost the same equation as $\mathbf{u}^\varepsilon$, and we must control some commutators.

$$\begin{aligned}\partial_t(x_k\mathbf{u}^\varepsilon)+\sum_{j=1}^n A_j(\mathbf{u}^\varepsilon)\partial_j(x_k\mathbf{u}^\varepsilon)+\sum_{j=1}^n B_j(\nabla\phi_{\rm eik})\partial_j(x_k\mathbf{u}^\varepsilon)+\\ +M\left(\nabla^2\phi_{\rm eik}\right)x_k\mathbf{u}^\varepsilon=\frac{\varepsilon}{2}L(x_k\mathbf{u}^\varepsilon)+A_k(\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon+B_k(\nabla\phi_{\rm eik})\mathbf{u}^\varepsilon+\frac{\varepsilon}{2}[x_k,L]\mathbf{u}^\varepsilon.\end{aligned}$$

The term $A_k(\mathbf{u}^\varepsilon)\mathbf{u}^\varepsilon$ is harmless. The term $B_k(\nabla\phi_{\rm eik})\mathbf{u}^\varepsilon$ is controlled by $\langle x\rangle\,\mathbf{u}^\varepsilon$, since $\phi_{\rm eik}$ is subquadratic: this is a (semi)linear perturbation. Finally,

$$[x_k,L]=\begin{pmatrix} 0 & 2\partial_k & 0 & \dots & 0\\ -2\partial_k & 0 & 0 & \dots & 0\\ 0 & 0 & & 0_{n\times n} & \end{pmatrix}.$$

Now we only have to notice that estimating $x_k\mathbf{u}^\varepsilon$ does not involve extra regularity or extra momenta. In the above computations, the first time we needed to consider momenta was for (5.4): we need exactly the same estimate now, since it is due to the symmetrizer, which remains the same. The same remark is valid for (5.6). For β a multi index of length $\leq s-1$, we find:

$$\begin{aligned}\frac{d}{dt}\Big(S\partial_x^\beta(x_k\mathbf{u}^\varepsilon),\partial_x^\beta(x_k\mathbf{u}^\varepsilon)\Big)&\leq\\ \leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s}+\|x\mathbf{u}^\varepsilon\|_{H^{s-1}}\right)&\left(\|\mathbf{u}^\varepsilon\|_{H^s}^2+\|x\mathbf{u}^\varepsilon\|_{H^{s-1}}^2\right).\end{aligned}\tag{5.8}$$

The term $\|\mathbf{u}^\varepsilon\|_{H^s}^2$ is due to the commutator $[x_k,L]$:

$$\begin{aligned}\left|\Big(S\partial_x^\beta\left([x_k,L]\mathbf{u}^\varepsilon\right),\partial_x^\beta\left(x_k\mathbf{u}^\varepsilon\right)\Big)\right|&=\left|\Big(\partial_x^\beta\left([x_k,L]\mathbf{u}^\varepsilon\right),\partial_x^\beta\left(x_k\mathbf{u}^\varepsilon\right)\Big)\right|\\&\leq\|\mathbf{u}^\varepsilon\|_{H^s}\|x\mathbf{u}^\varepsilon\|_{H^{s-1}}.\end{aligned}$$

Summing over the inequalities (5.7) and (5.8) yields a closed set of estimates, from which we infer the analogue of Proposition 4.1; note that the time T_s is not larger than T by construction, and may be strictly smaller than T, due to possible shocks for (5.2). We also mention the fact that the above analysis gives $v^\varepsilon=\nabla\phi^\varepsilon\in C([0,T_s];H^s)$, with $x\nabla\phi^\varepsilon\in C([0,T_s];H^{s-1})$: back to (5.1), this shows that $\phi^\varepsilon\in C([0,T_*];H^{s-1})$, but that we cannot claim that $x\phi^\varepsilon\in C([0,T_*];H^{s-1})$.

