The N-membranes problem with Neumann type boundary condition

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Abstract. We consider the problem of finding the equilibrium position of N membranes constrained not to pass through each other, under prescribed volumic forces and boundary tensions. This model corresponds to solve variationally a N-system for linear second order elliptic equations with sequential constraints. We obtain interior and boundary Lewy-Stampacchia type inequalities for the respective solution and we establish the conditions for stability in measure of the interior contact zones of the membranes.

1. Introduction

Let Ω be a bounded open subset of $\mathbb{R}^d$ with Lipschitz boundary Γ. Denote by $\boldsymbol{u} = (u_1, \ldots, u_N)$ the equilibrium displacements of N ($N \geq 2$) elastic membranes, each one constrained not to pass through the others, subject to external volumic forces $\mathbf{f} = (f_1, \ldots, f_N)$ and boundary tensions $\mathbf{g} = (g_1, \ldots, g_N)$. The problem consists of minimizing the energy functional

$$E(\boldsymbol{u}) = \int_\Omega \left(\frac{1}{2} \left(a(\boldsymbol{u}, \boldsymbol{u}) + c\, \boldsymbol{u} \cdot \boldsymbol{u} \right) - \boldsymbol{f} \cdot \boldsymbol{u} \right) + \int_\Gamma \left(\frac{1}{2} b\, \boldsymbol{u} \cdot \boldsymbol{u} - \boldsymbol{g} \cdot \boldsymbol{u} \right), \tag{1.1}$$

in the convex set

$$\mathbb{K}_N = \Big\{ \boldsymbol{v} = (v_1, \ldots, v_N) \in \left[H^1(\Omega) \right]^N : v_1 \geq \cdots \geq v_N \text{ a.e. in } \Omega \Big\}, \tag{1.2}$$

where $a(\boldsymbol{u}, \boldsymbol{v}) = \sum_{k=1}^{N} a(u_k, v_k)$, with $a(u, v) = a_{ij} u_{x_i} v_{x_j}$ (using the summation convention for $i, j = 1, \ldots, d$) and $\boldsymbol{u} \cdot \boldsymbol{v}$ denotes the usual internal product between $\boldsymbol{u}$ and $\boldsymbol{v}$.

The N-membranes problem attached to rigid supports was considered in [3] for N linear coercive elliptic operators of second order and extended in [1] to

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quasilinear operators, with smooth coefficients of p-Laplacian type. For general linear second order elliptic operators with measurable coefficients, see also [2].

Although Neumann boundary type problems can also be considered for more general operators, for simplicity, here we assume

$$\begin{cases} a_{ij} \in L^\infty(\Omega),\ a_{ij}=a_{ji}, \quad \exists\,\nu>0\,\forall\xi\in\mathbb{R}^d \quad a_{ij}\xi_i\xi_j \ge \nu|\xi|^2, \\ c\in L^\infty(\Omega),\ b\in L^\infty(\Gamma), \quad c\ge c_0\ge 0,\ b\ge b_0\ge 0, \quad c_0+b_0>0. \end{cases} \tag{1.3}$$

$$\begin{cases} f_1,\dots,f_N \in L^p(\Omega), \quad g_1,\dots,g_N\in L^q(\Gamma), \\ p\ge \frac{2d}{d+2} \quad \text{if } d\ge 3, \quad p>1 \text{ if } d=2, \\ q\ge \frac{2(d-1)}{d} \quad \text{if } d\ge 3, \quad q>1 \text{ if } d=2. \end{cases} \tag{1.4}$$

Here we use $\bigvee$ and $\bigwedge$ for the supremum and infimum, respectively, of two or more functions

$$\bigvee_{k=1}^N \xi_k = \sup\{\xi_1,\dots,\xi_N\}, \qquad \bigwedge_{k=1}^N \xi_k = \inf\{\xi_1,\dots,\xi_N\},$$

and, accordingly, we set $\xi^+ = \xi\vee 0$ and $\xi^- = -(\xi\wedge 0)$.

