Asymptotic Stability for Delayed Logistic Type Equations

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Abstract

We study the stability of scalar delayed equations of logistic type with a positive equilibrium and a linear logistic term. The global asymptotic stability of the positive equilibrium, called the carrying capacity, is proven imposing a condition on a negative feedback term without delay dominating the delayed effect. It turns out that this assumption is a necessary and sufficient condition for the linearized equation about the positive equilibrium to be asymptotically stable, globally in the delays. The global stability of more general scalar delay differential equations is also addressed.

Keywords: delayed logistic equation, delayed population model, global asymptotic stability.

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1. Introduction

For the last decades, delay differential equations, and among them delay equations of logistic type, have been extensively used as models in biology and other sciences, with particular emphasis in population dynamics. In this paper, we study the global attractivity of positive equilibria of delayed logistic differential equations which appear as models for the growth of a single species population, in ecology problems or in disease modelling.

There is an extensive literature dealing with scalar delayed logistic and Lotka-Volterra type equations. We refer the reader to the books of Gopalsamy [1] and Kuang [2], and references therein. The use of time-delays in differential equations arises naturally in mathematical models. The introduction of delays produces oscillations, which is a phenomenon in population biology observed from data. In general, large delays imply the loss of stability of equilibria, and even existence of unbounded solutions. To study the behaviour of solutions of delay differential equations, and in particular the stability of equilibria, one approach is to give conditions involving the size of the delays, such as the well-known 3/2-type conditions, so that the FDE is expected to behave similarly

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to an ordinary differential equation if the delays are small enough. This line of investigation goes back to the work of Wright [3], with remarkable extensions given by Yorke [4] , Yoneyama [5], So and Yu [6], to mention only a few authors (see also the recent work [7]).

However, in many situations, it does not seem realistic to assume that the delays are very small. Rather than considering conditions on the size of the delays and coefficients, another approach is to assume that there is a non-delayed negative feedback term which in some sense dominates the delay effect. This is the setting developed by e.g. Seifert [8], Kuang and Smith [9], Györi [10] and Faria and Liz [11] to study logistic models with delays, and pursued in the present paper.

Consider a scalar functional differential equation (FDE) of the general form,

$$\dot{x}(t) = f(t, x_t), \quad t \geq 0, \tag{1.1}$$

where $f : [0, \infty) \times C \to \mathbb{R}$ is continuous, and $C := C([-h, 0]; \mathbb{R})$ is the phase space of continuous functions from $[-r, 0]$ to $\mathbb{R}$, $r > 0$, with the sup norm $\|\varphi\| = \max_{-r\leq\theta\leq 0} |\varphi(\theta)|$. As usual, x_t denotes the function in C defined by $x_t(\theta) = x(t + \theta), -r \leq \theta \leq 0$. Eq. (1.1) often appears as a model for the growth of a single population species, where $x(t)$ denotes the population density at time t. For $f(t, \varphi) = b(t)\varphi(0)[1 - L(\varphi)]$ in (1.1), where $b : [0, \infty) \to (0, \infty)$ and $L : C([-r, 0]; \mathbb{R}) \to \mathbb{R}$ is a bounded linear operator, we obtain a general delayed logistic equation with autonomous linearity,

$$\dot{x}(t) = b(t)x(t)[1 - L(x_t)], \tag{1.2}$$

which was studied in [11]. In biological terms, it is natural to assume that there is a term without delay, which reflects the intrinsic growth of the species, and a delayed logistic term, which corresponds to the influence of crowding. We assume that $L(1) > 0$, so that (1.2) has a positive steady state $x_* = L(1)^{-1}$, usually called the carrying capacity. From the point of view of applications, it is particularly important to study the stability and attractivity of x_*.

The following criterion for the global asymptotic stability of the positive equilibrium x_* of (1.2) was established in [11] (where, actually, a hypothesis slightly stronger than (H1) below was assumed):

Theorem 1.1. *Assume that:*

(H1) (i) $b(t)$ is positive and uniformly bounded on $[0, \infty)$;

(ii) $\int^{\infty} b(t)dt = \infty$;

(H2) for all $\varphi \in C$ such that $\|\varphi\| = \varphi(0) > 0$, then $L(\varphi) > 0$.

Then all positive solutions $x(t)$ of (1.2) are defined for $t \geq 0$ and satisfy $x(t) \to x_*$ as $t \to \infty$, where $x_* = L(1)^{-1}$.

Note that (H2) is a theoretical condition of dominance of the delay effect on the time interval $[t - r, t]$ by the instant effect at time t. In the present paper, we shall follow the framework in [11], whose results are extended here in two directions. On one hand, the global asymptotical stability of the positive equilibrium x_* of (1.2) is proven with (H2) replaced by a weaker condition. On the other hand, the techniques used in [11] for the study of Eq. (1.2) are applied to more general situations

$$\dot{x}(t) = b(t)f(x_t), \tag{1.3}$$

with $b, f : C \to \mathbb{R}$ continuous and b positive.

When studying equations of logistic type, special attention will be given to equations of the form

$$\dot{x}(t) = b(t)x(t)[1 - b_0x(t) - L_0(x_t)], \quad t \geq 0. \tag{1.4}$$

where $b_0 \neq 0$ and $L_0 : C \to \mathbb{R}$ is a bounded linear operator. In Section 3, it will become clear why linear operators L, $L(\varphi) = b_0\varphi(0) + L_0(\varphi)$, that are atomic at zero are chosen. In [11], it was shown that (H2) is fulfilled if either $b_0 > \|L_0\|$, or $b_0 = \|L_0\|$ and L_0 is given by

$$L_0(\varphi) = \int_{-r}^{0} \varphi(\theta)d\eta_0(\theta), \quad \varphi \in C,$$

where $\eta_0 : [-r, 0] \to \mathbb{R}$ is non-decreasing (i.e., L_0 is a *positive* bounded operator), and the additional condition $\eta_0(\theta) < \eta_0(0)$, $-\varepsilon < \theta < 0$, holds for some $\varepsilon > 0$.

Motivated by several examples in the literature, our aim is to improve this result, by proving Theorem 1.1 for (1.4) under the weaker assumption $b_0 = \|L_0\|$. Moreover, we show that this condition is sharp, in a sense to be precised later.

Related to the results presented here, we mention some recent works dealing with general settings of delay differential equations of logistic type, where the analysis of stability is conducted by imposing a condition on a negative feedback term without delay dominating the delay effect. Györi [10] investigated the boundedness of solutions and global stability of non-autonomous logistic equations with several time-dependent discrete delays, and Berezansky et al. [12] continued this study adding a linear harvesting term. The global stability of a delayed logistic equation with a piecewise constant argument was studied by Gopalsamy and Liu [13]. In Huang [14], the author addressed the stability of the positive steady state to an autonomous delay reaction-diffusion equation of logistic type. Hadeler and Bocharov [15] used neutral retarded FDEs in population models. For important applications of logistic type equations with delays, we refer to e.g. the works of Cooke et al. [16] on epidemic models, Drasdo and Höme [17] on tumour growth modelling, and Dyson et al. [18], where the asymptotic behaviour of solutions to several classes of abstract logistic equations was analyzed, with applications to models of structured population dynamics. In fact, we should also mention that many authors use age-structured population models, instead of delays equations, to reproduce the realistic features of ecological or epidemic models (see e.g. [15, 18], also for relevant references).

