Steady-state Bingham flow with temperature dependent nonlocal parameters and friction

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Abstract. We consider Bingham incompressible flows with temperature dependent viscosity and plasticity threshold and with mixed boundary conditions, including a friction type boundary condition. The coupled system of motion and energy steady-state equations may be formulated through a variational inequality for the velocity and variational methods provide a weak solution to the model. In the asymptotic limit case of a high thermal conductivity, the temperature becames a constant solving an implicit total energy equation involving the viscosity function, the plasticity threshold and the friction yield coefficient. The limit model corresponds to a steady-state Bingham flow with nonlocal parameters, which has therefore at least one solution.

1. Introduction

In the sixties, Ladyzhenskaya [8] proposed a modified Navier-Stokes system with nonlocal viscosity. In [5], the authors proved that the nonlocal model, as well as other nonlocal non-Newtonian models, can be obtained as an asymptotic limit case of a very large thermal conductivity when the viscosities depend on temperature. In the present work, we extend some of those models for the nonlocal Bingham flow when the friction behavior on a part of the boundary is also taken into account. The principal difficulty is that the quadratic term due to the energy dissipation arising in the right hand side of the heat equation leads to the L^1-analysis of the partial differential equation. The new feature in the limit model is due to a Fourier type boundary condition, and consists in the appearance of a nonlocal energy term on the boundary part where friction is taken into account.

The Bingham viscoplastic fluid does not flow as a fluid unless the stress tensor achieves at least some critical shear stress η (the plasticity threshold):

$$D(\mathbf{u}) = 0 \quad \text{if} \quad |\tau| \leq \eta \tag{1.1}$$

$$D(\mathbf{u}) = \frac{|\tau| - \eta}{\mu|\tau|}\tau \quad \text{if} \quad |\tau| > \eta \tag{1.2}$$

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where $\mathbf{u}$ is the velocity vector, $D(\mathbf{u}) = (\nabla\mathbf{u} + (\nabla\mathbf{u})^T)/2$ the symmetric part of the gradient of the velocity vector, μ the viscosity and τ the deviator tensor of the Cauchy stress tensor σ, that is, $\sigma = -pI + \tau$ where p denotes the pressure and I is the identity matrix. The law (1.1)-(1.2) is an inverse form of the constitutive law [6]

$$\tau = \mu(\theta)D(\mathbf{u}) + \eta(\theta)\frac{D(\mathbf{u})}{|D(\mathbf{u})|} \text{ if } |D(\mathbf{u})| \neq 0$$

$$|\tau| \leq \eta(\theta) \text{ if and only if } |D(\mathbf{u})| = 0$$

considering the viscosity and the plasticity threshold dependent on the temperature θ, and $|D(\mathbf{u})| = (D_{ij}(\mathbf{u})D_{ij}(\mathbf{u}))^{1/2}$, with the convention on implicit summation over repeated indices.

Here, let Ω be a bounded open subset of $\mathbb{R}^n (n = 2, 3)$ with Lipschitz continuous boundary $\partial\Omega$, which is assumed to consist of two disjoint parts Γ_0 and Γ such that $\partial\Omega = \bar{\Gamma}_0 \cup \bar{\Gamma}$ and $\text{meas}(\Gamma_0) > 0$. The governing equations to the Bingham incompressible thermal flow at steady-state are given by

$$(\mathbf{u}\cdot\nabla)\mathbf{u} - \nabla\cdot\tau = -\nabla p + \mathbf{f} \text{ in } \Omega; \tag{1.3}$$

$$\nabla\cdot\mathbf{u} = \sum_{i=1}^{n}\frac{\partial u_i}{\partial x_i} = 0 \text{ in } \Omega; \tag{1.4}$$

$$\mathbf{u}\cdot\nabla\theta - \kappa\Delta\theta = \tau : D(\mathbf{u}) - \alpha\theta \text{ in } \Omega, \qquad (\alpha \geq 0), \tag{1.5}$$

where the density and the specific heat are assumed equal to one, $\mathbf{f}$ denotes the external forces, and κ is the thermal conductivity. Note that we admit a possible external heat source proportional to the temperature if $\alpha > 0$, in addition to the dissipation energy factor $\tau : D(\mathbf{u})$.