The minimization problem (1.1)-(1.2) is equivalent to the variational inequality

$$\begin{cases} \boldsymbol{u}\in\mathbb{K}_N : \\ \displaystyle\int_\Omega \big(a(\boldsymbol{u},\boldsymbol{v}-\boldsymbol{u}) + c\,\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u})\big) + \int_\Gamma b\,\boldsymbol{u}\cdot(\boldsymbol{v}-\boldsymbol{u}) \\ \qquad\qquad \displaystyle\ge \int_\Omega \boldsymbol{f}\cdot(\boldsymbol{v}-\boldsymbol{u}) + \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{v}-\boldsymbol{u}), \qquad \forall \boldsymbol{v}\in\mathbb{K}_N. \end{cases} \tag{1.5}$$

For $N=2$ this problem can be considered, when the solution is known, as two one obstacle problems. For $N\ge 3$, the upper and the lower membranes are of this type, but each membrane in between may be considered a solution of a two obstacles problem. This last problem corresponds to a variational inequality with the convex set given in the form

$$\mathbb{K}_\psi^\varphi = \{\xi\in H^1(\Omega):\ \psi\le\xi\le\varphi \text{ a.e. in } \Omega\},$$

where the given obstacles are such that $\psi\le\varphi$. For two obstacles, the Lewy--Stampacchia inequalities for the solution v are

$$f\wedge A\varphi \le Av \le f\vee A\psi \ \text{ a.e. in } \Omega, \qquad g\wedge B\varphi\le Bv\le g\vee B\psi \ \text{ a.e. on } \Gamma, \tag{1.6}$$

where A and B denote the associated differential and boundary operators, respectively,

$$Av = -\left(a_{ij}v_{x_i}\right)_{x_j} + cv, \qquad \text{in } \Omega, \tag{1.7}$$

$$Bv = a_{ij}v_{x_i}n_j + bv, \qquad \text{on } \Gamma, \tag{1.8}$$

$(n_1, \dots, n_d)$ denoting the unit outward normal vector to Γ.

The iteration of these inequalities yields the new set of N inequalities for the solution $\boldsymbol{u}$ of the N-membranes problem, both in Ω and on Γ

$$\bigwedge_{k=1}^{l} f_k \le Au_l \le \bigvee_{k=l}^{N} f_k, \qquad \text{a.e. in } \Omega, \qquad l = 1, \dots, N, \tag{1.9}$$

$$\bigwedge_{k=1}^{l} g_k \le Bu_l \le \bigvee_{k=l}^{N} g_k, \qquad \text{a.e. on } \Gamma, \qquad l = 1, \dots, N, \tag{1.10}$$

which allows to reduce the regularity of the solutions to the corresponding regularity of a system of equations, as shown in the next section. In particular, in the following special cases:

- $f_1 = \ldots = f_N = f$, the solution $\boldsymbol{u}$ of the variational inequality (1.5) satisfies the system of N equations $Au_k = f$ a.e. in Ω, $k = 1, \dots, N$;
- $g_1 = \ldots = g_N = g$, the solution $\boldsymbol{u}$ of the variational inequality (1.5) satisfies the Neumann boundary conditions $Bu_k = g$ a.e. on Γ, $i = 1, \dots, N$, although in the general case we only can say that $\boldsymbol{u}$ satisfies Signorini type boundary conditions.

Another interesting result is the stability of the $\frac{N(N-1)}{2}$ coincidence sets

$$I_{k,l} = \{x \in \Omega : u_k(x) = \cdots = u_l(x) \text{ for a.e. } x \in \Omega\}, \qquad 1 \le k < l \le N, \tag{1.11}$$

the sets of contact of $l - k + 1$ consecutive membranes. Given a subset A of Ω, we denote by χ_A (the characteristic function of A), i.e., $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \in \Omega \backslash A$. As we have shown in [1] this is a consequence of writing the solution of (1.5) as the solution of a semilinear system involving the characteristic functions $\chi_{I_{k,l}}$. We exemplify the argument in the simple case $N = 3$.

For $N = 2$ there is only one possible coincidence set, the contact of u_1 with u_2. If the two forces associated with the two membranes are almost everywhere different in Ω ($f_1 \ne f_2$ a.e. in Ω), then the characteristic function $\chi_{I_{1,2}}$ of $I_{1,2}$ is easily shown to converge strongly in any $L^s(\Omega)$, $1 < s < \infty$, for variations of the forces in $L^p(\Omega)$.

