Traveling Waves for Delayed Reaction-Diffusion Equations with Global Response

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Abstract: We develop a new approach to obtain the existence of traveling wave solutions for reaction diffusion equations with delayed non-local response. The approach is based on an abstract formulation of the wave profile as a solution of an operational equation in a certain Banach space, coupled with an index formula of the associated Fredholm operator and some careful estimation of the nonlinear perturbation. The general result relates the existence of traveling wave solutions to the existence of heteroclinic connecting orbits of a corresponding functional differential equation, and this result is illustrated by an application to a model describing the population growth when the species has two age classes and the diffusion of the individual during the maturation process leads to an interesting non-local and delayed response for the matured population.

1. Introduction

The purpose of this paper is to study the existence of traveling wave solutions for the following delayed reaction-diffusion equation with nonlocal interaction:

$$\frac{\partial u(x,t)}{\partial t} = D\Delta u(x,t) + F\Big(u(x,t), \int_{-r}^{0}\int_{\Omega} d\eta(\theta)d\mu(y)g\big(u(x+y,t+\theta)\big)\Big), \qquad (1.1)$$

where $x \in \mathbb{R}^m$ is the spatial variable, $t \geq 0$ is the time, $u(x,t) \in \mathbb{R}^n$, $D = \mathrm{diag}(d_1, \cdots, d_n)$ with positive constants d_i, $i = 1, \cdots, n$, $\Delta = \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2}$ is the Laplacian operator, r is a positive constant, $\eta : [-r, 0] \to \mathbb{R}^{n\times n}$ is of bounded variation, μ is a bounded measure on $\Omega \subset \mathbb{R}^m$ with values in $\mathbb{R}^{n\times n}$, $F : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $g : \mathbb{R}^n \to \mathbb{R}^n$ are given mappings with additional conditions to be specified later.

Equation (1.1) serves as a model for many physical, chemical, ecological, and biological problems. In particular, as will be shown in Section 6, Eq. (1.1) includes a model for the population growth where the species has an age-structure and a non-monotone birth function, and the spatial diffusion of the individuals during the maturation period leads to an interesting non-local delayed response. See Britton [1] and Gourley and Britton [9] for some earlier work on non-local delayed reaction diffusion equations.

Because of their significant role in governing the long time behavior of dynamical systems with a diffusion process, traveling wave solutions have been one of lasting interests, and a variety of methods for studying the existence of traveling wave solutions have been developed. In this paper, we develop a new approach to study the existence of traveling wave solutions for Eq. (1.1). This approach reflects a natural connection between the existence of a traveling wave solution for Eq. (1.1) and the existence of a heteroclinic solution for the corresponding ordinary delay differential equation on $\mathbb{R}^n$

$$\dot{u}(t) = F\Big(u(t), \int_{-r}^{0} d\eta(\theta)\mu_\Omega g\big(u(t+\theta)\big)\Big), \tag{1.2}$$

where $\mu_\Omega = \int_\Omega d\mu$.

Before giving a precise statement of our main result, we first formulate some assumptions about the nonlinearities F and g. Throughout the remaining part of this paper, we suppose that F and g are C^k-smooth functions, $k \geq 2$, and we let $F_u(u,v)$, $F_v(u,v)$ denote the partial derivatives of F with respect to the variables $u \in \mathbb{R}^n$ and $v \in \mathbb{R}^n$, respectively, and let $g_u(u)$ be the derivative of g with respect to the variable $u \in \mathbb{R}^n$. In addition, we suppose that Eq. (1.2) has two equilibria E_i, $i = 1, 2$, and we define

$$A_i = F_u\big(E_i, \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(E_i)\big), \quad B_i = F_v\big(E_i, \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(E_i)\big).$$

For a complex number λ we let

$$\Lambda_i(\lambda) = \det\Big[\lambda I - A_i - B_i \int_{-r}^{0} d\eta(\theta)\mu_\Omega g_u(E_i)e^{\lambda\theta}\Big].$$

We assume that the following hypotheses hold:

(H1) All eigenvalues corresponding to the equilibrium E_2 have negative real parts, that is, $\sup\{\Re\lambda : \ \Lambda_2(\lambda) = 0\} < 0$.

(H2) E_1 is hyperbolic and the unstable manifold at the equilibrium E_1 is M $(M \geq 1)$ dimensional. In other words, $\Lambda_1(iv) \neq 0$ for all $v \in \mathbb{R}$ and $\Lambda_1(\lambda) = 0$ has exactly M roots with positive real parts, where the multiplicities are taken into account.

(H3) Eq. (1.2) has a heteroclinic solution $u^* : \mathbb{R} \to \mathbb{R}^n$ from E_1 to E_2. Namely, Eq. (1.2) has a solution $u^*(t)$ defined for all $t \in \mathbb{R}$ such that

$$u^*(-\infty) := \lim_{t\to-\infty} u^*(t) = E_1, \quad u^*(\infty) := \lim_{t\to\infty} u^*(t) = E_2.$$

(H4) $\big\| \int_\Omega d|\mu|(y)\|y\|_{\mathbb{R}^m} \big\|_{\mathbb{R}^{n\times n}} < \infty$, where $|\mu| = \mu^+ - \mu^-$ with μ^+ and μ^- the positive and negative parts of μ, respectively.

Our main result is as follows:

Theorem 1.1 *Under assumptions* **(H1)**–**(H4)**, *there is a* $c^* > 0$ *such that*

(i) *for each fixed unit vector* $\nu \in \mathbb{R}^m$ *and* $c > c^*$, *Eq. (1.1) has a traveling wave solution* $u(x,t) = U(\nu \cdot x + ct)$ *connecting* E_1 *to* E_2 *(that is,* $U(-\infty) = E_1$ *and* $U(\infty) = E_2$*);*

(ii) *if restricted to a small neighborhood of the heteroclinic solution* $u^* : \mathbb{R} \to \mathbb{R}^n$ *in the space* $C(\mathbb{R}, \mathbb{R}^n)$ *of bounded continuous functions equipped with the sup-norm, then for each fixed* $c > c^*$ *and* $\nu \in \mathbb{R}^m$, *the set of all traveling wave solutions connecting* E_1 *to* E_2 *in this neighborhood forms a* M*-dimensional manifold* $\mathcal{M}_\nu(c)$*;*

(iii) $\mathcal{M}_\nu(c)$ *is a* C^{k-1}*-smooth manifold which is also* C^{k-1}*-smooth with respect to* c*. More precisely, there is a* C^{k-1}*-function* $h : U \times (c^*, \infty) \to C(\mathbb{R}, \mathbb{R}^n)$, *where* U *is an open set in* $\mathbb{R}^M$, *such that* $\mathcal{M}_\nu(c)$ *has the form*

$$\mathcal{M}_\nu(c) = \{\psi : \psi = h(z, c),\ z \in U\}.$$

Let $\nu \cdot x + ct = s \in \mathbb{R}$ and $u(x,t) = U(\nu \cdot x + ct)$. Then, upon a straightforward substitution, a traveling wave $U(s)$ satisfies the second order equation

$$c\dot{U}(s) = D\ddot{U}(s) + F\Big(U(s), \int_{-r}^{0}\int_{\Omega} d\eta(\theta)d\mu(y)g\big(U(s + \nu \cdot y + c\theta)\big)\Big), \quad s \in \mathbb{R}. \quad (1.3)$$

Writing $V(s) = U(cs)$ and $\epsilon = 1/c^2$, then (1.3) leads to

$$\dot{V}(s) = \epsilon D\ddot{V}(s) + F\Big(V(s), \int_{-r}^{0}\int_{\Omega} d\eta(\theta)d\mu(y)g\big(V(s + \sqrt{\epsilon}\nu \cdot y + \theta)\big)\Big), \quad s \in \mathbb{R}. \quad (1.4)$$

In the case where c is sufficiently large, ϵ is small and hence Eq. (1.4) is a singularly perturbed equation. Such an equation has been extensively investigated via both geometric and analytic methods where the main idea is to study the corresponding slow motion and fast motion. See, for example, Carpenter [2], Fenichel [6,7], Fife [8], Hoppensteadt [12], Jones [13], Lin [15], and Szmolyan [26]. The geometrical approach makes the connection of slow and fast motions by studying the intersection of the relevant invariant manifolds, while the analytic approach matches the slow and fast motion by using the asymptotic expansion of inner and outer layers. For both methods, to make a connection between slow and fast motions is far from being trivial. In addition, both methods work only on dynamical systems where the stable, unstable, and invariant manifolds play an essential role. It is very important to point out that the differential equation (1.4) does not generate a dynamical system, for there is no way an initial value problem can be formulated. In this paper, we take a different approach to avoid this difficulty. The central idea of our approach is to use a certain type of transformation to convert the singularly perturbed differential equation (1.4) into a regularly perturbed operational equation in a Banach space, that enables us to directly apply the Banach fixed point theorem and some existing results regarding the index of an associated Fredholm operator to prove the existence of traveling wave solutions. This approach also allows us to determine the number of traveling wave solutions as well as smooth dependence of traveling wave solutions on the wave speed c.

Theorem 1.1, relating the existence of traveling wave fronts for the reaction diffusion equation (1.1) with delay and non-local interaction to the existence of a connecting orbit between two hyperbolic equilibria of the associated ordinary delay differential equation (1.2), enables us to apply some existing results for invariant curves of semiflows generated by ordinary delay differential equations to derive systematically sharp sufficient conditions for the existence of traveling wave fronts of delayed reaction diffusion equations that, in turn, includes most of the existing results in the literature as special cases. In particular, as will be illustrated in Section 6 where a recently derived non-local delayed reaction diffusion equation for the population growth of a single species when the delayed birth function is not monotone in the considered range is considered, Theorem 1.1 allows us to apply the powerful monotone dynamical systems theory to obtain the existence of traveling waves.

This paper is organized as follows. In Section 2 we transform Eq. (1.4) into an operational integral equation involving a linear operator and a nonlinear perturbation. Section 3 is devoted to the study of the null space and range of the linear operator introduced in Section 2. The properties of the nonlinear function in the operational equation are studied in Section 4. The proof of our main theorem is given in Section 5. In the last section, we present applications of our main result to some population models, including a non-local delayed RD-system with non-monotone birth functions.

2. Operational Equations for Traveling Wave Solutions

In the sequel, we use more compact notations:

$$\zeta(\theta, y) = \eta(\theta)\mu(y), \qquad \int_{\Omega_r} d\zeta(\theta, y) = \int_{-r}^{0}\int_{\Omega} d\eta(\theta)d\mu(y),$$

with $\Omega_r = [-r, 0] \times \Omega$. We will also let $C = C(\mathbb{R}, \mathbb{R}^n)$ be the space of continuous and bounded functions from $\mathbb{R}$ to $\mathbb{R}^n$ equipped with the standard norm $\|\psi\|_C = \sup\{\|\psi(t)\| : t \in \mathbb{R}\}$.

Our main approach to study the existence of traveling wave solutions is to convert the differential equation for a traveling wave into an equivalent operational equation in a suitable Banach space. For this purpose, we further transform Eq. (1.4) by introducing the variable $w(s) = V(s) - u^*(s)$ for $s \in \mathbb{R}$. Then we obtain the equation for w as

$$\begin{aligned} \dot{w}(s) &= \epsilon D\ddot{w}(s) + \epsilon D\ddot{u}^*(s) \\ &\qquad + F\Big(w(s) + u^*(s), \int_{\Omega_r} d\zeta(\theta, y) g\big([w + u^*](s + \sqrt{\epsilon}\nu \cdot y + \theta)\big)\Big) \\ &\qquad - F\Big(u^*(s), \int_{\Omega_r} d\zeta(\theta, y) g\big(u^*(s + \theta)\big)\Big) \\ &= \epsilon D\ddot{w}(s) + P^0 w(s) + \mathcal{G}(\epsilon, s, w), \qquad\qquad s \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $[w + u^*](t) = w(t) + u^*(t)$ for $t \in \mathbb{R}$, and the linear operator $P^0 : C \to C$ is defined by

$$P^0 w(s) = A(s)w(s) + B(s) \int_{\Omega_r} d\zeta(\theta, y) g_u\big(u^*(s + \theta)\big) w(s + \theta), \quad s \in \mathbb{R}, \tag{2.2}$$

with

$$A(s) = F_u\Big(u^*(s), \int_{\Omega_r} d\zeta(\theta, y) g\big(u^*(s + \theta)\big)\Big), \quad s \in \mathbb{R}, \tag{2.3}$$

$$B(s) = F_v\Big(u^*(s), \int_{\Omega_r} d\zeta(\theta, y) g\big(u^*(s + \theta)\big)\Big), \quad s \in \mathbb{R}, \tag{2.4}$$

and

$$\begin{aligned} &\mathcal{G}(\epsilon, s, w) \\ &\quad = F\Big(w(s) + u^*(s), \int_{\Omega_r} d\zeta(\theta, y) g\big([w + u^*](s + \sqrt{\epsilon}\nu \cdot y + \theta)\big)\Big) \\ &\qquad - F\Big(u^*(s), \int_{\Omega_r} d\zeta(\theta, y) g\big(u^*(s + \theta)\big)\Big) - P^0 w(s) + \epsilon D\ddot{u}^*(s). \end{aligned} \tag{2.5}$$

Next we transform Eq. (2.1) into an integral equation as follows. We first write (2.1) as

$$\epsilon d_i \ddot{w}_i(s) - \dot{w}_i(s) - w_i(s) = -w_i(s) - P_i^0 w(s) - \mathcal{G}_i(\epsilon, s, w), \quad s \in \mathbb{R}, \tag{2.6}$$

for $i = 1, \cdots, n$, where i denotes the ith component for the corresponding functions or operators. We observe that the equation

$$\epsilon d_i z^2 - z - 1 = 0$$

has two real zeros α_i^ϵ and β_i^ϵ, with

$$-1 < \alpha_i^\epsilon = \frac{1 - \sqrt{1 + 4\epsilon d_i}}{2\epsilon d_i} < 0, \qquad \beta_i^\epsilon = \frac{1 + \sqrt{1 + 4\epsilon d_i}}{2\epsilon d_i} > 0.$$

Moreover, it is easy to verify that

$$\lim_{\epsilon\to 0^+}\alpha_i^\epsilon=-1,\qquad \lim_{\epsilon\to 0^+}\beta_i^\epsilon=+\infty. \tag{2.7}$$