Passing to the limit $\varepsilon\to 0$ in (5.1), it is natural to introduce the system:

$$\begin{aligned}&\partial_t\phi+\frac{1}{2}\left|\nabla\phi\right|^2+\nabla\phi_{\rm eik}\cdot\nabla\phi+f\left(|a|^2\right)=0,\\&\partial_t a+\nabla\phi\cdot\nabla a+\nabla\phi_{\rm eik}\cdot\nabla a+\frac{1}{2}a\Delta\phi+\frac{1}{2}a\Delta\phi_{\rm eik}=0,\\&\phi\big|_{t=0}=0\quad;\quad a\big|_{t=0}=a_0\,.\end{aligned}\tag{5.9}$$

The above analysis shows that this system has a unique solution in H^s with the first momentum in H^{s-1}, locally in time for $t\in[0,T_*]$, for some $T_*\in]0,T]$; T_* is independent of s, from the usual continuation principle, explained for instance in [30, Section 2.2]. We easily obtain the analogue of Proposition 4.2: mimicking the above computations, we can estimate the error $(a^\varepsilon-a,\phi^\varepsilon-\phi)$ in H^s, by first estimating $(a^\varepsilon-a,\nabla\phi^\varepsilon-\nabla\phi)$ *and its first momentum* in H^s. We deduce that $T_s\geq T_*$, and (1.8) follows.

The end of Theorem 1.3 can then be proved as in [23]: from the above analysis, the functions

$$\frac{a^\varepsilon-a}{\varepsilon},\ \frac{\nabla\phi^\varepsilon-\nabla\phi}{\varepsilon}$$

and their first momentum, are bounded in H^s for every $s\geq 0$. A subsequence converges to the linearization of (5.2), yielding a pair $(a^{(1)},\phi^{(1)})$. By uniqueness for the limit system, the whole sequence is convergent, and the analogue of Proposition 4.3 follows. This completes the proof of Theorem 1.3.

6. The case $0<\kappa<1$

When $0<\kappa<1$, we propose an analysis which can be considered as a generalization of the study led in Section 5. Throughout this paragraph, we suppose that Assumptions 1-3 are satisfied. Again, we write the exact solution as

$$u^\varepsilon = a^\varepsilon e^{i\Phi^\varepsilon/\varepsilon}, \quad \text{with } \Phi^\varepsilon = \phi_{\rm eik}+\phi^\varepsilon.$$

The unknown function is the pair $(a^\varepsilon,\phi^\varepsilon)$. We have two unknown functions to solve a single equation, (1.1). We can choose how to balance the terms: we resume the approach followed when $\kappa=0$. Note that this approach would also be efficient for the case $\kappa\geq 1$, with the serious drawback that we still assume $f'>0$, an assumption proven to be unnecessary when $\kappa\geq 1$ (see Section 3). We impose:

$$\begin{aligned}
&\partial_t\phi^\varepsilon + \frac{1}{2}\left|\nabla\phi^\varepsilon\right|^2 + \nabla\phi_{\rm eik}\cdot\nabla\phi^\varepsilon + \varepsilon^\kappa f\left(|a^\varepsilon|^2\right)=0,\\
&\partial_t a^\varepsilon + \nabla\phi^\varepsilon\cdot\nabla a^\varepsilon + \nabla\phi_{\rm eik}\cdot\nabla a^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi_{\rm eik} = i\frac{\varepsilon}{2}\Delta a^\varepsilon,\\
&\phi^\varepsilon\big|_{t=0}=0 \quad ; \quad a^\varepsilon\big|_{t=0}=a_0^\varepsilon\,.
\end{aligned}\tag{6.1}$$