The remainder of this paper is organized in three sections. Section 2 deals with delayed logistic equations (1.2) with a dominating negative feedback term without delay. After a brief overview of the literature, a criterion for the global asymptotic stability of its positive equilibrium is presented, which improves Theorem 1.1 as well as generalizes other known criteria. In Section 3, we show that the assumptions imposed on L in the previous section can be considered optimal, since they are necessary and sufficient conditions for the linear FDE $\dot{x}(t) = -L(x_t)$ to be exponentially asymptotically stable "independently of the delays", in the sense set in [19, 20]. Finally, in Section 4 we apply the method in [11] to study the asymptotic behaviour of scalar FDEs (1.3), and give sufficient conditions for the global asymptotical stability of equilibria. We illustrate the results with some well-known models.

2. Global stability for a delayed logistic equation with autonomous linearity

In the phase space $C = C([-r, 0]; \mathbb{R})$, consider the scalar delayed logistic equation

$$\dot{x}(t) = b(t)x(t)[1 - L(x_t)], \quad t \geq 0, \tag{2.1}$$

where $b : [0, \infty) \to (0, \infty)$ is a continuous function and $L : C \to \mathbb{R}$ is a non-zero bounded linear operator.

Due to the biological interpretation of model (2.1), only positive solutions are meaningful and therefore *admissible*. For (2.1), we only consider *admissible* initial conditions

$$x_0 = \varphi, \qquad \varphi \in C_0\,, \tag{2.2}$$

where C_α denotes the set

$$C_\alpha := \{\varphi \in C : \varphi(\theta) \geq \alpha \text{ for } \theta \in [-r, 0) \text{ and } \varphi(0) > \alpha\}, \tag{2.3}$$

for $\alpha \in \mathbb{R}$. The initial value problem (IVP) (2.1)-(2.2) has a unique solution, denoted by $x(\varphi)(t)$ or simply $x(t)$.

Suppose that $L(1) > 0$. Then $x_* = L(1)^{-1}$ is the unique positive equilibrium of equation (2.1). By the change of variables $y = (x - x_*)/x_*$, (2.1) is written in the form

$$\dot{y}(t) = -x_* b(t)(1 + y(t))L(y_t) \tag{2.4}$$

with C_{-1} as the set of admissible initial conditions. Clearly, if $y(t) = y(\varphi)(t)$ is a solution of (2.4) with initial condition $\varphi \in C_{-1}$, then $y(t) > -1$ for $t \geq 0$, so $y(t)$ is admissible.

We turn our attention to the particular situation of

$$\dot{x}(t) = b(t)x(t)[1 - b_0 x(t) - L_0(x_t)], \quad t \geq 0. \tag{2.5}$$

where $b_0 \neq 0$ and $L_0 : C \to \mathbb{R}$ is a bounded linear operator, i.e., the operator L in (2.1) is written as

$$L(\varphi) = b_0\varphi(0) + L_0(\varphi). \tag{2.6}$$

For a scalar FDE with a finite number of discrete delays, it is known that the result in Theorem 1.1 remains true if $b_0 = \|L_0\|$. In fact, consider the equation

$$\dot{x}(t) = x(t)[\alpha - b_0x(t) - \sum_{i=1}^{n} b_ix(t-r_i)], \tag{2.7}$$

where the constants $\alpha, b_0, r_1, \dots, r_n$ are positive and $b_1, \dots, b_n \in \mathbb{R}$, studied by Lenhart and Travis [21] and other authors. For the non-autonomous version of Eq. (2.7), see [10, 12].

Let $\sum_{i=0}^{n} b_i > 0$, so that there is a positive equilibrium $x_* = \alpha\big(\sum_{i=0}^{n} b_i\big)^{-1}$. If in addition $b_0 > \sum_{i=1}^{n} |b_i|$, then (H2) is satisfied, hence x_* is globally asymptotically stable. However, in [21] this result was obtained under the weaker hypothesis

$$\sum_{i=0}^{n} b_i > 0, \quad b_0 \geq \sum_{i=1}^{n} |b_i|, \tag{2.8}$$

by using Liapunov functional techniques. (See also [2, p. 35] for a correction of [21].)

Eq. (2.7) has the form (2.5) with $L_0(\varphi) = \sum_{i=1}^{n} b_i\varphi(-r_i)$, and is a particular case of the general scalar Lotka-Volterra equation with distributed delays

$$\dot{x}(t) = b(t)x(t)\left[1 - b_0x(t) - b_1\int_{-r}^{0} x(t+\theta)d\mu_1(\theta) + b_2\int_{-r}^{0} x(t+\theta)d\mu_2(\theta)\right], \tag{2.9}$$

where b_0, b_1, b_2 are non-negative constants and $\mu_i : [-r, 0] \to \mathbb{R}$ are non-decreasing functions, normalized so that $\int_{-r}^{0} d\mu_i(\theta) = 1, \ i = 1, 2$.

Observe that any linear bounded operator $L_0 : C \to \mathbb{R}$ can be written as the difference of two linear positive operators. In fact, let $BV[-r, 0]$ be the space of functions of bounded variation from $[-r, 0]$ to $\mathbb{R}$; for $\eta_0 \in BV[-r, 0]$ such that $L_0(\varphi) = \int_{-r}^{0} \varphi(\theta)d\eta_0(\theta), \ \varphi \in C$, we write $\eta_0 = \eta_1 - \eta_2$, where $\eta_i : [-r, 0] \to \mathbb{R}$ are non-decreasing, $i = 1, 2$. Thus $L_0(\varphi) = L_1(\varphi) - L_2(\varphi)$, where $L_i(\varphi) = \int_{-r}^{0} \varphi(\theta)d\eta_i(\theta), i = 1, 2$ are positive linear operators. Therefore, (2.5) can always be written in the form (2.9).

Eq. (2.9) with $b(t) \equiv \gamma > 0$ was investigated by Kuang [2, pp. 34-37]. Again with the use of a Liapunov functional, Kuang obtained the global asymptotic stability of $x_* = (b_0 + b_1 - b_2)^{-1}$ under conditions

$$b_2 - b_1 < b_0, \quad b_1 + b_2 \leq b_0. \tag{2.10}$$

It is clear that if (2.9) has the form (2.7), then (2.10) translates as (2.8). On the other hand, for L_0 in (2.5) given by $L_0 = L_1 - L_2$, with L_1, L_2 positive linear operators, then $\|L_i\| = L_i(1)$, and (2.10) reads as

$$b_0 + \|L_1\| > \|L_2\|, \quad b_0 \geq \|L_1\| + \|L_2\|. \tag{2.11}$$

Here, we generalize the results in [2, 21]: not only do we consider (2.5) with a general continuous function $b(t)$ (instead of a positive constant), but also condition $b_0 \geq \|L_1\| + \|L_2\|$ in (2.11) is replaced by the weaker condition $b_0 \geq \|L_0\|$. Moreover, our result follows without considering Liapunov functionals.

A preliminary lemma is required, which will be stated here in the general framework of scalar FDEs

$$\dot{y}(t) = f(t, y_t), \tag{2.12}$$

for which a set $S \subset C$ is chosen as the set of *admissible initial conditions*. As before, a solution $y(t)$ of (2.12) with initial conditions $y_0 = \varphi \in S$ is said to be *admissible* if $y_t \in S$, $t > 0$, whenever y_t is defined.