We introduce a thermal friction law on the part Γ of the boundary, keeping the no-slip condition on the other part Γ_0:

$$\text{on } \Gamma_0: \quad \mathbf{u} = 0 \tag{1.6}$$

$$\text{on } \Gamma: \quad u_N = 0 \text{ and} \tag{1.7}$$

$$|\sigma_T| < \nu(\theta) \Rightarrow \mathbf{u}_T = 0 \tag{1.8}$$

$$|\sigma_T| = \nu(\theta) \Rightarrow \exists\lambda \geq 0,\ \mathbf{u}_T = -\lambda\sigma_T. \tag{1.9}$$

Here the tangential and normal velocities and the components of the tangential stress tensor are given, respectively, by

$$\mathbf{u}_T = \mathbf{u} - u_N\mathbf{n}, \quad u_N = u_i n_i, \quad \sigma_{Ti} = \sigma_{ij}n_j - \sigma_N n_i$$

where $\mathbf{n} = (n_i)$ denotes the unit outward normal to $\partial\Omega$. In (1.8)-(1.9), we assume a temperature dependent function ν, $\nu \geq 0$, to represent the friction yield coefficient (see, for instance, [6] for solid-solid interface or [2, 3, 4] for liquid-solid interface).

Finally we consider a homogeneous Neumann boundary condition

$$\frac{\partial\theta}{\partial n} = 0 \text{ on } \Gamma_0, \tag{1.10}$$

and the Fourier boundary condition

$$\kappa\frac{\partial\theta}{\partial n}+\beta\theta=\nu(\theta)|\mathbf{u}_T| \text{ on } \Gamma, \quad (\beta\geq 0). \tag{1.11}$$

In the framework of Lebesgue and Sobolev spaces with $W^{1,2}(\Omega)=H^1(\Omega)$, we introduce

$$\begin{aligned}\mathcal{V} &= \{\mathbf{v}\in(C^\infty(\Omega))^n:\ \nabla\cdot\mathbf{v}=0 \text{ in } \Omega\};\\ H_s &= \{\mathbf{v}\in(L^s(\Omega))^n:\ \nabla\cdot\mathbf{v}=0 \text{ in } \Omega,\ v_N=0 \text{ on } \partial\Omega\}, \qquad (s>1);\\ V &= \{\mathbf{v}\in(H^1(\Omega))^n:\ \nabla\cdot\mathbf{v}=0 \text{ in } \Omega,\ \mathbf{v}=0 \text{ on } \Gamma_0,\ v_N=0 \text{ on } \Gamma\},\end{aligned}$$

endowed with the standard norm

$$\|\mathbf{v}\|_V=\|D(\mathbf{v})\|_{2,\Omega}=\|D(\mathbf{v})\|_{L^2(\Omega)}.$$

For fixed $\kappa>0$, we formulate the problem (1.1)-(1.11) in variational form [6]: find a weak solution $(\mathbf{u},\theta)\in V\times W^{1,q}(\Omega)$, for $1<q<n/(n-1)$, such that,

$$\int_\Omega\{\mu(\theta)D(\mathbf{u})-\mathbf{u}\otimes\mathbf{u}\}:D(\mathbf{v}-\mathbf{u})dx+J(\theta,\mathbf{v})-J(\theta,\mathbf{u})\geq \tag{1.12}$$
$$\geq\int_\Omega\mathbf{f}\cdot(\mathbf{v}-\mathbf{u})dx, \quad \forall\mathbf{v}\in V;$$

$$\kappa\int_\Omega\nabla\theta\cdot\nabla\phi dx-\int_\Omega\theta\mathbf{u}\cdot\nabla\phi dx+\alpha\int_\Omega\theta\phi dx+\beta\int_\Gamma\theta\phi ds= \tag{1.13}$$
$$=\int_\Omega\{\mu(\theta)|D(\mathbf{u})|^2+\eta(\theta)|D(\mathbf{u})|\}\phi dx+\int_\Gamma\nu(\theta)|\mathbf{u}_T|\phi ds, \quad \forall\phi\in W^{1,q'}(\Omega);$$

where $J:W^{1,1}(\Omega)\times V\to\mathbb{R}_0^+$ is defined by

$$J(\theta,\mathbf{v})=\int_\Omega\eta(\theta)|D(\mathbf{v})|dx+\int_\Gamma\nu(\theta)|\mathbf{v}_T|ds.$$

The main idea is to pass to the limit on κ ($\kappa\to+\infty$) in order to reformulate the local system (1.12)-(1.13) into a nonlocal problem with constant parameters for the viscosity, the plasticity threshold and the friction yield coefficient calculated at the constant homogenized temperature, which is implicitly given through a scalar equation. We notice that the argument used in this work is applicable to the Newtonian as well as non-Newtonian fluids, as shown in [5].