This is the same system as (5.1), with only f replaced by $\varepsilon^\kappa f$. Mimicking the analysis of Section 5, we work with the unknown $\mathbf{u}^\varepsilon$ given by the same definition: it solves the system (5.2), where only the matrices A_j have changed, and now depend on ε. The symmetrizer is the same as before, with f' replaced by $\varepsilon^\kappa f'$: the matrix $S=S^\varepsilon$ is not bounded as $\varepsilon\to 0$, but its inverse is. We see that (5.3) and (5.4) still hold, independent of κ. We claim that inequalities similar to (5.7) and (5.8) hold:

$$\frac{d}{dt}\left(S^\varepsilon\partial_x^\alpha\mathbf{u}^\varepsilon,\partial_x^\alpha\mathbf{u}^\varepsilon\right)\leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s}+\|x\mathbf{u}^\varepsilon\|_{H^{s-1}}\right)\sum_{|\gamma|\leq s}\left(S^\varepsilon\partial_x^\gamma\mathbf{u}^\varepsilon,\partial_x^\gamma\mathbf{u}^\varepsilon\right),$$

$$\begin{aligned}
\frac{d}{dt}\Big(S^\varepsilon\partial_x^\beta(x_k\mathbf{u}^\varepsilon),\partial_x^\beta(x_k\mathbf{u}^\varepsilon)\Big)\leq C\left(\|\mathbf{u}^\varepsilon\|_{H^s}+\|x\mathbf{u}^\varepsilon\|_{H^{s-1}}\right)\Big(&\sum_{|\gamma|\leq s}\left(S^\varepsilon\partial_x^\gamma\mathbf{u}^\varepsilon,\partial_x^\gamma\mathbf{u}^\varepsilon\right)\\
&+\sum_{\substack{|\gamma|\leq s-1\\1\leq j\leq n}}\left(S^\varepsilon\partial_x^\gamma(x_j\mathbf{u}^\varepsilon),\partial_x^\gamma(x_j\mathbf{u}^\varepsilon)\right)\Big),
\end{aligned}$$

where the map $C(\cdot)$ is locally bounded, *independent of* $\varepsilon\in]0,1]$. The fact that such estimates remain valid, with this dependence upon ε, stems essentially from the following reasons:

- The matrices S^ε and B_j are diagonal.
- The matrix M is block diagonal: the blocks correspond to the presence/absence of ε in S^ε.
- The matrices $S^\varepsilon A_j^\varepsilon$ are independent of $\varepsilon\in]0,1]$.
- The inverse of S^ε is uniformly bounded on compacts, as $\varepsilon\to 0$.

A continuity argument and Gronwall lemma then imply the analogue of Proposition 4.1: $\mathbf{u}^\varepsilon$ exists locally in time, with H^s-norm uniformly bounded as $\varepsilon \to 0$. Note that since $\phi^\varepsilon\big|_{t=0} = 0$, we have:

$$\left(S^\varepsilon \partial_x^\alpha \mathbf{u}^\varepsilon, \partial_x^\alpha \mathbf{u}^\varepsilon\right)\big|_{t=0} = \mathcal{O}(1),$$

and we infer more precisely:

$$\|a^\varepsilon\|_{L^\infty([0,T_*];H^s)} = \mathcal{O}(1) \quad ; \quad \|\nabla\phi^\varepsilon\|_{L^\infty([0,T_*];H^s)} = \mathcal{O}\left(\varepsilon^\kappa\right).$$

It seems natural to change unknown functions, and work with $\widetilde{\phi}^\varepsilon = \varepsilon^{-\kappa}\phi^\varepsilon$ instead of ϕ^ε. With this, we somehow correct the shift in the cascade of equations caused by the factor ε^κ in front of the nonlinearity. Then (6.1) becomes:

$$\begin{aligned} &\partial_t\widetilde{\phi}^\varepsilon + \frac{\varepsilon^\kappa}{2}\left|\nabla\widetilde{\phi}^\varepsilon\right|^2 + \nabla\phi_{\mathrm{eik}}\cdot\nabla\widetilde{\phi}^\varepsilon + f\left(|a^\varepsilon|^2\right) = 0,\\ &\partial_t a^\varepsilon + \varepsilon^\kappa\nabla\widetilde{\phi}^\varepsilon\cdot\nabla a^\varepsilon + \nabla\phi_{\mathrm{eik}}\cdot\nabla a^\varepsilon + \frac{\varepsilon^\kappa}{2}a^\varepsilon\Delta\widetilde{\phi}^\varepsilon + \frac{1}{2}a^\varepsilon\Delta\phi_{\mathrm{eik}} = i\frac{\varepsilon}{2}\Delta a^\varepsilon,\\ &\widetilde{\phi}^\varepsilon\big|_{t=0} = 0 \quad ; \quad a^\varepsilon\big|_{t=0} = a_0^\varepsilon\,. \end{aligned} \tag{6.2}$$