For $N = 3$ there are three possible coincidence sets, the sets $I_{1,2}$, $I_{2,3}$ and $I_{1,3} = I_{1,2} \cap I_{2,3}$. Setting $\chi_{k,l} = \chi_{I_{k,l}}$, $1 \le k < l \le 3$, the characteristic functions $\chi_{k,l}$ of the sets $I_{k,l}$ are shown to converge strongly in any $L^s(\Omega)$, $1 < s < \infty$, for variations of the forces f_1, f_2 and f_3 in $L^p(\Omega)$, as long as

$$f_1 \ne f_2, \qquad f_2 \ne f_3, \qquad f_1 \ne \frac{1}{2}(f_2 + f_3), \qquad \frac{1}{2}(f_1 + f_2) \ne f_3. \tag{1.12}$$

This is a consequence of the fact that the solution $\boldsymbol{u}$ of (1.5) satisfies the system a.e. in Ω,

$$
(1.13)\quad \begin{cases} Au_1 = f_1 + \frac{1}{2}(f_2 - f_1)\chi_{1,2} & + \frac{1}{6}(2f_3 - f_2 - f_1)\chi_{1,3} \\ Au_2 = f_2 - \frac{1}{2}(f_2 - f_1)\chi_{1,2} + \frac{1}{2}(f_3 - f_2)\chi_{2,3} + \frac{1}{6}(2f_2 - f_1 - f_3)\chi_{1,3} \\ Au_3 = f_3 \qquad - \frac{1}{2}(f_3 - f_2)\chi_{2,3} + \frac{1}{6}(2f_1 - f_2 - f_3)\chi_{1,3}. \end{cases}
$$

Notice that the system (1.13) contains the case $N = 2$, that reduces only to the two first equations of this system, with $I_{2,3} = \emptyset$ (so $\chi_{2,3} = \chi_{1,3} = 0$). Even in the more complicated situation of $N > 3$, the stability result can still be extended in the interior of Ω as we show in Section 3. However, the corresponding stability result on the boundary Γ is an open question. In this paper we have chosen to present only the Neumann case when $\Gamma = \partial\Omega$, but all the results are still valid, with simple adaptations, for the mixed problem where $\partial\Omega = \Gamma_0 \cup \Gamma_1$, with Dirichlet data on Γ_0 and Neumann data on Γ_1 (see [7], for instance).

2. The Lewy-Stampacchia inequalities

We begin this section recalling a theorem for the double obstacle problem:

Theorem 2.1. *Suppose that $\psi_1, \psi_2 \in H^1(\Omega)$, $f \in L^p(\Omega)$, $g \in L^q(\Gamma)$, p, q defined as in (1.4). Let u be the solution of the variational inequality*

$$
(2.1)\quad \int_\Omega \Big(a(u, v-u) + cu(v-u) \Big) + \int_\Gamma b(v-u) \geq \int_\Omega f(v-u) + \int_\Gamma g(v-u),
$$

with the assumptions (1.3), in the convex set

$$
(2.2)\quad \mathbb{K}_{\psi_1}^{\psi_2} = \{ v \in H^1(\Omega) : \psi_1 \leq v \leq \psi_2 \ a.e.\ in\ \Omega \}.
$$

If $\big(A\psi_1 - f\big)^+$, $\big(A\psi_2 - f\big)^- \in L^p(\Omega)$ *and* $\big(B\psi_1 - g\big)^+$, $\big(B\psi_2 - g\big)^- \in L^q(\Gamma)$, *then*

$$
(2.3)\quad f \wedge A\psi_1 \leq Au \leq f \vee A\psi_2, \qquad a.e.\ in\ \Omega,
$$

$$
(2.4)\quad g \wedge B\psi_1 \leq Bu \leq g \vee B\psi_2, \qquad a.e.\ on\ \Gamma.
$$

Proof. The proof of this theorem is a simple adaptation of the arguments used for the one obstacle problem with Neumann boundary condition (see, for instance, [9] or [7]). □

Remark 2.2. We observe that both the lower and the upper one obstacle variational inequalities (2.1) in the convex sets

$$
\mathbb{K}_{\psi_1} = \{ v \in H^1(\Omega) : v \geq \psi_1 \text{ a.e. in } \Omega \}
$$

and

$$
\mathbb{K}^{\psi_2} = \{ v \in H^1(\Omega) : v \leq \psi_2 \text{ a.e. in } \Omega \},
$$

can be regarded as particular cases of the double obstacle problem, corresponding formally to $\psi_2 = +\infty$ and $\psi_1 = -\infty$, respectively.