It is well known that $w:\mathbb{R}\to\mathbb{R}^n$ is a bounded solution of (2.6) if and only if $w(s)$ is a bounded solution of the integral equation

$$\begin{aligned}
w_i(s) &= \frac{1}{\epsilon d_i(\beta_i^\epsilon-\alpha_i^\epsilon)}\int_{-\infty}^{s} e^{\alpha_i^\epsilon(s-t)}[w_i(t)+P_i^0 w(t)]dt\\
&\quad+\frac{1}{\epsilon d_i(\beta_i^\epsilon-\alpha_i^\epsilon)}\int_{s}^{\infty} e^{\beta_i^\epsilon(s-t)}[w_i(t)+P_i^0 w(t)]dt\\
&\quad+\frac{1}{\epsilon d_i(\beta_i^\epsilon-\alpha_i^\epsilon)}\Big(\int_{-\infty}^{s} e^{\alpha_i^\epsilon(s-t)}\mathcal{G}_i(\epsilon,t,w)dt+\int_{s}^{\infty} e^{\beta_i^\epsilon(s-t)}\mathcal{G}_i(\epsilon,t,w)dt\Big)\\
&=\frac{1}{\sqrt{1+4\epsilon d_i}}\int_{-\infty}^{s} e^{\alpha_i^\epsilon(s-t)}[w_i(t)+P_i^0 w(t)]dt\\
&\quad+\frac{1}{\sqrt{1+4\epsilon d_i}}\int_{s}^{\infty} e^{\beta_i^\epsilon(s-t)}[w_i(t)+P_i^0 w(t)]dt\\
&\quad+\frac{1}{\sqrt{1+4\epsilon d_i}}\Big(\int_{-\infty}^{s} e^{\alpha_i^\epsilon(s-t)}\mathcal{G}_i(\epsilon,t,w)dt+\int_{s}^{\infty} e^{\beta_i^\epsilon(s-t)}\mathcal{G}_i(\epsilon,t,w)dt\Big),\\
&\quad i=1,\cdots,n.
\end{aligned} \tag{2.8}$$

Therefore, w is a bounded solution of (2.6) if and only if it solves

$$w(s)-\int_{-\infty}^{s} e^{-(s-t)}[w(t)+P^0 w(t)]dt=\mathcal{H}(s,w,\epsilon), \tag{2.9}$$

where $\mathcal{H}(s,w,\epsilon)=(\mathcal{H}_1(s,w,\epsilon),\cdots,\mathcal{H}_n(s,w,\epsilon))$ is defined as

$$\begin{aligned}
&\mathcal{H}_i(s,w,\epsilon)\\
&\quad=\int_{-\infty}^{s}\Big[\frac{e^{\alpha_i^\epsilon(s-t)}}{\sqrt{1+4\epsilon d_i}}-e^{-(s-t)}\Big]\big[w_i(t)+P_i^0 w(t)\big]dt\\
&\quad+\frac{1}{\sqrt{1+4\epsilon d_i}}\int_{s}^{\infty} e^{\beta_i^\epsilon(s-t)}[w_i(t)+P_i^0 w(t)]dt\\
&\quad+\frac{1}{\sqrt{1+4\epsilon d_i}}\Big(\int_{-\infty}^{s} e^{\alpha_i^\epsilon(s-t)}\mathcal{G}_i(\epsilon,t,w)dt+\int_{s}^{\infty} e^{\beta_i^\epsilon(s-t)}\mathcal{G}_i(\epsilon,t,w)dt\Big)
\end{aligned} \tag{2.10}$$

for $i=1,\cdots,n$.

In summary, we show that Eq. (1.3) has a solution $U:\mathbb{R}\to\mathbb{R}^n$ connecting E_1 to E_2 if and only if Eq. (2.9) has a solution w such that $\lim_{|s|\to\infty}w(s)=0$. Finally, we let L be the linear operator defined on the left hand side of Eq. (2.9), namely

$$[Lw](s)=w(s)-\int_{-\infty}^{s} e^{-(s-t)}[w(t)+P^0 w(t)]dt,\quad s\in\mathbb{R}. \tag{2.11}$$

Then we can write Eq. (2.9) as the operational equation

$$[Lw](s) = \mathcal{H}(s, w, \epsilon), \quad s \in \mathbb{R}. \tag{2.12}$$

So our goal is to show the existence of solutions of Eq. (2.12). We shall achieve this by using the Banach fixed point theorem. For this purpose, we need further detailed properties of the nonlinear function $\mathcal{H}$ and the linear operator L. In the next section, we shall show that, with an appropriate choice of the Banach space, the operator L is surjective, an essential property required in the proof of our main theorem.

3. The Kernel and Range of the Operator L

Let us first introduce some additional notations.

(i). For a vector $x \in \mathbb{R}^n$, $\|x\| = \|x\|_{\mathbb{R}^n}$, and for an $n \times n$ matrix A, $\|A\| = \|A\|_{\mathbb{R}^{n\times n}}$ denotes the norm of A as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^n$.

(ii). For a continuous function $w : [a-r, b] \to \mathbb{R}^n$, as usual we let $w_t \in C([-r, 0], \mathbb{R}^n)$, $t \in [a, b]$, be defined by $w_t(\theta) = w(t + \theta)$ for $\theta \in [-r, 0]$. Moreover, for $f \in C([-r, 0], \mathbb{R}^n)$ we denote the norm of f by $\|f\| = \sup_{\theta\in[-r,0]} \|f(\theta)\|$.

(iii). In a similar fashion, for a function $h : [a, b + r] \to \mathbb{R}^n$ we define the function $h^t : [0, r] \to \mathbb{R}^n$ by $h^t(\theta) = h(t + \theta)$ for $\theta \in [0, r]$ and $t \in [a, b]$.

(iv). Let $C^1 = C^1(\mathbb{R}, \mathbb{R}^n) = \{\psi \in C : \dot{\psi} \in C\}$ be the Banach space equipped with the standard norm $\|\psi\|_{C^1} = \|\psi\|_C + \|\dot{\psi}\|_C$.

(v). Let $C_0 = \{\psi \in C : \lim_{t\to\pm\infty} \psi(t) = 0\}$ and $C_0^1 = \{\psi \in C_0 : \dot{\psi} \in C_0\}$ equipped with the same norms as C and C^1, respectively.

Let $T : C^1 \to C$ be the linear operator obtained from the linearization of (1.2) around the heteroclinic solution u^*. That is,

$$(T\psi)(t) = \dot{\psi}(t) - P(t)\psi_t, \quad t \in \mathbb{R}, \tag{3.1}$$

where for $t \in \mathbb{R}$ the linear operator $P(t) : C([-r, 0], \mathbb{R}^n) \to \mathbb{R}^n$ is defined by

$$P(t)\xi = A(t)\xi(0) + B(t) \int_{-r}^{0} d\eta(\theta)\mu_\Omega g_u\big(u^*(t + \theta)\big)\xi(\theta), \tag{3.2}$$

with $A(t)$ and $B(t)$ defined in (2.3), (2.4). We remark that $P^0\psi(t) = P(t)\psi_t$ for $\psi \in C$ and $t \in \mathbb{R}$. Since $u^*(t) \to E_1$ and E_2 as $t \to -\infty$ and $+\infty$, respectively, we have

$$\begin{aligned} \lim_{t\to\infty} A(t) = A_2, &\quad \lim_{t\to\infty} B(t) = B_2, \\ \lim_{t\to-\infty} A(t) = A_1, &\quad \lim_{t\to-\infty} B(t) = B_1. \end{aligned} \tag{3.3}$$

Hypotheses **(H1)**, **(H2)**, and (3.3) imply that the linear operator T is asymptotically hyperbolic as $t \to \pm\infty$ in the sense of p. 12 of Mallet-Paret [16]. That is, the linear delay differential equations

$$\dot{\psi}(t) - P(+\infty)\psi_t = 0 \quad \text{and} \quad \dot{\psi}(t) - P(-\infty)\psi_t = 0,$$

where $P(+\infty), P(-\infty)$ are the limiting operators defined in the obvious way, are hyperbolic. We define the formal adjoint equation of $T\psi = 0$ as

$$\dot{\phi}(t) = -P^*(t)\phi^t, \quad t \in \mathbb{R}, \tag{3.4}$$

where for $\xi \in C([0,r], \mathbb{R}^n)$,

$$P^*(t)\xi = A^T(t)\xi(0) + \int_{-r}^{0} g_u^T\big(u^*(t)\big)\mu_\Omega^T d\eta^T(\theta)B^T(t-\theta)\xi(-\theta),$$

and for a matrix H, H^T denotes the transpose of H.

Lemma 3.1 If $\phi \in C$ is a solution of (3.4) and ϕ is C^1-smooth, then $\phi = 0$.

Proof. Let ϕ be a bounded solution of (3.4) and $h(t) = \phi(-t)$ for $t \in \mathbb{R}$. Then

$$\begin{aligned} \dot{h}(t) &= A^T(-t)h(t) + \int_{-r}^{0} g_u^T\big(u^*(-t)\big)\mu_\Omega^T d\eta^T(\theta)B^T(-t-\theta)h(t+\theta) \\ &:= Q(t)h_t. \end{aligned} \tag{3.5}$$

The limiting equation of (3.5) as $t \to -\infty$ is

$$\begin{aligned} \dot{\xi}(t) &= A_2^T\xi(t) + \int_{-r}^{0} g_u^T(E_2)\mu_\Omega^T d\eta^T(\theta)B_2^T\xi(t+\theta) \\ &:= Q(-\infty)\xi_t. \end{aligned} \tag{3.6}$$

Since the linear delay differential equation (3.6) and the linear delay differential equation

$$\dot{\zeta}(t) = A_2\zeta(t) + B_2\int_{-r}^{0} d\eta(\theta)\mu_\Omega g_u(E_2)\zeta(t+\theta)$$

share the same eigenvalues, we conclude that all eigenvalues of Eq. (3.6) have negative real parts by assumption **(H1)**. Let $\{J(t)\}_{t\geq 0}$ be the semigroup generated by the solutions of Eq. (3.6), that is, $J(t) : C([-r,0], \mathbb{R}^n) \to C([-r,0], \mathbb{R}^n)$ and $J(t)\xi_0$ is the solution of (3.6) with initial condition $\xi(\theta) = \xi_0(\theta)$ for $\theta \in [-r,0]$. Moreover, let $Z(t) : [0,\infty) \to \mathbb{R}^{n\times n}$ be the matrix solution of Eq. (3.6) with initial condition

$$Z(\theta) = \begin{cases} I, & \text{for } \theta = 0, \\ 0, & \text{for } \theta \in [-r,0), \end{cases}$$

where I is the $n \times n$ identity matrix. Then there are positive constants $\gamma > 0$ and $a > 0$ such that

$$\|J(t)\xi_0\| \le \gamma e^{-at}\|\xi_0\|, \quad \|Z(t)\| \le \gamma e^{-at}, \quad t \ge 0, \xi_0 \in C([-r,0], \mathbb{R}^n). \tag{3.7}$$

Let $\delta > 0$ be such that $\delta\gamma e^{ar} < a$. Since $Q(t) \to Q(-\infty)$ as $t \to -\infty$, there is a t^* such that

$$\|Q(t) - Q(-\infty)\| \le \delta, \quad t \le t^*. \tag{3.8}$$

Now we write (3.5) as

$$\dot{h}(t) = Q(-\infty)h_t + [Q(t) - Q(-\infty)]h_t. \tag{3.9}$$

By the variation of constants formula (see (2.2) on p.173 of Hale and Lunel [11]) solutions of (3.9) can be expressed as

$$h_t(\theta) = [J(t-s)h_s](\theta) + \int_s^{t+\theta} Z(t+\theta-\tau)[Q(\tau) - Q(-\infty)]h_\tau d\tau, \quad s \le t \tag{3.10}$$

for $\theta \in [-r, 0]$. Note that $\theta \le 0$ and $Z(\tau) = 0$ for $\tau < 0$. From (3.7), (3.8) and (3.10) we obtain

$$\|h_t\| \le \gamma e^{-a(t-s)}\|h_s\| + \delta\gamma e^{ar} \int_s^t e^{-a(t-\tau)}\|h_\tau\| d\tau \tag{3.11}$$

for $s \le t \le t^*$. Or equivalently,

$$e^{at}\|h_t\| \le \gamma e^{as}\|h_s\| + \delta\gamma e^{ar} \int_s^t e^{a\tau}\|h_\tau\| d\tau. \tag{3.12}$$

The Gronwall inequality applied to (3.12) yields that

$$e^{at}\|h_t\| \le \gamma e^{as}\|h_s\| e^{\delta\gamma e^{ar}(t-s)}.$$

From the last inequality we have

$$\|h_t\| \le \gamma e^{-(a-\delta\gamma e^{ar})(t-s)}\|h_s\|, \quad s \le t \le t^*. \tag{3.13}$$

Note that h_s is bounded. By letting $s \to -\infty$ in (3.13), we immediately have

$$\|h_t\| = 0, \quad t \le t^*.$$

Then the uniqueness of the solution of (3.9) implies that $h_t = 0$ for all $t \in \mathbb{R}$ and hence $\phi = 0$. ∎

Lemma 3.2 *$\mathcal{R}(T) = C$ and $\dim\mathcal{N}(T) = M$, where $\mathcal{R}(T)$ and $\mathcal{N}(T)$ denote the range and null space of T, respectively.*

Proof. It follows from assumptions **(H1)-(H2)** that the operator T is Fredholm (see p.7 of Chow, Lin, and Mallet-Paret [4]). Furthermore,

$$\begin{aligned} \mathrm{ind}T &= \dim\mathcal{N}(T) - \mathrm{codim}\mathcal{R}(T) \\ &= \text{dimension of unstable manifold of } E_1 \\ &\quad - \text{dimension of unstable manifold of } E_2 \\ &= M - 0 \\ &= M. \end{aligned} \tag{3.14}$$

Moreover, we have

$$\mathcal{R}(T) = \left\{ \psi \in C : \int_{-\infty}^{\infty} h(t)\psi(t)dt = 0 \text{ for every bounded solution } h(\cdot) \text{ of Eq.(3.4)} \right\}.$$

With the use of Lemma 3.1, one concludes that $\mathcal{R}(T) = C$ and hence $\mathrm{codim}\mathcal{R}(T) = 0$. Therefore (3.14) implies that $\dim\mathcal{N}(T) = M$. ∎

Lemma 3.3 *Let $y \in C_0$ be given. If ϕ is a bounded solution of the equation $T\phi = y$, then $\phi \in C_0^1$. In particular, $T\phi = 0$ implies that $\phi \in C_0^1$ and hence, $\mathcal{N}(T) \subset C_0^1$.*