The pair $(\widetilde{\phi}^\varepsilon, a^\varepsilon)$ is bounded in $C([0,T];H^s)$. Therefore, a subsequence is convergent, and the limit is given by:

$$\begin{aligned} &\partial_t\widetilde{\phi} + \nabla\phi_{\mathrm{eik}}\cdot\nabla\widetilde{\phi} + f\left(|a|^2\right) = 0\ ; \quad \widetilde{\phi}\big|_{t=0} = 0,\\ &\partial_t a + \nabla\phi_{\mathrm{eik}}\cdot\nabla a + \frac{1}{2}a\Delta\phi_{\mathrm{eik}} = 0\ ; \quad a\big|_{t=0} = a_0\,. \end{aligned}$$

We see that a solves (1.6); $\widetilde{\phi}$ is given by an ordinary differential equation along the rays associated to ϕ_{eik}, with a source term showing nonlinear effect: $f\left(|a|^2\right)$. By uniqueness, the whole sequence is convergent. Roughly speaking, we see that if

$$\mathbf{w}^\varepsilon = {}^t\left(\nabla\left(\widetilde{\phi}^\varepsilon - \widetilde{\phi}\right), a^\varepsilon - a\right),$$

then Gronwall lemma yields:

$$\left(S^\varepsilon\partial_x^\alpha\mathbf{w}^\varepsilon, \partial_x^\alpha\mathbf{w}^\varepsilon\right) \le C\left(\varepsilon + \varepsilon^\kappa\right) \le 2C\varepsilon^\kappa.$$

We infer:

Proposition 6.1. *Let $s > 2 + n/2$. Then* (6.1) *has a unique solution $(a^\varepsilon,\phi^\varepsilon) \in C([0,T];H^s)^2$, such that $x_k a^\varepsilon, x_k\partial_j\phi^\varepsilon \in C([0,T];H^s)$, for every $1 \le j,k \le n$ (T is given by Lemma 1.1). Moreover, there exists C_s independent of ε such that for every $0 \le t \le T$,*

$$\|a^\varepsilon(t) - a(t)\|_{H^s} \le C_s\varepsilon^\kappa \quad ; \quad \|\phi^\varepsilon(t) - \varepsilon^\kappa\widetilde{\phi}\|_{H^s} \le C_s\varepsilon^{2\kappa}t, \tag{6.3}$$

where a is given by (1.6)*.*

Three cases must be distinguished:

- If $1/2 < \kappa < 1$, then we can infer the analogue of (1.9).
- If $\kappa = 1/2$, then we can infer the analogue of (1.8) (but not yet of (1.9)).
- If $0 < \kappa < 1/2$, then we must pursue the analysis, and compute a corrector of order $\varepsilon^{2\kappa}$.

We shall not go further into detailed computations, but instead, discuss the whole analysis in a rather loose fashion. However, we note that all the ingredients have been given for a complete justification.