The next lemma is inspired by a result in [22] (see also [11]), and in particular gives a negative feedback condition for boundedness of solutions of (2.12).

Lemma 2.1. *Suppose that $f : [0, \infty) \times C \to \mathbb{R}$ is continuous, and*

(H2) for all $t \geq 0$ and $\varphi \in S$ such that $|\varphi(\theta)| < |\varphi(0)|$ for $\theta \in [-r, 0)$, then $\varphi(0) f(t, \varphi) < 0$.*

Then all admissible solutions of (2.12) are defined and bounded on $[0, \infty)$. Furthermore, if there is $t_0 \geq 0$ such that $|y(t)| \leq K$ for $t \in [t_0 - r, t_0]$, then $|y(t)| \leq K$ for $t \geq t_0$.

Proof. Let a solution $y(t)$ be defined on $[-r, a)$, $a > 0$ or $a = \infty$, and $|y(t)| \leq K$ for $t \in [t_0 - r, t_0]$ for some $t_0 \in [0, a)$. Suppose that there exists $t_1 > t_0$ such that $|y(t_1)| > K$. Define $T = \min\{t \in [t_0, t_1] : |y(t)| = \sup_{t_0 \leq s \leq t_1} |y(s)|\}$. Then $T \in (t_0, t_1]$ and $|y(s)| < |y(T)|$ for $t_0 < s < T$. Since $|y(s)| \leq K < |y(T)|$ for $s \in [t_0 - r, t_0]$, in particular we obtain $|y_T(\theta)| < |y(T)|$ for $-r \leq \theta < 0$. If $y(T) > 0$, then $y(s) < y(T)$ for $t_0 - r \leq s < T$, and we deduce that $\dot{y}(T) \geq 0$. But this leads to a contradiction, since (H2*) implies that

$$\dot{y}(T) = f(T, x_T) < 0.$$

The case $y(T) < 0$ is treated in a similar way. ∎

For the situation $f(t, \varphi) = -x_* b(t)(1 + \varphi(0)) L(\varphi)$ and $S = C_{-1}$ in (2.4), hypothesis (H2*) in Lemma 2.1 is clearly weaker than (H2) in Theorem 1.1, and translates simply as the following condition, still denoted by (H2*):

(H2*) for all $\varphi \in C$ such that $|\varphi(\theta)| < \varphi(0)$ for $\theta \in [-r, 0)$, then $L(\varphi) > 0$.

We now prove the main result in this section.

Theorem 2.2. *Consider Eq. (2.5), with $b(t)$ continuous, $b_0 > 0$ and $L_0 : C \to \mathbb{R}$ a bounded linear operator. Assume that (H1) holds and*

$$b_0 + L_0(1) > 0, \quad b_0 \geq \|L_0\|. \tag{2.13}$$

Then the positive equilibrium $x_* = (b_0 + L_0(1))^{-1}$ of (2.5) is globally asymptotically stable (in the set of all positive solutions).

Proof. Define L by $L(\varphi) = b_0\varphi(0) + L_0(\varphi)$, $\varphi \in C$, and set $l = L(1) = b_0 + L_0(1)$. Condition (2.13) implies that $L(1) > 0$, and L satisfies (H2*). In fact, consider $\varphi \in C$ such that $|\varphi(\theta)| < \varphi(0)$ for $\theta \in [-r, 0)$. For $m = \min_{\theta\in[-r,0]} \varphi(\theta)$, define $\varepsilon = (m + \varphi(0))/2$. We have $\|\varphi - \varepsilon\| = \varphi(0) - \varepsilon$, and from (2.13)

$$L(\varphi) = \varepsilon(b_0 + L_0(1)) + b_0(\varphi(0) - \varepsilon) + L_0(\varphi - \varepsilon) \geq \varepsilon(b_0 + L_0(1)) > 0,$$

and (H2*) holds. From Lemma 2.1, we deduce that all (admissible) solutions are defined, bounded and bounded away below from zero on $[0, \infty)$, and that x_* is uniformly stable.

It remains to prove that x_* is globally attractive. Translating x_* to the origin by setting $y = (x - x_*)/x_*$, Eq. (2.5) is written in the form (2.4) with L as above. Consider now a solution $y(t) = y(\psi)(t)$ of (2.4), for $\psi \in C_{-1}$.

If $y(t)$ is eventually monotone, one can show that $y(t) \to 0$ as $t \to \infty$ by adapting the arguments in [11], where the result was proven with (H1)(ii) replaced by the stronger condition $0 < \beta_0 \leq b(t)$, $t \in \mathbb{R}$. See also the proof of Theorem 4.1 in Section 4, for an equation more general than (2.4).

Now suppose that $y(t)$ is not eventually monotone, and define

$$\liminf_{t\to\infty} y(t) = -v, \quad \limsup_{t\to\infty} y(t) = u.$$

Note that u, v are well-defined and $-1 < -v \leq u < \infty$.

Fix $\varepsilon > 0$. There is $T = T(\varepsilon) > 0$ such that

$$-v - \varepsilon < y(t) < u + \varepsilon, \quad t \geq T - r. \tag{2.14}$$

Suppose that $|v| \leq u$ (the case $|u| \leq v$ is similar). In order to get a contradiction, let $u > 0$. With $u_\varepsilon := u + \varepsilon$, then

$$\|y_t\| < u_\varepsilon \quad \text{for} \quad t \geq T.$$

For $t \geq T$, we have

$$\dot{y}(t) = x_* b(t)(1 + y(t))\big(-b_0 y(t) + L_0(-y_t)\big) \leq x_* b_0 b(t)(1 + y(t))(u_\varepsilon - y(t)),$$

implying that

$$\frac{1 + y(t)}{u_\varepsilon - y(t)} \leq \frac{1 + y(t_0)}{u_\varepsilon - y(t_0)} \exp\left((1 + u_\varepsilon)x_* b_0 \int_{t_0}^{t} b(s)ds\right), \quad \text{for } t \geq t_0 \geq T. \tag{2.15}$$

Now consider a sequence (t_n) such that $t_n > T, \dot{y}(t_n) = 0, y(t_n) > 0, y(t_n) \to u$ and $t_n \to \infty$ as $n \to \infty$.

From (H1), we have $0 < b(t) \leq \beta$ on $[0, \infty)$ for some positive constant β, hence both $y(t)$ and $\dot{y}(t)$ are uniformly bounded on $[0, \infty)$, and $y(t)$ is uniformly continuous on $[0, \infty)$. We obtain that the sequence $(y_{t_n}) \subset C$ is uniformly bounded and equicontinuous, and thus there is a subsequence, still denoted by (y_{t_n}), that converges to a function φ on C. From $y(t_n) \to u, L(y_{t_n}) = 0$, we get $\varphi(0) = u, L(\varphi) = 0$. Since $\|y_{t_n}\| \leq u_\varepsilon$ and $\varepsilon > 0$ is arbitrary, then $\|\varphi\| \leq u$. This implies that $\varphi(\theta) = -u$ for some $\theta \in [-r, 0)$. Otherwise, with $\gamma = \big(u + \min_{\theta \in [-r,0]} \varphi(\theta)\big)/2 > 0$, it follows that $\|\varphi - \gamma\| = u - \gamma$, hence the contradiction $L(\varphi) \geq \gamma(b_0 + L_0(1)) > 0$. Clearly $y(t_n + \theta) \to -u$ as $n \to \infty$. We now apply (2.15) and get

$$(1 + y(t_n))(u_\varepsilon - y(t_n + \theta)) \leq (u_\varepsilon - y(t_n))(1 + y(t_n + \theta)) \exp\left((1 + u_\varepsilon)b_0\beta r\right).$$

By letting $n \to \infty$ and $\varepsilon \to 0^+$, we obtain

$$(1 + u)2u \leq 0,$$

which is not possible. Hence $u = 0$, thus also $v = 0$. This means that $y(t) \to 0$ as $t \to \infty$, and the proof is complete. ■

Under condition (2.11), the requirement of $b(t)$ uniformly bounded can be weakened.