Proof. We shall only prove $\lim_{t\to-\infty}\phi(t) = 0$. The convergence of $\phi(t)$ to 0 as $t \to \infty$ can be proved analogously. By the definition of the operator T, $T\phi = y$ implies that

$$\dot{\phi}(t) = P(t)\phi_t + y(t), \quad t \in \mathbb{R},$$

or

$$\dot{\phi}(t) = P(-\infty)\phi_t + z(t), \quad t \in \mathbb{R}, \tag{3.15}$$

with $z(t) = [P(t) - P(-\infty)]\phi_t + y(t)$. Consider the homogeneous equation

$$\dot{\phi}(t) = P(-\infty)\phi_t. \tag{3.16}$$

Recall that for $\xi \in C([-r, 0], \mathbb{R}^n)$,

$$P(-\infty)\xi = A_1\xi(0) + B_1 \int_{-r}^{0} d\tilde{\eta}(\theta)\xi(\theta),$$

where $\tilde{\eta}(\theta) = \eta(\theta)\mu_\Omega g_u(E_1)$, $\theta \in [-r, 0]$. By assumption **(H2)**, the generalized eigenfunction space U of Eq. (3.16) corresponding to eigenvalues with positive real part is M-dimensional. Let $\Phi = (\Phi^1, \cdots, \Phi^M)$ be a basis of U and $\Psi = (\Psi^1, \cdots, \Psi^M)^T$ be a basis of the generalized eigenfunction space of the formal adjoint equation of Eq. (3.16) associated with U, satisfying

$$(\Psi, \Phi) = [(\Psi^i, \Phi^j)]_{M\times M} = I,$$

where for $\xi \in C([-r,0], \mathbb{R}^n)$ and $\psi \in C([0,r], \mathbb{R}^n)$, (ξ, ψ) is defined by

$$(\psi, \xi) = \psi^T(0)\xi(0) - \int_{-r}^{0} \psi^T(\tau - \theta) B_1 d\tilde{\eta}(\theta)\xi(\tau)d\tau.$$

Let $K(t) : C([-r,0], \mathbb{R}^n) \to C([-r,0], \mathbb{R}^n), t \geq 0$, be the semigroup generated by solutions of Eq. (3.16). Define projections $K^U, K^S = (I - K^U) : C([-r,0], \mathbb{R}^n) \to C([-r,0], \mathbb{R}^n)$ with

$$K^U\xi = \Phi(\Psi, \xi), \quad \xi \in C([-r,0], \mathbb{R}^n).$$

Then there are positive constants $\alpha > 0$ and $\beta > 0$ such that for $\xi \in C([-r,0], \mathbb{R}^n)$

$$\|K(t)K^S\xi\| \leq \beta e^{-\alpha t}\|\xi\|, \quad t \geq 0, \tag{3.17}$$

$$\|K(t)K^U\xi\| \leq \beta e^{\alpha t}\|\xi\|, \quad t \leq 0, \tag{3.18}$$

where for $t \leq 0$, $K(t) : \mathcal{R}(K^U) \to \mathcal{R}(K^U)$ is the inverse of $K(-t)|_{\mathcal{R}(K^U)}$. Now let $\phi(t)$ be a bounded solution on $\mathbb{R}$ of Eq. (3.15). Then $\phi_t = K^U\phi_t + K^S\phi_t$. By the variation-of-constants formula (see p. 226-228 of Hale and Lunel [11]), we have

$$K^U\phi_t = K(t-s)K^U\phi_s + \int_s^t K(t-\tau)\Phi[\Psi(0)z(\tau)]d\tau, \quad t \geq s, \tag{3.19}$$

$$K^S\phi_t = K(t-s)K^S\phi_s + \int_s^t d_\tau\big[Y(t,\tau)^S\big]z(\tau), \qquad t \geq s, \tag{3.20}$$

where $Y(t,\tau)^S$ is defined as follows (see (9.10) on p. 228 of Hale and Lunel [11]):

$$\begin{aligned} Y(t,\tau)^S &= \int_{t-r-\tau}^{t-r} K(\theta)\big[X_r - \Phi(\Psi, X_r)\big]d\theta, \quad \text{if } \tau \leq t-r, \\ Y(t,\tau)^S &= \int_0^{t-r} K(\theta)\big[X_r - \Phi(\Psi, X_r)\big]d\theta \\ &\quad + \int_{t-r-\tau}^{0} X_{r+\theta}d\theta - \Phi\Big(\Psi, \int_{t-r-\tau}^{0} X_{r+\theta}d\theta\Big), \quad \text{if } \tau > t-r. \end{aligned} \tag{3.21}$$

Here we suppose $t-r \geq s$, and $X(t), t \geq -r$, is the matrix solution of the homogeneous equation (3.16) with initial condition $X(0)=I$ and $X(\theta)=0$ for $\theta \in [-r,0)$. Applying $K(s-t)$, the inverse of $K(t-s)$ on $\mathcal{R}(K^U)$, to (3.19) we obtain

$$\begin{aligned} K(s-t)K^U\phi_t &= K^U\phi_s + \int_s^t K(s-t)K(t-\tau)\Phi[\psi(0)z(\tau)]d\tau \\ &= K^U\phi_s + \int_s^t K(s-\tau)\Phi[\Psi(0)z(\tau)]d\tau, \quad t \geq s, \end{aligned}$$

or,

$$K^U\Phi_s = K(s-t)K^U\phi_t - \int_s^t K(s-\tau)\Phi[\Psi(0)z(\tau)]d\tau, \quad t \geq s. \tag{3.22}$$

Therefore, (3.18) and (3.22) imply that

$$\begin{aligned} \|K^U\phi_s\| &\leq \beta e^{\alpha(s-t)}\|\phi_t\| + \beta\int_s^t e^{\alpha(s-\tau)}\|\Phi[\Psi(0)z(\tau)]\|d\tau, \\ &\leq \beta e^{\alpha(s-t)}\|\phi_t\| + \beta\int_s^t e^{\alpha(s-\tau)}d\tau \sup_{s\leq\tau\leq t}\big\{\|\Phi[\Psi(0)z(\tau)]\|\big\}d\tau \\ &= \beta e^{a(s-t)}\|\phi_t\| + \frac{\beta}{\alpha}\big(1-e^{\alpha(s-t)}\big)\sup_{s\leq\tau\leq t}\big\{\|\Phi[\Psi(0)z(\tau)]\|\big\}. \end{aligned} \tag{3.23}$$

Since $\|\phi_t\|$ is bounded for $t \in \mathbb{R}$, by letting $s \to -\infty$ in (3.23), we obtain

$$\lim_{s\to-\infty}\|K^U\phi_s\| \leq \frac{\beta}{\alpha}\sup_{-\infty\leq\tau\leq t}\big\{\|\Phi[\Psi(0)z(\tau)]\|\big\}. \tag{3.24}$$

Notice that, by the definition of $z(t)$, we have $\lim_{t\to-\infty} z(t) = 0$. Thus by letting $t \to -\infty$ in (3.24) we obtain

$$\lim_{s\to-\infty}\|K^U\phi_s\| = 0. \tag{3.25}$$

Next, we remark that for fixed $t \in \mathbb{R}$, $Y(t,\tau)$ is continuous with respect to the variable τ (see (9.4) on p. 226 of Hale and Lunel [11]). From expression (3.21), one sees that $Y(t,\tau)^S$ is continuously differentiable with respect to τ except for a finite jump at $\tau = t-r$, and

$$\begin{aligned} \frac{\partial Y(t,\tau)^S}{\partial\tau} &= K(t-r-\tau)\big(X_r - \Phi(\Psi, X_r)\big), \quad \tau < t-r, \\ \frac{\partial Y(t,\tau)^S}{\partial\tau} &= X_{t-\tau} - \Phi(\Psi, X_{t-\tau}), \quad t-r < \tau < t. \end{aligned} \tag{3.26}$$

Therefore, (3.17), (3.20) and (3.31) yield that

$$
\begin{aligned}
\big\|K^S\phi_t\big\| &\le \big\|K(t-s)K^S\phi_s\big\| + \Big\|\int_{t-r}^{t} d_\tau\big[Y(t,\tau)^S\big]z(\tau)\Big\| \\
&\quad + \Big\|\int_{s}^{t-r} d_\tau\big[Y(t,\tau)^S\big]z(\tau)\Big\| \\
&\le \beta e^{-\alpha(t-s)}\|\phi_s\| + \sup_{t-r\le\tau\le t}\big\{\|X_{t-\tau}-\Phi(\Psi,X_{t-\tau})\|\|z(\tau)\|\big\} \\
&\quad + \beta\Big\|\int_{s}^{t-r} e^{-\alpha(t-r-\tau)}d\tau \sup_{s\le\tau\le t-r}\big\{\|X_r-\Phi(\Psi,X_r)\|\|z(\tau)\|\big\} \\
&\le \beta e^{-\alpha(t-s)}\|\phi_s\| + \sup_{t-r\le\tau\le t}\big\{\|X_{t-\tau}-\Phi(\Psi,X_{t-\tau})\|\|z(\tau)\|\big\} \\
&\quad + \frac{\beta}{\alpha}\sup_{s\le\tau\le t-r}\big\{\|X_r-\Phi(\Psi,X_r)\|\|z(\tau)\|\big\}, \quad s\le t.
\end{aligned} \tag{3.27}
$$

By letting $s\to-\infty$ in (3.27), we conclude that

$$
\begin{aligned}
\big\|K^S\phi_t\big\| \le &\sup_{t-r\le\tau\le t}\big\{\|X_{t-\tau}-\Phi(\Psi,X_{t-\tau})\|\|z(\tau)\|\big\} \\
&+ \frac{\beta}{\alpha}\sup_{-\infty\le\tau\le t-r}\big\{\|X_r-\Phi(\Psi,X_r)\|\|z(\tau)\|\big\}.
\end{aligned} \tag{3.28}
$$

Since $\|z(\tau)\|\to 0$ as $\tau\to-\infty$, it immediately follows from (3.28) that

$$
\lim_{t\to-\infty}\big\|K^s\phi_t\big\| = 0. \tag{3.29}
$$

Combining (3.25) and (3.29), we have

$$
\lim_{t\to-\infty}\phi_t = \lim_{t\to-\infty}\big(K^U\phi_t + K^S\phi_t\big) = 0.
$$

From (3.15), we also have that $\lim_{t\to-\infty}\dot{\phi}(t) = 0$. ∎

Let us return to the linear operator L defined in (2.11). It is obvious that if $w\in C_0$, then $Lw\in C_0$. Hence, we can consider L to be a linear operator from C_0 to C_0. For this operator, we have

Theorem 3.4 *$\dim\mathcal{N}(L) = M$ and $\mathcal{R}(L) = C_0$.*

Proof. By definition, $w\in C_0$ and $Lw=0$ if and only if

$$
w(s) = \int_{-\infty}^{s} e^{-(s-t)}[w(t)+P^0 w(t)]dt, \quad s\in\mathbb{R}.
$$

Hence, w is continuously differentiable. By differentiating the last equation one sees that $Lw = 0$ if and only if

$$\dot{w}(s) = P^0 w(s), \quad s \in \mathbb{R}.$$

Recall that for $z \in C_0$ and $s \in \mathbb{R}$,

$$P^0 z(s) = A(s)z(s) + B(s)\int_{-r}^{0} d\eta(\theta)\mu_\Omega g_u\big(u^*(s+\theta)\big)z(s+\theta) = P(s)z_s. \tag{3.30}$$

Thus, the above equation and Lemma 3.3 imply that $w \in C_0^1$ and $Tw = 0$. That is, $w \in \mathcal{N}(L)$ if and only if $w \in \mathcal{N}(T)$. Therefore, Lemmas 3.2 and 3.3 imply that $\dim\mathcal{N}(L) = \dim\mathcal{N}(T) = M$, with $\mathcal{N}(L) \subset C_0^1$. Next, we shall prove that $\mathcal{R}(L) = C_0$. That is, for each $z \in C_0$, we need to show that equation $Lw = z$, or equivalently,

$$w(s) - \int_{-\infty}^{s} e^{-(s-t)}[w(t) + P^0 w(t)]dt = z(s), \quad s \in \mathbb{R}, \tag{3.31}$$

has a solution in C_0. To this end, we let $\xi(s) = w(s) - z(s)$, $s \in \mathbb{R}$. Upon a substitution, we obtain the equation for ξ as

$$\xi(s) = \int_{-\infty}^{s} e^{-(s-t)}[\xi(t) + P^0\xi(t)]dt + \int_{-\infty}^{s} e^{-(s-t)}[z(t) + P^0 z(t)]dt.$$

Differentiating the above equation yields that

$$\dot{\xi}(s) = P^0\xi(s) + z(s) + P^0 z(s), \quad s \in \mathbb{R}. \tag{3.32}$$

Thus, (3.30) implies that (3.32) is equivalent to the equation

$$(T\xi)(s) = z(s) + P^0 z(s). \tag{3.33}$$

From the expression of $P^0 z(s)$ it follows that $z \in C_0$ implies that $P^0 z(\cdot) \in C_0$, and hence $z + P^0 z \in C_0$. Thus Lemmas 3.2 and 3.3 guarantee that Eq. (3.33) has a solution $\xi \in C_0^1$. Consequently, $w = \xi + z \in C_0$ is a solution of Eq. (3.31). ∎

4. Properties of the Nonlinearity $\mathcal{H}$

In order to complete the proof of Theorem 1.1, we need further information about the behavior of the nonlinearity $\mathcal{H}(\cdot, \psi, \epsilon)$ when $\epsilon > 0$ is small and ψ is near the origin. To simplify the presentation, we let $R^\epsilon : C \to C$ for small $\epsilon \geq 0$ be defined by

$$R^\epsilon \psi(s) = \int_{\Omega_r} d\zeta(\theta, y) g\big(\psi(s + \sqrt{\epsilon}\nu \cdot y + \theta)\big), \quad s \in \mathbb{R}. \tag{4.1}$$

With the above notation, we can rewrite the nonlinear function $\mathcal{G}$ defined in (2.5) as

$$\begin{aligned}
&\mathcal{G}(\epsilon, s, w) \\
&= F\Big(w(s) + u^*(s), \int_{\Omega_r} d\zeta(\theta, y) g\big([w + u^*](s + \sqrt{\epsilon}\nu \cdot y + \theta)\big)\Big) \\
&\quad - F\Big(u^*(s), \int_{\Omega_r} d\zeta(\theta, y) g\big(u^*(s + \theta)\big)\Big) - P^0 w(s) + \epsilon D\ddot{u}^*(s) \\
&= F(w(s) + u^*(s), R^\epsilon[w + u^*](s)) - F(u^*(s), R^\epsilon u^*(s)) \\
&\quad + F(u^*(s), R^\epsilon u^*(s)) - F(u^*(s), R^0 u^*(s)) \\
&\quad - P^0 w(s) + \epsilon D\ddot{u}^*(s) \\
&= P^\epsilon w(s) - P^0 w(s) + \epsilon D\ddot{u}^*(s) \\
&\quad + F(w(s) + u^*(s), R^\epsilon[w + u^*](s)) - F(u^*(s), R^\epsilon u^*(s)) - P^\epsilon w(s) \\
&\quad + F(u^*(s), R^\epsilon u^*(s)) - F(u^*(s), R^0 u^*(s)) \\
&= P^\epsilon w(s) - P^0 w(s) + G(\epsilon, s, w) + \Theta(\epsilon, s),
\end{aligned} \tag{4.2}$$

where for $\epsilon > 0$ the linear operator $P^\epsilon : C_0 \to C$ is defined by

$$\begin{aligned}
&P^\epsilon \psi(s) \\
&= A^\epsilon(s)\psi(s) + B^\epsilon(s) \int_{\Omega_r} d\zeta(\theta, y) g_u\big(u^*(s + \sqrt{\epsilon}\nu \cdot y + \theta)\big)\psi(s + \sqrt{\epsilon}\nu \cdot y + \theta)
\end{aligned} \tag{4.3}$$

for $s \in \mathbb{R}$, with

$$\begin{aligned}
A^\epsilon(s) &= F_u(u^*(s), R^\epsilon u^*(s)), \quad s \in \mathbb{R}, \\
B^\epsilon(s) &= F_v(u^*(s), R^\epsilon u^*(s)), \quad s \in \mathbb{R},
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
G(\epsilon, s, \psi) = F(&\psi(s) + u^*(s), R^\epsilon[\psi + u^*](s)) \\
&- F(u^*(s), R^\epsilon u^*(s)) - P^\epsilon \psi(s), \quad s \in \mathbb{R},
\end{aligned} \tag{4.5}$$

$$\Theta(\epsilon, s) = \epsilon D\ddot{u}^*(s) + F(u^*(s), R^\epsilon u^*(s)) - F(u^*(s), R^0 u^*(s)), \quad s \in \mathbb{R}. \tag{4.6}$$