Let $N=[1/\kappa]$, where $[r]$ is the largest integer not larger than $r>0$. We construct $a^{(1)},\dots,a^{(N)}$ and $\widetilde{\phi}^{(1)},\dots,\widetilde{\phi}^{(N)}$ such that:

$$\left\|a^\varepsilon - a-\varepsilon^\kappa a^{(1)}-\dots-\varepsilon^{N\kappa}a^{(N)}\right\|_{L^\infty([0,T];H^s)}+ \\ +\left\|\widetilde{\phi}^\varepsilon-\widetilde{\phi}-\varepsilon^\kappa\widetilde{\phi}^{(1)}-\dots-\varepsilon^{N\kappa}\widetilde{\phi}^{(N)}\right\|_{L^\infty([0,T];H^s)}=o\left(\varepsilon^{N\kappa}\right).$$

But since $N+1>1/\kappa$, we have:

$$\left\|\phi^\varepsilon-\varepsilon^\kappa\widetilde{\phi}-\varepsilon^{2\kappa}\widetilde{\phi}^{(1)}-\dots-\varepsilon^{N\kappa}\widetilde{\phi}^{(N-1)}\right\|_{L^\infty([0,T];H^s)}=\mathcal{O}\left(\varepsilon^{(N+1)\kappa}\right)=o(\varepsilon).$$

The analogue of (1.9) follows:

$$\left\|u^\varepsilon-ae^{i\phi_{\rm eik}/\varepsilon+i\phi^\varepsilon_{\rm app}}\right\|_{L^\infty([0,T];L^2\cap L^\infty)}=o(1),$$

where

$$\phi^\varepsilon_{\rm app}=\frac{\widetilde{\phi}}{\varepsilon^{1-\kappa}}+\frac{\widetilde{\phi}^{(1)}}{\varepsilon^{1-2\kappa}}+\dots+\frac{\widetilde{\phi}^{(N-1)}}{\varepsilon^{1-N\kappa}}.$$

Remark 6.2. In the case $\kappa=1$, $N=1$, and the above analysis shows that one phase shift factor appears: we retrieve the function G of Proposition 1.2 (under the unnecessary assumption $f'>0$). If $\kappa>1$, then $N=0$, and we see that $ae^{i\phi_{\rm eik}/\varepsilon}$ is a good approximation for u^ε.

Appendix A. Gross-Pitaevskii equation

To conclude, we discuss the supercritical case $\kappa=0$ for (1.1), with different assumptions concerning the initial amplitude: the initial profile is not in L^2, but in Zhidkov spaces: for $k\geq 1$, they are defined as

$$X^k(\mathbb{R}^n)=\{u\in L^\infty(\mathbb{R}^n)\ ;\ \nabla u\in H^{k-1}(\mathbb{R}^n)\}.$$

Zhidkov spaces were introduced in the one-dimensional case in [42] (see also [43]), and their study was generalized in the multidimensional case by C. Gallo [20]. They make it possible to consider solutions to (1.1) whose modulus has a non-zero limit as $|x|\to\infty$.

When the nonlinearity f is exactly $f(|z|^2)=|z|^2-1$, a natural space to study (1.1) is the energy space

$$E=\{u\in H^1_{\rm loc}(\mathbb{R}^n)\ ;\ \nabla u\in L^2(\mathbb{R}^n),\ |u|^2-1\in L^2(\mathbb{R}^n)\}.$$

Recently, P. Gérard [22] solved the Cauchy problem for the Gross-Pitaevskii equation in E. We refer to [29], where a semi-classical limit is considered with this nonlinearity.

Assumption 5. *The nonlinearity is smooth, defocusing, and cubic at the origin. The initial amplitude and its first momentum converge in Zhidkov spaces:*

- $f\in C^\infty(\mathbb{R};\mathbb{R})$, *with* $f'>0$.
- *There exists* $a_0,a_1\in X^\infty=\cap_{k\geq 1}X^k$, *such that* $xa_0,xa_1\in X^\infty$, *and:*

$$a_0^\varepsilon=a_0+\varepsilon a_1+o(\varepsilon)\ \textit{and}\ xa_0^\varepsilon=xa_0+\varepsilon xa_1+o(\varepsilon)\ \textit{in}\ X^k,\ \textit{for any}\ k\geq 1.$$