Theorem 2.3. *Consider Eq. (2.5), with $b(t)$ a positive continuous function, $b_0 > 0$ and $L_0(\varphi) = L_1(\varphi) - L_2(\varphi)$, for $L_i : C \to \mathbb{R}$ positive linear operators, $i = 1, 2$. Assume (2.11) and*

$$\sup_{t \geq r} \int_{t-r}^{t} b(s)\, ds < \infty, \qquad \int^{\infty} b(t) dt = \infty. \tag{2.16}$$

Then the positive equilibrium $x_* = (b_0 + L_0(1))^{-1}$ of (2.1) is globally asymptotically stable.

Proof. For L defined as above and satisfying (2.11), we can always assume that $L_1 \neq 0$. Otherwise, for $L_1 = 0$ write $L(\varphi) = \bar{b}_0\varphi(0) + \bar{L}_1(\varphi) - L_2(\varphi)$, where $\bar{b}_0 = b_0 - \alpha, \bar{L}_1(\varphi) = \alpha\varphi(0)$ and $\alpha = (b_0 - \|L_2\|)/2 > 0$, with $\bar{b}_0, \bar{L}_1, L_2$ satisfying (2.11).

As in the above proof, set $y = (x - x_*)/x_*$. Consider now an admissible solution $y(t) = y(\varphi)(t)$ of (2.4), for some $\varphi \in C_{-1}$. Clearly, conditions (2.11) imply (2.13), and all the hypotheses of Lemma 2.1 and Theorem 2.2 are satisfied, except that $b(t)$ is not required to be uniformly bounded. Thus only the treatment of the case of $y(t)$ not eventually monotone has to be adjusted to this situation.

Suppose that $y(t)$ is not eventually monotone. With the notation in the above proof, consider the case $|v| \leq u$ (the case $|u| \leq v$ is similar). If $u > 0$, fix $\varepsilon > 0$ with $u - \varepsilon > 0$, and let $T = T(\varepsilon) > 0$ be such that

$$\|y_t\| < u + \varepsilon \quad \text{for} \quad t \geq T.$$

Consider a sequence $(y(t_n))$ of local maxima, $t_n \to \infty$ as $n \to \infty$, with $|y(t_n) - u| < \varepsilon$ and $t_n > T$. Define $\varepsilon_0 = 2\varepsilon\|L_2\|/\|L_1\|$ and $m_n = \min\limits_{[-r,0]} y_{t_n}$. Then,

$$\begin{aligned}\dot{y}(t_n) = 0 = L(y_{t_n}) &\geq b_0 y(t_n) + m_n\|L_1\| - (u+\varepsilon)\|L_2\| \\ &\geq (y(t_n) + m_n)\|L_1\| + (y(t_n) - u - \varepsilon)\|L_2\| \geq (y(t_n) + m_n - \varepsilon_0)\|L_1\|,\end{aligned}$$

implying that $m_n \leq -y(t_n) + \varepsilon_0$. For each $n \in \mathbb{N}$, let $\theta_n \in [-r, 0]$ with

$$-u - \varepsilon < y(t_n + \theta_n) \leq -y(t_n) + \varepsilon_0,$$

hence $y(t_n + \theta_n) \to -u$ as $n \to \infty$. From (2.15) with $t = t_n$ and $t_0 = t_n + \theta_n$, we obtain

$$(1 + y(t_n))(u_\varepsilon - y(t_n + \theta_n)) \leq (u_\varepsilon - y(t_n))(1 + y(t_n + \theta_n)) \exp\left((1 + u_\varepsilon) b_0 \int_{t_n - r}^{t_n} b(s)\, ds\right).$$

Since $\sup_{t \geq r} \int_{t-r}^{t} b(s)\, ds < \infty$, by letting $n \to \infty$ and $\varepsilon \to 0^+$, we obtain the contradiction $(1+u)2u \leq 0$. Thus $u = 0$, and $y(t) \to 0$ as $t \to \infty$. ■

Corollary 2.4. *Consider Eq. (2.9), where b_0, b_1, b_2 are non-negative constants, μ_i are non-decreasing functions with $\int_{-r}^{0} d\mu_i(\theta) = 1$, $i = 1, 2$, and define $L_0(\varphi) = b_1 \int_{-r}^{0} \varphi(\theta)\, \mu_1(\theta) - b_2 \int_{-r}^{0} \varphi(\theta)\, \mu_2(\theta)$, $\varphi \in C$. If either (H1) and (2.13), or (2.10) and (2.16), are satisfied, then the equilibrium $x_* = (b_0 + b_1 - b_2)^{-1}$ is globally asymptotically stable.*

3. Linear FDEs and stability globally in the delays

This section is devoted to the study of necessary and sufficient conditions for scalar linear FDEs to be globally asymptotically stable independently of the delays (in a sense to be precised later), as well as the relationship between these conditions and the assumptions on L given in Theorem 2.2.

To motivate this study, consider (2.1) with $b(t) \equiv 1$, and suppose that $L(1) > 0$. Without loss of the generality, set $L(1) = 1$. Translating the unique positive equilibrium $x_* = 1$ to the origin, the equation becomes

$$\dot{y}(t) = -L(y_t) + f(y_t), \tag{3.1}$$

where $f(\varphi) = -\varphi(0)L(\varphi)$, $\varphi \in C$, which can be interpreted as a perturbation of the linear FDE

$$\dot{y}(t) = -L(y_t). \tag{3.2}$$

It is therefore natural to relate the global stability of the equilibrium x_* of (2.1) to the global asymptotical stability of the trivial solution of (3.2).

As in (2.6), write $L(\varphi) = b_0\varphi(0) + L_0(\varphi)$, $\varphi \in C$, and

$$L_0 = L_1 - L_2, \quad L_i(\varphi) = b_i \int_{-r}^{0} \varphi(\theta)\, d\mu_i(\theta), \quad \varphi \in C, \tag{3.3}$$

with $b_i \geq 0$ and

$$\mu_i : [-r, 0] \to \mathbb{R} \quad \text{nondecreasing}, \quad \mu_i(0) - \mu_i(-r) = 1, \quad i = 1, 2. \tag{3.4}$$

From Lenhart and Travis' results in [23], for given constants $b_0 \in \mathbb{R}, b_1, b_2 \geq 0$, it turns out that (2.11), or in other words (2.10), is a necessary and sufficient condition for all the characteristic roots of (3.2) to have negative real parts, whenever μ_i $(i = 1, 2)$ satisfy (3.4). So, (2.11) is equivalent to saying that the zero solution of (3.2) is "absolutely" exponentially asymptotically stable, in the sense that this remains true if the positive linear operators L_1, L_2 are replaced by others with the same norm. Since $f \in C^1$ with $f(0) = 0, Df(0) = 0$, from [24, p. 281] we also conclude that the zero solution of (3.1) is exponentially asymptotically stable, independently of μ_i.