From the above notations and (2.10), we can express $\mathcal{H}_i(s,w,\epsilon)$ as

$$\begin{aligned}\mathcal{H}_i(s,w,\epsilon) = &\int_{-\infty}^{s} \big[\frac{e^{\alpha_i^\epsilon(s-t)}}{\sqrt{1+4\epsilon d_i}} - e^{-(s-t)}\big]\big[w_i(t)+P_i^0 w(t)\big]dt \\ &+ \frac{1}{\sqrt{1+4\epsilon d_i}}\int_s^\infty e^{\beta_i^\epsilon(s-t)}[w_i(t)+P_i^0w(t)]dt \\ &+ \frac{1}{\sqrt{1+4\epsilon d_i}}\int_{-\infty}^s e^{\alpha_i^\epsilon(s-t)}[P_i^\epsilon w(t)-P_i^0w(t)]dt \\ &+ \frac{1}{\sqrt{1+4\epsilon d_i}}\int_s^\infty e^{\beta_i^\epsilon(s-t)}[P_i^\epsilon w(t)-P_i^0w(t)]dt \\ &+ \frac{1}{\sqrt{1+4\epsilon d_i}}\Big(\int_{-\infty}^s e^{\alpha_i^\epsilon(s-t)}G_i(\epsilon,t,w)dt + \int_s^\infty e^{\beta_i^\epsilon(s-t)}G_i(\epsilon,t,w)dt\Big) \\ &+ \frac{1}{\sqrt{1+4\epsilon d_i}}\Big(\int_{-\infty}^s e^{\alpha_i^\epsilon(s-t)}\Theta_i(\epsilon,t)dt + \int_s^\infty e^{\beta_i^\epsilon(s-t)}\Theta_i(\epsilon,t)dt\Big).\end{aligned} \tag{4.7}$$

Thus, we can rewrite $\mathcal{H}(s,w,\epsilon)$ as

$$\mathcal{H}(s,w,\epsilon) = W(s,\epsilon) + \sum_{j=1}^{4} H^j(s,w,\epsilon),$$

where for $i=1,2,\cdots,n$, $w\in C_0$, and $s\in\mathbb{R}$,

$$\begin{aligned}H_i^1(s,w,\epsilon) &= \int_{-\infty}^{s} \big[\frac{e^{\alpha_i^\epsilon(s-t)}}{\sqrt{1+4\epsilon d_i}} - e^{-(s-t)}\big]\big[w_i(t)+P_i^0 w(t)\big]dt \\ &\quad + \frac{1}{\sqrt{1+4\epsilon d_i}}\int_s^\infty e^{\beta_i^\epsilon(s-t)}[w_i(t)+P_i^0w(t)]dt, \\ H_i^2(s,w,\epsilon) &= \frac{1}{\sqrt{1+4\epsilon d_i}}\int_{-\infty}^s e^{\alpha_i^\epsilon(s-t)}[P_i^\epsilon w(t)-P_i^0w(t)]dt, \\ H_i^3(s,w,\epsilon) &= \frac{1}{\sqrt{1+4\epsilon d_i}}\int_s^\infty e^{\beta_i^\epsilon(s-t)}[P_i^\epsilon w(t)-P_i^0w(t)]dt, \\ H_i^4(s,w,\epsilon) &= \frac{1}{\sqrt{1+4\epsilon d_i}}\Big[\int_{-\infty}^s e^{\alpha_i^\epsilon(s-t)}G_i(\epsilon,t,w)dt + \int_s^\infty e^{\beta_i^\epsilon(s-t)}G_i(\epsilon,t,w)dt\Big], \\ W_i(s,\epsilon) &= \frac{1}{\sqrt{1+4\epsilon d_i}}\Big[\int_{-\infty}^s e^{\alpha_i^\epsilon(s-t)}\Theta_i(\epsilon,t)dt + \int_s^\infty e^{\beta_i^\epsilon(s-t)}\Theta_i(\epsilon,t)dt\Big].\end{aligned} \tag{4.8}$$

In what follows, we shall give a detailed analysis of the behavior of functions $H^1(\cdot,w,\epsilon)$, …, $H^4(\cdot,w,\epsilon)$ and $W(\cdot,\epsilon)$ for small $w\in C_0$ and $\epsilon\geq 0$.

Lemma 4.1 *Let $\alpha\in C$ be given so that $\lim_{s\to\pm\infty}\alpha(s)=\alpha(\pm\infty)$ exist. Then for each $\epsilon\geq 0$,*

$$\lim_{s\to\pm\infty}\int_{\Omega_r} d\zeta(\theta,y)\alpha(s+\sqrt{\epsilon}\nu\cdot y+\theta) = \int_{\Omega_r} d\zeta(\theta,y)\alpha(\pm\infty).$$

Proof. We shall prove Lemma 4.1 only for the case when $s \to \infty$. The proof for the case where $s \to -\infty$ is analogous. For a positive integer j, let B_j be the open ball in $\mathbb{R}^m$ with radius j and center at the origin. Then

$$\lim_{j\to\infty} \int_{B_j\cap\Omega} d|\mu| = \int_\Omega d|\mu|,$$

and hence the boundedness of $\int_\Omega d|\mu|$ implies that

$$\lim_{j\to\infty} \int_{(\mathbb{R}^m\setminus B_j)\cap\Omega} d|\mu| = 0.$$

Therefore, for any $\sigma > 0$, there is a sufficiently large J such that

$$\Big\| \int_{(\mathbb{R}^m\setminus B_J)\cap\Omega} d|\mu| \Big\| < \sigma. \tag{4.9}$$

Now $\lim_{s\to\infty} \|\alpha(s) - \alpha(\infty)\| = 0$ implies that there is a $t^* > 0$ such that

$$\|\alpha(t) - \alpha(\infty)\| < \sigma, \quad t \geq t^*. \tag{4.10}$$

Note that if $s > t^* + \sqrt{\epsilon}J + r$, then for all $y \in B_J \cap \Omega$ and $\theta \in [-r, 0]$,

$$s + \sqrt{\epsilon}\nu \cdot y + \theta > t^* + \sqrt{\epsilon}J + r - \sqrt{\epsilon}\|y\| - |\theta| \geq t^*.$$

Hence, for $s > t^* + \sqrt{\epsilon}J + r$, we have

$$\|\alpha(s + \sqrt{\epsilon}\nu \cdot y + \theta) - \alpha(\infty)\| < \sigma, \quad y \in B_J,\ \theta \in [-r, 0]. \tag{4.11}$$

It follows from (4.9) - (4.11) that for all $s > t^* + \sqrt{\epsilon}J + r$ and $\theta \in [-r, 0]$,

$$\begin{aligned}
&\Big\| \int_\Omega d\mu(y)[\alpha(s + \sqrt{\epsilon}\nu \cdot y + \theta) - \alpha(\infty)] \Big\| \\
&\qquad \leq \Big\| \int_{B_J\cap\Omega} d\mu(y)[\alpha(s + \sqrt{\epsilon}\nu \cdot y + \theta) - \alpha(\infty)] \Big\| \\
&\qquad\quad + \Big\| \int_{(\mathbb{R}^m\setminus B_J)\cap\Omega} d\mu(y)[\alpha(s + \sqrt{\epsilon}\nu \cdot y + \theta) - \alpha(\infty)] \Big\| \\
&\qquad \leq \sigma \Big\| \int_{B_J\cap\Omega} d|\mu| \Big\| + 2\sigma\|\alpha\|_C \\
&\qquad \leq \sigma \Big(\Big\| \int_\Omega d|\mu| \Big\| + 2\|\alpha\|_C \Big).
\end{aligned} \tag{4.12}$$

Since $\sigma > 0$ is arbitrary, (4.12) implies that

$$\lim_{s\to\infty} \Big\| \int_{\Omega} d\mu(y)[\alpha(s+\sqrt{\epsilon}\nu\cdot y+\theta)-\alpha(\infty)]\Big\| = 0 \tag{4.13}$$

uniformly for $\theta \in [-r, 0]$. Consequently, we have

$$\lim_{s\to\infty} \Big\| \int_{\Omega_r} d\zeta(\theta,y)[\alpha(s+\sqrt{\epsilon}\nu\cdot y+\theta)-\alpha(\infty)]\Big\| = 0. \qquad \blacksquare$$

Corollary 4.2 *For each* $\epsilon \geq 0$ *and each* $w \in C_0$, $\mathcal{H}(\cdot, w, \epsilon) \in C_0$. *In other words,* $\mathcal{H}(\cdot, C_0, \epsilon) \subseteq C_0$ *for each* $\epsilon \geq 0$.

Proof. For $w \in C_0$ and $\epsilon \geq 0$, if we let $\alpha(t) = g_u\big(u^*(t)\big)w(t)$, $t \in \mathbb{R}$, then $\alpha \in C$ and $\alpha(s) \to 0$ as $|s| \to \infty$. It follows from the definition of P^ϵ and Lemma 4.1 that $P^\epsilon w(s) \to 0$ as $|s| \to \infty$. Therefore, $H^i(s, w, \epsilon) \to 0$ as $|s| \to \infty$ for $i = 1, 2, 3$. Next, by the definition of R^ϵ given in (4.1) and Lemma 4.1 we have

$$\lim_{|s|\to\infty} R^\epsilon[w+u^*](s) = \lim_{|s|\to\infty} R^\epsilon u^*(s).$$

The above equality yields that $G(\epsilon, s, w) \to 0$ as $|s| \to \infty$, and so does for the function $H^4(s, w, \epsilon)$. Similarly, we obtain that $W(s, \epsilon) \to 0$ as $|s| \to \infty$. $\blacksquare$

Proposition 4.3 For $w \in C_0$ and small $\epsilon \geq 0$, $H^1(\cdot, w, \epsilon) = O(\epsilon)\|w\|_{C_0}$.

Proof. For $s \in \mathbb{R}$ and $\epsilon \geq 0$ we have

$$\begin{aligned}
&\int_{-\infty}^{s} \big|e^{\alpha_i^\epsilon(s-t)} - \sqrt{1+4\epsilon d_i}e^{-(s-t)}\big| dt \\
&\quad= \int_{-\infty}^{s} \big|e^{\alpha_i^\epsilon(s-t)}(1-\sqrt{1+4\epsilon d_i}) + \sqrt{1+4\epsilon d_i}\big(e^{\alpha_i^\epsilon(s-t)} - e^{-(s-t)}\big)\big| dt \\
&\quad\leq \big|1-\sqrt{1+4\epsilon d_i}\big| \int_{-\infty}^{s} e^{\alpha_i^\epsilon(s-t)} dt \\
&\qquad + \sqrt{1+4\epsilon d_i} \int_{-\infty}^{s} \big|e^{\alpha_i^\epsilon(s-t)} - e^{-(s-t)}\big| dt.
\end{aligned} \tag{4.14}$$

Since $\alpha_i^\epsilon > -1$, for $t \leq s$ we have

$$\big|e^{\alpha_i^\epsilon(s-t)} - e^{-(s-t)}\big| = e^{\alpha_i^\epsilon(s-t)} - e^{-(s-t)}, \tag{4.15}$$

and (4.14) and (4.15) yield that

$$\begin{aligned}
&\int_{-\infty}^{s} \big|e^{\alpha_i^\epsilon(s-t)} - \sqrt{1+4\epsilon d_i}e^{-(s-t)}\big| dt \\
&\qquad \leq \big|1-\sqrt{1+4\epsilon d_i}\big|\Big(-\frac{1}{\alpha_i^\epsilon}\Big) + \sqrt{1+4\epsilon d_i}\Big[-\frac{1}{\alpha_i^\epsilon} - 1\Big].
\end{aligned}$$

Noticing that $\alpha_i^\epsilon \to -1$ as $\epsilon \to 0^+$, we obtain from the above inequality that

$$\lim_{\epsilon\to 0^+} \int_{-\infty}^{s} \left| e^{\alpha_i^\epsilon (s-t)} - \sqrt{1+4\epsilon d_i}\, e^{-(s-t)} \right| dt = 0. \tag{4.16}$$

Next let $K = 1 + \|P^0\|_{\mathcal{L}(C_0,C_0)}$. Then

$$\begin{aligned}
&\Big| \int_{-\infty}^{s} \Big[\frac{e^{\alpha_i^\epsilon (s-t)}}{\sqrt{1+4\epsilon d_i}} - e^{-(s-t)} \Big] \big[w_i(t) + P_i^0 w(t) \big] dt \Big| \\
&\quad \le K \int_{-\infty}^{s} \Big| \frac{e^{\alpha_i^\epsilon (s-t)}}{\sqrt{1+4\epsilon d_i}} - e^{-(s-t)} \Big| dt\, \|w\|_{C_0} \\
&\quad = \frac{K}{\sqrt{1+4\epsilon d_i}} \int_{-\infty}^{s} \left| e^{\alpha_i^\epsilon (s-t)} - \sqrt{1+4\epsilon d_i}\, e^{-(s-t)} \right| dt\, \|w\|_{C_0} \\
&\quad = O(\epsilon) \|w\|_{C_0}.
\end{aligned} \tag{4.17}$$