2. The main result

Let us state the weak nonlocal formulation to the problem (1.1)-(1.4) and (1.6)-(1.9) corresponding formally to the limit model $\kappa=\infty$.

PROBLEM. Find $(\mathbf{u},\Theta)\in V\times\mathbb{R}$ satisfying

$$\mu(\Theta)\int_\Omega D(\mathbf{u}):D(\mathbf{v}-\mathbf{u})dx-\int_\Omega \mathbf{u}\otimes\mathbf{u}:D(\mathbf{v}-\mathbf{u})dx+ \\ +\eta(\Theta)\int_\Omega\{|D(\mathbf{v})|-|D(\mathbf{u})|\}dx+\nu(\Theta)\int_\Gamma\{|\mathbf{v}_T|-|\mathbf{u}_T|\}ds \\ \geq\int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})dx,\quad \forall \mathbf{v}\in V, \tag{2.1}$$

where Θ is a solution to the implicit scalar equation

$$(\alpha|\Omega|+\beta|\Gamma|)\,\Theta=\mu(\Theta)\int_\Omega|D(\mathbf{u})|^2dx+\eta(\Theta)\int_\Omega|D(\mathbf{u})|dx+\nu(\Theta)\int_\Gamma|\mathbf{u}_T|ds. \tag{2.2}$$

Remark 2.1. Notice that the antisymmetry of the convective term $\int_\Omega(\mathbf{u}\cdot\nabla)\mathbf{u}\cdot\mathbf{v}$ is valid by the incompressibility property (1.4) and the boundary condition $u_N=0$ on $\partial\Omega$ given by (1.6)-(1.7).

We assume

$$\mu\in C^0(\mathbb{R}):\quad \exists\mu_*,\mu^*>0,\quad \mu_*\leq\mu(s)\leq\mu^*,\quad \forall s\in\mathbb{R}; \tag{2.3}$$

$$\eta\in C^0(\mathbb{R}):\quad \exists\eta^*>0,\quad 0\leq\eta(s)\leq\eta^*,\quad \forall s\in\mathbb{R}; \tag{2.4}$$

$$\nu\in C^0(\mathbb{R}):\quad \exists\nu^*>0,\quad 0\leq\nu(s)\leq\nu^*,\quad \forall s\in\mathbb{R}; \tag{2.5}$$

$$\alpha,\beta\geq 0:\quad \alpha+\beta>0; \tag{2.6}$$

$$\mathbf{f}\in V'. \tag{2.7}$$

The main result of this work is the following theorem.

Theorem 2.2. *Under the assumptions (2.3)-(2.7), there exists $(\mathbf{u},\Theta)\in V\times\mathbb{R}$ a solution to the problem (2.1)-(2.2), which can be obtained as a limit in $V\times W^{1,q}(\Omega)$, $1<q<n/(n-1)$, as $\kappa\to\infty$ of solutions $(\mathbf{u}_\kappa,\theta_\kappa)$ of (1.12)-(1.13).*

3. Auxiliary existence results

The following propositions are essential in the proof of the theorem 2.2.

Proposition 3.1. *For every $\mathbf{w}\in H_s$, $s\geq n$, and $\xi\in W^{1,1}(\Omega)$ there exists a unique solution $\mathbf{u}=\mathbf{u}(\mathbf{w},\xi)\in V$ to the variational inequality*

$$\int_\Omega\{\mu(\xi)D(\mathbf{u})-\mathbf{w}\otimes\mathbf{u}\}:D(\mathbf{v}-\mathbf{u})dx+J(\xi,\mathbf{v})-J(\xi,\mathbf{u})\geq \\ \geq\int_\Omega \mathbf{f}\cdot(\mathbf{v}-\mathbf{u})dx,\quad\forall\mathbf{v}\in V, \tag{3.1}$$

and it satisfies the estimate

$$\|\mathbf{u}\|_V\leq\frac{\|\mathbf{f}\|_{V'}}{\mu_*}. \tag{3.2}$$

Moreover, if $\mathbf{w}_m$ *and* ξ_m *are sequences in* H_s *and* $W^{1,1}(\Omega)$*, respectively, such that* $\mathbf{w}_m \to \mathbf{w}$ *in* H_s*,* $\xi_m \to \xi$ *in* $L^1(\Omega)$ *and* $\xi_m \to \xi$ *in* $L^1(\Gamma)$*, and* $\mathbf{u}_m = \mathbf{u}(\mathbf{w}_m, \xi_m)$ *are the corresponding solutions satisfying (3.1), then there exists* $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi) \in V$ *the solution to (3.1) such that* $\mathbf{u}_m \to \mathbf{u}$ *in* V*.*