Theorem A.1. *Let Assumptions 1 and 5 be satisfied. Let* $\kappa = 0$. *Then there exists* $T_* > 0$ *independent of* $\varepsilon \in]0,1]$ *and a unique solution* $u^\varepsilon \in C([0,T_*]; X^k)$ *for any* $k > n/2+1$ *to* (1.1)-(1.2). *Moreover, there exist* $a, \phi \in C([0,T_*]; X^k)$ *for every* $k \geq 1$, *such that:*

$$\limsup_{\varepsilon \to 0} \left\| u^\varepsilon - a e^{i(\phi+\phi_{\rm eik})/\varepsilon} \right\|_{L^\infty} = \mathcal{O}(t) \quad \text{as } t \to 0.$$

There exists $\phi^{(1)} \in C([0,T_*]; X^k)$ *for every* $k \geq 1$, *real-valued, such that:*

$$\limsup_{\varepsilon \to 0} \sup_{0 \leq t \leq T_*} \left\| u^\varepsilon - a e^{i\phi^{(1)}} e^{i(\phi+\phi_{\rm eik})/\varepsilon} \right\|_{L^\infty} = 0.$$

The phase shift $\phi^{(1)}$ *is a nonlinear function of* $\phi_{\rm eik}, a_0$ *and* a_1.

Sketch of the proof. Uniqueness was established in [20]. Existence consists in resuming the computations of Section 5, working in X^k instead of H^s. The usual iterative scheme is easily adapted to this framework (see e.g. the proof in [1, Chap. III.B.1]).

We must just explain how to do without L^2-estimates of the functions. Resume the notations of Section 5. First, we have to estimate the H^{k-1}-norms of $\nabla \mathbf{u}^\varepsilon$. For this, we resume the computations of Section 5, with $1 \leq |\alpha| \leq k$. It is easy to check that every time we wrote $C(\|\mathbf{u}^\varepsilon\|_{H^s})$, what really appears is $C(\|\nabla \mathbf{u}^\varepsilon\|_{H^{s-1}})$, $C(\|\mathbf{u}^\varepsilon\|_{L^\infty})$, or $C(\|\nabla \mathbf{u}^\varepsilon\|_{L^\infty})$. The norms appearing in the first two terms can be estimated by the X^k-norm of $\mathbf{u}^\varepsilon$. For the last norm, it is controlled by $\|\nabla \mathbf{u}^\varepsilon\|_{H^s}$ for $s > n/2$ from Gagliardo-Nirenberg inequalities, and this brings us back to Zhidkov spaces. The same remains true when $\mathbf{u}^\varepsilon$ is replaced by $x\mathbf{u}^\varepsilon$. Thus, we see that we can estimate the H^{k-1}-norms of $\nabla \mathbf{u}^\varepsilon$ and $\nabla(x_j \mathbf{u}^\varepsilon)$ in Zhidkov spaces.

Finally, the L^∞-estimates for $\mathbf{u}^\varepsilon$ follow by integrating (5.2) in time and using Gronwall lemma. We proceed in a similar fashion for the equation for $x\mathbf{u}^\varepsilon$. □

Remark A.2 (Defocusing nonlinearities not necessarily cubic at the origin). We could also consider nonlinearities $f \in C^\infty(\mathbb{R};\mathbb{R})$ such that $f' > 0$ on $\mathbb{R}_+ \setminus \{0\}$, like $f(|z|^2) = |z|^4$ for instance, provided that the modulus of the initial amplitude is bounded away from zero:

$$\exists \delta > 0,\ |a_0(x)| \geq \delta,\ \forall x \in \mathbb{R}^n.$$

Such an assumption appears also in [29]. Then up to considering smaller times, and $0 < \varepsilon \leq \varepsilon_0 \ll 1$, we can assume that $|a^\varepsilon(t,x)| \geq \delta/2$. The previous proof can then be mimicked, with the same symmetrizer.