Next, since $\beta_i^\epsilon \to \infty$ as $\epsilon \to 0$, $1/\beta_i^\epsilon = O(\epsilon)$ as $\epsilon \to 0$. This yields that

$$\begin{aligned}
&\Big| \frac{1}{\sqrt{1+4\epsilon d_i}} \int_{s}^{\infty} e^{\beta_i^\epsilon (s-t)} [w_i(t) + P_i^0 w(t)] dt \Big| \\
&\qquad \le \frac{K}{\sqrt{1+4\epsilon d_i}} \int_{s}^{\infty} e^{\beta_i^\epsilon (s-t)} dt \|w\|_{C_0} \\
&\qquad = \frac{K}{\beta_i^\epsilon \sqrt{1+4\epsilon d_i}} \|w\|_{C_0} \\
&\qquad = O(\epsilon) \|w\|_{C_0}, \qquad \text{as} \quad \epsilon \to 0.
\end{aligned} \tag{4.18}$$

The proposition therefore follows from (4.17) and (4.18). ∎

Lemma 4.4 *For $\epsilon > 0$ and $(s, y, \theta) \in \mathbb{R} \times \mathbb{R}^m \times [-r, 0]$,*

$$\begin{aligned}
&\Big\| \int_{\Omega_r} d\zeta(\theta, y) \big[g\big(u^*(s + \sqrt{\epsilon}\nu \cdot y + \theta)\big) - g\big(u^*(s+\theta)\big) \big] \Big\| \\
&\quad \le \sqrt{\epsilon} \|\eta\| \Big\| \int_{\Omega} d|\mu|(y) \|y\| \Big\| \|g_u\| \|\dot{u}^*\|_C,
\end{aligned}$$

where $\|\eta\| = V_{[-r,0]}\eta$ and

$$\|g_u\| = \sup \Big\{ \|g_u(\lambda u^*(t) + (1-\lambda) u^*(\tau))\| : (\lambda, t, \tau) \in [0,1] \times \mathbb{R} \times \mathbb{R} \Big\}.$$

Proof. Since g is differentiable, for $(s, y, \theta) \in \mathbb{R} \times \mathbb{R}^m \times [-r, 0]$, we have

$$\begin{aligned}
&g\big(u^*(s + \sqrt{\epsilon}\nu \cdot y + \theta)\big) - g\big(u^*(s+\theta)\big) \\
&\qquad = \int_0^1 g_u\big(\lambda u^*(s+\theta) + (1-\lambda) u^*(s + \sqrt{\epsilon}\nu \cdot y + \theta)\big) d\lambda \\
&\qquad\quad \times \big[u^*(s + \sqrt{\epsilon}\nu \cdot y + \theta) - u^*(s+\theta) \big].
\end{aligned}$$

The above equality yields that for $(s, y, \theta) \in \mathbb{R} \times \mathbb{R}^m \times [-r, 0]$,

$$\|g\big(u^*(s+\sqrt{\epsilon}\nu \cdot y+\theta)\big) - g\big(u^*(s+\theta)\big)\| \le \sqrt{\epsilon}\|y\|\|g_u\|\|\dot{u}^*\|_C. \tag{4.19}$$

Recalling that $\int_{\Omega_r} d\zeta(\theta, y) = \int_{-r}^0 \int_\Omega d\eta(\theta)d\mu(y)$, as an immediate consequence of (4.19) we have, for $s \in \mathbb{R}$ and $\epsilon > 0$,

$$\begin{aligned}&\Big\| \int_{\Omega_r} d\zeta(\theta, y)\big[g\big(u^*(s+\sqrt{\epsilon}\nu \cdot y+\theta)\big) - g_u\big(u^*(s+\theta)\big)\big]\Big\| \\ &\le \sqrt{\epsilon}\|\eta\|\Big\| \int_\Omega d|\mu|(y)\|y\|\Big\|\|g_u\|\|\dot{u}^*\|_C. \qquad \blacksquare\end{aligned}$$

Proposition 4.5 *There exist $\epsilon_0 > 0$ and $M_0 > 0$ such that for all $\epsilon \in [0, \epsilon_0]$ and $\psi \in C_0$,*

$$\|H^2(\cdot, \psi, \epsilon)\|_{C_0} \le \sqrt{\epsilon}M_0\|\psi\|_{C_0}.$$

Proof. Let $h(s, y, \theta) = g_u\big(u^*(s+\sqrt{\epsilon}\nu \cdot y+\theta)\big)$. From the definitions of $P^\epsilon\psi$ and $P^0\psi$, we have

$$\begin{aligned}&[P^\epsilon - P^0]\psi(s) \\ &= [A^\epsilon(s) - A(s)]\psi(s) \\ &\quad + [B^\epsilon(s) - B(s)]\int_{\Omega_r} d\zeta(\theta, y)h(s, y, \theta)\psi(s+\sqrt{\epsilon}\nu \cdot y+\theta) \\ &\quad + B(s)\int_{\Omega_r} d\zeta(\theta, y)\big[h(s, y, \theta) - h(s, 0, \theta)\big]\psi(s+\sqrt{\epsilon}\nu \cdot y+\theta) \\ &\quad + B(s)\int_{\Omega_r} d\zeta(\theta, y)h(s, 0, \theta)\big[\psi(s+\sqrt{\epsilon}\nu \cdot y+\theta) - \psi(s+\theta)\big].\end{aligned} \tag{4.20}$$

For $\epsilon > 0$, let

$$E^\epsilon(s) = \mathrm{diag}\Big(\frac{e^{\alpha_1^\epsilon s}}{\sqrt{1+4\epsilon d_1}}, \frac{e^{\alpha_2^\epsilon s}}{\sqrt{1+4\epsilon d_2}}, \cdots, \frac{e^{\alpha_n^\epsilon s}}{\sqrt{1+4\epsilon d_n}}\Big), \quad s \in \mathbb{R}.$$

Then

$$\dot{E}^\epsilon(s) = \mathrm{diag}\Big(\frac{\alpha_1^\epsilon e^{\alpha_1^\epsilon s}}{\sqrt{1+4\epsilon d_1}}, \frac{\alpha_2^\epsilon e^{\alpha_2^\epsilon s}}{\sqrt{1+4\epsilon d_2}}, \cdots, \frac{\alpha_n^\epsilon e^{\alpha_n^\epsilon s}}{\sqrt{1+4\epsilon d_n}}\Big), \quad s \in \mathbb{R}.$$

Since $\alpha_i^\epsilon \to -1$ as $\epsilon \to 0$ for $i = 1, \cdots, n$, there are $\epsilon_0 > 0$ and $K_0 > 0$ such that for $\epsilon \in [0, \epsilon_0]$,

$$\|E^\epsilon(0)\| \le K_0, \quad \int_{-\infty}^s \|E^\epsilon(s-t)\|dt \le K_0, \quad \int_{-\infty}^s \|\dot{E}^\epsilon(s-t)\|dt \le K_0. \tag{4.21}$$

By the definition of H^2 and (4.20), we have

$$\begin{aligned}
&H^2(s,\psi,\epsilon)\\
&=\int_{-\infty}^{s} E^\epsilon(s-t)[P^\epsilon - P^0]\psi(t)dt\\
&=\int_{-\infty}^{s} E^\epsilon(s-t)[A^\epsilon(t)-A(t)]\psi(t)dt\\
&\quad+\int_{-\infty}^{s} E^\epsilon(s-t)[B^\epsilon(t)-B(t)]\int_{\Omega_r} d\zeta(\theta,y)h(t,y,\theta)\psi(t+\sqrt{\epsilon}\nu\cdot y+\theta)dt\\
&\quad+\int_{-\infty}^{s} E^\epsilon(s-t)B(t)\int_{\Omega_r} d\zeta(\theta,y)\big[h(t,y,\theta)-h(t,0,\theta)\big]\psi(t+\sqrt{\epsilon}\nu\cdot y+\theta)dt\\
&\quad+\int_{-\infty}^{s} E^\epsilon(s-t)B(t)\int_{\Omega_r} d\zeta(\theta,y)h(t,0,\theta)\big[\psi(t+\sqrt{\epsilon}\nu\cdot y+\theta)-\psi(t+\theta)\big]dt.
\end{aligned} \tag{4.22}$$

Let

$$\hat{u}^*_\epsilon(t) = R^\epsilon u^*(t) = \int_{-r}^{0}\int_\Omega d\eta(\theta)d\mu(y)g\big(u^*(t+\sqrt{\epsilon}\nu\cdot y+\theta)\big), \quad t\in \mathbb{R}. \tag{4.23}$$

Then, from the definitions of $A^\epsilon(t)$, $A(t)$ and $\hat{u}^*_\epsilon(t)$, it follows that

$$\begin{aligned}
A^\epsilon(t)-A(t) &= F_u\big(u^*(t),\hat{u}^*_\epsilon(t)\big) - F_u\big(u^*(t),\hat{u}^*_0(t)\big)\\
&=\int_0^1 F_{uv}\big(u^*(t),\hat{u}^*_0(t)+\tau[\hat{u}^*_\epsilon(t)-\hat{u}^*_0(t)]\big)d\tau\big[\hat{u}^*_\epsilon(t)-\hat{u}^*_0(t)\big].
\end{aligned} \tag{4.24}$$

Since F is C^2-smooth, there is a constant $K_1>0$ such that

$$\Big\|\int_0^1 F_{uv}\big(u^*(t),\hat{u}^*_0(t)+\tau[\hat{u}^*_\epsilon(t)-\hat{u}^*_0(t)]\big)d\tau\Big\| \le K_1, \quad t\in\mathbb{R}, \epsilon\in[0,\epsilon_0]. \tag{4.25}$$

Lemma 4.4 and (4.25) therefore yield that

$$\big\|A^\epsilon(t)-A(t)\| \le \sqrt{\epsilon}K_1\|\eta\|\Big\|\int_\Omega d|\mu|(y)\|y\|\Big\|\|g_u\|\|\dot{u}^*\|_{C_0}, \quad t\in\mathbb{R}, \epsilon\in[0,\epsilon_0]. \tag{4.26}$$

For all $s\in\mathbb{R}$ and $\epsilon\in[0,\epsilon_0]$, (4.21) and (4.26) imply that

$$\begin{aligned}
&\Big\|\int_{-\infty}^{s} E^\epsilon(s-t)[A^\epsilon(t)-A(t)]\psi(t)dt\Big\|\\
&\qquad\le \sqrt{\epsilon}K_1\|\eta\|\Big\|\int_\Omega d|\mu|(y)\|y\|\Big\|\|g_u\|\|\dot{u}^*\|_{C_0}\int_{-\infty}^{s}\|E^\epsilon(s-t)\|dt\|\psi\|_{C_0}\\
&\qquad\le \sqrt{\epsilon}M_1\|\psi\|_{C_0},
\end{aligned} \tag{4.27}$$

where

$$M_1 = K_0 K_1 \|\eta\| \Big\| \int_\Omega d|\mu|(y) \|y\| \Big\| \|g_u\| \|\dot{u}^*\|_{C_0}$$

and K_0 is defined in (4.21).

Arguing in the same way as above, we obtain that there is a constant $K_2 > 0$ such that

$$\big\|B^\epsilon(t) - B(t)\big\| \leq \sqrt{\epsilon} K_2 \|\eta\| \Big\| \int_\Omega d|\mu|(y)\|y\| \Big\| \|g_u\| \|\dot{u}^*\|_{C_0}, \quad s \in \mathbb{R}.$$

Thus, for all $s \in \mathbb{R}$,

$$\begin{aligned} &\Big\| \int_{-\infty}^s E^\epsilon(s-t)[B^\epsilon(t) - B(t)] \int_{\Omega_r} d\zeta(\theta, y) h(t, y, \theta)\psi(t + \sqrt{\epsilon}\nu \cdot y + \theta) dt \Big\| \\ &\leq \sqrt{\epsilon} M_2 \|\psi\|_{C_0} \end{aligned} \tag{4.28}$$

with $M_2 = K_0 K_2 \|\eta\| \int_\Omega d|\mu|(y)\|y\| \| \|g_u\| \|\dot{u}^*\|_{C_0}$.

It is also clear that for $s \in \mathbb{R}$,

$$\begin{aligned} &\Big\| \int_{-\infty}^s E^\epsilon(s-t) B(t) \int_{\Omega_r} d\zeta(\theta, y) \big[h(t, y, \theta) - h(t, 0, \theta)\big] \psi(t + \sqrt{\epsilon}\nu \cdot y + \theta) dt \Big\| \\ &\leq \sqrt{\epsilon} M_3 \|\psi\|_{C_0} \end{aligned} \tag{4.29}$$

with

$$M_3 = 2K_0 \sup_{t \in \mathbb{R}} \big\{\|B(t)\|\big\} \|\eta\| \Big\| \int_\Omega d|\mu|(y) \Big\| \|g_u\|.$$

Next, if $\psi \in C_0^1$, by exchanging the order of integration and integration by parts we have

$$\begin{aligned} &\int_{-\infty}^s E^\epsilon(s-t)B(t)\Big[\int_{\Omega_r} d\zeta(\theta, y)h(t, 0, \theta)[\psi(t + \sqrt{\epsilon}\nu \cdot y + \theta) - \psi(t + \theta)]\Big] dt \\ &= \int_{-\infty}^s E^\epsilon(s-t)B(t)\Big[\int_{\Omega_r} d\zeta(\theta, y)h(t, 0, \theta) \int_0^1 \dot{\psi}(t + \tau\sqrt{\epsilon}\nu \cdot y + \theta)\sqrt{\epsilon}(\nu \cdot y) d\tau\Big] dt \\ &= \sqrt{\epsilon} \int_0^1 \Big[\int_{-\infty}^s E^\epsilon(s-t)B(t) \int_{\Omega_r} d\zeta(\theta, y)(\nu \cdot y)h(t, 0, \theta)\dot{\psi}(t + \tau\sqrt{\epsilon}\nu \cdot y + \theta) dt\Big] d\tau \\ &= \sqrt{\epsilon} \int_0^1 \Big(\Big[E^\epsilon(s-t)B(t) \int_{\Omega_r} d\zeta(\theta, y)(\nu \cdot y)h(t, 0, \theta)\psi(t + \tau\sqrt{\epsilon}\nu \cdot y + \theta)\Big]\Big|_{t=-\infty}^{t=s}\Big) d\tau \\ &\quad + \sqrt{\epsilon} \int_0^1 \Big\{\int_{-\infty}^s \Big[\dot{E}^\epsilon(s-t)B(t) - E^\epsilon(s-t)\dot{B}(t)\Big] \times \\ &\quad \int_{\Omega_r} d\zeta(\theta, y)(\nu \cdot y)h(t, 0, \theta)\psi(t + \tau\sqrt{\epsilon}\nu \cdot y + \theta) dt\Big\} d\tau \\ &\quad - \sqrt{\epsilon} \int_0^1 \Big[\int_{-\infty}^s E^\epsilon(s-t)B(t) \times \\ &\quad \int_{\Omega_r} d\zeta(\theta, y)(\nu \cdot y)\frac{\partial h(t, 0, \theta)}{\partial t}\psi(t + \tau\sqrt{\epsilon}\nu \cdot y + \theta)\Big] dt d\tau. \end{aligned}$$