Given N functions $\varphi_1, \ldots, \varphi_N$, we define, for $1 \leq k < l \leq N$, the average of $\varphi_k, \ldots, \varphi_l$ as

$$\langle \varphi \rangle_{k,l} = \frac{\varphi_k + \cdots + \varphi_l}{l - k + 1}. \tag{2.5}$$

Denote

$$\xi_0 = \max \left\{ \langle f \rangle_{1,k} : k = 1, \ldots, N \right\}, \qquad \eta_0 = \max \left\{ \langle g \rangle_{1,k} : k = 1, \ldots, N \right\} \tag{2.6}$$

and, for $k = 1, \ldots, N$,

$$\xi_k = k\big(\xi_0 - \langle f \rangle_{1,k}\big) \qquad \eta_k = k\big(\eta_0 - \langle g \rangle_{1,k}\big) \tag{2.7}$$

We may approximate the solution of (1.5) by the solution of the penalized problem given by the semilinear system with Neumann boundary conditions, for $k = 1, \ldots, N$,

$$\begin{cases} Au_k^\varepsilon + \xi_k \theta_\varepsilon(u_k^\varepsilon - u_{k+1}^\varepsilon) - \xi_{k-1}\theta_\varepsilon(u_{k-1}^\varepsilon - u_k^\varepsilon) \;=\; f_k & \text{in } \Omega, \\ \\ Bu_k^\varepsilon + \eta_k \theta_\varepsilon(u_k^\varepsilon - u_{k+1}^\varepsilon) - \eta_{k-1}\theta_\varepsilon(u_{k-1}^\varepsilon - u_k^\varepsilon) = g_k & \text{on } \Gamma, \end{cases} \tag{2.8}$$

with the conventions $u_0^\varepsilon = +\infty$, $u_{N+1}^\varepsilon = -\infty$, where for $\varepsilon > 0$, θ_ε is defined by $\theta_\varepsilon(s) = -1$ if $s \leq -\varepsilon$, $\theta_\varepsilon(s) = -\frac{s}{\varepsilon}$, if $-\varepsilon < s < 0$ and $\theta_\varepsilon(s) = 0$ for $s \geq 0$.

Proposition 2.3. *With the assumptions (1.3) and (1.4), problem (2.8) has a unique solution* $(u_1^\varepsilon, \ldots, u_N^\varepsilon)$*, bounded independently of* ε *in* $\left[H^1(\Omega)\right]^N$*. Besides that,* $A\boldsymbol{u}^{\boldsymbol{\varepsilon}}$ *and* $B\boldsymbol{u}^{\boldsymbol{\varepsilon}}$ *are bounded independently of* ε *in* $\left[L^p(\Omega)\right]^N$ *and in* $\left[L^q(\Gamma)\right]^N$*, respectively.*

Proof. Consider the monotone operator

$$\begin{aligned} \langle \Psi_\varepsilon(\boldsymbol{v}), \boldsymbol{w} \rangle \;=\; & \sum_{k=1}^N \int_\Omega \Big(\xi_k \theta_\varepsilon(v_k - v_{k+1}) - \xi_{k-1}\theta_\varepsilon(v_{k-1} - v_k) \Big) w_k \\ & + \sum_{k=1}^N \int_\Gamma \Big(\eta_k \theta_\varepsilon(v_k - v_{k+1}) - \eta_{k-1}\theta_\varepsilon(v_{k-1} - v_k) \Big) w_k \end{aligned} \tag{2.9}$$

The problem (2.8) is equivalent to the semilinear variational problem

$$\begin{cases} \boldsymbol{u}^{\boldsymbol{\varepsilon}} \in \left[H^1(\Omega)\right]^N : \\ \displaystyle\int_\Omega \left(a(\boldsymbol{u}^{\boldsymbol{\varepsilon}}, \boldsymbol{v}) + c\, \boldsymbol{u}^{\boldsymbol{\varepsilon}} \cdot \boldsymbol{v} \right) + \int_\Gamma b\, \boldsymbol{u}^{\boldsymbol{\varepsilon}} \cdot \boldsymbol{v} + \langle \Psi_\varepsilon(\boldsymbol{u}^{\boldsymbol{\varepsilon}}), \boldsymbol{v} \rangle \\ \qquad\qquad = \displaystyle\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} + \int_\Gamma \boldsymbol{g} \cdot \boldsymbol{v}, \qquad \forall \boldsymbol{v} \in \left[H^1(\Omega)\right]^N \end{cases} \tag{2.10}$$

and this problem has a unique solution, by standard monotone methods.