PROOF. The existence and uniqueness of the solution are consequences of classical results (for instance, see [9]) on variational inequalities with convex continuous functionals. The estimate (3.2) follows by choosing $\mathbf{v} = 0$ as a test function in (3.1).

Let $\mathbf{w}_m, \xi_m, \mathbf{u}_m = \mathbf{u}(\mathbf{w}_m, \xi_m)$ be sequences in the conditions of the Proposition. From estimate (3.2) we have $\mathbf{u}_m \rightharpoonup \mathbf{u}$ in V for a subsequence of $\mathbf{u}_m$, still denoted by $\mathbf{u}_m$, and consequently

$$\mathbf{u}_m \to \mathbf{u} \text{ in } H_s, \qquad \text{for } s < 2n/(n-2) \tag{3.3}$$

$$\text{and in } L^r(\Gamma), \qquad \text{for } r < 2(n-1)/(n-2). \tag{3.4}$$

The convective term $\mathbf{w}_m \otimes \mathbf{u}_m : D(\mathbf{v})$ easily passes to the limit in m. Since $\xi_m \to \xi$ a.e. in Ω and on Γ, the functions μ, η and ν are continuous, and due to the sequential weak lower semicontinuity of the continuous and convex functional J, we obtain as in [7]

$$\int_\Omega \{\mu(\xi) D(\mathbf{u}) - \mathbf{w} \otimes \mathbf{u}\} : D(\mathbf{v}) dx + J(\xi, \mathbf{v}) - \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \mathbf{u}) dx \geq$$

$$\geq \liminf_{m \to +\infty} \int_\Omega \mu(\xi_m) |D(\mathbf{u}_m)|^2 dx + \liminf_{m \to +\infty} J(\xi_m, \mathbf{u}_m) \geq \int_\Omega \mu(\xi) |D(\mathbf{u})|^2 dx + J(\xi, \mathbf{u})$$

So $\mathbf{u}$ is a solution to (3.1), and its uniqueness is due to the standard variational argument.

Choosing $\mathbf{v} = (\mathbf{u}_m + \mathbf{u})/2$ as a test function in (3.1) for the solutions $\mathbf{u}_m$ and $\mathbf{u}$, and subtrating the obtained inequalities, it results

$$\mu_* \int_\Omega |D(\mathbf{u}_m - \mathbf{u})|^2 dx + \int_\Omega \{\eta(\xi_m) - \eta(\xi)\} |D(\mathbf{u}_m)| dx + \int_\Gamma \{\nu(\xi_m) - \nu(\xi)\} |\mathbf{u}_m| ds$$

$$\leq \int_\Omega (\mathbf{w} - \mathbf{w}_m) \otimes \mathbf{u}_m : D(\mathbf{u}) dx + \int_\Omega \{\mu(\xi) - \mu(\xi_m)\} D(\mathbf{u}) : D(\mathbf{u}_m - \mathbf{u}) dx +$$

$$+ \int_\Omega \{\eta(\xi_m) - \eta(\xi)\} |D(\mathbf{u})| dx + \int_\Gamma \{\nu(\xi_m) - \nu(\xi)\} |\mathbf{u}_T| ds.$$

Applying Fatou lemma to the second and third terms on the left hand side of the above inequality and using Lebesgue theorem to the convergences on the right hand side, the required strong convergence holds. □