Therefore,

$$
\begin{aligned}
&\int_{-\infty}^{s} E^{\epsilon}(s-t)B(t)\Big[\int_{\Omega_r} d\zeta(\theta,y)h(t,0,\theta)[\psi(t+\sqrt{\epsilon}\nu\cdot y+\theta)-\psi(t+\theta)]\Big]dt \\
&= \sqrt{\epsilon}E^{\epsilon}(0)B(s)\int_0^1 \Big[\int_{\Omega_r} d\zeta(\theta,y)(\nu\cdot y)h(s,0,\theta)\psi(s+\tau\sqrt{\epsilon}\nu\cdot y+\theta)\Big]d\tau \\
&\quad + \sqrt{\epsilon}\int_0^1 \Big\{\int_{-\infty}^{s}\Big[\dot{E}^{\epsilon}(s-t)B(t)-E^{\epsilon}(s-t)\dot{B}(t)\Big]\times \\
&\quad \int_{\Omega_r} d\zeta(\theta,y)(\nu\cdot y)h(t,0,\theta)\psi(t+\tau\sqrt{\epsilon}\nu\cdot y+\theta)dt\Big\}d\tau \\
&\quad - \sqrt{\epsilon}\int_0^1 \Big[\int_{-\infty}^{s} E^{\epsilon}(s-t)B(t)\times \\
&\quad \int_{\Omega_r} d\zeta(\theta,y)(\nu\cdot y)\frac{\partial h(t,0,\theta)}{\partial t}\psi(t+\tau\sqrt{\epsilon}\nu\cdot y+\theta)dt\Big]d\tau, \qquad s\in \mathbb{R}.
\end{aligned}
\tag{4.30}
$$

Recalling that $h(t,0,\theta)=g_u\big(u^*(t+\theta)\big)$, we have

$$
\frac{\partial h(t,0,\theta)}{\partial t}=\frac{\partial g_u\big(u^*(t+\theta)\big)}{\partial t}=g_{uu}\big(u^*(t+\theta)\big)\dot{u}^*(t+\theta).
$$

Therefore, (4.30) implies that for all $s\in\mathbb{R}$,

$$
\begin{aligned}
&\Big\|\int_{-\infty}^{s} E^{\epsilon}(s-t)B(t)\Big[\int_{\Omega_r} d\zeta(\theta,y)h(t,0,\theta)[\psi(t+\sqrt{\epsilon}\nu\cdot y+\theta)-\psi(t+\theta)]\Big]dt\Big\| \\
&\leq \sqrt{\epsilon}M_4\|\psi\|_{C_0},
\end{aligned}
\tag{4.31}
$$

where

$$
M_4 = 2K_0\sup_{t\in\mathbb{R}}\{\|B(t)\|+\|\dot{B}(t)\|\}\|(\|g_u\|+\|\tilde{g}_{uu}\|\|\dot{u}^*\|_C)\eta\|\Big\|\int_{\Omega} d\mu(y)|y|^+\Big\|_{\mathbb{R}^m}
$$

and $\|\tilde{g}_{uu}\|=\sup_{t\in\mathbb{R}}\|g_{uu}(u^*(t)\|\}$. It therefore follows from (4.27) - (4.29) and (4.31) that for $\epsilon\in[0,\epsilon_0]$ and $\psi\in C_0^1$,

$$
\|H^2(\cdot,\psi,\epsilon)\|_{C_0}\leq\sqrt{\epsilon}M_0\|\psi\|_{C_0}, \quad \text{with} \quad M_0=\sum_{j=1}^{4}M_j. \tag{4.32}
$$

Since $H^2(\cdot,\cdot,\epsilon):C_0\to C_0$ is a bounded linear operator and C_0^1 is dense in C_0, the inequality (4.32) holds for all $\psi\in C_0$. ∎

Proposition 4.6 *For $\epsilon>0$ and $\psi\in C_0$, $H^3(\cdot,\psi,\epsilon)=O(\epsilon)\|\psi\|_{C_0}$ as $\epsilon\to 0$.*

Proof. Since $\beta_i^\epsilon \to \infty$ as $\epsilon \to 0$ for $i = 1, \cdots, n$, one obtains that for all $s \in \mathbb{R}$,

$$\int_s^\infty e^{\beta_i^\epsilon (s-t)} dt = \frac{1}{\beta_i^\epsilon} \to 0 \quad \text{as} \quad \epsilon \to 0, \quad i = 1, \cdots, n. \tag{4.33}$$

Thus Proposition 4.6 follows from (4.33) and the definition of H^3. ∎

Proposition 4.7 $H^4(\cdot, 0, \epsilon) = 0$ and for each $\delta > 0$, there is a $\sigma > 0$ such that

$$\|H^4(\cdot, \phi, \epsilon) - H^4(\cdot, \psi, \epsilon)\|_{C_0} \le \delta \|\phi - \psi\|_{C_0}$$

for all $\epsilon \in [0, 1]$ and all $\phi, \psi \in B(\sigma)$, where $B(\sigma)$ is the ball in C_0 with radius σ and center at the origin.

Proof. It is apparent that, from the definition of $G(\epsilon, \cdot, \psi)$ (see (4.5)), $G_\psi(\epsilon, \cdot, \psi)$ and $G_{\psi\psi}(\epsilon, \cdot, \psi)$ are continuous for $\epsilon \in [0, 1]$ and for ψ in a neighborhood of the origin in C_0. Moreover, we have $G_\psi(\epsilon, \cdot, 0) \equiv 0$ for $\epsilon \in [0, 1]$. It therefore follows that

$$\|G(\epsilon, \cdot, \psi)\|_{C_0} = O(\|\psi\|_{C_0}^2) \quad \text{as} \quad \|\psi\|_{C_0} \to 0 \tag{4.34}$$

uniformly for $\epsilon \in [0, 1]$, and the proposition follows from the definition of H^4 and (4.34). ∎

Proposition 4.8 $\|W(\cdot, \epsilon)\|_{C_0} = O(\epsilon)$ *as* $\epsilon \to 0$.

Proof. We note that $\ddot{u}^*(\cdot)$ is bounded in C_0 and

$$F(u^*(\cdot), R^\epsilon u^*(\cdot)) - F(u^*(\cdot), R^0 u^*(\cdot)) = O(\epsilon)\|\dot{u}^*\|_{C_0} \quad \text{as} \quad \epsilon \to 0,$$

by Lemma 4.4. Therefore, we obtain Proposition 4.8 from the expression of $W(\cdot, \epsilon)$ given in (4.8). ∎

5. Proof of the Main Theorem

We shall complete the proof of our main Theorem 1.1 in this section. To do so we need a final auxiliary result. By Theorem 3.4 we have $\dim \mathcal{N}(L) = M$. Therefore there are functions $w_1, \cdots, w_M \in C_0$ which give a basis of $\mathcal{N}(L)$. Hence there exist linear functionals $h_1, \cdots, h_M : C_0 \to \mathbb{R}$, such that

$$h_i(w_i) = 1, \quad h_i(w_j) = 0, \quad i \ne j, \quad i, j = 1, \cdots, M.$$

Lemma 5.1 *Let* $X = \{\phi \in C_0 : \ h_i(\phi) = 0, i = 1, \cdots, M\}$. *Then*

$$C_0 = X \oplus \mathcal{N}(L).$$

Proof. We note first that this result is not new. Nevertheless, we give a short proof here for the sake of completion. For each $\psi \in C_0$, let $\phi = \psi - \sum_{i=1}^{M} h_i(\psi)w_i$. Then we have $h_i(\phi) = 0$, $i = 1, \cdots, M$, and $\psi = \phi + \sum_{i=1}^{M} h_i(\psi)w_i$. That is, each $\psi \in C_0$ can be expressed as the sum of an element of X and an element of $\mathcal{N}(L)$. Moreover, let $\psi \in X \cap \mathcal{N}(L)$. Thus there are constants c_i, $i = 1, \cdots, M$ such that

$$\psi = \sum_{i=1}^{M} c_i w_i.$$

The definition of X and h_i imply that

$$0 = h_i(\psi) = c_i h_i(w_i) = c_i, \quad i = 1, \cdots, M.$$

Hence $\psi = 0$ and thus $X \cap \mathcal{N}(L) = 0$. This proves the lemma. ∎

It is clear that $X \subset C_0$ is a Banach space. If we let $S = L|_X$ be the restriction of L on X, then $S : X \to C_0$ is one-to-one and onto, since $\mathcal{R}(L) = C_0$ by Theorem 3.4. Therefore, S has an inverse $S^{-1} : C_0 \to X$ which is a bounded linear operator.

Proof of Theorem 1.1 For each $\psi \in C_0$, there are unique $\xi \in \mathcal{N}(L)$ and $\phi \in X$ such that $\psi = \phi + \xi$. Hence ψ is a solution of Eq. (2.12) if and only if

$$L\phi = \mathcal{H}(\cdot, \xi + \phi, \epsilon), \tag{5.1}$$

or equivalently, if and only if ϕ is a solution of the equation

$$\phi = S^{-1}\mathcal{H}(\cdot, \phi + \xi, \epsilon). \tag{5.2}$$

Let $\|S^{-1}\| = \|S^{-1}\|_{\mathcal{L}(C_0, X)}$. It follows from Propositions 4.3, 4.5 - 4.8 that there are $\sigma > 0$, $\epsilon^* > 0$, and $0 < \rho < 1$ such that for all $\epsilon \in (0, \epsilon^*]$ and $\psi, \varphi \in \overline{B(\sigma)} \subset C_0$,

$$\|\mathcal{H}(\cdot, \psi, \epsilon)\|_{C_0} \leq \frac{1}{3\|S^{-1}\|}\Big(\|\psi\|_{C_0} + \sigma\Big), \tag{5.3}$$

$$\|\mathcal{H}(\cdot, \psi, \epsilon) - \mathcal{H}(\cdot, \varphi, \epsilon)\|_{C_0} \leq \frac{\rho}{\|S^{-1}\|}\|\psi - \varphi\|_{C_0}. \tag{5.4}$$

For each fixed $\xi \in \mathcal{N}(L) \cap \overline{B(\sigma)}$, (5.3) implies that

$$\begin{gathered} \|S^{-1}\mathcal{H}(\cdot, \phi + \xi, \epsilon)\|_{C_0} \leq \frac{1}{3}\big(\|\phi + \xi\|_{C_0} + \sigma\big) \leq \sigma \\ \text{for} \quad \epsilon \in (0, \epsilon^*], \quad \phi \in X \cap \overline{B(\sigma)}. \end{gathered} \tag{5.5}$$

Hence, together with (5.4) we see that the mapping

$$\mathcal{F} : \big(X \cap \overline{B(\sigma)}\big) \times \big(\mathcal{N}(L) \cap \overline{B(\sigma)}\big) \times (0, \epsilon^*) \to X \cap \overline{B(\sigma)}$$

given by

$$\mathcal{F}(\phi, \xi, \epsilon) = S^{-1}\mathcal{H}(\cdot, \phi + \xi, \epsilon)$$

is a uniform contraction mapping of $\phi \in X \cap \overline{B(\sigma)}$. Hence, for each $(\xi, \epsilon) \in \big(\mathcal{N}(L) \cap \overline{B(\sigma)}\big) \times (0, \epsilon^*)$ there is a unique fixed point $\phi_{(\xi,\epsilon)} \in X \cap \overline{B(\sigma)}$ of the mapping $\mathcal{F}(\cdot, \xi, \epsilon)$. In other words, $\phi_{(\xi,\epsilon)}$ is the unique solution in $X \cap \overline{B(\sigma)}$ of Eq. (5.2). Thus, for $\epsilon \in (0, \epsilon^*)$ fixed, $\psi_{(\xi,\epsilon)} = \phi_{(\xi,\epsilon)} + \xi$ is a solution of (2.12). Notice that $\mathcal{N}(L) \cap \overline{B(\sigma)}$ is M-dimensional. It follows that for each $\epsilon \in (0, \epsilon^*)$ and for each unit vector $\nu \in \mathbb{R}^m$, the set

$$\Gamma_\nu(\epsilon) = \big\{\psi_{(\xi,\epsilon)} : \xi \in \mathcal{N}(L) \cap \overline{B(\sigma)}\big\}$$

is an M-dimensional manifold. This proves claims (i) and (ii) in the statement of the theorem.

To prove (iii), we first note that if F, g are C^k $(k \geq 2)$, then $\mathcal{H}(\cdot, \psi, \epsilon)$ is continuous on (ψ, ϵ) and C^{k-1}-smooth with respect to ψ. Hence $\mathcal{F}(\phi, \xi, \epsilon)$ is continuous on (ϕ, ξ, ϵ) and C^{k-1}-smooth with respect to ϕ and ξ. The uniform contraction mapping principle (see p. 25-26 of Chow and Hale [3]) implies that the fixed point $\phi_{(\xi,\epsilon)}$ is a continuous mapping on (ξ, ϵ) and C^{k-1} on ξ. Therefore, in addition we conclude that for each $\epsilon \in (0, \epsilon^*)$ and for each unit vector $\nu \in \mathbb{R}^m$, $\Gamma_\nu(\epsilon)$ is a C^{k-1} manifold. It is locally given as the graph of a C^{k-1} mapping that is also continuous with respect to c.

Let $c = 1/\sqrt{\epsilon}$ with $\epsilon \in (0, \epsilon^*)$ and

$$\mathcal{M}_\nu(c) = \big\{U : U(s) = \psi_\xi(s/c) + u^*(s/c),\ s \in \mathbb{R},\ \psi_\xi \in \Gamma_\nu(s/c^2)\big\}.$$

Then $\mathcal{M}_\nu(c)$ is an M-dimensional manifold in a neighborhood of u^* consisting of traveling wave solutions of Eq. (1.1) with wave speed c and direction ν. Moreover, for each $c > c^*$ and each unit vector $\nu \in \mathbb{R}^m$, $\mathcal{M}_\nu(c)$ is a C^{k-1} manifold that is given by the graph of a C^{k-1}-mapping that is continuous on c.

It remains to prove that the above fixed point $\phi_{(\xi,\epsilon)}$ is also C^{k-1}-smooth on ϵ. We will achieve this in several steps.

Assume the functions F, g in (1.1) are C^k $(k \geq 2)$. For $p \in \mathbb{N}$, define $\mathcal{X}_0^p$ as the space of the functions $\phi : \mathbb{R} \to \mathbb{R}^n$ such that $\phi \in C_0$ and ϕ is C^p-smooth.

Claim 1. From the definition of P^0 in (2.2), it is clear that $P^0 : C_0 \to C_0$ is linear bounded and that $P_0(\mathcal{X}_0^p) \subset \mathcal{X}_0^p$, for $1 \leq p \leq k-1$.

Claim 2. From the definition of L in (2.11), $L : C_0 \to C_0$ is linear bounded and $L(\mathcal{X}_0^p) \subset \mathcal{X}_0^p$, for $1 \leq p \leq k-1$.