The same type of stability is deduced when L satisfies (H2). In fact, for $\lambda = a + ib \in \mathbb{C}$ and $a \geq 0$, for $\varphi(\theta) = e^{a\theta} \cos b\theta$, $-r \leq \theta \leq 0$, from (H2) we obtain $a + L(\varphi) \geq L(\varphi) > 0$, so that again all the characteristic roots of (3.2) have negative real parts. However, rather than (H2) or (2.11), we note that our main result Theorem 2.2 was proven under the weaker assumption (2.13).

In the case of discrete delays $L(\varphi) = b_0\varphi(0) + \sum_{i=1}^{n} b_i\varphi(-r_i)$, (2.11) and (2.13) are equivalent, and translate as (2.8). It is easy to see that condition (2.8) is also equivalent to saying that $L(1) > 0$ and L satisfies (H2*). On the other hand, condition (2.8) holds if and only if the *linear* FDE $\dot{y}(t) = -L(y_t)$ is exponentially asymptotically stable independently of the delays $r_i, i = 1, \ldots n$ (cf. [19, 20, 23]).

This raises the following interesting question. Consider $b_0 \in \mathbb{R}$ and a bounded linear operator $L_0 = L_1 - L_2$ defined by (3.3)-(3.4), and suppose that all the characteristic roots of equation $\dot{y}(t) = -b_0 y(y) - L_0(y_t)$ have negative real parts, independently of the choices of μ_i satisfying (3.4). Assuming also that $b(t)$ satisfies (H1), the question is whether this is sufficient for the positive equilibrium $x_* = (b_0 + L_0(1))^{-1}$ of (2.5) to be globally asymptotically stable. We shall prove that this is true, and analyse condition (2.13) in this setting.

The next lemmas not only justify our choice of operators L in the form (2.6), but also show that hypothesis (2.13) is a natural one. Recall that a linear bounded operator $L \in \mathcal{L}(C; \mathbb{R})$, $L(\varphi) = \int_{-r}^{0} \varphi(\theta)\, d\eta(\theta)$ with $\eta \in BV[-r, 0]$, is said to be *atomic at zero* if η is atomic at zero, i.e., $\eta(0) - \eta(0^-) \neq 0$.

Lemma 3.1. *Let $L : C \to \mathbb{R}$ be a linear bounded operator such that $L(1) > 0$ and L satisfies (H2*). Then L is atomic at zero. Moreover, L is written in the form (2.6) with $b_0 > 0$, L_0 non-atomic at zero, and $\|L_0\| \leq b_0$.*

Proof. Let $L(\varphi) = \int_{-r}^{0} \varphi(\theta)\, d\eta(\theta),\ \varphi \in C$, for some $\eta : [-r, 0] \to \mathbb{R}$ of bounded variation, with $\eta(0) - \eta(-r) = L(1) > 0$.

Suppose that L is non-atomic at zero. Fix $\varepsilon > 0$, with $\varepsilon < L(1)/3$. Then there is $\delta \in (0, r)$ with $Var_{[-\delta,0]}\eta < \varepsilon$ (cf. [24, p. 256]).

Consider $\varphi : [-r, 0] \to \mathbb{R}$ continuous given by

$$\varphi(\theta) = -\frac{1}{2}, \quad \theta \in [-r, -\delta] \qquad \text{and} \qquad \varphi(\theta) = \frac{3}{2\delta}\theta + 1, \quad \theta \in [-\delta, 0]\,.$$

Clearly, $|\varphi(\theta)| < \varphi(0)$ for $\theta \in [-r, 0)$, hence (H2*) implies that $L(\varphi) > 0$. But this is a contradiction, since

$$\begin{aligned} L(\varphi) &\leq -\frac{1}{2}(\eta(-\delta) - \eta(-r)) + Var_{[-\delta,0]}\eta \\ &= -\frac{1}{2}(\eta(0) - \eta(-r)) + \frac{1}{2}(\eta(0) - \eta(-\delta)) + Var_{[-\delta,0]}\eta \\ &\leq -\frac{1}{2}L(1) + \frac{3}{2}Var_{[-\delta,0]}\eta < 0. \end{aligned}$$

Then L is atomic at zero, and can be written in the form (2.6), where $b_0 = \eta(0) - \eta(0^-) \neq 0$ and L_0 is non-atomic at zero and given by $L_0(\varphi) = \int_{-r}^{0} \varphi(\theta)\, d\eta_0(\theta)$, for $\eta_0(\theta) = \eta(\theta)$ for $\theta \in [-r, 0)$ and $\eta_0(0) = \eta(0^-)$. We prove now that $b_0 > 0$. Let $\varepsilon > 0$ be given, and choose $\varphi_n \in C$ as $\varphi_n(\theta) = 0$ for $\theta \in [-r, -r/n]$, $\varphi_n(0) = 1$ and φ_n linear on $[-r/n, 0]$. Since η_0 is non-atomic at zero and (H2*) holds, we have $0 < L(\varphi_n) \leq b_0 + \varepsilon$ for n large, proving that $b_0 \geq 0$. Hence $b_0 > 0$.

Since η_0 is non-atomic at zero, there is a continuous non-decreasing function $\lambda : [0, r] \to [0, \infty)$ such that $\lambda(0) = 0$ and

$$\Big| \int_{-s}^{0} \varphi(\theta)\, d\eta_0(\theta) \Big| \leq \lambda(s) \max_{-s \leq \theta \leq 0} |\varphi(\theta)|, \quad \varphi \in C, s \in [0, r].$$

If $\|L_0\| > b_0$, there is $\psi \in C$ with $\|\psi\| \leq 1$ and $-b_1 := L_0(\psi) < -b_0$. For each $s \in (0, r)$ fixed, we have

$$\int_{-r}^{-s} \psi(\theta)\, d\eta_0(\theta) \leq -b_1 + \lambda(s).$$

Let $\varepsilon > 0$, and $s = s(\varepsilon) \in (0, r)$ be such that $\lambda(s) < \varepsilon$ Now, consider the continuous function ψ_ε defined by $\psi_\varepsilon(\theta) = \psi(\theta)$ for $\theta \in [-r, -s]$, $\psi_\varepsilon(0) = 1 + \varepsilon$, ψ_ε linear on $[-s, 0]$. Since $\psi_\varepsilon(0) > |\psi_\varepsilon(\theta)|$ for $\theta \in [-r, 0)$, (H2*) implies that $b_0(1 + \varepsilon) + L_0(\psi_\varepsilon) > 0$. On the other hand,

$$L_0(\psi_\varepsilon) \leq \int_{-r}^{-s} \psi(\theta)\, d\eta_0(\theta) + \lambda(s)(1 + \varepsilon) < -b_1 + \varepsilon(2 + \varepsilon),$$

and we obtain a contradiction by letting $\varepsilon \to 0^+$. ∎

Lemma 3.2. *Suppose that $L(\varphi) = b_0\varphi(0) + L_0(\varphi)$, $\varphi \in C$, where $b_0 > 0$ and $L_0 : C \to \mathbb{R}$ is a linear bounded operator. If (2.13) holds, then the zero solution of $\dot{x}(t) = -L(x_t)$ is exponentially asymptotically stable.*