Proposition 3.2. *Let* $\mathbf{u} = \mathbf{u}(\mathbf{w}, \xi)$ *be the solution given by Proposition 3.1. Then there exists* $\theta = \theta(\mathbf{u}, \xi) \in W^{1,q}(\Omega)$ *a solution to the variational problem*

$$\int_\Omega (\kappa \nabla \theta - \theta \mathbf{u}) \cdot \nabla \phi dx + \alpha \int_\Omega \theta \phi dx + \beta \int_\Gamma \theta \phi ds = \tag{3.5}$$

$$= \int_\Omega \{\mu(\xi) |D(\mathbf{u})|^2 + \eta(\xi) |D(\mathbf{u})|\} \phi dx + \int_\Gamma \nu(\xi) |\mathbf{u}_T| \phi ds, \quad \forall \phi \in W^{1,q'}(\Omega),$$

that satisfies the estimate

$$\alpha\|\theta\|_{q,\Omega} + \beta\|\theta\|_{q,\Gamma} + \sqrt{\kappa}\|\nabla\theta\|_{q,\Omega} \leq \mathcal{F}\left(\|\mathbf{f}\|_{V'}, \frac{\mu^*}{\mu_*}, \eta^*, \nu^*\right) \tag{3.6}$$

for an arbitrary $1 < q < n/(n-1)$, *and* $\mathcal{F}$ *is a positive function. Moreover, let* $\mathbf{w}_m$ *and* ξ_m *be sequences in* H_s *and* $W^{1,1}(\Omega)$, *respectively, such that* $\mathbf{w}_m \to \mathbf{w}$ *in* H_s, $\xi_m \to \xi$ *in* $L^1(\Omega)$ *and* $\xi_m \to \xi$ *in* $L^1(\Gamma)$, *and* $\mathbf{u}_m = \mathbf{u}(\mathbf{w}_m, \xi_m)$ *be the corresponding solutions given by Proposition 3.1. If* $\theta_m = \theta(\mathbf{u}_m, \xi_m)$ *are solutions satisfying (3.5), then there exists* $\theta = \theta(\mathbf{u}, \xi)$ *a solution to (3.5) such that* $\theta_m \to \theta$ *in* $W^{1,q}(\Omega) -$ *weak,* $L^1(\Omega) -$ *strong and* $L^1(\Gamma) -$ *strong.*

Remark 3.3. In (3.5), the terms on the right hand side have sense, since $\phi \in W^{1,q'}(\Omega)) \hookrightarrow C(\bar{\Omega})$ for $q' > n$, that is, $q < n/(n-1)$, and the term $\int_\Omega \theta\mathbf{u}\cdot\nabla\phi$ has meaning for $\theta \in W^{1,q}(\Omega)$, $\mathbf{u} \in H_s$ with $s \geq n$, and $\phi \in W^{1,q'}(\Omega)$.

Proof. Let us define $F = \mu(\xi)|D(\mathbf{u})|^2 + \eta(\xi)|D(\mathbf{u})|$ and $G = \nu(\xi)|\mathbf{u}_T| \in L^r(\Omega)$ for r as in (3.4), and, for each $m \in \mathbb{N}$, take

$$F_m = \frac{mF}{m + |F|} \in L^\infty(\Omega).$$

From the Lax-Milgram theorem, there exists a unique solution $\theta_m \in H^1(\Omega)$ to the following variational problem

$$\int_\Omega (\kappa\nabla\theta_m - \theta_m\mathbf{u}) \cdot \nabla\phi dx + \alpha\int_\Omega \theta_m\phi dx + \beta\int_\Gamma \theta_m\phi ds = \tag{3.7}$$
$$= \int_\Omega F_m\phi dx + \int_\Gamma G\phi ds, \quad \forall\phi \in H^1(\Omega).$$

From L^1-data theory (see, for instance, [5] or [10]), the estimate (3.6) follows for θ_m. Indeed, choosing

$$\phi = \mathrm{sign}(\theta_m)[1 - 1/(1 + |\theta_m|)^\varsigma] \in W^{1,2}(\Omega) \cap L^\infty(\Omega), \quad \text{for } \varsigma > 0,$$

as a test function in (3.7) it follows

$$\kappa\int_\Omega \frac{\varsigma|\nabla\theta_m|^2}{(1 + |\theta_m|)^{\varsigma+1}}dx + \beta C(\varsigma)\int_\Gamma |\theta_m|ds \leq \|F\|_{1,\Omega} + \|G\|_{r,\Gamma}.$$