References

1. S. Alinhac and P. Gérard, *Opérateurs pseudo-différentiels et théorème de Nash-Moser*, Savoirs Actuels, InterEditions, Paris, 1991.
2. H. Bahouri and J.-Y. Chemin, *Équations d'ondes quasilinéaires et effet dispersif*, Internat. Math. Res. Notices (1999), no. 21, 1141–1178.
3. ———, *Équations d'ondes quasilinéaires et estimations de Strichartz*, Amer. J. Math. **121** (1999), no. 6, 1337–1377.
4. W. Bao, S. Jin, and P. A. Markowich, *Numerical study of time-splitting spectral discretizations of nonlinear Schrödinger equations in the semiclassical regimes*, SIAM J. Sci. Comput. **25** (2003), no. 1, 27–64.

5. N. Burq, P. Gérard, and N. Tzvetkov, *Multilinear eigenfunction estimates and global existence for the three dimensional nonlinear Schrödinger equations*, Ann. Sci. École Norm. Sup. (4) **38** (2005), no. 2, 255–301.
6. N. Burq and M. Zworski, *Instability for the semiclassical non-linear Schrödinger equation*, Comm. Math. Phys. **260** (2005), no. 1, 45–58.
7. R. Carles, *Geometric optics and instability for semi-classical Schrödinger equations*, preprint, `arXiv:math.AP/0505468`, 2005.
8. R. Carles and Y. Nakamura, *Nonlinear Schrödinger equations with Stark potential*, Hokkaido Math. J. **33** (2004), no. 3, 719–729.
9. T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics, vol. 10, New York University Courant Institute of Mathematical Sciences, New York, 2003.
10. J.-Y. Chemin, *Dynamique des gaz à masse totale finie*, Asymptotic Anal. **3** (1990), no. 3, 215–220.
11. C. Cheverry, *Cascade of phases in turbulent flows*, Bull. Soc. Math. France (2005), to appear. Preprint version: `arXiv:math.AP/0402408`.
12. C. Cheverry and O. Guès, *A step in supercritical WKB analysis and some applications*, preprint, 2005.
13. M. Christ, J. Colliander, and T. Tao, *Ill-posedness for nonlinear Schrödinger and wave equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2005), see also `arXiv:math.AP/0311048`.
14. F Dalfovo, S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Theory of Bose-Einstein condensation in trapped gases*, Rev. Mod. Phys. **71** (1999), no. 3, 463–512.
15. J. Dereziński and C. Gérard, *Scattering theory of quantum and classical N-particle systems*, Texts and Monographs in Physics, Springer Verlag, Berlin Heidelberg, 1997.
16. J. J. Duistermaat, *Oscillatory integrals, Lagrange immersions and unfolding of singularities*, Comm. Pure Appl. Math. **27** (1974), 207–281.
17. N. Dunford and J. T. Schwartz, *Linear operators. Part II: Spectral theory. Self adjoint operators in Hilbert space*, With the assistance of William G. Bade and Robert G. Bartle, Interscience Publishers John Wiley & Sons New York-London, 1963.
18. D. Fujiwara, *A construction of the fundamental solution for the Schrödinger equation*, J. Analyse Math. **35** (1979), 41–96.
19. ______, *Remarks on the convergence of the Feynman path integrals*, Duke Math. J. **47** (1980), no. 3, 559–600.
20. C. Gallo, *Schrödinger group on Zhidkov spaces*, Adv. Differential Equations **9** (2004), no. 5-6, 509–538.
21. P. Gérard, *Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire*, Séminaire sur les Équations aux Dérivées Partielles, 1992–1993, École Polytech., Palaiseau, 1993, pp. Exp. No. XIII, 13.
22. ______, *The Cauchy problem for the Gross-Pitaevskii equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire (2005), to appear.
23. E. Grenier, *Semiclassical limit of the nonlinear Schrödinger equation in small time*, Proc. Amer. Math. Soc. **126** (1998), no. 2, 523–530.
24. S. Jin, C. D. Levermore, and D. W. McLaughlin, *The semiclassical limit of the defocusing NLS hierarchy*, Comm. Pure Appl. Math. **52** (1999), no. 5, 613–654.
25. S. Kamvissis, K. D. T.-R. McLaughlin, and P. D. Miller, *Semiclassical soliton ensembles for the focusing nonlinear Schrödinger equation*, Annals of Mathematics Studies, vol. 154, Princeton University Press, Princeton, NJ, 2003.
26. E. B. Kolomeisky, T. J. Newman, J. P. Straley, and X. Qi, *Low-dimensional Bose liquids: Beyond the Gross-Pitaevskii approximation*, Phys. Rev. Lett. **85** (2000), no. 6, 1146–1149.
27. L. Landau and E. Lifschitz, *Physique théorique ("Landau-Lifchitz"). Tome III: Mécanique quantique. Théorie non relativiste*, Éditions Mir, Moscow, 1967, Deuxième édition, Traduit du russe par Édouard Gloukhian.
28. D. Lannes and J. Rauch, *Validity of nonlinear geometric optics with times growing logarithmically*, Proc. Amer. Math. Soc. **129** (2001), no. 4, 1087–1096.