Proof. Consider an eigenvalue λ of $\dot{x}(t) = -L(x_t)$, i.e., λ is a root of the characteristic equation

$$\lambda + b_0 + L_0(e^{\lambda\theta}) = 0, \tag{3.5}$$

where we abuse the notation and write $L_0(\varphi(\theta))$ for $L_0(\varphi)$, $\varphi \in C$. For $\lambda = a + ib$, from (3.5) we have

$$a + b_0 + L_0(e^{a\theta}\cos b\theta) = 0, \quad b + L_0(e^{a\theta}\sin b\theta) = 0. \tag{3.6}$$

If $a \geq 0$, $|L_0(e^{\lambda\theta})| \leq \|L_0\| \leq b_0$, i.e.,

$$\big(L_0(e^{a\theta}\cos b\theta)\big)^2 + \big(L_0(e^{a\theta}\sin b\theta)\big)^2 \leq b_0^2.$$

If $L_0(e^{a\theta}\sin b\theta) = 0$, from (3.6) we get $b = 0$, thus $\lambda = a$ and $0 = a + b_0 + L_0(e^{a\theta}) \geq a$, hence $a = 0$, and $b_0 + L_0(1) = 0$, a contradiction. If $L_0(e^{a\theta}\sin b\theta) \neq 0$, then $|L_0(e^{a\theta}\cos b\theta)| < b_0$, and from (3.6) we obtain $a < 0$, a contradiction. Hence all the roots λ of (3.5) have negative real parts. ∎

It is well-known that in many situations the stability of a linear FDE $\dot{x}(t) = -L(x_t)$ depends on the size of the delay r. In the case of discrete delays, $L(\varphi) = b_0\varphi(0) + \sum_{i=1}^{n} b_i\varphi(-r_i)$, we say that the zero solution of $\dot{x}(t) = -L(x_t)$ is *absolutely globally asymptotically stable* if it is globally asymptotically stable for all values of the delays $r_i, i = 1, \dots, n$. However, for distributed delays it is not always clear how to translate the idea of stability depending on the delay. Here, the notation of [19] is followed. See also [20].

Consider a general bounded linear operator $L : C([-r, 0]; \mathbb{R}) \to \mathbb{R}$, or equivalently a function $\eta \in BV[-r, 0]$, with $L(\varphi) = \int_{-r}^{0} \varphi(\theta)\, d\eta(\theta)$. By considering $\mu(\theta) = \eta(r\theta)$, it is sufficient to consider μ in $BV[-1, 0]$, and $L(\varphi) = \int_{-1}^{0} \varphi(r\theta)\, d\mu(\theta)$. On the other hand, for $r \in C_+[-1, 0]$, the set of non-negative functions $r(\theta)$ defined on $[-1, 0]$, and $\mu \in BV[-1, 0]$, the operator $L(\varphi) = \int_{-1}^{0} \varphi(-r(\theta))\, d\mu(\theta)$ is a bounded linear operator from $C([-r_0, 0]; \mathbb{R})$ to $\mathbb{R}$, for any choice of $r_0 \geq \max_{\theta\in[-1,0]} r(\theta)$. Following Cooke and Ferreira [19], some definitions are given below.

Definition 3.1. Let $b_0 \in \mathbb{R}$, $\mu \in BV[-1, 0]$ be given. We say that the linear FDE

$$\dot{x}(t) = -b_0(t)x(t) - \int_{-1}^{0} x(t - r(\theta))\, d\mu(\theta)\,, \tag{3.7}$$

is *exponentially asymptotically stable globally in the delays* if it is exponentially asymptotically stable for all $r \in C_+[-1, 0]$. Conversely, for a given $r \in C_+[-1, 0]$, the *region of stability* for equation (3.7) is defined as $S_0(r) = \{(b_0, \mu) \in \mathbb{R} \times BV[-1, 0] :$ Eq. (3.7) is exponentially asymptotically stable$\}$. The *region of global stability in the delays* is defined as $S_0 = \cap_{r\in C_+} S_0(r)$.

Theorem 3.3. *Let $b_0 \in \mathbb{R}$, $\mu \in BV[-1,0]$ be given, with μ non-atomic at zero. Then, the following conditions are equivalent:*

(a) $(b_0, \mu) \in S_0$;

(b) $b_0 + \mu(0) - \mu(-1) > 0$ and $b_0 \geq Var_{[-1,0]}\mu$;

(c) $L(1) > 0$ and L satisfies (H2), for every $r \in C_+[-1,0]$, where $L = L(r)$ is defined by*

$$L(\varphi) = b_0\varphi(0) + \int_{-1}^{0} \varphi(-r(\theta))\, d\mu(\theta), \quad \varphi \in C([-r_0, 0]; \mathbb{R}), \tag{3.8}$$

with $r_0 \geq \max_{\theta\in[-1,0]} r(\theta)$.

Proof. For any $r \in C_+[-1,0]$, define $L = L(r)$ as in (3.8), and let $L_0(\varphi) = \int_{-1}^{0} \varphi(-r(\theta))\, d\mu(\theta)$. Hence $\|L_0\| \leq Var_{[-1,0]}\mu$. If (b) holds, then $L = L(r)$ satisfies (2.13), and consequently (H2*). Reciprocally, from Lemma 3.1 it follows that (c) implies (b). On the other hand, from Lemma 3.2 we conclude that (b) implies (a). It remains to prove that (a) implies (b).

For the scalar situation (3.7), from [19, Theorem 3.3] we have that $(b_0, \mu) \in S_0$ if and only if

(i) $b_0 + \mu(0) - \mu(-1) > 0$,

(ii) $b_0 > 0$,

(iii) $\left|(iy + b_0)^{-1} \int_{-1}^{0} \lambda(\theta)\, d\mu(\theta)\right| < 1$, for every $y \neq 0$ and every $\lambda \in C([-1,0]; \Gamma)$,

where Γ denotes the unit circle in the complex plane.

Suppose that (i)–(iii) are satisfied, and consider $L_0 \in \mathcal{L}(C([-1,0]; \mathbb{R}); \mathbb{R})$ given by $L_0(\varphi) = \int_{-1}^{0} \varphi(\theta)\, d\mu(\theta)$ for $\varphi \in C([-1,0]; \mathbb{R})$, so that $\|L_0\| = Var_{[-1,0]}\mu$. If $Var_{[-1,0]}\mu > b_0$, then there is $\varphi \in C, \|\varphi\| \leq 1$, such that $\left|\int_{-1}^{0} \varphi(\theta)\, d\mu(\theta)\right| > b_0$. Defining $\lambda(\theta) = \varphi(\theta) + i\sqrt{1 - \varphi^2(\theta)}$, we have $\lambda \in C([-1,0]; \Gamma)$ and $\left|\int_{-1}^{0} \lambda(\theta)\, d\mu(\theta)\right| > b_0$, which contradicts (iii). This completes the proof. ∎

By a scaling, one can always consider the delay $r = 1$, in which case hypothesis (2.13) translates exactly as condition (b) above. From Theorems 2.2, 3.3 and Lemma 3.1, we conclude the following:

Corollary 3.4. *Let $L : C([-1,0]; \mathbb{R}) \to \mathbb{R}$ be a linear bounded operator. Then, L satisfies $L(1) > 0$ and (H2*) if and only if it has the form*

$$L(\varphi) = b_0\varphi(0) + \int_{-1}^{0} \varphi(\theta)\, d\mu(\theta), \quad \varphi \in C([-1,0]; \mathbb{R}),$$

with $(b_0, \mu) \in S_0$ and μ non-atomic at zero. Furthermore, if $b(t)$ is a continuous functions satisfying (H1), then the positive equilibrium of

$$\dot{x}(t) = b(t)x(t)\Big[1 - b_0x(t) - \int_{-1}^{0} x(t - r(\theta))\, d\mu(\theta)\Big]$$

is globally asymptotically stable, for every $r \in C_+[-1,0]$.