Arguing as in [10] and [5] we conclude, for $q < n/(n-1)$, that

$$\int_\Omega |\nabla\theta_m|^q dx \leq \left(\frac{\|F\|_{1,\Omega} + \|G\|_{r,\Gamma}}{\kappa\varsigma}\right)^{q/2} (\varepsilon\left(\int_\Omega |\theta_m|^{qn/(n-q)}\right)^{(2-q)/2} + C(\varepsilon))$$

for arbitrary $\varepsilon > 0$. If $\beta > 0$, using a Poincaré-Sobolev type inequality we obtain

$$\|\theta_m\|_{qn/(n-q),\Omega} \leq C\left(\|\nabla\theta_m\|_{q,\Omega} + \beta\int_\Gamma |\theta_m|ds\right) \leq \tag{3.8}$$
$$\leq C\sqrt{\frac{\|F\|_{1,\Omega} + \|G\|_{r,\Gamma}}{\varsigma}}(\varepsilon^{1/q}\|\theta_m\|_{qn/(n-q),\Omega}^{n(2-q)/[2(n-q)]} + C'(\varepsilon)) \quad \text{for } \kappa > 1.$$

If $\beta = 0$, the assumption (2.6) implies that $\alpha > 0$. Choosing $\phi \equiv 1$ as a test function in (3.7) we get

$$0 \leq \int_\Omega \theta_m = \int_\Omega F_m dx + \int_\Gamma G ds \leq \|F\|_{1,\Omega} + \|G\|_{1,\Gamma};$$

and instead of (3.8) we obtain

$$\|\theta_m\|_{qn/(n-q),\Omega} \leq C\|\nabla\theta_m\|_{q,\Omega} + |\Omega|^{(n-q)/(nq)}|\fint_\Omega \theta_m| \leq$$

$$\leq C\sqrt{\frac{\|F\|_{1,\Omega} + \|G\|_{r,\Gamma}}{\varsigma}}(\varepsilon^{1/q}\|\theta_m\|^{n(2-q)/[2(n-q)]}_{qn/(n-q),\Omega} + C'(\varepsilon)) \quad \text{for } \kappa > 1,$$

where C is a constant depending on Ω, and $\fint_\Omega$ denotes $\frac{1}{|\Omega|}\int_\Omega$.

Consequently, for ε sufficiently small it follows

$$\|\theta_m\|_{qn/(n-q),\Omega} \leq C \text{ (independent of } m \text{ and } \kappa),$$

and then θ_m satisfies the estimate (3.6). Thus, we can extract a subsequence of θ_m, still denoted by θ_m, such that it weakly converges to θ in $W^{1,q}(\Omega)$, where θ solves the limit problem (3.5).

Let $\mathbf{w}_m, \xi_m$ and $\mathbf{u}_m = \mathbf{u}(\mathbf{w}_m, \xi_m)$ be sequences in the conditions of Proposition 3.1, that is, $\mathbf{u}_m$ is such that $\mathbf{u}_m \to \mathbf{u}$ in V. In order to pass to the limit in (3.5) for solutions $\theta_m = \theta(\mathbf{u}_m, \xi_m)$ when m tends to infinity, from estimate (3.6), we can extract a subsequence of θ_m, still denoted by θ_m, such that it converges to θ, which is the solution to (3.5). Note that by (3.3) and $\theta_m \rightharpoonup \theta$ in $L^{qn/(n-q)}(\Omega)$ we obtain $\theta_m\mathbf{u}_m \rightharpoonup \theta\mathbf{u}$ in $L^{sqn/[qn+s(n-q)]}(\Omega) \hookrightarrow L^q(\Omega)$ for $n = 2, 3$. □

4. Proof of Theorem 2.2

This proof is divided in two parts.

4.1. Existence for the coupled system (1.12)-(1.13)

Consider the multivalued mapping $\mathcal{L}$ defined on

$$K := \{(\mathbf{w}, \xi) \in V \times W^{1,q}(\Omega) : \|\mathbf{w}\|_V \leq R_1 \text{ and } \|\xi\|_{W^{1,q}(\Omega)} \leq R_2\},$$