Since

$$A\boldsymbol{u}^{\boldsymbol{\varepsilon}} = \boldsymbol{f} - \big(\xi_k \theta_\varepsilon(u_k^\varepsilon - u_{k+1}^\varepsilon) - \xi_{k-1}\theta_\varepsilon(u_{k-1}^\varepsilon - u_k^\varepsilon) \big)_{k=1,\ldots,N},$$

$-1 \leq \theta_\varepsilon \leq 0$ and $\boldsymbol{f}, \boldsymbol{\xi} \in [L^p(\Omega)]^N$, it follows that $\{A\boldsymbol{u}^{\boldsymbol{\varepsilon}} : 0 < \varepsilon < 1\}$ belongs to a bounded subset of $[L^p(\Omega)]^N$. Analogously, after integration by parts, the set $\{B\boldsymbol{u}^{\boldsymbol{\varepsilon}} : 0 < \varepsilon < 1\}$ is bounded in $[L^q(\Gamma)]^N$. □

Proposition 2.4. *Under the assumptions (1.3) and (1.4), let $\boldsymbol{u}^{\boldsymbol{\varepsilon}}$ be the solution of problem (2.8) and $\boldsymbol{u}$ the solution of the variational inequality (1.5). Then*

$$u_k^\varepsilon \leq u_{k-1}^\varepsilon + \varepsilon, \qquad k = 2, \dots, N, \tag{2.11}$$

and, when $\varepsilon \to 0$,

$$\boldsymbol{u}^{\boldsymbol{\varepsilon}} \longrightarrow \boldsymbol{u} \qquad \text{in } \left[H^1(\Omega)\right]^N,$$

$$A\boldsymbol{u}^{\boldsymbol{\varepsilon}} \rightharpoonup A\boldsymbol{u} \text{ in } [L^p(\Omega)]^N\text{-weak}, \qquad B\boldsymbol{u}^{\boldsymbol{\varepsilon}} \rightharpoonup B\boldsymbol{u} \text{ in } [L^q(\Gamma)]^N\text{-weak}.$$

Proof. We begin noticing that,

$$\xi_k \geq 0 \ \ (k \geq 1), \qquad \big(\xi_{k-1} - \xi_{k-2}\big) - \big(\xi_k - \xi_{k-1}\big) = f_k - f_{k-1} \ \ (k \geq 2),$$

$$\eta_k \geq 0 \ \ (k \geq 1), \qquad \big(\eta_{k-1} - \eta_{k-2}\big) - \big(\eta_k - \eta_{k-1}\big) = g_k - g_{k-1} \ \ (k \geq 2).$$

To prove (2.11), we multiply the $k-$th equation of (2.8) by $(u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+$ and integrate on Ω. Using that $\theta_\varepsilon(u_{k-1}^\varepsilon - u_k^\varepsilon)(u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+ = -(u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+$ and $\theta_\varepsilon(u_k^\varepsilon - u_{k+1}^\varepsilon) \geq -1$, we obtain

$$\begin{aligned}\int_\Omega A\, u_k^\varepsilon (u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+ \quad \leq \quad & \int_\Omega [f_k + \xi_k - \xi_{k-1}]\, (u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+ \\ & + \int_\Gamma [g_k + \eta_k - \eta_{k-1}]\, (u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+.\end{aligned} \tag{2.12}$$

With similar arguments, if we multiply, for $k \geq 2$, the $(\,k-1)-$th equation of (2.8) by $(u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+$ and integrate on Ω we obtain,

$$\begin{aligned}\int_\Omega A\, u_{k-1}^\varepsilon (u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+ \quad \geq \quad & \int_\Omega [f_{k-1} + \xi_{k-1} - \xi_{k-2}]\, (u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+ \\ & + \int_\Gamma [g_{k-1} + \eta_{k-1} - \eta_{k-2}]\, (u_k^\varepsilon - u_{k-1}^\varepsilon - \varepsilon)^+.\end{aligned} \tag{2.13}$$

Subtracting equation (2.13) from (2.12), using the assumptions (1.3), the conclusion (2.11) follows.

The strong convergence in $\left[H^1(\Omega)\right]^N$ of $\boldsymbol{u}^{\boldsymbol{\varepsilon}}$ to the solution $\boldsymbol{u}$ of the variational inequality (1.5), when $\varepsilon \to 0$, follows by a standard argument.