4. Scalar populations models $\dot{y}(t) = b(t)f(y_t)$

We now study the asymptotic behaviour of (admissible) solutions of (1.1) for the particular case of separable variables t and φ, i.e., for equations of the form

$$\dot{y}(t) = b(t)f(y_t), \quad t \geq 0, \tag{4.1}$$

where $b : [0, \infty) \to (0, \infty)$ and $f : C \to \mathbb{R}$ are continuous functions. For (4.1), choose $S = C_\alpha$ as the set of admissible initial conditions, for some $\alpha < 0$ and C_α defined by (2.3), and assume that solutions $y(t) = y(\varphi)(t)$ of (4.1) with initial conditions $y_0 = \varphi \in S$ are admissible.

In the sequel, we suppose that b satisfies (H1) and impose the following generalization of hypothesis (H2) in Theorem 1.1:

(H2) (i) f is bounded on bounded sets of $\overline{S}$;
(ii) for all $\varphi \in \overline{S}$ such that $\|\varphi\| = |\varphi(0)| > 0$, then $\varphi(0)f(\varphi) < 0$.

Note that (H2)(ii) implies the negative feedback condition $cf(c) < 0$ for $c \neq 0$. In particular, $y = 0$ is the unique equilibrium of (4.1).

Eq. (2.4) takes the form (4.1) with $f(\varphi) = -x_*(1 + \varphi(0))L(\varphi)$ and $S = C_{-1}$, and L a linear bounded operator. In this case, (H2)(i) is clearly satisfied, and hypothesis (H2)(ii) above is the direct translation of (H2) in Theorem 1.1.

The generalization of Theorem 1.1 is as follows:

Theorem 4.1. *Assume (H1) and (H2). Then the zero solution of (4.1) is globally asymptotically stable (in the set of all admissible solutions).*

Proof. Since (H2) implies (H2*), from Lemma 2.1 we have that all admissible solutions are defined and bounded on $[0, \infty)$, and that the zero solution is uniformly stable. We now prove that the zero solution is a global attractor.

Let $y(t)$ be a solution of (4.1). If $y(t)$ is eventually monotone, set $c = \lim_{t\to\infty} y(t)$. If $c > 0$, then $f(c) < 0$. On the other hand, $y_t \to c$ as $t \to \infty$ in C and f is continuous, thus $f(y_t) < f(c)/2$ for t large. From (4.1), we obtain, for $t \geq t_0$ and t_0 large,

$$y(t) \leq y(t_0) + \frac{f(c)}{2} \int_{t_0}^{t} b(s)\, ds \to -\infty \quad \text{as} \quad t \to \infty,$$

a contradiction. The case $c > 0$ is treated in a similar way. This proves that $y(t) \to 0$ as $t \to \infty$, for all eventually monotone solutions $y(t)$.

Now, consider the case of $y(t)$ not eventually monotone, and define

$$\liminf_{t\to\infty} y(t) = -v, \quad \limsup_{t\to\infty} y(t) = u.$$

Clearly, either $|v| \le u$ or $|u| \le v$. Suppose that $|v| \le u$ (the case $|u| \le v$ is analogous).

Suppose $u > 0$, and fix $\varepsilon > 0$. For t large, we have $\|y_t\| \le u + \varepsilon$. Consider a sequence (t_n) of local maximum points, $\dot{y}(t_n) = 0, y(t_n) > 0, y(t_n) \to u$ and $t_n \to \infty$ as $n \to \infty$. From (H1)(i) and (H2)(i), and since $y(t)$ is uniformly bounded on $[0, \infty)$, we have $\dot{y}(t)$ also uniformly bounded on $[0, \infty)$. Thus the sequence $(y_{t_n}) \subset S$ is uniformly bounded and equicontinuous, and there is a subsequence, still denoted by (y_{t_n}), that converges to a function φ on $\overline{S}$. From $y(t_n) \to u, f(y_{t_n}) = 0, \|y_{t_n}\| \le u + \varepsilon$, and since $\varepsilon > 0$ is arbitrary, we get $\varphi(0) = u, f(\varphi) = 0, \|\varphi\| \le u$. But this contradicts (H2)(ii). Hence $u = 0$, which implies that $v = 0$, and the conclusion follows. ∎

In applications, scalar delayed population models often take the form (see e.g. [1, 2])

$$\dot{x}(t) = b(t)\big[f(L(x_t)) - g(x(t))\big], \tag{4.2}$$

or

$$\dot{x}(t) = b(t)x(t)\big[f(L(x_t)) - g(x(t))\big], \tag{4.3}$$

where $b, f, g : [0, \infty) \to \mathbb{R}$ are continuous and positive for $x > 0$, and $L : C \to \mathbb{R}$ is a *positive* linear operator normalized so that $\|L\| = 1$. For these equations, assume that there is a unique positive steady state x_*, $f(x_*) = g(x_*)$. For biological reasons, only positive solutions are to be considered, hence C_0 is taken as the set of admissible initial conditions. The change $y(t) = (x(t) - x_*)/x_*$ transforms (4.2) and (4.3) into $\dot{y}(t) = b(t)F(y_t)$, where

$$F(\varphi) = \frac{1}{x_*}\Big[f\Big(x_*(1 + L(\varphi))\Big) - g\Big(x_*(1 + \varphi(0))\Big)\Big] \tag{4.4}$$

and

$$F(\varphi) = (1 + \varphi(0))\Big[f\Big(x_*(1 + L(\varphi))\Big) - g\Big(x_*(1 + \varphi(0))\Big)\Big], \tag{4.5}$$

respectively, with $S = C_{-1}$ as the set of admissible initial conditions.

Theorem 4.2. *Consider equations (4.2) and (4.3), where $b(t)$ satisfies (H1), $f, g : [0, \infty) \to \mathbb{R}$ are continuous and positive for $x > 0$, and $g(0) = 0$. Assume that there is a unique positive x_* such that $f(x_*) = g(x_*)$. For H defined by*

$$H(x, y) = g(x_*(1 + x)) - f(x_*(1 + y)), \quad x > -1, y \ge -1, \tag{4.6}$$

suppose that

$$|y| \le |x| \ne 0 \Rightarrow xH(x, y) > 0\,, \quad x > -1, y \ge -1. \tag{4.7}$$

Then, the solutions $x(t)$ with initial conditions $\psi \in C_0$ are positive for $t \ge 0$, and satisfy $x(t) \to x_*$ as $t \to \infty$.