Claim 3. From the definition of $\mathcal{H}$ in (2.10) and (2.5), we have $\mathcal{H}(\cdot, \mathcal{X}_0^{p-1}, \epsilon) \subset \mathcal{X}_0^p$ for $\epsilon > 0$, $p = 1, \ldots, k-1$, where $\mathcal{X}_0^0 = C_0$.

Claim 4. $\mathcal{N}(L) \subset \mathcal{X}_0^{k-1}$.

In fact, from Theorem 3.4 we have $\mathcal{N}(L) = \mathcal{N}(T) = \{\phi \in C^1 : \dot{\phi}(t) = P^0\phi(t), t \in \mathbb{R}\}$. From claim 1, by induction we conclude that $\mathcal{N}(T) \subset \mathcal{X}_0^{k-1}$.

Claim 5. For each $(\xi, \epsilon) \in \big(\mathcal{N}(L) \cap \overline{B(\sigma)}\big) \times (0, \epsilon^*)$, the fixed point $\phi^* := \phi_{(\xi,\epsilon)} \in \mathcal{X}_0^1$.

To prove this claim, we fix $(\xi, \epsilon) \in \big(\mathcal{N}(L) \cap \overline{B(\sigma)}\big) \times (0, \epsilon^*)$, and define $\psi^* = \phi^* + \xi$. From $\phi^* = \mathcal{F}(\phi^*, \xi, \epsilon)$, we obtain

$$L\psi^* = \mathcal{H}(\cdot, \psi^*, \epsilon),$$

or equivalently,

$$\psi^*(s) = \mathcal{H}(s, \psi^*, \epsilon) + \int_{-\infty}^{s} e^{-(s-t)}[\psi^*(t) + P^0\psi^*(t)]dt, \quad s \in \mathbb{R}.$$

Hence $\psi^* \in \mathcal{X}_0^1$. From claim 4, we conclude that $\phi^* \in \mathcal{X}_0^1$.

Claim 6. The fixed point $\phi^* = \phi_{(\xi,\epsilon)}$ is C^1-smooth with respect to ϵ.

Consider $\mathcal{F}$ restricted to $\phi \in X \cap \overline{B(\sigma)} \cap \mathcal{X}_0^1$; more precisely, using claims 2 and 3 we consider

$$\mathcal{F}^1 : \big(X \cap \overline{B(\sigma)} \cap \mathcal{X}_0^1\big) \times \big(\mathcal{N}(L) \cap \overline{B(\sigma)}\big) \times (0, \epsilon^*) \to X \cap \overline{B(\sigma)} \cap \mathcal{X}_0^1,$$
$$\mathcal{F}^1(\phi, \xi, \epsilon) = \mathcal{F}(\phi, \xi, \epsilon).$$

Notice that $\mathcal{F}^1$ is a uniform contraction of $\phi \in X \cap \overline{B(\sigma)} \cap \mathcal{X}_0^1$ for the norm $\|\cdot\|_{C_0}$, and that $\mathcal{F}^1$ is a C^1-mapping on (ϕ, ξ, ϵ). In fact, for $\psi(s) = \phi(s) + \xi(s)$ C^1-smooth on s, from the definition of $\mathcal{H}$ and $\mathcal{G}$ in (2.10) and (2.5), we conclude that $\frac{\partial \mathcal{H}}{\partial \epsilon}(s, \psi, \epsilon)$ exists and is continuous. In claim 5, we have proven that there exists a fixed point $\phi^* = \phi_{(\xi,\epsilon)}$ of $\mathcal{F}^1$. By repeating the arguments used to prove the differentiability of the fixed point in the uniform contraction principle (see e.g. p. 25-26 of Chow and Hale [3]), we conclude that $\phi_{(\xi,\epsilon)}$ is a C^1-smooth mapping on (ξ, ϵ).

Claim 7. The fixed point $\phi^* = \phi_{(\xi,\epsilon)}$ is C^{k-1}-smooth with respect to ϵ.

As in claim 5, by induction we prove that $\phi_{(\xi,\epsilon)}(\cdot) \in \mathcal{X}_0^p$, $p = 2, \ldots, k-1$. By using claims 2 and 3, we consider now

$$\mathcal{F}^p : \big(X \cap \overline{B(\sigma)} \cap \mathcal{X}_0^p\big) \times \big(\mathcal{N}(L) \cap \overline{B(\sigma)}\big) \times (0, \epsilon^*) \to X \cap \overline{B(\sigma)} \cap \mathcal{X}_0^p,$$
$$\mathcal{F}^p(\phi, \xi, \epsilon) = \mathcal{F}(\phi, \xi, \epsilon), \quad p = 2, \ldots, k-1.$$

As in the proof of the uniform contraction principle, by an inductive argument we conclude that $\phi^* = \phi_{(\xi,\epsilon)}$ is C^{k-1}-smooth with respect to ϵ. ∎

Remark 5.2 In some applications, the diffusion process does not apply to all state variables and thus the model is of a mixed type such as

$$\begin{cases} \dfrac{\partial u}{\partial t} = D\Delta u + F\Big(W, \displaystyle\int_{\Omega_r} d\alpha(\theta, y) f\big(W(x+y, t+\theta)\big)\Big), \\ \dfrac{\partial v}{\partial t} = G\Big(W, \displaystyle\int_{\Omega_r} d\beta(\theta, y) g\big(W(x+y, t+\theta)\big)\Big) \end{cases} \tag{5.6}$$

with $u \in \mathbb{R}^m$, $v \in \mathbb{R}^m$, and $W = (u, v)^T$. This system can be regarded as a special case of Eq. (1.1) if we allow some of the diffusion coefficients $d_i's$ to be zero. We remark that under the same assumptions **(H1)**–**(H4)** on the nonlinearities F,f, G and g, Theorem 1.1 remains true for system (5.6). In fact, if for some index i, the diffusion coefficient d_i is zero in Eq. (1.1), then we have $\alpha_i^\epsilon = -1$ and $\beta_i^\epsilon = \infty$. Consequently, the nonlinear function $\mathcal{H}_i$ (see (2.10) in Section 2) in the equation for the variable w_i will be reduced to

$$\mathcal{H}_i(s, w, \epsilon)(s) = \int_{-\infty}^{s} e^{-(s-t)} \mathcal{G}_i(\epsilon, t, w) dt.$$

It is apparent that all results presented so far remain valid without any change.

6. Applications to a Non-local Delayed RD-System with Non-Monotone Birth Functions

Our main result, Theorem 1.1, relates the existence of traveling wave fronts for the reaction diffusion equation (1.1) with delay and non-local interaction to the existence of a connecting orbit between two hyperbolic equilibria of the associated ordinary delay differential equation (1.2). This enables us to apply some existing results for invariant curves of order-preserving semiflows generated by ordinary delay differential equations to derive systematically sharp sufficient conditions for the existence of traveling wave fronts of delayed reaction diffusion equations that, in turn, include most of the existing results in the literature as special cases. In this section, we illustrate this by a recently derived non-local delayed reaction diffusion equation for the population growth of a single species when the delayed birth function is not monotone in the considered range.

We start with a short review of relevant results for the existence of heteroclinic orbits in monotone dynamical systems. Let X be an ordered Banach space with a closed cone K. For $u, v \in X$ we write $u \geq v$ if $u - v \in K$, and $u > v$ if $u \geq v$ but $u \neq v$.

Lemma 6.1 *Let U be a subset of X and $\Phi : [0,\infty) \times U \to U$ be a semiflow such that*

(i) *Φ is strictly order-preserving, i.e., $\Phi(t,u) > \Phi(t,v)$ for $t \geq 0$ and for all $u, v \in U$ with $u > v$;*

(ii) *for some $t_0 > 0$, $\Phi(t_0, \cdot) : U \to U$ is set-condensing with respect to a measure of noncompactness.*

Suppose $u_2 > u_1$ are two equilibria of Φ and assume $[u_1, u_2] := \{u : u_2 \geq u \geq u_1\}$ contains no other equilibria. Then there exists a full orbit connecting u_1 and u_2. Namely, there is a continuous function $\phi : \mathbb{R} \to U$ such that $\Phi(t, \phi(s)) = \phi(t+s)$ for all $t \geq 0$ and all $s \in \mathbb{R}$, and either (a). $\phi(t) \to u_1$ as $t \to \infty$ and $\phi(t) \to u_2$ as $t \to -\infty$; or (b). $\phi(t) \to u_1$ as $t \to -\infty$ and $\phi(t) \to u_2$ as $t \to \infty$.

In applications, one can easily distinguish the above cases (a) and (b) by looking at the stability of the equilibria. For detailed discussions and related references, see Wu, Freedman and Miller [29], Matano [17], Polacik [19], Dance and Hess [5] and Smith [22,23].

Returning to systems (1.1) and (1.2), we use the standard phase space for (1.2). In this section, C will denote the Banach space $C = C([-r,0];\mathbb{R}^n)$ of continuous $\mathbb{R}^n$-valued functions on $[-r, 0]$ with the usual supremum norm. Under the smoothness condition on F, system (1.2) generates a (local) semiflow on C given by

$$\Phi(t,\phi) = u(\phi)(t+\cdot), \quad t \geq 0, \phi \in C,$$

for all those t for which a unique solution $u(\phi)$ of (1.2) with $u(\phi)(\theta) = \phi(\theta)$ for $\theta \in [-r, 0]$ is defined. Let B be an $n \times n$ quasipositive matrix, that is, $B + \lambda I \geq 0$ for all sufficiently large λ. Here and in what follows, we write $A \geq B$ for $m \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ if and only if $a_{ij} \geq b_{ij}$ for $1 \leq i \leq m, 1 \leq j \leq n$. Define

$$K_B = \{\phi \in C : \phi \geq 0, \phi(t) \geq e^{B(t-s)}\phi(s), -r \leq s \leq t \leq 0\}.$$

Then K_B is a closed cone in C and this induces a partial order on C, denoted by $\geq_B$. Namely, $\phi \geq_B \psi$ if and only if $\phi - \psi \in K_B$.

We will need the following conditions:

(O_B) $\hat{E}_2 \geq_B \hat{E}_1$, here $\hat{E}_i$ is the constant mapping on $[-r, 0]$ with the value E_i, $i = 1, 2$.

(M_B) Whenever $\phi, \psi \in C$ with $\phi \geq_B \psi$, then

$$F(\phi(0), \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(\phi(\theta))) - F(\psi(0), \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(\psi(\theta))) \geq B[\phi(0) - \psi(0)].$$

Under the above assumptions, Smith and Thieme [25] proved the following

Lemma 6.2 *Assume that there exists an $n \times n$ quasipositive matrix B such that (O_B) and (M_B) are satisfied. Then*

(i) $[E_1, E_2]_B := \{\phi \in C : \hat{E}_2 \geq_B \phi \geq_B \hat{E}_1\}$ *is positively invariant for the semiflow* Φ*;*

(ii) *the semiflow* $\Phi : [0, \infty) \times [E_1, E_2]_B \to [E_1, E_2]_B$ *is strictly monotone with respect to* $\geq_B$ *in the sense that if* $\phi, \psi \in [E_1, E_2]_B$ *with* $\phi >_B \psi$*, then* $\Phi(t, \psi) >_B \Phi(t, \psi)$ *for all* $t \geq 0$.

In Smith and Thieme [25], it was also shown that (M_B) holds if for all $u, v \in \mathbb{R}^n$ with $\hat{u}, \hat{v} \in [E_1, E_2]_B$ the following is satisfied:

$$\begin{cases} F_u\big(u, \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(v)\big) \geq B, \\ [F_u\big(u, \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(v)\big) - B]e^{Br} + F_v\big(u, \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(v)\big)g'(v) \geq 0. \end{cases}$$

In the case where $n = 1$, it was shown in Smith and Thieme [24] that (M_B) holds for some $B < 0$ if

(S_B) $L_2 < 0, L_1 + L_2 < 0, r|L_2| < 1, rL_1 - \ln(r|L_2|) > 1$, where

$$L_1 = \inf_{E_1 \leq u, v \leq E_2} F_u(u, \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(v))$$

and

$$L_2 = \inf_{E_1 \leq u, v \leq E_2} F_v(u, \int_{-r}^{0} d\eta(\theta)\mu_\Omega g(v))g'(v).$$

Note also that $[E_1, E_2]_B$ is a bounded set in C and that $\Phi(t, \cdot) : C \to C$ is compact for $t > r$. Therefore, for $t_0 > r$, the mapping $\Phi(t_0, \cdot) : [E_1, E_2]_B \to [E_1, E_2]_B$ is compact, and hence is set-condensing. This observation allows us to derive from Lemmas 6.1, 6.2 and Theorem 1.1 the following general result:

Theorem 6.3 *Assume that*

(i) **(H1)**, **(H2)** *and* **(H4)** *are satisfied;*

(ii) *there exists an $n \times n$ quasipositive matrix B such that (O_B) and (M_B) are satisfied;*

(iii) *there exist no other equilibria in* $[E_1, E_2]_B$.

Then the conclusions of Theorem 1.1 hold.

We now apply Theorem 6.3 to a reaction diffusion equation with time delay and nonlocal effect, recently derived by So, Wu and Zou [27], for the total mature population of a single species population with two age classes and a fixed maturation period living in a spatially unbounded environment. In So, Wu and Zou [27], the existence of a traveling wave front was established for the special case when the birth function is the one which appears in the well–known Nicholson's blowflies equation and when the birth function remains monotonically increasing in the interval between the trivial equilibrium and the positive equilibrium representing the maximal capacity of the environment. However, as will be shown below, in a wide range of parameter values, this monotonicity condition is not satisfied and the method developed there cannot be applied. Theorem 6.3 enables us to address the existence of traveling waves when this monotonicity is not satisfied.