The uniform boundedeness of $\{A\boldsymbol{u}^{\boldsymbol{\varepsilon}} : 0 < \varepsilon < 1\}$ in $[L^p(\Omega)]^N$ implies the weak convergence of $A\boldsymbol{u}^{\boldsymbol{\varepsilon}}$ to $A\boldsymbol{u}$ in $[L^p(\Omega)]^N$, and, analogously, the boundedeness of $\{B\boldsymbol{u}^{\boldsymbol{\varepsilon}} : 0 < \varepsilon < 1\}$ in $[L^q(\Gamma)]^N$ implies the weak convergence of $B\boldsymbol{u}^{\boldsymbol{\varepsilon}}$ to $B\boldsymbol{u}$ in $[L^q(\Gamma)]^N$. □

We are now able to prove the following result:

Theorem 2.5. *Under the assumptions* (1.3) *and* (1.4)*, the solution* $\boldsymbol{u}$ *of the problem (1.5) satisfies the following Lewy-Stampacchia type inequalities*

$$
\tag{2.14}
\left.\begin{array}{rcccl}
f_1 & \leq & A\,u_1 & \leq & f_1 \vee \cdots \vee f_N \\
f_1 \wedge f_2 & \leq & A\,u_2 & \leq & f_2 \vee \cdots \vee f_N \\
 & & \vdots & & \\
f_1 \wedge \cdots \wedge f_{N-1} & \leq & A\,u_{N-1} & \leq & f_{N-1} \vee f_N \\
f_1 \wedge \cdots \wedge f_N & \leq & A\,u_N & \leq & f_N
\end{array}\right\} \quad \textit{a.e. in } \Omega
$$

and

$$
\tag{2.15}
\left.\begin{array}{rcccl}
g_1 & \leq & B\,u_1 & \leq & g_1 \vee \cdots \vee g_N \\
g_1 \wedge g_2 & \leq & B\,u_2 & \leq & g_2 \vee \cdots \vee g_N \\
 & & \vdots & & \\
g_1 \wedge \cdots \wedge g_{N-1} & \leq & B\,u_{N-1} & \leq & g_{N-1} \vee g_N \\
g_1 \wedge \cdots \wedge g_N & \leq & B\,u_N & \leq & g_N
\end{array}\right\} \quad \textit{a.e. on } \Gamma.
$$

Proof. If $(v, u_2, \ldots, u_N) \in \mathbb{K}_N$, with $v \in \mathbb{K}_{u_2}$, we see that $u_1 \in \mathbb{K}_{u_2}$ solves the variational inequality (1.5) with $f = f_1$. Observing that $Au_2 \in L^p(\Omega)$ and that $Bu_2 \in L^q(\Gamma)$, by (2.3) and (2.4) we have

$$f_1 \leq A\,u_1 \leq f_1 \vee A\,u_2 \quad \text{a.e. in } \Omega$$

$$g_1 \leq B\,u_1 \leq g_1 \vee B\,u_2 \quad \text{a.e. in } \Gamma.$$

Since $u_k \in \mathbb{K}_{u_{k+1}}^{u_{k-1}}$ solves the two obstacles problem (2.1) with $f = f_k$, $k = 2, \ldots, N-1$, and satisfies, by (2.3) and (2.4),

$$f_k \wedge A\,u_{k-1} \leq A\,u_k \leq f_k \vee A\,u_{k+1} \quad \text{a.e. in } \Omega,$$

$$g_k \wedge B\,u_{k-1} \leq B\,u_k \leq g_k \vee B\,u_{k+1} \quad \text{a.e. in } \Gamma.$$

As $u_N \in \mathbb{K}^{u_{N-1}}$ satisfies

$$f_N \wedge A\,u_{N-1} \leq A\,u_N \leq f_N \quad \text{a.e. on } \Omega,$$

$$g_N \wedge B\,u_{N-1} \leq B\,u_N \leq g_N \quad \text{a.e. on } \Gamma,$$

(2.14) and (2.15) are easily obtained by simple iterations. □

Remark 2.6. The Lewy-Stamppachia inequalities appeared first in [6] for the obstacle problem with Dirichlet boundary conditions and were extended to the Neumann case in [5] (see also [9] and [8]).