29. F. Lin and P. Zhang, *Semiclassical limit of the Gross-Pitaevskii equation in an exterior domain*, Arch. Rational Mech. Anal. **179** (2005), no. 1, 79–107.
30. A. Majda, *Compressible fluid flow and systems of conservation laws in several space variables*, Applied Mathematical Sciences, vol. 53, Springer-Verlag, New York, 1984.
31. T. Makino, S. Ukai, and S. Kawashima, *Sur la solution à support compact de l'équation d'Euler compressible*, Japan J. Appl. Math. **3** (1986), no. 2, 249–257.
32. P. D. Miller, *Asymptotics of semiclassical soliton ensembles: rigorous justification of the WKB approximation*, Int. Math. Res. Not. (2002), no. 8, 383–454.
33. L. Pitaevskii and S. Stringari, *Bose-Einstein condensation*, International Series of Monographs on Physics, vol. 116, The Clarendon Press Oxford University Press, Oxford, 2003.
34. J. Rauch and M. Keel, *Lectures on geometric optics*, Hyperbolic equations and frequency interactions (Park City, UT, 1995), Amer. Math. Soc., Providence, RI, 1999, pp. 383–466.
35. M. Reed and B. Simon, *Methods of modern mathematical physics. II. Fourier analysis, self-adjointness*, Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1975.
36. J. T. Schwartz, *Nonlinear functional analysis*, Gordon and Breach Science Publishers, New York, 1969, Notes by H. Fattorini, R. Nirenberg and H. Porta, with an additional chapter by Hermann Karcher, Notes on Mathematics and its Applications.
37. C. Sulem and P.-L. Sulem, *The nonlinear Schrödinger equation, self-focusing and wave collapse*, Springer-Verlag, New York, 1999.
38. F.-R. Tian and J. Ye, *On the Whitham equations for the semiclassical limit of the defocusing nonlinear Schrödinger equation*, Comm. Pure Appl. Math. **52** (1999), no. 6, 655–692.
39. A. Tovbis, S. Venakides, and X. Zhou, *On semiclassical (zero dispersion limit) solutions of the focusing nonlinear Schrödinger equation*, Comm. Pure Appl. Math. **57** (2004), no. 7, 877–985.
40. K. Yajima and G. Zhang, *Smoothing property for Schrödinger equations with potential superquadratic at infinity*, Comm. Math. Phys. **221** (2001), no. 3, 573–590.
41. ———, *Local smoothing property and Strichartz inequality for Schrödinger equations with potentials superquadratic at infinity*, J. Differential Equations **202** (2004), no. 1, 81–110.
42. P. E. Zhidkov, *The Cauchy problem for a nonlinear Schrödinger equation*, JINR Commun., P5-87-373, Dubna (1987), (in Russian).
43. ———, *Korteweg-de Vries and nonlinear Schrödinger equations: qualitative theory*, Lecture Notes in Mathematics, vol. 1756, Springer-Verlag, Berlin, 2001.

MAB, Université de Bordeaux 1, 351 cours de la Libération, 33405 Talence cedex, France

E-mail address: Remi.Carles@math.cnrs.fr