Let $u(t, a, x)$ denote the density of the population of the species under consideration at time $t \geq 0$, age $a \geq 0$ and location $x \in R$. It is natural to assume

$$|u(t, a, \pm\infty)| < \infty, \quad \text{for} \quad t \geq 0, \quad a \geq 0. \tag{6.1}$$

A standard argument on population dynamics with age structure and diffusion (c.f. Metz and Diekmann [18]) gives

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial a} = D(a)\frac{\partial^2 u}{\partial x^2} - d(a)u, \tag{6.2}$$

where $D(a)$ and $d(a)$ are the diffusion rate and death rate respectively, at age a. Let $r \geq 0$ be the maturation time for the species. Then the total matured population at time t and location x is given by

$$w(t, x) = \int_r^\infty u(t, a, x)\, da,$$

and using (6.2) and the biologically realistic assumption

$$u(t, \infty, x) = 0, \tag{6.3}$$

we can get

$$\frac{\partial w}{\partial t} = u(t, r, x) + \int_r^\infty \left[D(a)\frac{\partial^2 u}{\partial x^2} - d(a)u\right] da.$$

We assume that the diffusion and death rates for the mature population are age independent, that is, $D(a) = D_m$ and $d(a) = d_m$ for $a \in [r, \infty)$, where D_m and d_m are constants. Furthermore, since only the mature can reproduce, we have

$$u(t, 0, x) = b(w(t, x)), \tag{6.4}$$

where $b(.)$ is the birth function. Then

$$\frac{\partial w}{\partial t} = u(t,r,x) + D_m \frac{\partial^2 w}{\partial x^2} - d_m w. \tag{6.5}$$

Denote by D_I and d_I the diffusion and death rates of the immature respectively, i.e., $D(a) = D_I(a)$ and $d(a) = d_I(a)$ for $a \in [0,r]$. In So, Wu and Zou [27], it was shown that provided

$$\alpha := \int_0^r D_I(a)\, da > 0, \tag{6.6}$$

the term $u(t,r,x)$ can be explicitly written, using a combination of integration along characterisitcs, method of separation of variables and Fourier tranformation, as

$$u(t,r,x) = \frac{e^{-\int_0^r d_I(a)\, da}}{\sqrt{4\pi\alpha}} \int_{-\infty}^{\infty} b(w(t-r,y)) e^{\frac{-(x-y)^2}{4\alpha}}\, dy. \tag{6.7}$$

Hence $w(t,x)$ satisfies

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \frac{e^{-\int_0^r d_I(\theta)\, d\theta}}{\sqrt{4\pi\alpha}} \int_{-\infty}^{\infty} b(w(t-r,y)) e^{\frac{-(x-y)^2}{4\alpha}}\, dy, \quad \text{for } t > r. \tag{6.8}$$

Let

$$\varepsilon = e^{-\int_0^r d_I(a)\, da} \quad \text{and} \quad f_\alpha(x) = \frac{1}{\sqrt{4\pi\alpha}} e^{\frac{-x^2}{4\alpha}}.$$

Then, $0 < \varepsilon \leq 1$ and (6.8) becomes

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \varepsilon \int_{-\infty}^{\infty} b(w(t-r,y)) f_\alpha(x-y)\, dy. \tag{6.9}$$

Eq. (6.9) is a reaction diffusion equation with time delays and nonlocal effects, with ε reflecting the impact of the death rate for immature and α representing the effect of the dispersal rate of the immature on the matured population.

When $\alpha \to 0$, that is, as the immature become immobile, (6.9) reduces to

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \varepsilon b(w(t-r,x)), \tag{6.10}$$

and the nonlocal effect disappears. If we further let $\varepsilon \to 1$, that is, all immatures live to maturity, then Eq. (6.10) becomes

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + b(w(t-r,x)), \tag{6.11}$$

which has been widely studied for different choices of the birth function $b(\cdot)$. In particular, So, Wu and Zou [27] considered a particular birth function for Eq. (6.9) given by $b(w) = pwe^{-aw}$. This function has been used in the well–studied Nicholson's blowflies equation (see Gurney, Blythe and Nisbet [10]). In the discrete case, it is commonly known as the Ricker's model (c.f. Ricker [20]). With this birth function, Eq. (6.9) becomes

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + \varepsilon p \int_{-\infty}^{\infty} w(t-r,y) e^{-aw(t-r,y)} f_\alpha(x-y)\, dy. \tag{6.12}$$

For the case when $D_I(\theta) \equiv 0$ and $d_I(\theta) \equiv 0$, i.e. $\alpha = 0$, $\varepsilon = 1$, (6.12) reduces to

$$\frac{\partial w}{\partial t} = D_m \frac{\partial^2 w}{\partial x^2} - d_m w + pw(t-r,y) e^{-aw(t-r,x)}, \tag{6.13}$$

which was studied in So and Zou [28], where the monotone iteration scheme and the method of upper–lower solutions in Wu and Zou [30,31] were used to show that a traveling wave front exists when $1 < \frac{\epsilon p}{d_m} \le e$. This result was extended to equation (6.12). More precisely, So, Wu and Zou [27] proved the following

Theorem 6.4 *If $1 < \frac{\varepsilon p}{d_m} \le e$, then there exists a $c^* > 0$ such that for every $c > c^*$, (6.12) has a traveling wave front solution, which connects the trivial equilibrium $w_1 = 0$ to the positive equilibrium $w_2 = \frac{1}{a} \ln \frac{\varepsilon p}{d_m}$.*

Unfortunately, in the case when $\frac{\epsilon p}{d_m} > e$, the method developed in So, Wu and Zou [27] cannot be used as the involved iteration scheme is no longer monotone. It is suspected that the method developed in Wu and Zou [30] for traveling waves of reaction diffusion equations without local effects and based on a non–standard exponential ordering could be utilized to this case but the construction of a pair of upper-lower solutions seems to be a highly nontrivial task. We are now in the position to confirm this existence by using Theorem 6.3.

We first notice that the associated ordinary differential equation of (6.12) is

$$\frac{dw}{dt} = -d_m w(t) + \epsilon b(w(t-r)) \tag{6.14}$$

with $b(w) = pwe^{-aw}$. If $\frac{\epsilon p}{d_m} > 1$, then (6.14) has exactly two nonnegative equilibria

$$E_1 = 0, \qquad E_2 = \frac{1}{a} \ln \frac{\epsilon p}{d_m}.$$

The corresponding characteristic equations are

$$\Lambda_1(\lambda) := \lambda + d_m - \epsilon p e^{-\lambda r} = 0$$

and

$$\Lambda_2(\lambda) := \lambda + d_m - \epsilon b'(E_2) e^{-\lambda r} = 0,$$

where

$$b'(E_2) = \frac{d_m}{\epsilon}(1 - \ln\frac{\epsilon p}{d_m}).$$

As $\epsilon p > d_m$, we can easily show that the unstable manifold for E_1 is at least one-dimensional. Furthermore, E_1 is hyperbolic for $r \neq r_n,\ n \in \mathbb{N}_0$, where

$$r_n = \frac{2\pi - \arccos(\frac{d_m}{\epsilon p})}{\sqrt{\epsilon^2 p^2 - d_m^2}} + 2n\pi.$$

We now claim that if $e < \frac{\epsilon p}{b d_m} \leq e^2$, then E_2 is asymptotically stable. In fact, in this case,

$$|\epsilon b'(E_2)| = |d_m(1 - \ln\frac{\epsilon p}{d_m})| \leq d_m$$

and hence all zeros of $\Lambda_2(\lambda)$ have negative real parts.

In the case where $\frac{\epsilon p}{d_m} > e^2$, the asymptotical stability of E_2 holds only when the delay r is sufficiently small. Namely, in $\Lambda_2(\lambda) = 0$, we let $\lambda = i\omega$ to get

$$i\omega = -d_m + d_m(1 - \ln\frac{\epsilon p}{d_m})[\cos(\omega r) - i\sin(\omega r)], \tag{6.15}$$

from which we can find the minimal $\hat{r} > 0$ so that (6.15) has a solution $\omega > 0$. This is given by

$$\hat{r} = \frac{\pi - \arccos\frac{1}{\ln\frac{\epsilon p}{d_m} - 1}}{d_m\sqrt{(\ln\frac{\epsilon p}{d_m} - 1)^2 - 1}}. \tag{6.16}$$

It then follows that if $\frac{\epsilon p}{d_m} > e^2$ and $0 \leq r < \hat{r}$ then E_2 is asymptotically stable.

We now choose $B < 0$ so that (S_B) holds. Recall that

$$b'(w) = pe^{-aw}(1 - aw)$$

and

$$b''(w) = pae^{-aw}(aw - 2).$$

Therefore, $b'(w)$ is decreasing on $[0, \frac{2}{a})$ and increasing on $[\frac{2}{a}, \infty)$. Consequently, on $[E_1, E_2]$, we have

$$b'(w) \geq b'_{min} = \begin{cases} b'(E_2) = \dfrac{d_m}{\epsilon}(1 - \ln\dfrac{\epsilon p}{d_m}),\ E_2 < \dfrac{2}{a}, \\ b'(\dfrac{2}{a}) = -\dfrac{p}{e^2},\ E_2 \geq \dfrac{2}{a}. \end{cases} \tag{6.17}$$

For (1.2), we have

$$F(u, v) = -d_m u + \epsilon v, g(w) = b(w) = pwe^{-aw}, \int_{-r}^{0} d\eta(\theta) = 1, \mu_\Omega = 1.$$

Therefore, for L_1, L_2 as in (S_B),

$$L_1 = \inf_{0 \le u,v \le E_2} F_u(u, b(v)) = -d_m < 0$$

and

$$L_2 = \inf_{0 \le u,v \le E_2} F_v(u, b(v))b'(v) = \epsilon b'_{min} < 0.$$

Therefore, (S_B) (and hence (M_B)) holds if

$$r\epsilon|b'_{min}| < 1 \tag{6.18}$$

and

$$\frac{e^{-rd_m}}{r\epsilon|b'_{min}|} > e. \tag{6.19}$$

The latter is equivalent to

$$re^{rd_m}e\epsilon|b'_{min}| < 1. \tag{6.20}$$

Clearly, if (6.20) holds so does (6.18). Therefore, we conclude that (M_B) holds if

$$0 < r < \tilde{r}, \text{ where } \tilde{r} \text{ is the unique solution of } re^{rd_m}e\epsilon|b'_{min}| = 1. \tag{6.21}$$

As $B < 0$, we also have that (O_B) holds. Therefore, from Theorem 6.3, we have

Theorem 6.5 *If $\frac{\varepsilon p}{d_m} > e$, then there exist $r^* > 0$ and $c^* > 0$ such that if $r \in [0, r^*)$ then for every $c > c^*$, (6.12) has a traveling wave, which connects the trivial equilibrium $w_1 = 0$ to the positive equilibrium $w_2 = \frac{1}{a}\ln\frac{\varepsilon p}{d_m}$, where*

$$r^* = \begin{cases} \min\{\hat{r}, \tilde{r}, r_0\}, & \dfrac{\epsilon p}{d_m} > e^2, \\ \min\{\tilde{r}, r_0\}, & \dfrac{\epsilon p}{d_m} \le e^2. \end{cases}$$

As a final remark, we note that in order to apply Theorem 6.3 for specific systems (1.1), all we need to do is to choose the quasipositive matrix B and to verify the hyperbolicity of the two equilibria. It turns out that much of the known results can be obtained as a special case of Theorem 6.3. For example, consider the following Fisher-KPP equation with delay

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x,t)[1 - u(x, t-r)]. \tag{6.22}$$

Using Theorem 6.3, we can get

Corollary 6.6 *There exists $c^* > 0$ such that if $0 \le r \le e^{-1}$ then for any $c > c^*$, equation (6.22) has a traveling wave front with wave speed c.*

To prove the corollary, we note that the corresponding ordinary delay differential equation is

$$\frac{d}{dt}u(t) = u(t)[1 - u(t-r)] := F(u, u(t-r)), \tag{6.23}$$

for which $E_1 = 0$ and $E_2 = 1$. When $u, v \in [0,1]$ we have $F_u(u,v) = 1 - v \ge 0$ and $F_v(u,v) = -u$. Therefore,

$$[F_u(u,v) - B]e^{Br} + F_v(u,v) = [1-v-B]e^B - u = (1-v)e^{Br} - Be^{Br} - u \ge -Be^r - 1 \ge 0$$

as long as $f(B) := -Be^{Br} \ge 1$. This is possible if $r \le e^{-1}$. In this case, we can choose $B = -r^{-1}$ so that $f(B) = r^{-1}e^{-1} = 1$. This verifies (M_B). (O_B) follows from $1 - e^{B(t-s)} \ge 0$ if $-r \le s \le t \le 0$. Note that $\Lambda_1(\lambda) = \lambda - 1$ and $\Lambda_2(\lambda) = \lambda + e^{-\lambda r}$. Thus, E_1 is hyperbolic and its unstable manifold is one-dimensional, and all eigenvalues corresponding to E_2 have negative real parts if $r \le e^{-1} < \frac{\pi}{2}$. This proves Corollary 6.6.

In Wu and Zou [30], it was shown that for any $c > 2$, there exists $r^*(c) > 0$ such that if $0 \le r \le r^*(c)$, then (6.22) has a traveling wave front with wave speed c. Their argument was based on an iterative scheme, coupled with the construction of a pair of upper and lower solutions. Note that our claim above gives an explicit form for r^*.

There is another way to incorporate the time delay to a logistic equation, such as

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + u(x, t-r)[1 - u(x,t)], \tag{6.24}$$

which was also derived by Kobayashi [14] from a branching process. The existence of traveling wave of (6.24) can be obtained by using the general theory of Schaaf [21] or the general monotone iteration technique developed in Zou and Wu [30,31]. It is interesting to note that this existence result becomes a trivial application of our Theorem 6.3 by choosing $B = -1$, since the corresponding $F(u,v) = v(1-u)$ satisfies $F_u(u,v) = -v \ge -1$ and $F_v(u,v) = 1 - u \ge 0$ for all $u, v \in [0,1]$. It is also clear that $E_2 = 1$ is asymptotically stable, and that $E_1 = 0$ is hyperbolic for $r \ne r_n$, where $r_n = (2n - 1/2)\pi$, $n \in \mathbb{N}$, and its unstable manifold is at least one-dimensional.

Remark 6.7 We consider the nonlinear reaction term F to be of the form given in Equation (1.1) in order to cover sufficiently large classes of equations and, at the same time, to keep the notations relatively in a minimum of complexity. A straightforward extension of the reaction term that has its application can be of the form

$$F(u(x,t), \int_{-r}^{0}\int_{\Omega} d\eta(\theta)d\mu(y)K(\theta,y)g(u(x+y,t+\theta))),$$

where K is a continuous and bounded function from $[-r, 0] \times \Omega$ to $\mathbb{R}^{n\times n}$. In this case, the corresponding reaction equation (1.2) becomes

$$\dot{u}(t) = F(u(t), \int_{-r}^{0} d\eta(\theta)\mu_\Omega(\theta)g(u(t+\theta)))$$

with $\mu_\Omega(\theta) = \int_\Omega d\mu(y)K(\theta, y)$. One can see that all arguments developed in the paper are still valid and Theorem 1.1 remains true for this more general form.

Remark 6.8 Our focus in this paper is on the existence of traveling waves for the delayed reaction-diffusion equation (1.1) in the neighborhood of a heteroclinic orbit of the corresponding ordinary delay differential equation (1.2). Whether some qualitative properties of the heteroclinic orbits such as monotonicity can be inherited by the traveling waves remains to be an interesting problem. We note, however, that if (1.2) is a monotone system that has a monotone heteroclinic solution u^* connecting E_1 and E_2, then we are able to use a traveling wave solution $V(t)$ of (1.4) near u^* to construct a monotone increasing lower and a monotone increasing upper solution for an integral equation equivalent to (1.4). Thus a further monotone iteration argument (see [30, 31]) can be applied to obtain a monotone traveling wave.

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