Proof. For $\psi \in C_0$, all solutions $x(t) = x(\psi)(t)$ of (4.2) and (4.3) are positive for $t \ge 0$ (see e.g. [2, p. 149]). Effect the change $y(t) = (x(t) - x_*)/x_*$, and consider $y = L(\varphi)$ for $\varphi \in C_{-1}$, and $x = \varphi(0)$. One can see that (4.7) implies that F in both (4.4) and (4.5) satisfies condition (H2). ∎

We remark that condition (4.7) in Theorem 4.2 is only used to prove that solutions $x(t)$ that are not eventually monotone go to x_* as $t \to \infty$.

Next result generalizes the criteria in [2], where $b(t)$ was taken constant and slightly stronger assumptions on functions f and g were assumed.

Corollary 4.3. *Assume that f, g are positive for $x > 0$ and:*

(i) $b(t)$ satisfies (H1);

(ii) $g(0) = 0$ and g is strictly increasing;

(iii) either (a) f is strictly increasing, or (b) f is strictly decreasing, or (c) there is $x_M > 0$ such that f is strictly increasing in $[0, x_M]$ and strictly decreasing for $x > x_M$;

(iv) there is a unique $x_* > 0$ such that

$$f(x) > g(x) \quad \text{for } x \in (0, x_*), \quad f(x) < g(x) \quad \text{for } x > x_*, \qquad \text{and} \tag{4.8}$$

$$|g^{-1}(f(y)) - x_*| < |y - x_*|, \quad y > 0, y \neq x_*. \tag{4.9}$$

Then the equilibrium x_* of (4.2) and (4.3) is globally asymptotically stable.

Proof. Clearly x_* is the unique positive solution of $f(x) = g(x)$. Let f be strictly increasing. Note that in this case (4.8) implies (4.9). For $x > -1, y \geq -1$, $|y| \leq |x| \neq 0$, we have

$$\begin{aligned} f(x_*(1+y)) &\leq f(x_*(1+x))) < g(x_*(1+x)), \quad \text{if } x > 0 \\ f(x_*(1+y)) &\geq f(x_*(1+x)) > g(x_*(1+x)), \quad \text{if } x < 0, \end{aligned}$$

hence (4.7) holds. The cases (iii)(b) and (iii)(c) are proven in a similar way, by noting that (4.9) implies (4.7). ∎

As an illustration, we consider the Nicholson's blowflies equation

$$x'(t) = b(t)[-\delta x(t) + px(t-r)e^{-ax(t-r)}], \quad t \geq 0, \tag{4.10}$$

where $b : [0, \infty) \to (0, \infty)$ is continuous and the constants r, δ, p, a are positive. By a scaling, we may assume $a = 1$. Suppose that $p/\delta > 1$, so that (4.10) has two equilibria: 0 and $x_* = \log(p/\delta)$. For $f(x) = pxe^{-x}$, f has a unique maximum point $x_M = 1$, and $x_* < x_M$ if and only $1 < p/\delta < e$.

The next result shows that the positive equilibrium of (4.10) is globally attractive if $1 < p/\delta \leq e^2$. This result is not new: in fact it was obtained in [25], where also relevant references can be found. However, we use it here to illustrate how several criteria in the literature follow from the general setting developed above.

Corollary 4.4. *Consider (4.10) with* $1 < \beta := p/\delta \le e^2$*, and suppose that* $b(t)$ *satisfies (H1). Then the positive equilibrium* x_* *of (4.10) is absolutely globally asymptotically stable.*

Proof. The function H defined in (4.6) reads as $H(x,y) = \delta x_* H_0(x,y)$ where

$$H_0(x,y) = (1+x) - (1+y)e^{-x_* y}, \quad x > -1, y \ge -1, \tag{4.11}$$

and $x_* = \log \beta$. Define also the auxiliary functions $h(y) = (1+y)\big[e^{-x_* y} - 1\big]$, $\Phi(y) = 1 - y - (1+y)e^{-x_* y}$ for $y \ge -1$. One can easily verify that $h(-1) = 0$, $yh(y) < 0$ for $y > -1, y \ne 0$. If $x_* \ge 2$, then $\Phi'(y) < 0$ for $y \ne 0$, thus $y\Phi(y) < 0$ for $y \ge -1, y \ne 0$.

We now prove that $x_* \ge 2$ implies (4.7). Let $x > -1, x \ne 0$, and $y \ge -1$. If $y = 0$, $xH_0(x,0) = x^2 > 0$. If $0 < y \le x$ or $x \le y < 0$, then $xH_0(x,y) = x[(x-y) - h(y)] > 0$. If $-x \le y < 0$ or $0 < y \le -x$, then $xH_0(x,y) = x[(x+y) + \Phi(y)] > 0$. ∎

We conclude by remarking that the techniques used to prove Theorem 4.1 are applicable to address a more general case of scalar FDEs that can be seen as perturbations of limiting equations (4.1). For $g : [0,\infty) \times C \to \mathbb{R}$ continuous, suppose that

(H3) there is a continuous function $f : C \to \mathbb{R}$ such that $g(t_n, \varphi_n) \to f(\varphi)$ for all sequences $(t_n) \subset [0,\infty)$, $(\varphi_n) \subset \overline{S}$ with $t_n \to \infty$, $\varphi_n \to \varphi$;

(H4) $g(t,\varphi)$ is bounded on sets $[0,\infty) \times K$, with $K \subset \overline{S}$ bounded.

Theorem 4.5. *Assume (H1), (H3), (H4) with the function* f *in (H3) satisfying (H2)(ii). If* $x(t)$ *is a solution of*

$$\dot{x}(t) = b(t) g(t, x_t),$$

defined and bounded on $[0,\infty)$*, then* $x(t) \to 0$ *as* $t \to \infty$*.*

Proof. The proof follows by arguing as in the proof of Theorem 4.1 and is omitted. ∎

The above setting allows consideration of non-autonomous delayed logistic equations which are perturbations of logistic models (2.1). In fact, consider an equation

$$\dot{x}(t) = b(t)x(t)[a(t) - L(t, x_t)], \quad t \ge 0, \tag{4.12}$$

where $a, b : [0,\infty) \to \mathbb{R}, L : [0,\infty) \times C \to \mathbb{R}$ are continuous functions, and $L(t,\cdot)$ is a bounded linear operator for all $t \ge 0$. If

$$a(t) \to a_0 > 0, \quad L(t,\varphi) \to L(\varphi), \quad \text{as} \quad t \to \infty \tag{4.13}$$

for all $\varphi \in C$, Theorems 4.2 establishes conditions for the global attractivity of x_* (in the set of positive solutions), where $x_* = a_0/L(1)$ is the positive equilibrium of the "limiting equation"

$$\dot{x}(t) = b(t)x(t)[a_0 - L(x_t)].$$

In fact, if $b(t)$ satisfies (H1) and the linear operator L in (4.13) satisfies (H2), then we deduce that all bounded positive solutions $x(t)$ of (4.12) tend to x_* as $t \to \infty$. For non-autonomous linearities $L(t, \cdot)$, note that x_* is not a solution of (4.12). For the asymptotic study of (4.12) and related examples, see also [11] and references therein.

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