From (2.14) and (2.15) the following corollary is immediate:

Corollary 2.7. *Let* $\boldsymbol{u}$ *be the solution of the variational inequality (1.5). We have*

$$\textit{if } \boldsymbol{f} = (f, \ldots, f), \ \textit{then } A\boldsymbol{u} = \boldsymbol{f} \ \textit{in } \Omega, \qquad \textit{if } \boldsymbol{g} = (g, \ldots, g), \ \textit{then } B\boldsymbol{u} = \boldsymbol{g} \ \textit{on } \Gamma.$$

From the linear elliptic regularity theory (see [4] or [8], for instance) we have

Corollary 2.8. *Under the assumptions (1.3) and (1.4), the solution* $\boldsymbol{u}$ *of (1.5) is in* $\left[C^{0,\alpha}(\overline{\Omega})\right]^N$, *for some* $0<\alpha<1$. *Besides that, if* $a_{ij}\in C^{0,1}(\overline{\Omega})$ *then* $\boldsymbol{u}\in \left[W^{2,p}_{loc}(\Omega)\right]^N$ *and* $\boldsymbol{u}\in\left[C^{1,\beta}(\Omega)\right]^N$ *if* $0<\beta=1-\frac{d}{p}<1$; *if in addition* $\Gamma\in C^{1,1}$, $b\in C^{0,1}(\Gamma)$ *and* $\boldsymbol{f}\in\left[L^2(\Omega)\right]^N$, $\boldsymbol{g}\in\left[L^2(\Gamma)\right]^N$ *then* $\boldsymbol{u}\in\left[W^{3/2,2}(\Omega)\right]^N$; *finally, if also* $g_1=\cdots=g_N\in W^{1-\frac{1}{p},p}(\Gamma)$, *then* $\boldsymbol{u}\in\left[W^{2,p}(\Omega)\right]^N$.

3. The stability of the coincidence sets

Let $\boldsymbol{u_n}$ be the solution of the N-membranes problem (1.5), under the assumptions (1.3), with given data $\boldsymbol{f_n}$ and $\boldsymbol{g_n}$ satisfying (1.4). Assuming that $\boldsymbol{f_n}$ converges to $\boldsymbol{f}$ in $[L^p(\Omega)]^N$ and that $\boldsymbol{g_n}$ converges to $\boldsymbol{g}$ in $[L^q(\Gamma)]^N$, we shall extend now the following stability result in $L^s(\Omega)$ $(1\le s<\infty)$ of [1] for the corresponding coincidence sets (defined in (1.11)),

$$\chi_{\{u^n_k=\cdots=u^n_l\}}\xrightarrow[n]{}\chi_{\{u_k=\cdots=u_l\}},\qquad\text{for } 1\le k<l\le N.$$

Recalling the inequalities (2.14), $A\boldsymbol{u}=\boldsymbol{F}$ a.e. in Ω, for some function $\boldsymbol{F}\in[L^p(\Omega)]^N$, as in Lemma 2 of [8], we have

$$Au_k=Au_{k+1}\qquad\text{a.e. in } \{x\in\Omega: u_k(x)=u_{k+1}(x)\}$$

and so we can characterize a.e. in Ω each F_k in terms of f_l and the characteristic functions $\chi_{\{u_r=\cdots=u_s\}}$, $1\le l\le N$, $1\le r<s\le N$.

In what follows, we use, as before, the convention, $u_0=+\infty$ and $u_{N+1}=-\infty$. We define the following sets

$$\Theta_{k,l}=\{x\in\Omega: u_{k-1}(x)>u_k(x)=\cdots=u_l(x)>u_{l+1}(x)\},\tag{3.1}$$

the sets of contact of exactly the membranes $u_k,\dots,u_l$.

Proposition 3.1. *If* $k,l\in\mathbb{N}$ *are such that* $1\le k\le l\le N$ *, we have*

1. $Au_r=\begin{cases}\langle f\rangle_{k,l} & \textit{a.e. in } \Theta_{k,l} \quad \textit{if } r\in\{k,\dots,l\},\\ f_r & \textit{a.e. in } \Theta_{k,l}\quad \textit{if } r\notin\{k,\dots,l\}.\end{cases}$
2. *If* $k<l$ *then for all* $r\in\{k,\dots,l\}$ $\langle f\rangle_{r+1,l}\ge\langle f\rangle_{k,r}$ *a.e. in* $\Theta_{k,l}$.

Proof. Because of the regularity result $A\boldsymbol{u}\in[L^p(\Omega)]^N$, the proof of this proposition is the same as for the case with boundary Dirichlet condition, done in [1], since it was done locally at a.e. point $x\in\Omega$. □

Remark 3.2. It is well known that a necessary condition for existing contact in the case of two membranes u_1 and u_2, subject to external forces f_1 and f_2 respectively, is that $f_2\ge f_1$. Depending on the boundary conditions, this condition may be (or not) sufficient for contact.