taking $R_1 \geq \|\mathbf{f}\|_{V'}/\mu_*$ and R_2 conveniently choosen from estimate (3.6), such that $\mathcal{L}$ applies $(\mathbf{w}, \xi)$ into the nonempty convex set $\{(\mathbf{u}, \theta)\} \subset K$, where $\mathbf{u}$ and θ are the solutions given at Propositions 3.1 and 3.2, respectively. Thus the Tychonof-Kakutani-Glicksberg fixed point theorem (see [1, pages 218-220]) garantees a solution, $(\mathbf{u}, \theta) \in \mathcal{L}(\mathbf{u}, \theta)$, to (1.12)-(1.13) still satisfying the estimates (3.2) and (3.6), provided $\mathcal{L}(\mathbf{w}, \xi)$ is a closed set and $\mathcal{L}$ is upper semicontinuous for the weak topology in $V \times W^{1,q}(\Omega)$, for $1 < q < n/(n-1)$. From the closed graph theorem [1, page 413], it remains therefore to prove that if $(\mathbf{w}_m, \xi_m) \rightharpoonup (\mathbf{w}, \xi)$ in $V \times W^{1,q}(\Omega)$ and $(\mathbf{u}_m, \theta_m) \in \mathcal{L}(\mathbf{w}_m, \xi_m)$ then

$$(\mathbf{u}_m, \theta_m) \rightharpoonup (\mathbf{u}, \theta) \in \mathcal{L}(\mathbf{w}, \xi). \tag{4.1}$$

By Rellich-Kondrachof imbeddings

$$V \hookrightarrow\hookrightarrow H_s, \qquad \text{for } n \leq s < 2n/(n-2);$$
$$W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega), \qquad \text{and} \qquad W^{1,q}(\Omega) \hookrightarrow\hookrightarrow L^1(\Gamma),$$

the final assertion (4.1) derives from Propositions 3.1 and 3.2.

4.2. Passage to the limit on κ

Let $(\mathbf{u}_\kappa, \theta_\kappa)$ be a solution to (1.12)-(1.13), corresponding to each $\kappa > 0$ and let $\kappa \to +\infty$. From the estimates (3.2) and (3.6), we can extract a subsequence of $(\mathbf{u}_\kappa, \theta_\kappa)$, still denoted by $(\mathbf{u}_\kappa, \theta_\kappa)$, satisfying

$$\nabla\theta_\kappa \to 0 \text{ in } L^q(\Omega),$$
$$\theta_\kappa \to \Theta = \text{ constant in } W^{1,q}(\Omega).$$

We can proceed as in the proof of Proposition 3.1 to get a strong convergence of $\mathbf{u}_\kappa$ to $\mathbf{u}$ in $H^1(\Omega)$. Then, we can pass to the limit (1.13) on κ ($\kappa \to +\infty$), taking $\phi \equiv 1$ to obtain (2.2). Now, taking the limit $\kappa \to +\infty$ in (1.12), it follows that the limit $\mathbf{u}$ solves the nonlocal problem (2.1). □

References

[1] C. Baiocchi and A. Capelo, *Variational and quasivariational inequalities: Applications to free boundary problems.* Wiley-Interscience, Chichester-New York, 1984.

[2] D. Cioranescu, *Sur quelques équations aux derivées partielles posées par la mécanique des milieux continus.* Thèse de Doctorat, Université Pierre et Marie Curie, Paris **6**, 1977.

[3] L. Consiglieri, Stationary solutions for a Bingham flow with nonlocal friction. In *Mathematical topics in fluid mechanics*, J.F.Rodrigues and A.Sequeira (eds), Pitman Res. Notes in Math., Longman (1992), 237-243.

[4] L. Consiglieri, A nonlocal friction problem for a class of non-Newtonian flows, *Portugaliae Mathematica* **60** :2 (2003), 237-252.

[5] L. Consiglieri and J. F. Rodrigues, *On stationary flows with energy dependent nonlocal viscosities.* Zapiski Nauchn. Seminarov POMI **295** (2003), 99–117.

[6] G. Duvaut and J. L. Lions, *Les inéquations en mécanique et en physique.* Dunod, 1972.

[7] G. Duvaut and J. L. Lions, *Transfert de chaleur dans un fluide de Bingham dont la viscosité dépend de la température.* Journal of Functional Analysis **11** (1972), 93–110.

[8] O. A. Ladyzhenskaya, *On new equations for the description of the motion of viscous incompressible fluids and on the solvability "in large" of some boundary-value problems for them.* Proceedings of Steklov Math. Inst. **102** (1967), 85–104.

[9] J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires.* Dunod et Gauthier-Villars, 1969.

[10] A. Prignet, *Conditions aux limites non homogènes pour des problèmes elliptiques avec second membre mesure.* Ann. Fac. Sci. Toulouse Math. **6** (1997), 297–318.

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