We would like to emphasize that condition 2. of the preceding proposition is a necessary condition for the first $r-k$ membranes $(k<r\le l)$ to be in contact

with the other $l-r+1$ membranes. We can interpret physically the condition 2. by regarding the first $r-k$ membranes as one membrane where a force with the intensity of the average of the forces $f_k,\dots,f_r$ is applied and all the other $l-r+1$ as another one where it was applied a force with the intensity equal to the average of the remaining forces $f_{r+1},\dots,f_l$.

As for the boundary Dirichlet condition case, we may characterize the variational inequality (1.5) as a system of N equations, coupled through the characteristic functions of the coincidence sets $I_{k,l}$. In (1.13) we presented the system for $N=3$, containing as a special case $N=2$. The next theorem presents the general case.

Theorem 3.3. *Under the assumptions (1.3), let $\boldsymbol{u}$ be the solution of the problem (1.5) with data $\boldsymbol{f}$ and $\boldsymbol{g}$ satisfying (1.4). Then*

$$Au_r = f_r + \sum_{1\le k<l\le N,\ k\le r\le l} b_r^{k,l}\,\chi_{k,l} \quad a.e.\ in\ \Omega, \tag{3.2}$$

where

$$b_r^{k,l}[f] = \begin{cases} \langle f\rangle_{k,l} - \langle f\rangle_{k,l-1} & \text{if } r=l \\ \langle f\rangle_{k,l} - \langle f\rangle_{k+1,l} & \text{if } r=k \\ \frac{2}{(l-k)(l-k+1)}\left(\langle f\rangle_{k+1,l-1} - \frac{1}{2}(f_k+f_l)\right) & \text{if } k<r<l. \end{cases}$$

Also exactly as in [1], using the variational convergence $\boldsymbol{u_n} \longrightarrow \boldsymbol{u}$ in $\left[H^1(\Omega)\right]^N$, we may prove the continuous dependence of the coincidence sets with respect to the external data.

Theorem 3.4. *Assuming (1.3) and given $n\in\mathbb{N}$, let $\boldsymbol{u_n}$ denote the solution of problem* (1.5) *with given data $\boldsymbol{f_n}\in[L^p(\Omega)]^N$, $\boldsymbol{g_n}\in[L^q(\Gamma)]^N$, with p,q as in* (1.4)*.*

Suppose that

$$\boldsymbol{f_n} \xrightarrow[n]{} \boldsymbol{f} \ \ in\ \ [L^p(\Omega)]^N, \qquad \boldsymbol{g_n} \xrightarrow[n]{} \boldsymbol{g} \ \ in\ \ [L^q(\Gamma)]^N.$$

Then

$$\boldsymbol{u_n} \xrightarrow[n]{} \boldsymbol{u} \ \ in\ \ \left[H^1(\Omega)\right]^N. \tag{3.3}$$

If, in addition, the limit forces satisfy

$$\langle f\rangle_{k,r} \ne \langle f\rangle_{r+1,l} \qquad for\ all\ k,r,l\in\{1,\dots,N\}\ with\ k\le r<l, \tag{3.4}$$

then, for any $1\le s<\infty$, $\forall\, k,l\in\{1,\dots,N\}$, $k<l$,

$$\chi_{\{u_k^n=\cdots=u_l^n\}} \xrightarrow[n]{} \chi_{\{u_k=\cdots=u_l\}} \qquad in\ \ L^s(\Omega). \tag{3.5}$$

Remark 3.5. The condition (3.4) for the stability of the coincidence sets for $N=2$ is simply $f_2\ne f_1$ and for $N=3$, the condition (1.12) (see [2] for a direct proof).

Remark 3.6. It would be interesting to prove a condition analogous to the system (3.2) for the boundary operator B (under additional regularity of the solution $\boldsymbol{u}$), i.e., to find sufficient conditions for some coefficients $\gamma_r^{j,k}$ involving the averages $\langle g\rangle_{k,l}$ such that, if $\hat{I}_{k,l} = \{x \in \Gamma : u_k(x) = \cdots = u_l(x)\}$, then

$$Bu_r = g_r + \sum_{1\le k<l\le N,\ k\le r\le l} \gamma_r^{k,l}\ \chi_{\hat{I}_{k,l}} \quad \text{a.e. on } \Gamma.$$

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